Isometry groups of k-curvature homogeneous pseudo-Riemannian manifolds

by

Peter Gilkey and Stana Nikcevic

Preprint no.: 56  2005
ISOMETRY GROUPS OF $k$-CURVATURE HOMOGENEOUS PSEUDO-RIEMANNIAN MANIFOLDS

P. GILKEY AND S. NIKCEVIĆ

Abstract. We study the isometry groups of a family of complete $p + 2$-curvature homogeneous pseudo-Riemannian metrics on $\mathbb{R}^{6+4p}$ which have neutral signature $(3 + 2p, 3 + 2p)$, and which are 0-curvature modeled on an indecomposable symmetric space.

1. Introduction

Let $\mathcal{M} := (M, g)$ be a pseudo-Riemannian manifold of signature $(p, q)$. Let $g_P := g|_{T_PM}$ (resp. $\nabla^i R_P := \nabla^i R|_{T_PM}$) be the restriction of the metric (resp. the $i$th covariant derivative of the curvature tensor) to the tangent space at $P \in M$. We define the $k$-model of $M$ at $P$ by setting:

$$\mathfrak{M}_k(M, P) := (T_PM, g_P, R_P, \ldots, \nabla^k R_P).$$

One says that $\phi : \mathfrak{M}_k(M_1, P_1) \to \mathfrak{M}_k(M_2, P_2)$ is an isomorphism from the $k$-model of $M_1$ at $P_1$ to the $k$-model of $M_2$ at $P_2$ if $\phi$ is a linear isomorphism from $T_{P_1}M_1$ to $T_{P_2}M_2$ with

$$\phi^* g_{2, P_2} = g_{1, P_1} \quad \text{and} \quad \phi^* \nabla^i_{2, P_2} R_{M_2, P_2} = \nabla^i_{1, P_1} R_{M_1, P_1} \quad \text{for} \quad 0 \leq i \leq k.$$ 

One says that $M$ is $k$-curvature homogeneous if the $k$-models $\mathfrak{M}_k(M, P)$ and $\mathfrak{M}_k(M, Q)$ are isomorphic for any $P, Q \in M$.

In the Riemannian setting ($p = 0$), Takagi [14] constructed 0-curvature homogeneous complete non-compact Riemannian manifolds; compact examples were exhibited subsequently by Ferus, Karcher, and Münzer [5]. Although many other examples have been constructed, there are no known Riemannian manifolds which are 1-curvature homogeneous but not locally homogeneous and it is natural to conjecture that any 1-curvature homogeneous Riemannian manifold is locally homogeneous.

In the Lorentzian setting ($p = 1$), curvature homogeneous manifolds which are not locally homogeneous were constructed by Cahen et al. [4]; 1-curvature homogeneous Lorentzian manifolds which are not locally homogeneous have been exhibited by Bueken and Djorić [2] and by Bueken and Vanhecke [3]. One could conjecture that a 2-curvature homogeneous Lorentzian manifold must be locally homogeneous.

It is clear that local homogeneity implies $k$-curvature homogeneity for any $k$. The following result, due to Singer [11] in the Riemannian setting and to F. Podesta and A. Spiro [10] in the general context, provides a partial converse:

**Theorem 1.1** (Singer, Podesta-Spiro). There exists an integer $k_{p, q}$ so that if $M$ is a complete simply connected pseudo-Riemannian manifold of signature $(p, q)$ which is $k_{p, q}$-curvature homogeneous, then $(M, g)$ is homogeneous.
Sekigawa, Suga, and Vanhecke [12, 13] showed any 1-affine homogeneous complete simply connected Riemannian manifold of dimension $m < 5$ is homogeneous; thus $k_{0,2} = k_{0,3} = k_{0,4} = 1$. The estimate $k_{0,m} < \frac{3}{2}m - 1$ was established by Gromov [9]. Results of [6] can be used to show $k_{p,q} \geq \min(p,q)$; we conjecture $k_{p,q} = \min(p,q) + 1$.

If $\mathcal{H}$ is a homogeneous space, let $\mathfrak{M}_k(\mathcal{H}) := \mathfrak{M}_k(\mathcal{H}, Q)$ for any point $Q \in H$; the isomorphism class of $\mathfrak{M}_k(\mathcal{H})$ is independent of the point $Q \in H$. We say that $\mathcal{M}$ is $k$-modeled on $\mathcal{H}$ and that $\mathfrak{M}_k(\mathcal{H})$ is a $k$-model for $\mathcal{M}$ if $\mathfrak{M}_k(\mathcal{H})$ and $\mathfrak{M}_k(\mathcal{M}, P)$ are isomorphic for any $P \in \mathcal{M}$.

Throughout this paper, we shall adopt the notational convention that $p \geq 1$.

In [7], we exhibited complete metrics on $\mathbb{R}^{6+4p}$ of neutral signature $(3 + 2p, 3 + 2p)$ which are $(p+2)$-curvature homogeneous, which are 0-modeled on an indecomposable symmetric space, but which are not $(p+3)$-curvature homogeneous; these examples show that the constants $k_{p,q} \rightarrow \infty$ as $(p, q) \rightarrow \infty$. The proof of Theorem 1.1 rested on a careful analysis of the isometry groups of the model spaces. In this paper, we continue our study of the manifolds introduced in [7] by examining their isometry groups and the isometry groups of their $k$-models.

We recall the definition of the metrics on $\mathbb{R}^{6+4p}$ which were introduced in [7]. We will be defining a number of tensors in this paper and, in the interests of brevity, we shall only give the non-zero components up to the usual symmetries. Let $x = (x_1, ..., x_m)$ be the usual coordinates on $\mathbb{R}^m$. Let $\{x, y, z_1, ..., z_p, \tilde{y}, \tilde{z}_1, ..., \tilde{z}_p, x^*, y^*, z_1^*, ..., z_p^*, \tilde{y}^*, \tilde{z}_1^*, ..., \tilde{z}_p^*\}$ be coordinates on $\mathbb{R}^{6+4p}$. Let $F = F(y, z_1, ..., z_p) \in C^\infty(\mathbb{R}^{p+1})$. Let

$$\mathcal{M}_{6+4p,F} := (\mathbb{R}^{6+4p}, g_{6+4p,F})$$

where $g_{6+4p,F}$ is the metric of neutral signature $(3 + 2p, 3 + 2p)$ on $\mathbb{R}^{6+4p}$ with:

$$g_{6+4p,F}(\partial_x, \partial_x) = -2\{F(y, z_1, ..., z_p) + y\tilde{y} + z_1\tilde{z}_1 + ... + z_p\tilde{z}_p\},$$

$$g_{6+4p,F}(\partial_y, \partial_y^*) = g_{6+4p,F}(\partial_{y^*}, \partial_{y^*}) = g_{6+4p,F}(\partial_{y}, \partial_{y^*}) = 1,$$

$$g_{6+4p,F}(\partial_z, \partial_{z^*}) = g_{6+4p,F}(\partial_{z^*}, \partial_z) = 1.$$

**Theorem 1.2 (Gilkey-Nikčević [7].)** Let $\mathcal{M} = \mathcal{M}_{6+4p,F}$. Then:

1. All geodesics in $\mathcal{M}$ extend for infinite time.
2. $\exp_p : T_p\mathbb{R}^{6+4p} \rightarrow \mathbb{R}^{6+4p}$ is a diffeomorphism for all $P \in \mathbb{R}^{6+4p}$.
3. $\nabla^k R(\partial_x, \partial_{y^*}, \partial_{y^*}, \partial_{y^*}) = -\frac{1}{2}(\partial_{y}, \partial_{y^*}, ..., \partial_{y^*})g_{6+4p,F}(\partial_y, \partial_x)$ are the non-zero components of $\nabla^k R$ where $\xi_i \in \{y, z_1, ..., z_p, \tilde{y}, \tilde{z}_1, ..., \tilde{z}_p\}$.
4. All scalar Weyl invariants of $\mathcal{M}$ vanish.
5. $\mathcal{M}$ is a symmetric space if and only if $F$ is at most quadratic.

**1.1. The manifolds $\mathcal{M}_{6+4p,k} = (\mathbb{R}^{6+4p}, g_{6+4p,k})$**. We can specialize this construction as follows. Let $g_{6+4p,k}$ be defined by setting $F = f_{p,k}$ where we let:

$$f_{p,0}(y, z_1, ..., z_p) := 0,$$

$$f_{p,k}(y, z_1, ..., z_p) := z_1y^2 + ... + z_ky^{k+1} \quad \text{if} \quad 1 \leq k \leq p.$$

As exceptional cases, we set:

$$f_{p,p+1}(y, z_1, ..., z_p) := z_1y^2 + ... + z_p y^{p+1} + y^{p+3},$$

$$f_{p,p+2}(y, z_1, ..., z_p) := z_1y^2 + ... + z_p y^{p+1} + e^y.$$

**Theorem 1.3 (Gilkey-Nikčević [7].)** Let $1 \leq k \leq p + 2$.

1. $\mathcal{M}_{6+4p,0}$ is an indecomposable symmetric space.
2. $\mathcal{M}_{6+4p,k}$ is an indecomposable homogeneous space which is not symmetric.
1.2. The manifolds $\mathcal{N}_{6+4p,\psi} = (\mathbb{R}^{6+4p}, g_{6+4p,\psi})$. Let $\psi = \psi(y)$ be a real analytic function of one variable such that $\psi^{(p+3)} > 0$, $\psi^{(p+4)} > 0$, and $\psi^{(p+3)} \neq ae^{bp}$.

Define a metric $g_{6+4p,\psi}$ on $\mathbb{R}^{6+4p}$ by taking $F = f_\psi$ where

$$f_\psi(y, z_1, ..., z_p) := \psi(y) + z_1 y^2 + ... + z_p y^{p+1}.$$ 

The following result shows that the geometry of a homogeneous pseudo Riemannian manifold need not determined by the $k$-model:

**Theorem 1.4** (Gilkey-Nikčević [7]). Let $0 \leq j < k \leq p + 2$.

1. $\mathcal{M}_{6+4p,k}$ is $j$-modeled on $\mathcal{M}_{6+4p,j}$; $\mathcal{M}_{6+4p,j}$ is not $k$-modeled on $\mathcal{M}_{6+4p,k}$.
2. $\mathcal{N}_{6+4p,\psi}$ is $p + 2$-curvature homogeneous and $p + 2$-modeled on $\mathcal{M}_{6+4p,p+2}$.
3. $\mathcal{N}_{6+4p,\psi}$ is not $p + 3$-curvature homogeneous and not locally homogeneous.

1.3. Isometry groups. Let $G(\mathcal{M})$ (resp. $G(\mathfrak{M}_k)$) be the isometry group of a pseudo-Riemannian manifold $\mathcal{M}$ (resp. of a k-model $\mathfrak{M}_k$). In this paper, we study the groups $G(\mathcal{M}_{6+4p,k}), G(\mathcal{N}_{6+4p,\psi})$, and $G(\mathfrak{M}_k(\mathcal{M}_{6+4p,k}, P))$ for any point $P$ of $\mathbb{R}^{6+4p}$. A byproduct of our study is the following result that shows, not surprisingly, that the symmetric space $\mathcal{M}_{6+4p,0}$ has the largest isometry group.

**Theorem 1.5.** Let $1 \leq k \leq p$. Let $n_p := (6 + 4p) + (p + 1)(3 + 2p) + (2p + 3)$.

1. $\dim \{G(\mathcal{M}_{6+4p,0})\} = n_p + (p + 1)(2p + 1)$.
2. $\dim \{G(\mathcal{M}_{6+4p,k})\} = n_p + (2p + 2) + \frac{1}{2}(2p - k)(2p - k) - 1$.
3. $\dim \{G(\mathcal{M}_{6+4p,p+1})\} = \dim \{G(\mathcal{M}_{6+4p,p})\} - 1$.
4. $\dim \{G(\mathcal{M}_{6+4p,p+2})\} = \dim \{G(\mathcal{M}_{6+4p,p+1})\} - 1$.
5. $\dim \{G(\mathcal{N}_{6+4p,\psi})\} = \dim \{G(\mathcal{M}_{6+4p,p+2})\} - 1$.

Here is a brief outline to the remainder of this paper. In Section 2, we review some results from [7]. In Section 3, we reduce the proof of Theorem 1.5 to a purely algebraic problem by showing for any $P \in \mathbb{R}^{6+4p}$ that for $0 \leq k \leq p + 2$, we have:

$$\dim \{G(\mathcal{M}_{6+4p,k})\} = 6 + 4p + \dim \{G(\mathfrak{M}_k(\mathcal{M}_{6+4p,k}, P))\},$$

$$\dim \{G(\mathcal{N}_{6+4p,\psi})\} = 5 + 4p + \dim \{G(\mathfrak{M}_{p+2}(\mathcal{M}_{6+4p,p+2}, P))\}.$$ 

In Section 4, we complete the proof by determining $\dim \{G(\mathfrak{M}_k(\mathcal{M}_{6+4p,k}, P))\}$ for $0 \leq k \leq p + 2$.

2. Models

It is convenient to work in the purely algebraic setting. Let

$$\mathfrak{M}_\nu := (V, \langle \cdot, \cdot \rangle, A^0, ..., A^\nu)$$

where $\langle \cdot, \cdot \rangle$ is a non-degenerate inner product of signature $(p, q)$ on a finite dimensional vector space $V$ of dimension $m = p + q$ and where $A^\mu \in \otimes^{4+\mu} V^*$ satisfies the appropriate symmetries of the covariant derivatives of the curvature tensor for $0 \leq \mu \leq \nu$; if $\nu = \infty$, then the sequence is infinite. We say that $\mathfrak{M}_\nu$ is a $\nu$-model for a pseudo-Riemannian manifold $\mathcal{M} = (M, g)$ if for each point $P \in M$, there is an isomorphism $\phi_P : T_PM \to V$ so that

$$\phi_P^* \langle \cdot, \cdot \rangle = g_P \quad \text{and} \quad \phi_P^* A^\mu = \nabla^\mu R_P \quad \text{for } 0 \leq \mu \leq \nu.$$ 

Clearly $\mathcal{M}$ is $\nu$-curvature homogeneous if and only if it admits a $\nu$-model.
2.1. Models for the manifolds $\mathcal{M}_{6+4p,k}$ and $\mathcal{N}_{6+4p,\psi}$. Let
\[ B = \{ X, Y, Z_1, ..., Z_p, \tilde{Y}, \tilde{Z}_1, ..., \tilde{Z}_p, X^*, Y^*, Z_1^*, ..., Z_p^*, \tilde{Y}^*, \tilde{Z}_1^*, ..., \tilde{Z}_p^* \} \]
be a basis for $\mathbb{R}^{6+4p}$. Define a hyperbolic inner-product on $\mathbb{R}^{6+4p}$ by pairing ordinary variables with the corresponding dual $\star$-variables:
\[
(X, X^*) = (Y, Y^*) = (Z_i, Z_i^*) = (\tilde{Z}_i, \tilde{Z}_i^*) = 1.
\]
Define $A^0 \in \otimes^4(\mathbb{R}^{6+4p})^\ast$ with non-zero components:
\[
A^0(X, Y, \tilde{Y}, X) = A^0(X, Z_i, \tilde{Z}_i, X) = 1.
\]
Define tensors $A^i \in \otimes^{4+i}(\mathbb{R}^{6+4p})^\ast$ for $1 \leq i \leq p$ with non-zero components:
\[
A^i(X, Y, Z_i, X; Y, ..., Y) = 1, \\
A^i(X, Y, Y, X; Z_i, ..., Y) = 1, ..., \\
A^i(X, Y, Y, X; Y, ..., Y, Z_i) = 1.
\]
Finally define $A^{p+1} \in \otimes^{5+p}(\mathbb{R}^{6+4p})^\ast$ and $A^{p+2} \in \otimes^{5+p}(\mathbb{R}^{6+4p})^\ast$ by setting
\[
A^{p+1}(X, Y, Y, X; Y, ..., Y) = 1, \\
A^{p+2}(X, Y, Y, X; Y, ..., Y) = 1.
\]
Define models:
\[
\mathcal{M}_{6+4p,k} := (\mathbb{R}^{6+4p}, \langle \cdot, \cdot \rangle, A^0, ..., A^k) \text{ for } 0 \leq k \leq p + 2.
\]

Lemma 2.1 (Gilkey-Nikčević [7]). Let $0 \leq k \leq p + 2$.
(1) $\mathcal{M}_{6+4p,k}$ is a $k$-model for $\mathcal{M}_{6+4p,k}$.
(2) $\mathcal{M}_{6+4p,p+2}$ is a $p + 2$-model for $\mathcal{N}_{6+4p,\psi}$.

3. ISOMETRY GROUPS IN THE GEOMETRIC SETTING

In this section we will reduce the proof of Theorem 1.5 to a purely algebraic problem by showing:

Theorem 3.1. Let $0 \leq k \leq p + 2$.
(1) $\dim\{G(\mathcal{M}_{6+4p,k})\} = 6 + 4p + \dim\{G(\mathcal{M}_{6+4p,k})\}$.
(2) $\dim\{G(\mathcal{N}_{6+4p,\psi})\} = 5 + 4p + \dim\{G(\mathcal{M}_{6+4p,p+2})\}$.

The proof of Theorem 3.1 will be based on several Lemmas. In Lemma 3.2, we review a basic result about group actions. In Lemma 3.3, we relate the full isometry group $G(\cdot)$ to the isotropy subgroup. In Lemma 3.4, we relate the isotropy subgroup to the isometry group of the $\infty$-model. In Lemma 3.5, we relate isometry group of the $\infty$-model to the isometry group of an appropriate finite model.

The following result is well known.

Lemma 3.2. Let $G$ be a Lie group which acts continuously on a metric space $X$. If $x \in X$, let $G \cdot x$ be the orbit and let $G_x = \{ g \in G : gx = x \}$ be the isotropy subgroup.
(1) We have a smooth principle bundle $G_x \to G \to G \cdot x$.
(2) $\dim\{G\} = \dim\{G_x\} + \dim\{G \cdot x\}$.

We can relate $\dim\{G(\mathcal{M})\}$ to $\dim\{G_P(\mathcal{M})\}$ for $\mathcal{M} = \mathcal{M}_{6+4p,k}$ or $\mathcal{M} = \mathcal{N}_{6+4p,\psi}$.

Lemma 3.3. Let $P \in \mathbb{R}^{6+4p}$. Let $0 \leq k \leq p + 2$.
(1) $\dim\{G(\mathcal{M}_{6+4p,k})\} = 6 + 4p + \dim\{G_P(\mathcal{M}_{6+4p,k})\}$.
(2) $\dim\{G(\mathcal{N}_{6+4p,\psi})\} = 6 + 4p - 1 + \dim\{G_P(\mathcal{N}_{6+4p,\psi})\}$.
We apply Lemma 3.2 to the canonical action of $G(M)$ on $\mathbb{R}^{6+4p}$. Assertion (1) follows as $M_{6+4p,k}$ is a homogeneous space. Let $\nu \geq 2$. Set
\[ \alpha_{6+4p,\nu}(\psi) := \psi^{(\nu+p+3)}(\psi^{(p+3)})^{\nu-1}(\psi^{(p+4)})^{-\nu}. \]
We showed \cite{7} that if $B$ is a basis satisfying the normalizations of Section 2.1, then the only non-zero components of $\nabla^{\nu+p+1} R$ are given by:
\[ \nabla^{\nu+p+1} R(X, Y, Y, Y, \ldots, Y) = \alpha_{6+4p,\nu}(\psi). \]
We also showed that the following assertions are equivalent:
\begin{enumerate}[1)
\item $\alpha_{6+4p,\nu}(\psi_1)(P_1) = \alpha_{6+4p,\nu}(\psi_2)(P_2)$ for all $\nu \geq 2$.
\item There exists an isometry $\phi : N_{6+4p,\psi_1} \to N_{6+4p,\psi_2}$ with $\phi(P_1) = P_2$.
\end{enumerate}
The functions $\alpha_{6+4p,\nu}(\psi)$ are constant on the hyperplanes $y = c$; thus the group of isometries acts transitively on such a hyperplane. Consequently
\[ \dim\{G(N_{6+4p,\psi})\} \geq \dim\{G_{P}(N_{6+4p,\psi})\} + 6 + 4p - 1. \]
Since $N_{6+4p,\psi}$ is not a homogeneous space, equality holds.

Let $P \in M$. We can show that $G_{P}(M)$ is isomorphic to $G(\mathfrak{M}_\infty(M, P))$ under certain circumstances.

Lemma 3.4.
\begin{enumerate}[1)
\item Let $M_1 := (M_1, g_1)$ and $M_2 := (M_2, g_2)$ be real analytic. Assume for $g = 1,2$ that there are points $P_0 \in M_0$ so $\exp_{P_0} : T_{P_0}M_0 \to M_0$ is a diffeomorphism. If $\phi : T_{P_1}M_1 \to T_{P_2}M_2$ induces an isomorphism from $\mathfrak{M}_\infty(M_1, P_1)$ to $\mathfrak{M}_\infty(M_2, P_2)$, then $\Phi := \exp_{P_2} \circ \phi \circ \exp_{P_1}^{-1}$ is an isometry from $M_1$ to $M_2$.
\item If $M = M_{6+4p,k}$ or if $M = N_{6+4p,\psi}$, then $G_{P}(M) = G(\mathfrak{M}_\infty(M, P))$ for any point $P \in \mathbb{R}^{6+4p}$.
\end{enumerate}

Proof. Belger and Kowalski \cite{1} note about analytic pseudo-Riemannian metrics that the “metric $g$ is uniquely determined, up to local isometry, by the tensors $R, \nabla R, \ldots, \nabla^k R, \ldots$ at one point.”; see also Gray \cite{8} for related work. The first assertion now follows; the second follows immediately from the first and from Theorem 1.2.

We now replace the infinite model by a finite model:

Lemma 3.5. Let $P \in \mathbb{R}^{6+4p}$. Let $0 \leq k \leq p + 2$. Then:
\begin{enumerate}[1)
\item $G(\mathfrak{M}_\infty(M_{6+4p,k}, P)) = G(\mathfrak{M}_{6+4p,k})$.
\item $G(\mathfrak{M}_\infty(N_{6+4p,\psi}, P)) = G(\mathfrak{M}_{6+4p,p+2})$.
\end{enumerate}

Proof. If $M$ is a pseudo-Riemannian manifold, restriction induces an injective map
\[ r : G(\mathfrak{M}_\infty(M, P)) \to G(\mathfrak{M}_k(M, P)). \]
Suppose that $M = M_{4p+k}$ for $k < p + 2$. Then $\nabla^j R = 0$ for $j > k$; consequently any isomorphism of the $k$-model is an isomorphism of the $\infty$-model; this proves Assertion (1) for $0 \leq k \leq p + 1$.

To deal with the remaining cases, we suppose that $\psi^{(p+3)}$ and $\psi^{(p+4)}$ are always positive, but drop the restriction that $\psi^{(p+3)} = ae^{by}$. Choose a basis $B$ for $T_P M$ satisfying the normalizations of Section 2.1. If $g \in G(\mathfrak{M}_{p+2}(M_{6+4p,p+2}, P))$, then $g B$ also satisfies the normalizations of Section 2.1. We may then apply Equation (3.a) to see that $g$ is in fact an isomorphism of the $\infty$-model since $g$ preserves $\nabla^k R$ for any $k > p + 2$. The first assertion with $k = p + 2$ and the second assertion of the Lemma now follow; this also completes the proof of Theorem 3.1.

\[ \square \]
4. ISOMETRY GROUPS OF THE MODELS

Let \( \mathbb{R}^{3+2p} := \text{Span}\{X, Y, Z_1, ..., Z_p, \tilde{Y}, \tilde{Z}_1, ..., \tilde{Z}_p\} \) and let \( B' \in \otimes^{4+i}(\mathbb{R}^{3+2p})^* \) be the restriction of \( A' \) to \( \mathbb{R}^{3+2p} \). We introduce the affine models by restricting the domain and suppressing the metric:

\[
\mathfrak{A}_{3+2p,k} := (\mathbb{R}^{3+2p}, B^0, ..., B^k) .
\]

**Lemma 4.1.** \( \dim\{G(\mathfrak{M}_{6+4p,k})\} = \dim\{G(\mathfrak{A}_{3+2p,k})\} + (p+1)(3+2p) \).

**Proof.** Let \( \mathfrak{o}(s) \) be Lie algebra of skew-symmetric \( s \times s \) real matrices. Set

\[
S := (S_1, ..., S_{3+2p}) = (X, Y, Z_1, ..., Z_p, \tilde{Y}, \tilde{Z}_1, ..., \tilde{Z}_p),
\]

\[
S^* := (S_1^*, ..., S_{3+2p}^*) = (X^*, Y^*, Z_1^*, ..., Z_p^*, \tilde{Y}^*, \tilde{Z}_1^*, ..., \tilde{Z}_p^*),
\]

\[
\mathcal{K} := \{\xi \in \mathbb{R}^{6+4p} : A^0(\xi,\eta_1,\eta_2,\eta_3) = 0 \forall \eta_i \in \mathbb{R}^{6+4p}\}
\]

\[
\mathcal{K} = \text{Span}\{S_1^*, ..., S_{3+2p}^*\} .
\]

Let \( g \in G(\mathfrak{M}_{6+4p,k}) \). The space \( \mathcal{K} \) is preserved by \( g \). Thus

\[
gS_i = \sum_{i,j} (g_{0,ij}S_j + g_{1,ij}S_j^*) \quad \text{and} \quad gS_i^* = \sum_{i,j} (g_{2,ij}S_j^*) .
\]

By Equation (2.a), \((gS_i, gS_j) = 0 \) and \((gS_i, gS_j^*) = \delta_{ij} \). Thus

\[
\sum_k (g_{0,ik}g_{1,jk} + g_{1,ik}g_{0,jk}) = 0 \quad \text{and} \quad \sum_k (g_{0,ik}g_{2,jk}) = \delta_{ij} .
\]

for all \( i, j \). Set \( \gamma := g_0g_1^t \). One then has

\[
(4.a) \quad g_0 \in G(\mathfrak{A}_{3+2p,k}), \quad \gamma + \gamma^t = 0, \quad \text{and} \quad g_0g_2^t = \text{id} .
\]

Conversely, if Equation (4.a) is satisfied then \( g \in G(\mathfrak{M}_{6+4p,k}) \). The map \( g \rightarrow (g_0, \gamma) \) yields an identification of

\[
G(\mathfrak{M}_{6+4p,k}) = G(\mathfrak{A}_{3+2p,k}) \times \mathfrak{o}(3+2p)
\]

as a twisted product. The Lemma follows as \( \dim\{\mathfrak{o}(3+2p)\} = \frac{1}{2}(3+2p)(2+2p) \). \( \square \)

There is a natural action of \( G(\mathfrak{A}_{3+2p,k}) \) on \( \mathbb{R}^{3+2p} \). We continue our study by relating \( G(\mathfrak{A}_{3+2p,k}) \) and the isotropy subgroup \( G_X(\mathfrak{A}_{3+2p,k}) \).

**Lemma 4.2.**

1. \( \dim\{G(\mathfrak{A}_{3+2p,k})\} = \dim\{G_X(\mathfrak{A}_{3+2p,k})\} + 2p + 3 \) for \( k \leq p + 1 \).
2. \( \dim\{G(\mathfrak{A}_{3+2p,p+2})\} = \dim\{G_X(\mathfrak{A}_{3+2p,p+2})\} + 2p + 2 \).

**Proof.** Lemma 4.2 will follow from Lemma 3.2 and the following relations:

\[
(4.b) \quad G(\mathfrak{A}_{3+2p,k})X = \{\xi \in \mathbb{R}^{3+2p} : (\xi, X^*) \neq 0\} \quad \text{if} \quad k \leq p + 1,
\]

\[
G(\mathfrak{A}_{3+2p,p+2})X = \{\xi \in \mathbb{R}^{3+2p} : (\xi, X^*) = \pm 1\} .
\]

We first show \( \supset \) holds in Equation (4.b). Let \( \xi \in \mathbb{R}^{3+2p} \). Assume that

\[
a := \langle \xi, X^* \rangle \neq 0 .
\]

Set \( gX = \xi \) and set

\[
\varepsilon_0 := (a^2)^{-1/(p+3)}, \quad gY := \varepsilon_0 Y, \quad g\tilde{Y} := a^{-2}\varepsilon_0^{-1}\tilde{Y},
\]

\[
\varepsilon_1 := \{a^2\varepsilon_0^t\}^{-1}, \quad gZ_i := \varepsilon_1 Z_i, \quad gZ_i^* := \varepsilon_1^{-1}a^{-2}Z_i^* .
\]
The non-zero components of $\nabla^i R$ for $1 \leq i \leq p + 2$ are then given by
\[
R(gX, gY, g\tilde{Y}, gX) = a^2 \varepsilon_0 a^{-2} \varepsilon_0^{-1} = 1,
\]
\[
R(gX, gZ_i, g\tilde{Z}_i, gX) = a^2 \varepsilon_1 \varepsilon_1^{-1} a^{-2} = 1,
\]
\[
\nabla R(gX, gY, gZ_i, gX; gY) = \nabla R(gX, gY, gY, gX; gZ_i) = a^2 \varepsilon_2 \varepsilon_1 = 1, ...
\]
\[
\nabla^p R(gX, gY, gZ_p, gX; gY, ..., gY) = \nabla^p R(gX, gY, gY, gX; gZ_p, gY, ..., gY) = ...
\]
\[
= \nabla^p R(gX, gY, gY, gX; gY, ..., gY, gZ_p) = a^2 \varepsilon_{p+1} \varepsilon_p = 1,
\]
\[
\nabla^{p+1} R(gX, gY, gY, gX; gY, ..., gY) = a^2 \varepsilon_{p+3} = 1,
\]
\[
\nabla^{p+2} R(gX, gY, gY, gX; gY, ..., gY) = a^2 \varepsilon_{p+4} = \varepsilon_0.
\]
Thus $g \in G(\mathfrak{a}_{3+2,p+1})$. Furthermore, $g \in G(\mathfrak{a}_{3+2,p+2})$ if $a^2 = 1$. Consequently:
\[
\{ \xi \in \mathbb{R}^{3+2p} : \langle \xi, X^* \rangle \neq 0 \} \subset G(\mathfrak{a}_{3+2,p,k}) \cdot X \quad \text{for} \quad k \leq p + 1,
\]
\[
\{ \xi \in \mathbb{R}^{3+2p} : \langle \xi, X^* \rangle = \pm 1 \} \subset G(\mathfrak{a}_{3+2,p+2}) \cdot X.
\]
We must establish the reverse inclusions to complete the proof. Let $\xi \in \mathbb{R}^{3+2p}$.
Let $J_\xi(\eta_1, \eta_2) := R(\xi, \eta_1, \eta_2, \xi)$ be the Jacobi form. Adopt the Einstein convention and sum over repeated indices to expand
\[
\xi = aX + b^iZ_i + \tilde{b}^i\tilde{Z}_i
\]
where $a = \langle \xi, X^* \rangle$. We have the following cases
(1) If $a = 0$, then $J_\xi = 0$ on $\text{Span}\{Y, \tilde{Y}, Z_i, \tilde{Z}_i\}$ so $\text{Rank}(J_\xi) \leq 1$.
(2) If $a \neq 0$, then $J_\xi(Y, \tilde{Y}) \neq 0$ so $\text{Rank}(J_\xi) \geq 2$.
If $g \in G(\mathfrak{a}_{3+2,p,k})$, then $\text{Rank}(J_\xi) = \text{Rank}(J_{g\xi})$. Consequently
\[
\langle \xi, X^* \rangle = 0 \iff \text{Rank}(J_\xi) \leq 1 \iff \text{Rank}(J_{g\xi}) \leq 1 \iff \langle g\xi, X^* \rangle = 0
\]
Consequently we have
\[
G(\mathfrak{a}_{3+2,p,k}) \cdot X \subset \{ \xi \in \mathbb{R}^{3+2p} : \langle \xi, X^* \rangle \neq 0 \},
\]
\[
G(\mathfrak{a}_{3+2,p,k}) \cdot \text{Span}\{Y, Z_i, \tilde{Z}_i\} = \text{Span}\{Y, Z_i, \tilde{Z}_i\}.
\]
Suppose $k = p + 2$. Since $\text{Rank}(J_Y) = 0$, $\text{Rank}(J_{gY}) = 0$ so $\langle gY, X^* \rangle = 0$. Expand
\[
gX = aX + a_0Y + \bar{a_0}\tilde{Y} + a^iZ_i + \bar{a}^i\tilde{Z}_i,
\]
\[
gY = b^0Y + \tilde{b}^0\tilde{Y} + b^iZ_i + \tilde{b}^i\tilde{Z}_i.
\]
Then
\[
1 = \nabla^{p+1} R(gX, gY, gY, gX; gY, ..., gY) = a^2(b^0)^{p+3},
\]
\[
1 = \nabla^{p+2} R(gX, gY, gY, gY; gY, ..., gY) = a^2(b^0)^{p+4}.
\]
This shows that $a^2 = 1$ and $b^0 = 1$ so
\[
G(\mathfrak{a}_{3+2,p+2}) X \subset \{ \xi \in \mathbb{R}^{3+2p} : \langle \xi, X^* \rangle = \pm 1 \},
\]
\[
G(\mathfrak{a}_{3+2,p+2}) Y \subset \{ \xi \in \mathbb{R}^{3+2p} : \langle \xi, X^* \rangle = 0, \text{ and } \langle \xi, Y^* \rangle = 1 \}.
\]
Equations (4.c), (4.d), and (4.e) now imply Equation (4.b); the Lemma follows. \(\square\)

We now consider the double isotropy group
\[
G_{X,Y}(\mathfrak{a}_{3+2,p,k}) = \{ g \in G(\mathfrak{a}_{3+2,p,k}) : gX = X \text{ and } gY = Y \}.
\]

**Lemma 4.3.**

1. $\dim\{G_X(\mathfrak{a}_{3+2,p,0})\} = (p + 1)(2p + 1)$.
2. $\dim\{G_X(\mathfrak{a}_{3+2,p,k})\} = \dim\{G_{X,Y}(\mathfrak{a}_{3+2,p,k})\} + 2p + 2$ for $1 \leq k \leq p$.
3. $\dim\{G_X(\mathfrak{a}_{3+2,p,k})\} = \dim\{G_{X,Y}(\mathfrak{a}_{3+2,p,k})\} + 2p + 1$ for $k = p + 1, p + 2$.
4. $G_{X,Y}(\mathfrak{a}_{3+2,p,p}) = G_{X,Y}(\mathfrak{a}_{3+2,p+1}) = G_{X,Y}(\mathfrak{a}_{3+2,p+2})$. 

Proof. As noted above, the Jacobi form $J_X(\cdot, \cdot) = R(X, \cdot, \cdot, X)$ defines a non-singular bilinear form of signature $(p+1, p+1)$ on $W := \text{Span}\{Y, Z_1, \ldots, Z_p, \tilde{Y}, \tilde{Z}_1, \ldots, \tilde{Z}_p\} = \{\xi : \text{Rank}(J_\xi) \leq 1\}$. Let $O(W, J_X)$ be the associated orthogonal group. If $g \in G_X(\mathfrak{A}_{3+2p,k})$, then we have $gW = W$ by Equation (4.d). Since $gX = X$, we may safely identify $g$ with $g|_W$. Furthermore, $J_X(\xi, \eta) = J_X(g\xi, g\eta) = J_X(g\xi, g\eta)$ so $G_X(\mathfrak{A}_{3+2p,k}) \subset O(W, J_X)$. Conversely, if $g$ is a linear map of $W$ which preserves $J_X$, we may extend $g$ to $\mathbb{R}^{3+2p}$ by defining $gX = X$ and thereby obtain an element of $G_X(\mathfrak{A}_{3+2p,0})$. Thus $G_X(\mathfrak{A}_{3+2p,0}) = O(W, J_X)$. Assertion (1) now follows since $\dim(O(W, J_X)) = \frac{1}{2} \dim W (\dim W - 1) = \frac{1}{2}(1 + 2p)(2 + 2p)$.

Assertions (2) and (3) will follow from Lemma 3.2 and from the relations:

\begin{equation}
\begin{aligned}
G_X(\mathfrak{A}_{3+2p,k}) \cdot Y &= \{\xi \in W : (\xi, Y^*) \neq 0\} \quad \text{for } 1 \leq k \leq p, \\
G_X(\mathfrak{A}_{3+2p,p+1}) \cdot Y &= \{\xi \in W : (\xi, Y^*)^{p+3} = 1\}, \\
G_X(\mathfrak{A}_{3+2p,p+2}) \cdot Y &= \{\xi \in W : (\xi, Y^*) = 1\}.
\end{aligned}
\end{equation}

If $\xi \in W$, let $S_\xi(\eta) := \nabla R(X, \xi, \xi; \eta)$. Expand $\xi = b^0Y + b^0\tilde{Y} + b^iZ_i + b^i\tilde{Z}_i$.

We then have that

\begin{equation}
\begin{aligned}
S_\xi(X) &= 0, \quad S_\xi(Z_i) = 0, \quad S_\xi(Y) = 2b^0b^1, \\
S_\xi(Z_1) &= (b^0)^2, \quad \text{and} \quad S_\xi(Z_i) = 0 \quad \text{for } i \geq 2.
\end{aligned}
\end{equation}

Thus $S_\xi = 0$ if and only if $b^0 = (\xi, Y^*) = 0$. It now follows that for $k \geq 1$ we have

\begin{equation}
\begin{aligned}
G_X(\mathfrak{A}_{3+2p,k})Y &\subset \{\xi \in W : (\xi, Y^*) \neq 0\}, \\
G_X(\mathfrak{A}_{3+2p,k}) \text{Span}\{Z_i, \tilde{Y}, \tilde{Z}_i\} &\subset \text{Span}\{Z_i, \tilde{Y}, \tilde{Z}_i\}.
\end{aligned}
\end{equation}

Since $a = 1$, the analysis used to prove Lemma 4.2 shows $(b^0)^{p+3} = 1$ if $k = p + 1$ and $b^0 = 1$ if $k = p + 2$. This establishes the inclusions $\subset$ in Equation (4.f).

We complete the proof by establishing the reverse inclusions in Equation (4.f). Expand $\xi$ in the form given in Equation (4.g). Assume $b^0 \neq 0$. Let $gX = X$, $gY = \xi$, $g\tilde{Y} = (b^0)^{-1}Y$, $gZ_i := \varepsilon_i\{Z_i - (b^0)^{-1}b^i\tilde{Y}\}$ and $g\tilde{Z}_i := \varepsilon_i^{-1}\{\tilde{Z}_i - (b^0)^{-1}b^\tilde{Y}\}$.

The possibly non-zero components of $R$ are then given by

\begin{equation}
\begin{aligned}
R(gX, gY, gZ_i, gX) &= 1, \\
R(gX, gY, gZ_i, gX) &= \varepsilon_i\{b^i - (b^0)(b^0)^{-1}b^i\} = 0, \\
R(gX, gY, g\tilde{Z}_i, gX) &= \varepsilon_i^{-1}\{b^\tilde{i} - (b^0)(b^0)^{-1}b^\tilde{i}\} = 0, \\
R(gX, gZ_i, g\tilde{Z}_i, gX) &= \varepsilon_i^{-1}\varepsilon_i = 1.
\end{aligned}
\end{equation}

The non-zero components of $\nabla^i R$ for $1 \leq i \leq p$ are given by

\begin{equation}
\begin{aligned}
\nabla^i R(gX, gY, gZ_i, gX; gY; gY, \ldots, gY) &= \ldots \\
\nabla^i R(gX, gY, gZ_i, gX; gY; gY, \ldots, gZ_i) &= (b^0)^{i+1}\varepsilon_i.
\end{aligned}
\end{equation}

We therefore set $\varepsilon_i = (b^0)^{-i-1}$ for $1 \leq i \leq p$ to ensure $g \in G(\mathfrak{A}_{3+2p,p})$.

The non-zero components of $\nabla^i R$ for $i = p + 1, p + 2$ are

\begin{equation}
\begin{aligned}
\nabla^i R(gX, gY, gZ_i, gX; gY, \ldots, gY; gY, \ldots, gZ_i) &= (b^0)^{i+2}.
\end{aligned}
\end{equation}

If $(b^0)^{p+3} = 1$, then $g \in G(\mathfrak{A}_{3+2p,p+1})$; if $b^0 = 1$, then $g \in G(\mathfrak{A}_{3+2p,p+2})$. This establishes the reverse inclusions in Equation (4.f) and completes the proof of Assertions (2) and (3); Assertion (4) is immediate. □
Let $W(p) := \text{Span}\{Z_1, ..., Z_p, \tilde{Z}_1, ..., \tilde{Z}_p\}$. Let $\{\beta_1, ..., \beta_p, \tilde{\beta}_1, ..., \tilde{\beta}_p\}$ be the corresponding dual basis for the dual space $W^*(p) := (W(p))^*$. The curvature tensor $R(X, \cdot, \cdot, X)$ defines a non-degenerate form $\langle \cdot, \cdot \rangle$ on $W(p)$; dually on $W(p)$ we have:

$$
\langle \beta_i, \beta_j \rangle = \langle \tilde{\beta}_i, \tilde{\beta}_j \rangle = 0, \quad \langle \beta_i, \tilde{\beta}_j \rangle = \delta_{ij}.
$$

Let $O(p)$ be the associated orthogonal group on $W(p)$. Let

$$
O(p, k) := \{h \in O(p) : h\beta_i = \beta_i \text{ for } 1 \leq i \leq k\}
$$

be the simultaneous isotropy group. We set $O(p, 0) = O(p)$. Theorem 1.5 will now follow from the following result:

**Lemma 4.4.** Let $1 \leq k \leq p$.

1. $G_{X,Y}(\mathfrak{a}_{3+2p,k}) = O(p, k)$.
2. $O_{\beta_i}(p, k) = O(p - 1, k - 1)$.
3. $\dim\{O(p, k)\} = \dim\{O(p - 1, k - 1)\} + 2p - k - 1$.
4. $\dim\{O(p, k)\} = \frac{3}{2}(2p - k)(2p - k - 1)$.

**Proof.** Let $g \in G_{X,Y}(\mathfrak{a}_{3+2p,k})$. Let $\xi \in \text{Span}\{Z_1, ..., Z_p, \tilde{Y}, \tilde{Z}_1, ..., \tilde{Z}_p\}$. We may use Equation (4.1.12) and the relation $R(X, Y, g\xi, X) = R(X, Y, \xi, X)$, to see

$$
g\tilde{Y} = \tilde{Y} + a^i Z_i + a^I \tilde{Z}_i, \quad gZ_i = a^i Z_i + a^I \tilde{Z}_i, \quad g\tilde{Z}_i = a^i Z_i + a^I \tilde{Z}_i.
$$

Consequently $\text{Span}_{1 \leq i \leq k}\{gZ_i, g\tilde{Z}_i\} = \text{Span}_{1 \leq i \leq k}\{Z_i, \tilde{Z}_i\}$ and the relation

$$
R(X, gZ_i, g\tilde{Y}, X) = R(X, g\tilde{Z}_i, g\tilde{Y}, X) = 0
$$

implies $a^i = a^I = 0$. Thus $g\tilde{Y} = \tilde{Y}$ and $g : W(p) \to W(p)$; this shows that $g$ is determined by its restriction to $W(p)$. Let $h := h^* g$ denote the dual action of $g$ on $W^*(p)$. The isomorphism of Assertion (1) now follows as:

$$
R(X, g\xi_1, g\xi_2, R) = R(X, \xi_1, \xi_2, X) \forall \xi_1, \xi_2 \Leftrightarrow h \in O(p), \quad \nabla^i R(X, Y, g\xi, X; Y, ..., Y) = \nabla^i R(X, Y, \xi, X; Y, ..., Y) \forall \xi \Leftrightarrow h\beta_i = \beta_i.
$$

If $h(\tilde{\beta}_1) = \beta_1$ and $h(\tilde{\beta}_i) = \tilde{\beta}_i$, then $h$ preserves

$$
\text{Span}\{\beta_1, \tilde{\beta}_1\} = \text{Span}\{\beta_2, ..., \beta_p, \tilde{\beta}_2, ..., \tilde{\beta}_p\}.
$$

The isomorphism of Assertion (2) now follows by restricting $h$ to this subspace and by renumbering the variables appropriately.

We set

$$
\mathcal{W}(p, k) := \{\xi \in \mathcal{W}(p) : \langle \xi, \xi \rangle = 0, \quad \langle \xi, \beta_1 \rangle = 1, \quad \langle \xi, \beta_i \rangle = 0 \text{ for } 2 \leq i \leq k\}.
$$

If $h \in O(p, k)$, then $h$ preserves $\langle \cdot, \cdot \rangle$ and $h$ preserves $\{\beta_1, ..., \beta_k\}$. Consequently $h\tilde{\beta}_1$ satisfies these relations. Conversely, $\xi \in \mathcal{W}(p, k)$ if and only if

$$
\xi = \beta^1 \beta_1 + \sum_{1 \leq i < k} \beta^i \beta_i + \tilde{\beta}^1 \tilde{\beta}_1 + \sum_{k < j} \tilde{\beta}^j \tilde{\beta}_j \quad \text{where} \quad \beta^1 + \sum \tilde{\beta}^i = 0.
$$

Since the variables $\{\beta^2, ..., \beta^p, \tilde{\beta}^{k+1}, ..., \tilde{\beta}^p\}$ can be chosen arbitrarily,

$$
\mathcal{W}(p, k) = \mathbb{R}^{p-1+p-k} \quad \text{so} \quad \dim \mathcal{W}(p, k) = 2p - k - 1.
$$

We show that $\xi \in O(p, k)\tilde{\beta}_1$ by finding $h \in O(p, k)$ so $h\tilde{\beta}_1 = \xi$. Set:

$$
h\beta_i = \beta_i \quad \text{for } 1 \leq i \leq k, \quad h\beta_i = \beta_i - \tilde{\beta}^i \beta_1 \quad \text{for } k < i, \quad h\tilde{\beta}_1 = \xi, \quad h\tilde{\beta}_i = \tilde{\beta}_i - \beta^i \beta_1 \quad \text{for } 1 < i.
$$

This shows $O(p, k)\tilde{\beta}_1 = \mathcal{W}(p, k)$. Assertion (3) now follows from Assertion (2) and from Lemma 3.2.

As $\dim\{O(p-k)\} = \frac{1}{2}(2p-2k)(2p-2k-1)$, Assertion (4) follows by induction. □
Acknowledgments

Research of P. Gilkey partially supported by the Max Planck Institute in the Mathematical Sciences (Leipzig). Research of S. Nikčević partially supported by MM 1646 (Srbija).

References


PG: Mathematics Department, University of Oregon, Eugene OR 97403 USA.
Email: gilkey@darkwing.uoregon.edu

Email: stanan@mi.sanu.ac.yu