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Isometry groups of k-curvature homogeneous pseudo-Riemannian manifolds

by

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ISOMETRY GROUPS OF k-CURVATURE HOMOGENEOUS PSEUDO-RIEMANNIAN MANIFOLDS

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Abstract. We study the isometry groups of a family of complete $p+2$-curvature homogeneous pseudo-Riemannian metrics on $\mathbb{R}^{6+4p}$ which have neutral signature $(3+2p, 3+2p)$, and which are 0-curvature modeled on an indecomposible symmetric space.

1. Introduction

Let $\mathcal{M} := (M, g)$ be a pseudo-Riemannian manifold of signature (p, q). Let $g_P := g|_{T_PM}$ (resp. $\nabla^i R_P := \nabla^i R|_{T_PM}$) be the restriction of the metric (resp. the i^{th} covariant derivative of the curvature tensor) to the tangent space at $P \in M$. We define the *k-model of $\mathcal{M}$ at P* by setting:

$$\mathfrak{M}_k(\mathcal{M}, P) := (T_PM, g_P, R_P, ..., \nabla^k R_P)\,.$$

One says that $\phi : \mathfrak{M}_k(\mathcal{M}_1, P_1) \to \mathfrak{M}_k(\mathcal{M}_2, P_2)$ is an *isomorphism* from the k-model of $\mathcal{M}_1$ at P_1 to the k-model of $\mathcal{M}_2$ at P_2 if ϕ is a linear isomorphism from $T_{P_1}M_1$ to $T_{P_2}M_2$ with

$$\phi^* g_{2,P_2} = g_{1,P_1} \quad \text{and} \quad \phi^* \nabla_2^i R_{\mathcal{M}_2,P_2} = \nabla_1^i R_{\mathcal{M}_1,P_1} \quad \text{for} \quad 0 \le i \le k\,.$$

One says that $\mathcal{M}$ is *k-curvature homogeneous* if the k-models $\mathfrak{M}_k(\mathcal{M}, P)$ and $\mathfrak{M}_k(\mathcal{M}, Q)$ are isomorphic for any $P, Q \in M$.

In the Riemannian setting ($p = 0$), Takagi [14] constructed 0-curvature homogeneous complete non-compact Riemannian manifolds; compact examples were exhibited subsequently by Ferus, Karcher, and Münzer [5]. Although many other examples have been constructed, there are no known Riemannian manifolds which are 1-curvature homogeneous but not locally homogeneous and it is natural to conjecture that any 1-curvature homogeneous Riemannian manifold is locally homogeneous.

In the Lorentzian setting ($p = 1$), curvature homogeneous manifolds which are not locally homogeneous were constructed by Cahen et. al. [4]; 1-curvature homogeneous Lorentzian manifolds which are not locally homogeneous have been exhibited by Bueken and Djorić [2] and by Bueken and Vanhecke [3]. One could conjecture that a 2-curvature homogeneous Lorentzian manifold must be locally homogeneous.

It is clear that local homogeneity implies k-curvature homogeneity for any k. The following result, due to Singer [11] in the Riemannian setting and to F. Podesta and A. Spiro [10] in the general context, provides a partial converse:

Theorem 1.1 (Singer, Podesta-Spiro). *There exists an integer $k_{p,q}$ so that if $\mathcal{M}$ is a complete simply connected pseudo-Riemannian manifold of signature (p, q) which is $k_{p,q}$-curvature homogeneous, then (M, g) is homogeneous.*

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Sekigawa, Suga, and Vanhecke [12, 13] showed any 1-curvature homogeneous complete simply connected Riemannian manifold of dimension $m < 5$ is homogeneous; thus $k_{0,2} = k_{0,3} = k_{0,4} = 1$. The estimate $k_{0,m} < \frac{3}{2}m - 1$ was established by Gromov [9]. Results of [6] can be used to show $k_{p,q} \geq \min(p,q)$; we conjecture $k_{p,q} = \min(p,q) + 1$.

If $\mathcal{H}$ is a homogeneous space, let $\mathfrak{M}_k(\mathcal{H}) := \mathfrak{M}_k(\mathcal{H}, Q)$ for any point $Q \in H$; the isomorphism class of $\mathfrak{M}_k(\mathcal{H})$ is independent of the point $Q \in H$. We say that $\mathcal{M}$ is *k-modeled on* $\mathcal{H}$ and that $\mathfrak{M}_k(\mathcal{H})$ is *a k-model for* $\mathcal{M}$ if $\mathfrak{M}_k(\mathcal{H})$ and $\mathfrak{M}_k(\mathcal{M}, P)$ are isomorphic for any $P \in M$.

Throughout this paper, we shall adopt the notational convention that

$$p \geq 1\,.$$

In [7], we exhibited complete metrics on $\mathbb{R}^{6+4p}$ of neutral signature $(3+2p, 3+2p)$ which are $(p+2)$-curvature homogeneous, which are 0-modeled on an indecomposible symmetric space, but which are not $(p+3)$-curvature homogeneous; these examples show that the constants $k_{p,q} \to \infty$ as $(p,q) \to \infty$. The proof of Theorem 1.1 rested on a careful analysis of the isometry groups of the model spaces. In this paper, we continue our study of the manifolds introduced in [7] by examining their isometry groups and the isometry groups of their k-models.

We recall the definition of the metrics on $\mathbb{R}^{6+4p}$ which were introduced in [7]. We will be defining a number of tensors in this paper and, in the interests of brevity, we shall only give the non-zero components up to the usual symmetries. Let $x = (x_1, ..., x_m)$ be the usual coordinates on $\mathbb{R}^m$. Let

$$\{x, y, z_1, ..., z_p, \tilde{y}, \tilde{z}_1, ..., \tilde{z}_p, x^*, y^*, z_1^*, ..., z_p^*, \tilde{y}^*, \tilde{z}_1^*, ..., \tilde{z}_p^*\}$$

be coordinates on $\mathbb{R}^{6+4p}$. Let $F = F(y, z_1, ..., z_p) \in C^\infty(\mathbb{R}^{p+1})$. Let

$$\mathcal{M}_{6+4p,F} := (\mathbb{R}^{6+4p}, g_{6+4p,F})$$

where $g_{6+4p,F}$ is the metric of neutral signature $(3+2p, 3+2p)$ on $\mathbb{R}^{6+4p}$ with:

$$\begin{aligned}
&g_{6+4p,F}(\partial_x, \partial_x) = -2\{F(y, z_1, ..., z_p) + y\tilde{y} + z_1\tilde{z}_1 ... + z_p\tilde{z}_p\},\\
&g_{6+4p,F}(\partial_x, \partial_{x^*}) = g_{6+4p,F}(\partial_y, \partial_{y^*}) = g_{6+4p,F}(\partial_{\tilde{y}}, \partial_{\tilde{y}^*}) = 1,\\
&g_{6+4p,F}(\partial_{z_i}, \partial_{z_i^*}) = g_{6+4p,F}(\partial_{\tilde{z}_i}, \partial_{\tilde{z}_i^*}) = 1\,.
\end{aligned}$$

Theorem 1.2 (Gilkey-Nikčević [7])**.** *Let* $\mathcal{M} = \mathcal{M}_{6+4p,F}$*. Then:*

1. *All geodesics in* $\mathcal{M}$ *extend for infinite time.*
2. $\exp_P : T_P\mathbb{R}^{6+4p} \to \mathbb{R}^{6+4p}$ *is a diffeomorphism for all* $P \in \mathbb{R}^{6+4p}$.
3. $\nabla^k R(\partial_x, \partial_{\xi_1}, \partial_{\xi_2}, \partial_x; \partial_{\xi_3}, ..., \partial_{\xi_{k+2}}) = -\frac{1}{2}(\partial_{\xi_1} \cdots \partial_{\xi_{k+2}}) g_{6+4p,F}(\partial_x, \partial_x)$ *are the non-zero components of* $\nabla^k R$ *where* $\xi_i \in \{y, z_1, ..., z_p, \tilde{y}, \tilde{z}_1, ..., \tilde{z}_p\}$.
4. *All scalar Weyl invariants of* $\mathcal{M}$ *vanish.*
5. $\mathcal{M}$ *is a symmetric space if and only if* F *is at most quadratic.*

1.1. **The manifolds** $\mathcal{M}_{6+4p,k} = (\mathbb{R}^{6+4p}, g_{6+4p,k})$**.** We can specialize this construction as follows. Let $g_{6+4p,k}$ be defined by setting $F = f_{p,k}$ where we let:

$$\begin{aligned}
&f_{p,0}(y, z_1, ..., z_p) := 0,\\
&f_{p,k}(y, z_1, ..., z_p) := z_1 y^2 + ... + z_k y^{k+1} \quad \text{if} \quad 1 \leq k \leq p\,.
\end{aligned}$$

As exceptional cases, we set:

$$\begin{aligned}
&f_{p,p+1}(y, z_1, ..., z_p) := z_1 y^2 + ... + z_p y^{p+1} + y^{p+3},\\
&f_{p,p+2}(y, z_1, ..., z_p) := z_1 y^2 + ... + z_p y^{p+1} + e^y\,.
\end{aligned}$$

Theorem 1.3 (Gilkey-Nikčević [7])**.** *Let* $1 \leq k \leq p+2$.

1. $\mathcal{M}_{6+4p,0}$ *is an indecomposible symmetric space.*
2. $\mathcal{M}_{6+4p,k}$ *is an indecomposible homogeneous space which is not symmetric.*

1.2. **The manifolds $\mathcal{N}_{6+4p,\psi} = (\mathbb{R}^{6+4p}, g_{6+4p,\psi})$.** Let $\psi = \psi(y)$ be a real analytic function of one variable such that

$$\psi^{(p+3)} > 0, \quad \psi^{(p+4)} > 0, \quad \text{and} \quad \psi^{(p+3)} \neq ae^{by}\,.$$

Define a metric $g_{6+4p,\psi}$ on $\mathbb{R}^{6+4p}$ by taking $F = f_\psi$ where

$$f_\psi(y, z_1, ..., z_p) := \psi(y) + z_1 y^2 + ... + z_p y^{p+1}\,.$$

The following result shows that the geometry of a homogeneous pseudo Riemannian manifold need not determined by the k-model:

Theorem 1.4 (Gilkey-Nikčević [7]). *Let $0 \leq j < k \leq p+2$.*

1. *$\mathcal{M}_{6+4p,k}$ is j-modeled on $\mathcal{M}_{6+4p,j}$; $\mathcal{M}_{6+4p,j}$ is not k-modeled on $\mathcal{M}_{6+4p,k}$.*
2. *$\mathcal{N}_{6+4p,\psi}$ is $p+2$-curvature homogeneous and $p+2$-modeled on $\mathcal{M}_{6+4p,p+2}$.*
3. *$\mathcal{N}_{6+4p,\psi}$ is not $p+3$-curvature homogeneous and not locally homogeneous.*

1.3. **Isometry groups.** Let $G(\mathcal{M})$ (resp. $G(\mathfrak{M}_k)$) be the isometry group of a pseudo-Riemannian manifold $\mathcal{M}$ (resp. of a k-model $\mathfrak{M}_k$). In this paper, we study the groups $G(\mathcal{M}_{6+4p,k})$, $G(\mathcal{N}_{6+4p,\psi})$, and $G(\mathfrak{M}_k(\mathcal{M}_{6+4p,k}, P))$ for any point P of $\mathbb{R}^{6+4p}$. A byproduct of our study is the following result that shows, not surprisingly, that the symmetric space $\mathcal{M}_{6+4p,0}$ has the largest isometry group.

Theorem 1.5. *Let $1 \leq k \leq p$. Let $n_p := (6+4p) + (p+1)(3+2p) + (2p+3)$.*

1. $\dim\{G(\mathcal{M}_{6+4p,0})\} = n_p + (p+1)(2p+1)$.
2. $\dim\{G(\mathcal{M}_{6+4p,k})\} = n_p + (2p+2) + \frac{1}{2}(2p-k)(2p-k-1)$.
3. $\dim\{G(\mathcal{M}_{6+4p,p+1})\} = \dim\{G(\mathcal{M}_{6+4p,p})\} - 1$.
4. $\dim\{G(\mathcal{M}_{6+4p,p+2})\} = \dim\{G(\mathcal{M}_{6+4p,p+1})\} - 1$.
5. $\dim\{G(\mathcal{N}_{6+4p,\psi})\} = \dim\{G(\mathcal{M}_{6+4p,p+2})\} - 1$.

Here is a brief outline to the remainder of this paper. In Section 2, we review some results from [7]. In Section 3, we reduce the proof of Theorem 1.5 to a purely algebraic problem by showing for any $P \in \mathbb{R}^{6+4p}$ that for $0 \leq k \leq p+2$, we have:

$$\begin{aligned}\dim\{G(\mathcal{M}_{6+4p,k})\} &= 6 + 4p + \dim\{G(\mathfrak{M}_k(\mathcal{M}_{6+4p,k}, P))\},\\ \dim\{G(\mathcal{N}_{6+4p,\psi})\} &= 5 + 4p + \dim\{G(\mathfrak{M}_{p+2}(\mathcal{M}_{6+4p,p+2}, P))\}\,.\end{aligned}$$

In Section 4, we complete the proof by determining $\dim\{G(\mathfrak{M}_k(\mathcal{M}_{6+4p,k}, P))\}$ for $0 \leq k \leq p+2$.

2. Models

It is convenient to work in the purely algebraic setting. Let

$$\mathfrak{M}_\nu := (V, \langle\cdot,\cdot\rangle, A^0, ..., A^\nu)$$

where $\langle\cdot,\cdot\rangle$ is a non-degenerate inner product of signature (p, q) on a finite dimensional vector space V of dimension $m = p + q$ and where $A^\mu \in \otimes^{4+\mu} V^*$ satisfies the appropriate symmetries of the covariant derivatives of the curvature tensor for $0 \leq \mu \leq \nu$; if $\nu = \infty$, then the sequence is infinite. We say that $\mathfrak{M}_\nu$ is a *ν-model* for a pseudo-Riemannian manifold $\mathcal{M} = (M, g)$ if for each point $P \in M$, there is an isomorphism $\phi_P : T_P M \to V$ so that

$$\phi_P^* \langle\cdot,\cdot\rangle = g_P \quad \text{and} \quad \phi_P^* A^\mu = \nabla^\mu R_P \quad \text{for} \quad 0 \leq \mu \leq \nu\,.$$

Clearly $\mathcal{M}$ is ν-curvature homogeneous if and only if it admits a ν-model.

2.1. **Models for the manifolds $\mathcal{M}_{6+4p,k}$ and $\mathcal{N}_{6+4p,\psi}$.** Let

$$\mathcal{B} = \{X, Y, Z_1..., Z_p, \tilde{Y}, \tilde{Z}_1, ..., \tilde{Z}_p, X^*, Y^*, Z_1^*, ..., Z_p^*, \tilde{Y}^*, \tilde{Z}_1^*, ..., \tilde{Z}_p^*\}$$

be a basis for $\mathbb{R}^{6+4p}$. Define a hyperbolic inner-product on $\mathbb{R}^{6+4p}$ by pairing ordinary variables with the corresponding dual $\star$-variables:

$$\langle X, X^*\rangle = \langle Y, Y^*\rangle = \langle \tilde{Y}, \tilde{Y}^*\rangle = \langle Z_i, Z_i^*\rangle = \langle \tilde{Z}_i, \tilde{Z}_i^*\rangle = 1\,. \tag{2.a}$$

Define $A^0 \in \otimes^4(\mathbb{R}^{6+4p})^*$ with non-zero components:

$$A^0(X, Y, \tilde{Y}, X) = A^0(X, Z_i, \tilde{Z}_i, X) = 1\,.$$

Define tensors $A^i \in \otimes^{4+i}(\mathbb{R}^{6+4p})^*$ for $1 \le i \le p$ with non-zero components:

$$\begin{aligned}&A^i(X, Y, Z_i, X; Y, ..., Y) = 1,\\&A^i(X, Y, Y, X; Z_i, Y, ..., Y) = 1, ...,\\&A^i(X, Y, Y, X; Y, ..., Y, Z_i) = 1\,.\end{aligned}$$

Finally define $A^{p+1} \in \otimes^{5+p}(\mathbb{R}^{6+4p})^*$ and $A^{p+2} \in \otimes^{6+p}(\mathbb{R}^{6+4p})^*$ by setting

$$\begin{aligned}&A^{p+1}(X, Y, Y, X; Y, ..., Y) = 1,\\&A^{p+2}(X, Y, Y, X; Y, ..., Y) = 1\,.\end{aligned}$$

Define models:

$$\mathfrak{M}_{6+4p,k} := (\mathbb{R}^{6+4p}, \langle\cdot,\cdot\rangle, A^0, ..., A^k) \quad \text{for} \quad 0 \le k \le p+2\,.$$

Lemma 2.1 (Gilkey-Nikčević [7]). *Let $0 \le k \le p+2$.*

(1) *$\mathfrak{M}_{6+4p,k}$ is a k-model for $\mathcal{M}_{6+4p,k}$.*
(2) *$\mathfrak{M}_{6+4p,p+2}$ is a $p+2$-model for $\mathcal{N}_{6+4p,\psi}$.*

3. Isometry groups in the geometric setting

In this section we will reduce the proof of Theorem 1.5 to a purely algebraic problem by showing:

Theorem 3.1. *Let $0 \le k \le p+2$.*

(1) $\dim\{G(\mathcal{M}_{6+4p,k})\} = 6 + 4p + \dim\{G(\mathfrak{M}_{6+4p,k})\}$.
(2) $\dim\{G(\mathcal{N}_{6+4p,\psi})\} = 5 + 4p + \dim\{G(\mathfrak{M}_{6+4p,p+2})\}$.

The proof of Theorem 3.1 will be based on several Lemmas. In Lemma 3.2, we review a basic result about group actions. In Lemma 3.3, we relate the full isometry group $G(\cdot)$ to the isotropy subgroup. In Lemma 3.4, we relate the isotropy subgroup to the isometry group of the ∞-model. In Lemma 3.5, we relate isometry group of the ∞-model to the isometry group of an appropriate finite model.

The following result is well known.

Lemma 3.2. *Let G be a Lie group which acts continuously on a metric space X. If $x \in X$, let $G \cdot x$ be the orbit and let $G_x = \{g \in G : gx = x\}$ be the isotropy subgroup.*

(1) *We have a smooth principle bundle $G_x \to G \to G \cdot x$.*
(2) $\dim\{G\} = \dim\{G_x\} + \dim\{G \cdot x\}$.

We can relate $\dim\{G(\mathcal{M})\}$ to $\dim\{G_P(\mathcal{M})\}$ for $\mathcal{M} = \mathcal{M}_{6+4p,k}$ or $\mathcal{M} = \mathcal{N}_{6+4p,\psi}$.

Lemma 3.3. *Let $P \in \mathbb{R}^{6+4p}$. Let $0 \le k \le p+2$.*

(1) $\dim\{G(\mathcal{M}_{6+4p,k})\} = 6 + 4p + \dim\{G_P(\mathcal{M}_{6+4p,k})\}$.
(2) $\dim\{G(\mathcal{N}_{6+4p,\psi})\} = 6 + 4p - 1 + \dim\{G_P(\mathcal{N}_{6+4p,\psi})\}$.

Proof. We apply Lemma 3.2 to the canonical action of $G(\mathcal{M})$ on $\mathbb{R}^{6+4p}$. Assertion (1) follows as $\mathcal{M}_{6+4p,k}$ is a homogeneous space. Let $\nu \geq 2$. Set

$$\alpha_{6+4p,\nu}(\psi) := \psi^{(\nu+p+3)}\{\psi^{(p+3)}\}^{\nu-1}\{\psi^{(p+4)}\}^{-\nu}\,.$$

We showed [7] that if $\mathcal{B}$ is a basis satisfying the normalizations of Section 2.1, then the only non-zero components of $\nabla^{\nu+p+1}R$ are given by:

$$\nabla^{\nu+p+1}R(X,Y,Y,X;Y,...,Y) = \alpha_{6+4p,\nu}(\psi)\,. \tag{3.a}$$

We also showed that the following assertions are equivalent:

1. $\alpha_{6+4p,\nu}(\psi_1)(P_1) = \alpha_{6+4p,\nu}(\psi_2)(P_2)$ for all $\nu \geq 2$.
2. There exists an isometry $\phi : \mathcal{N}_{6+4p,\psi_1} \to \mathcal{N}_{6+4p,\psi_2}$ with $\phi(P_1) = P_2$.

The functions $\alpha_{6+4p,\nu}(\psi)$ are constant on the hyperplanes $y = c$; thus the group of isometries acts transitively on such a hyperplane. Consequently

$$\dim\{G(\mathcal{N}_{6+4p,\psi})\} \geq \dim\{G_P(\mathcal{N}_{6+4p,\psi})\} + 6 + 4p - 1\,.$$

Since $\mathcal{N}_{6+4p,\psi}$ is not a homogeneous space, equality holds. □

Let $P \in M$. We can show that $G_P(\mathcal{M})$ is isomorphic to $G(\mathfrak{M}_\infty(\mathcal{M},P))$ under certain circumstances.

Lemma 3.4.

1. *Let $\mathcal{M}_1 := (M_1, g_1)$ and $\mathcal{M}_2 := (M_2, g_2)$ be real analytic. Assume for $\varrho = 1,2$ that there are points $P_\varrho \in M_\varrho$ so $\exp_{P_\varrho} : T_{P_\varrho}M_\varrho \to M_\varrho$ is a diffeomorphism. If $\phi : T_{P_1}M_1 \to T_{P_2}M_2$ induces an isomorphism from $\mathfrak{M}_\infty(\mathcal{M}_1,P_1)$ to $\mathfrak{M}_\infty(\mathcal{M}_2,P_2)$, then $\Phi := \exp_{P_2}\circ\phi\circ\exp_{P_1}^{-1}$ is an isometry from $\mathcal{M}_1$ to $\mathcal{M}_2$.*
2. *If $\mathcal{M} = \mathcal{M}_{6+4p,k}$ or if $\mathcal{M} = \mathcal{N}_{6+4p,\psi}$, then $G_P(\mathcal{M}) = G(\mathfrak{M}_\infty(\mathcal{M},P))$ for any point $P \in \mathbb{R}^{6+4p}$.*

Proof. Belger and Kowalski [1] note about analytic pseudo-Riemannian metrics that the "metric g is uniquely determined, up to local isometry, by the tensors R, ∇R, ..., $\nabla^k R$, ... at one point."; see also Gray [8] for related work. The first assertion now follows; the second follows immediately from the first and from Theorem 1.2. □

We now replace the infinite model by a finite model:

Lemma 3.5. *Let $P \in \mathbb{R}^{6+4p}$. Let $0 \leq k \leq p+2$. Then:*

1. $G(\mathfrak{M}_\infty(\mathcal{M}_{6+4p,k},P)) = G(\mathfrak{M}_{6+4p,k})$.
2. $G(\mathfrak{M}_\infty(\mathcal{N}_{6+4p,\psi},P)) = G(\mathfrak{M}_{6+4p,p+2})$.

Proof. If $\mathcal{M}$ is a pseudo-Riemannian manifold, restriction induces an injective map

$$r : G(\mathfrak{M}_\infty(\mathcal{M},P)) \to G(\mathfrak{M}_k(\mathcal{M},P))\,.$$

Suppose that $\mathcal{M} = \mathcal{M}_{4p+6,k}$ for $k < p+2$. Then $\nabla^j R = 0$ for $j > k$; consequently any isomorphism of the k-model is an isomorphism of the ∞-model; this proves Assertion (1) for $0 \leq k \leq p+1$.

To deal with the remaining cases, we suppose that $\psi^{(p+3)}$ and $\psi^{(p+4)}$ are always positive, but drop the restriction that $\psi^{(p+3)} \neq ae^{by}$. Choose a basis $\mathcal{B}$ for T_PM satisfying the normalizations of Section 2.1. If $g \in G(\mathfrak{M}_{p+2}(\mathcal{M}_{6+4p,p+2},P))$, then $g\mathcal{B}$ also satisfies the normalizations of Section 2.1. We may then apply Equation (3.a) to see that g is in fact an isomorphism of the ∞-model since g preserves $\nabla^k R$ for any $k > p+2$. The first assertion with $k = p+2$ and the second assertion of the Lemma now follow; this also completes the proof of Theorem 3.1. □

4. Isometry groups of the models

Let $\mathbb{R}^{3+2p} := \text{Span}\{X, Y, Z_1, ..., Z_p, \tilde{Y}\tilde{Z}_1, ..., \tilde{Z}_p\}$ and let $B^i \in \otimes^{4+i}(\mathbb{R}^{3+2p})^*$ be the restriction of A^i to $\mathbb{R}^{3+2p}$. We introduce the affine models by restricting the domain and suppressing the metric:

$$\mathfrak{A}_{3+2p,k} := (\mathbb{R}^{3+2p}, B^0, ..., B^k)\,.$$

Lemma 4.1. $\dim\{G(\mathfrak{M}_{6+4p,k})\} = \dim\{G(\mathfrak{A}_{3+2p,k})\} + (p+1)(3+2p)$.

Proof. Let $\mathfrak{o}(s)$ be Lie algebra of skew-symmetric $s \times s$ real matrices. Set

$$\begin{aligned}
\mathcal{S} : &= (S_1, ..., S_{3+2p}) = (X, Y, Z_1..., Z_p, \tilde{Y}, \tilde{Z}_1..., \tilde{Z}_p),\\
\mathcal{S}^* : &= (S_1^*, ..., S_{3+2p}^*) = (X^*, Y^*, Z_1^*, ..., Z_p^*, \tilde{Y}^*, \tilde{Z}_1^*, ..., \tilde{Z}_p^*),\\
\mathcal{K} : &= \{\xi \in \mathbb{R}^{6+4p} : A^0(\xi, \eta_1, \eta_2, \eta_3) = 0\ \forall\ \eta_i \in \mathbb{R}^{6+4p}\}\\
&= \text{Span}\{S_1^*, ..., S_{3+2p}^*\}\,.
\end{aligned}$$

Let $g \in G(\mathfrak{M}_{6+4p,k})$. The space $\mathcal{K}$ is preserved by g. Thus

$$gS_i = \textstyle\sum_{i,j}\{g_{0,ij}S_j + g_{1,ij}S_j^*\} \quad \text{and} \quad gS_i^* = \textstyle\sum_{i,j}\{g_{2,ij}S_j^*\}\,.$$

By Equation (2.a), $\langle gS_i, gS_j\rangle = 0$ and $\langle gS_i, gS_j^*\rangle = \delta_{ij}$. Thus

$$\textstyle\sum_k\{g_{0,ik}g_{1,jk} + g_{1,ik}g_{0,jk}\} = 0 \quad \text{and} \quad \textstyle\sum_k\{g_{0,ik}g_{2,jk}\} = \delta_{ij}\,.$$

for all i, j. Set $\gamma := g_0 g_1^t$. One then has

$$g_0 \in G(\mathfrak{A}_{3+2p,k}), \quad \gamma + \gamma^t = 0, \quad \text{and} \quad g_0 g_2^t = \text{id}\,. \tag{4.a}$$

Conversely, if Equation (4.a) is satisfied then $g \in G(\mathfrak{M}_{6+4p,k})$. The map $g \to (g_0, \gamma)$ yields an identification of

$$G(\mathfrak{M}_{6+4p,k}) = G(\mathfrak{A}_{3+2p,k}) \times \mathfrak{o}(3+2p)$$

as a twisted product. The Lemma follows as $\dim\{\mathfrak{o}(3+2p)\} = \frac{1}{2}(3+2p)(2+2p)$. □

There is a natural action of $G(\mathfrak{A}_{3+2p,k})$ on $\mathbb{R}^{3+2p}$. We continue our study by relating $G(\mathfrak{A}_{3+2p,k})$ and the isotropy subgroup $G_X(\mathfrak{A}_{3+2p,k})$.

Lemma 4.2.

1. $\dim\{G(\mathfrak{A}_{3+2p,k})\} = \dim\{G_X(\mathfrak{A}_{3+2p,k})\} + 2p + 3$ *for* $k \le p+1$.
2. $\dim\{G(\mathfrak{A}_{3+2p,p+2})\} = \dim\{G_X(\mathfrak{A}_{3+2p,p+2})\} + 2p + 2$.

Proof. Lemma 4.2 will follow from Lemma 3.2 and the following relations:

$$\begin{aligned}
&G(\mathfrak{A}_{3+2p,k})X = \{\xi \in \mathbb{R}^{3+2p} : \langle \xi, X^*\rangle \neq 0\} \ \text{ if } \ k \le p+1,\\
&G(\mathfrak{A}_{3+2p,p+2})X = \{\xi \in \mathbb{R}^{3+2p} : \langle \xi, X^*\rangle = \pm 1\}\,.
\end{aligned} \tag{4.b}$$

We first show $\supset$ holds in Equation (4.b). Let $\xi \in \mathbb{R}^{3+2p}$. Assume that

$$a := \langle \xi, X^*\rangle \neq 0\,.$$

Set $gX = \xi$ and set

$$\begin{aligned}
&\varepsilon_0 := (a^2)^{-1/(p+3)}, \quad gY := \varepsilon_0 Y, \quad g\tilde{Y} := a^{-2}\varepsilon_0^{-1}\tilde{Y},\\
&\varepsilon_i := \{a^2\varepsilon_0^{i+1}\}^{-1}, \quad gZ_i := \varepsilon_i Z_i, \quad gZ_i^* := \varepsilon_i^{-1}a^{-2}\tilde{Z}_i\,.
\end{aligned}$$

The non-zero components of $\nabla^i R$ for $1 \le i \le p+2$ are then given by

$$\begin{aligned}
&R(gX,gY,g\tilde{Y},gX) = a^2\varepsilon_0 a^{-2}\varepsilon_0^{-1} = 1,\\
&R(gX,gZ_i,g\tilde{Z}_i,gX) = a^2\varepsilon_i\varepsilon_i^{-1}a^{-2} = 1,\\
&\nabla R(gX,gY,gZ_1,gX;gY) = \nabla R(gX,gY,gY,gX;gZ_1) = a^2\varepsilon_0^2\varepsilon_1 = 1, ...\\
&\nabla^p R(gX,gY,gZ_p,gX;gY,...,gY) = \nabla^p R(gX,gY,gY,gX;gZ_p,gY,...,gY) = ...\\
&\quad = \nabla^p R(gX,gY,gY,gX;gY,...,gY,gZ_p) = a^2\varepsilon_0^{p+1}\varepsilon_p = 1,\\
&\nabla^{p+1} R(gX,gY,gY,gX;gY,...,gY) = a^2\varepsilon_0^{p+3} = 1,\\
&\nabla^{p+2} R(gX,gY,gY,gX;gY,...,gY) = a^2\varepsilon_0^{p+4} = \varepsilon_0\,.
\end{aligned}$$

Thus $g \in G(\mathfrak{A}_{3+2p,p+1})$. Furthermore, $g \in G(\mathfrak{A}_{3+2p,p+2})$ if $a^2 = 1$. Consequently:

$$\begin{aligned}
&\{\xi \in \mathbb{R}^{3+2p} : \langle \xi, X^*\rangle \neq 0\} \subset G(\mathfrak{A}_{3+2p,k})\cdot X \quad \text{for} \quad k \le p+1,\\
&\{\xi \in \mathbb{R}^{3+2p} : \langle \xi, X^*\rangle = \pm 1\} \subset G(\mathfrak{A}_{3+2p,p+2})\cdot X\,.
\end{aligned} \tag{4.c}$$

We must establish the reverse inclusions to complete the proof. Let $\xi \in \mathbb{R}^{3+2p}$. Let $J_\xi(\eta_1,\eta_2) := R(\xi,\eta_1,\eta_2,\xi)$ be the *Jacobi form.* Adopt the Einstein convention and sum over repeated indices to expand

$$\xi = aX + b^i Z_i + \tilde{b}^i \tilde{Z}_i$$

where $a = \langle \xi, X^*\rangle$. We have the following cases

(1) If $a = 0$, then $J_\xi = 0$ on $\operatorname{Span}\{Y,\tilde{Y},Z_i,\tilde{Z}_i\}$ so $\operatorname{Rank}(J_\xi) \le 1$.
(2) If $a \neq 0$, then $J_\xi(Y,\tilde{Y}) \neq 0$ so $\operatorname{Rank}(J_\xi) \ge 2$.

If $g \in G(\mathfrak{A}_{3+2p,k})$, then $\operatorname{Rank}\{J_\xi\} = \operatorname{Rank}\{J_{g\xi}\}$. Consequently

$$\langle \xi, X^*\rangle = 0 \Leftrightarrow \operatorname{Rank}(J_\xi) \le 1 \Leftrightarrow \operatorname{Rank}(J_{g\xi}) \le 1 \Leftrightarrow \langle g\xi, X^*\rangle = 0$$

Consequently we have

$$\begin{aligned}
&G(\mathfrak{A}_{3+2p,k})\cdot X \subset \{\xi \in \mathbb{R}^{3+2p} : \langle \xi, X^*\rangle \neq 0\},\\
&G(\mathfrak{A}_{3+2p,k})\cdot \operatorname{Span}\{Y,Z_i,\tilde{Z}_i\} = \operatorname{Span}\{Y,Z_i,\tilde{Z}_i\}\,.
\end{aligned} \tag{4.d}$$

Suppose $k = p+2$. Since $\operatorname{Rank}(J_Y) = 0$, $\operatorname{Rank}(J_{gY}) = 0$ so $\langle gY, X^*\rangle = 0$. Expand

$$\begin{aligned}
gX &= aX + a_0 Y + \tilde{a}_0\tilde{Y} + a^i Z_i + \tilde{a}^i\tilde{Z}_i,\\
gY &= \qquad\quad b^0 Y + \tilde{b}^0\tilde{Y} + b^i Z_i + \tilde{b}^i\tilde{Z}_i\,.
\end{aligned}$$

Then

$$\begin{aligned}
1 &= \nabla^{p+1}R(gX,gY,gY,gX;gY,...,gY) = a^2(b^0)^{p+3},\\
1 &= \nabla^{p+2}R(gX,gY,gY,gX;gY,...,gY) = a^2(b^0)^{p+4}\,.
\end{aligned}$$

This shows that $a^2 = 1$ and $b^0 = 1$ so

$$\begin{aligned}
&G(\mathfrak{A}_{3+2p,p+2})X \subset \{\xi \in \mathbb{R}^{3+2p} : \langle \xi, X^*\rangle = \pm 1\},\\
&G(\mathfrak{A}_{3+2p,p+2})Y \subset \{\xi \in \mathbb{R}^{3+2p} : \langle \xi, X^*\rangle = 0, \text{ and } \langle \xi, Y^*\rangle = 1\}\,.
\end{aligned} \tag{4.e}$$

Equations (4.c), (4.d), and (4.e) now imply Equation (4.b); the Lemma follows. □

We now consider the double isotropy group

$$G_{X,Y}(\mathfrak{A}_{3+2p,k}) = \{g \in G(\mathfrak{A}_{3+2p,k}) : gX = X \text{ and } gY = Y\}\,.$$

Lemma 4.3.

(1) $\dim\{G_X(\mathfrak{A}_{3+2p,0})\} = (p+1)(2p+1)$.
(2) $\dim\{G_X(\mathfrak{A}_{3+2p,k})\} = \dim\{G_{X,Y}(\mathfrak{A}_{3+2p,k})\} + 2p + 2$ *for* $1 \le k \le p$.
(3) $\dim\{G_X(\mathfrak{A}_{3+2p,k})\} = \dim\{G_{X,Y}(\mathfrak{A}_{3+2p,k})\} + 2p + 1$ *for* $k = p+1, p+2$.
(4) $G_{X,Y}(\mathfrak{A}_{3+2p,p}) = G_{X,Y}(\mathfrak{A}_{3+2p,p+1}) = G_{X,Y}(\mathfrak{A}_{3+2p,p+2})$.

Proof. As noted above, the Jacobi form $J_X(\cdot,\cdot) = R(X,\cdot,\cdot,X)$ defines a non-singular bilinear form of signature $(p+1,p+1)$ on

$$W := \operatorname{Span}\{Y, Z_1, ..., Z_p, \tilde{Y}, \tilde{Z}_1, ..., \tilde{Z}_p\} = \{\xi : \operatorname{Rank}(J_\xi) \le 1\}\,.$$

Let $O(W, J_X)$ be the associated orthogonal group. If $g \in G_X(\mathfrak{A}_{3+2p,k})$, then we have $gW = W$ by Equation (4.d). Since $gX = X$, we may safely identify g with $g|_W$. Furthermore,

$$J_X(\xi,\eta) = J_{gX}(g\xi, g\eta) = J_X(g\xi, g\eta) \quad \text{so} \quad G_X(\mathfrak{A}_{3+2p,k}) \subset O(W, J_X)\,.$$

Conversely, if g is a linear map of W which preserves J_X, we may extend g to $\mathbb{R}^{3+2p}$ by defining $gX = X$ and thereby obtain an element of $G_X(\mathfrak{A}_{3+2p,0})$. Thus $G_X(\mathfrak{A}_{3+2p,0}) = O(W, J_X)$. Assertion (1) now follows since

$$\dim\{O(W, J_X)\} = \tfrac{1}{2}\dim W(\dim W - 1) = \tfrac{1}{2}(1+2p)(2+2p)\,.$$

Assertions (2) and (3) will follow from Lemma 3.2 and from the relations:

$$\begin{aligned} &G_X(\mathfrak{A}_{3+2p,k}) \cdot Y = \{\xi \in W : \langle \xi, Y^*\rangle \ne 0\} \quad \text{for} \quad 1 \le k \le p,\\ &G_X(\mathfrak{A}_{3+2p,p+1}) \cdot Y = \{\xi \in W : \langle \xi, Y^*\rangle^{p+3} = 1\},\\ &G_X(\mathfrak{A}_{3+2p,p+2}) \cdot Y = \{\xi \in W : \langle \xi, Y^*\rangle = 1\}\,. \end{aligned} \tag{4.f}$$

If $\xi \in W$, let $S_\xi(\eta) := \nabla R(X, \xi, \xi, X; \eta)$. Expand

$$\xi = b^0 Y + \tilde{b}^0 \tilde{Y} + b^i Z_i + \tilde{b}^i \tilde{Z}_i\,. \tag{4.g}$$

We then have that

$$\begin{aligned} &S_\xi(X) = 0, \quad S_\xi(\tilde{Z}_i) = 0, \quad S_\xi(Y) = 2b^0 b^1,\\ &S_\xi(Z_1) = (b^0)^2, \quad \text{and} \quad S_\xi(Z_i) = 0 \quad \text{for} \quad i \ge 2\,. \end{aligned}$$

Thus $S_\xi = 0$ if and only if $b^0 = \langle \xi, Y^*\rangle = 0$. It now follows that for $k \ge 1$ we have

$$\begin{aligned} &G_X(\mathfrak{A}_{3+2p,k})Y \subset \{\xi \in W : \langle \xi, Y^*\rangle \ne 0\},\\ &G_X(\mathfrak{A}_{3+2p,k}) \operatorname{Span}\{Z_i, \tilde{Y}, \tilde{Z}_i\} \subset \operatorname{Span}\{Z_i, \tilde{Y}, \tilde{Z}_i\}\,. \end{aligned} \tag{4.h}$$

Since $a = 1$, the analysis used to prove Lemma 4.2 shows $(b^0)^{p+3} = 1$ if $k = p+1$ and $b^0 = 1$ if $k = p+2$. This establishes the inclusions $\subset$ in Equation (4.f).

We complete the proof by establishing the reverse inclusions in Equation (4.f). Expand ξ in the form given in Equation (4.g). Assume $b^0 \ne 0$. Let $gX = X$, $gY = \xi$, $g\tilde{Y} = (b^0)^{-1}\tilde{Y}$,

$$gZ_i := \varepsilon_i\{Z_i - (b^0)^{-1}\tilde{b}^i\tilde{Y}\} \quad \text{and} \quad g\tilde{Z}_i := \varepsilon_i^{-1}\{\tilde{Z}_i - (b^0)^{-1}b^i\tilde{Y}\}\,.$$

The possibly non-zero components of R are then given by

$$\begin{aligned} &R(gX, gY, g\tilde{Y}, gX) = 1,\\ &R(gX, gY, gZ_i, gX) = \varepsilon_i\{\tilde{b}^i - (b^0)(b^0)^{-1}\tilde{b}^i\} = 0,\\ &R(gX, gY, g\tilde{Z}_i, gX) = \varepsilon_i^{-1}\{b^i - (b^0)(b^0)^{-1}b^i\} = 0,\\ &R(gX, gZ_i, g\tilde{Z}_i, gX) = \varepsilon_i^{-1}\varepsilon_i = 1\,. \end{aligned}$$

The non-zero components of $\nabla^i R$ for $1 \le i \le p$ are given by

$$\begin{aligned} &\nabla^i R(gX, gY, gZ_i, gX; gY, ..., gY) = ...\\ = \quad &\nabla^i R(gX, gY, gY, gX; gY, ..., gZ_i) = (b^0)^{i+1}\varepsilon_i\,. \end{aligned}$$

We therefore set $\varepsilon_i = (b^0)^{-i-1}$ for $1 \le i \le p$ to ensure $g \in G(\mathfrak{A}_{3+2p,p})$.

The non-zero components of $\nabla^i R$ for $i = p+1, p+2$ are

$$\nabla^i R(gX, gY, gY, gX; gY, ..., gY) = (b^0)^{i+2}\,.$$

If $(b^0)^{p+3} = 1$, then $g \in G(\mathfrak{A}_{3+2p,p+1})$; if $b^0 = 1$, then $g \in G(\mathfrak{A}_{3+2p,p+2})$. This establishes the reverse inclusions in Equation (4.f) and completes the proof of Assertions (2) and (3); Assertion (4) is immediate. $\square$

Let $W(p) := \text{Span}\{Z_1, ..., Z_p, \tilde{Z}_1, ..., \tilde{Z}_p\}$. Let $\{\beta_1, ..., \beta_p, \tilde{\beta}_1, ..., \tilde{\beta}_p\}$ be the corresponding dual basis for the dual space $\mathcal{W}(p) := W(p)^*$. The curvature tensor $R(X, \cdot, \cdot, X)$ defines a non-degenerate form $\langle \cdot, \cdot \rangle$ on $W(p)$; dually on $\mathcal{W}(p)$ we have:

$$\langle \beta_i, \beta_j \rangle = \langle \tilde{\beta}_i, \tilde{\beta}_j \rangle = 0, \quad \langle \beta_i, \tilde{\beta}_j \rangle = \delta_{ij} \,.$$

Let $\mathcal{O}(p)$ be the associated orthogonal group on $\mathcal{W}(p)$. Let

$$\mathcal{O}(p, k) := \{h \in \mathcal{O}(p) : h\beta_i = \beta_i \quad \text{for} \quad 1 \le i \le k\}$$

be the simultaneous isotropy group. We set $\mathcal{O}(p, 0) = \mathcal{O}(p)$. Theorem 1.5 will now follow from the following result:

Lemma 4.4. *Let* $1 \le k \le p$.

1. $G_{X,Y}(\mathfrak{A}_{3+2p,k}) = \mathcal{O}(p, k)$.
2. $\mathcal{O}_{\tilde{\beta}_1}(p, k) = \mathcal{O}(p - 1, k - 1)$.
3. $\dim\{\mathcal{O}(p, k)\} = \dim\{\mathcal{O}(p - 1, k - 1)\} + 2p - k - 1$.
4. $\dim\{\mathcal{O}(p, k)\} = \frac{1}{2}(2p - k)(2p - k - 1)$.

Proof. Let $g \in G_{X,Y}(\mathfrak{A}_{3+2p,k})$. Let $\xi \in \text{Span}\{Z_1, ..., Z_p, \tilde{Y}, \tilde{Z}_1, ..., \tilde{Z}_p\}$. We may use Equation (4.h) and the relation $R(X, Y, g\xi, X) = R(X, Y, \xi, X)$, to see

$$g\tilde{Y} = \tilde{Y} + a^i Z_i + a^{\tilde{i}} \tilde{Z}_i, \quad gZ_i = a_i^j Z_j + a_i^{\tilde{j}} \tilde{Z}_{\tilde{j}}, \quad g\tilde{Z}_{\tilde{i}} = a_{\tilde{i}}^j Z_j + a_{\tilde{i}}^{\tilde{j}} \tilde{Z}_{\tilde{j}} \,.$$

Consequently $\text{Span}_{1 \le i \le p}\{gZ_i, g\tilde{Z}_{\tilde{i}}\} = \text{Span}_{1 \le i \le p}\{Z_i, \tilde{Z}_{\tilde{i}}\}$ and the relation

$$R(X, gZ_i, g\tilde{Y}, X) = R(X, g\tilde{Z}_{\tilde{i}}, g\tilde{Y}, X) = 0$$

implies $a^i = a^{\tilde{i}} = 0$. Thus $g\tilde{Y} = \tilde{Y}$ and $g : W(p) \to W(p)$; this shows that g is determined by its restriction to $W(p)$. Let $h := {}^*g$ denote the dual action of g on $\mathcal{W}(p)$. The isomorphism of Assertion (1) now follows as:

$$\begin{aligned}
&R(X, g\xi_1, g\xi_2, R) = R(X, \xi_1, \xi_2, X) \ \forall \xi_1, \xi_2 \Leftrightarrow h \in \mathcal{O}(p) \,,\\
&\nabla^i R(X, Y, g\xi, X; Y, ..., Y) = \nabla^i R(X, Y, \xi, X; Y, ..., Y) \ \forall \xi \Leftrightarrow h\beta_i = \beta_i \,.
\end{aligned}$$

If $h(\beta_1) = \beta_1$ and $h(\tilde{\beta}_1) = \tilde{\beta}_1$, then h preserves

$$\text{Span}\{\beta_1, \tilde{\beta}_1\}^\perp = \text{Span}\{\beta_2, ..., \beta_p, \tilde{\beta}_2, ..., \tilde{\beta}_p\} \,.$$

The isomorphism of Assertion (2) now follows by restricting h to this subspace and by renumbering the variables appropriately.

We set

$$\mathcal{W}(p, k) := \{\xi \in \mathcal{W}(p) : \langle \xi, \xi \rangle = 0, \ \langle \xi, \beta_1 \rangle = 1, \ \langle \xi, \beta_i \rangle = 0 \text{ for } 2 \le i \le k\} \,.$$

If $h \in \mathcal{O}(p, k)$, then h preserves $\langle \cdot, \cdot \rangle$ and h preserves $\{\beta_1, ..., \beta_k\}$. Consequently $h\tilde{\beta}_1 \in \mathcal{W}(p, k)$ as $\tilde{\beta}_1$ satisfies these relations. Conversely, $\xi \in \mathcal{W}(p, k)$ if and only if

$$\xi = b^1 \beta_1 + \sum_{1 < i} b^i \beta_i + \tilde{\beta}_1 + \sum_{k < i} \tilde{b}^i \tilde{\beta}_i \quad \text{where} \quad b^1 + \sum_{k < i} b^i \tilde{b}^i = 0 \,.$$

Since the variables $\{b^2, ..., b^p, \tilde{b}^{k+1}, ..., \tilde{b}^p\}$ can be chosen arbitrarily,

$$\mathcal{W}(p, k) = \mathbb{R}^{p-1+p-k} \quad \text{so} \quad \dim \mathcal{W}(p, k) = 2p - k - 1 \,.$$

We show that $\xi \in \mathcal{O}(p, k)\tilde{\beta}_1$ by finding $h \in \mathcal{O}(p, k)$ so $h\tilde{\beta}_1 = \xi$. Set:

$$\begin{aligned}
&h\beta_i = \beta_i \quad \text{for } 1 \le i \le k, \quad && h\beta_i = \beta_i - \tilde{b}^i \beta_1 \quad \text{for } k < i,\\
&h\tilde{\beta}_1 = \xi, && h\tilde{\beta}_i = \tilde{\beta}_i - b^i \beta_1 \quad \text{for } 1 < i \,.
\end{aligned}$$

This shows $\mathcal{O}(p, k) \cdot \tilde{\beta}_1 = \mathcal{W}(p, k)$. Assertion (3) now follows from Assertion (2) and from Lemma 3.2.

As $\dim\{\mathcal{O}(p-k)\} = \frac{1}{2}(2p-2k)(2p-2k-1)$, Assertion (4) follows by induction. □

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