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manifolds with Zaremba boundary conditions

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HEAT CONTENT ASYMPTOTICS FOR RIEMANNIAN MANIFOLDS WITH ZAREMBA BOUNDARY CONDITIONS

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ABSTRACT. The existence of a full asymptotic expansion for the heat content asymptotics of an operator of Laplace type with classical Zaremba boundary conditions on a smooth manifold is established. The first three coefficients in this asymptotic expansion are determined in terms of geometric invariants; partial information is obtained about the fourth coefficient.

1. INTRODUCTION

Let (M, g) be a smooth compact m -dimensional manifold with smooth boundary ∂M and let V be a smooth vector bundle over M . Let

$$D = -(a^{ij} \text{Id} \cdot \partial_i \partial_j + b^k \partial_k + c)$$

be a smooth second order operator over M with scalar leading symbol; we adopt the Einstein convention and sum over repeated indices. We assume the matrix $\{a^{ij}\}$ is positive definite and use the inverse matrix g_{ij} to define a Riemannian metric on M .

We can write the operator D invariantly as follows. There is a unique connection ∇ on V and a unique endomorphism E of V so that

$$D = D(\nabla, E) = -(a^{ij} \nabla_i \nabla_j + E).$$

The connection 1 form and endomorphism E are given in terms of the derivatives of the total symbol of D and the Christoffel symbols Γ by:

$$(1.a) \quad \begin{aligned} \omega_i &= \frac{1}{2} g_{ij} b^j + \frac{1}{2} a^{kl} \Gamma_{kli} \\ E &= c - a^{ij} (\partial_i \omega_j + \omega_i \omega_j - \omega_k \Gamma_{ij}{}^k). \end{aligned}$$

The boundary conditions we shall impose are at the heart of the matter. We assume given a decomposition $\partial M = C_{\mathcal{R}} \cup C_{\mathcal{D}}$ as the union of two closed submanifolds with common smooth boundary $C_{\mathcal{R}} \cap C_{\mathcal{D}} = \Sigma$. Let $\phi_{;m}$ denote the covariant derivative of ϕ with respect to the inward unit normal on ∂M . Let S be an auxiliary endomorphism of $V|_{C_{\mathcal{R}}}$. We take Robin boundary conditions on $C_{\mathcal{R}}$ and Dirichlet boundary conditions on $C_{\mathcal{D}}$ arising from the boundary operator:

$$(1.b) \quad \mathcal{B}\phi := (\phi_{;m} + S\phi)|_{\{C_{\mathcal{R}} - \Sigma\}} \oplus \phi|_{C_{\mathcal{D}}}.$$

We refer to Seeley [17, 18] for a more general formalism; see also related work of Avramidi [1], Dowker [7, 8], and Jakobson et al. [11].

Let $e^{-tD_{\mathcal{B}}}$ be the fundamental solution of the heat equation; $u = e^{-tD_{\mathcal{B}}}\phi$ is then characterized by the equations:

$$(1.c) \quad (\partial_t + D)u = 0, \quad u(x; 0) = \phi(x), \quad \mathcal{B}u(x, t) = 0 \text{ for } t > 0.$$

The equality $u(x; 0) = \phi(x)$ is to be taken in the L^2 sense where $\phi \in C^\infty(M; V)$ is a smooth section to V which gives the initial temperature distribution. Let $\phi^* \in C^\infty(M; V^*)$ be a smooth section to the dual bundle V^* which gives the specific heat of the manifold. We denote the natural pairing between V and V^* by

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$\langle \cdot, \cdot \rangle$. Let dx , dx' , and dz be the Riemannian measures on M , on ∂M , and on Σ , respectively. We define the *total heat energy content* of the manifold by setting:

$$\beta(\phi, \phi^*, D, \mathcal{B})(t) := \int_M \langle e^{-tD_{\mathcal{B}}} \phi, \phi^* \rangle dx.$$

It is worth putting this in a more classical framework in the special case where $D = \Delta$ is the Laplacian and $S = 0$. Let $W^{1,2}(M)$ be the closure of $C^\infty(M)$ with respect to the Sobolev norm

$$\|\phi\|_1^2 = \int_M \{|\nabla\phi|^2 + |\phi|^2\} dx.$$

Let $W_{0,C_D}^{1,2}(M)$ be the closure of the set

$$\{\phi \in W^{1,2}(M) : \text{supp}(\phi) \cap C_D = \emptyset\}.$$

Thus, for example, $W_{0,\emptyset}^{1,2}(M) = W^{1,2}(M)$. Let $W_0^{1,2}(M) = W_{0,\partial M}^{1,2}(M)$. For $\lambda > 0$, let

$$N(M, C_D, \lambda) = \sup(\dim E_\lambda)$$

where the supremum is taken over all subspaces $E_\lambda \subset W_{0,C_D}^{1,2}(M)$ such that

$$\|\nabla\phi\|_{L^2(M)} < \lambda \|\phi\|_{L^2(M)}, \quad \forall \phi \in E_\lambda.$$

Then $N(M, \emptyset, \lambda)$ is the spectral counting function for the Neumann Laplacian on M , $N(M, \partial M, \lambda)$ is the spectral counting function for the Dirichlet Laplacian on M , and $N(M, C_D, \lambda)$ is the spectral counting function for the Laplacian acting in $L^2(M)$ with Dirichlet conditions on C_D and Neumann conditions on $\partial M - C_D$.

It is well known, see for example McKean and Singer [16], that since M is compact and ∂M is smooth, $N(M, \emptyset, \lambda)$ is finite. By the variational principle,

$$N(M, \partial M, \lambda) \leq N(M, C_D, \lambda) \leq N(M, \emptyset, \lambda);$$

consequently $N(M, C_D, \lambda)$ is finite and counts the number of eigenvalues less than λ with the Zaremba boundary condition defined by C_D . Let $\lambda_1 \leq \lambda_2 \leq \dots$ be the eigenvalues counted by $N(M, C_D, \cdot)$ and let $\{\phi_i\}$ be a corresponding orthonormal basis of eigenfunctions in $L^2(M)$. If ϕ and ϕ^* are smooth, we can express the heat content in terms of the Fourier coefficients:

$$(1.d) \quad \beta(\phi, \phi^*, D, \mathcal{B})(t) = \sum_i e^{-t\lambda_i} \langle \phi, \phi_i \rangle_{L^2(M)} \langle \phi^*, \phi_i \rangle_{L^2(M)}.$$

By Parseval's identity, the series converges for $\phi = \phi^* = 1$ and for all $t > 0$:

$$\begin{aligned} \beta(1, 1, D, \mathcal{B})(t) &= \sum_i e^{-t\lambda_i} \{\langle 1, \phi_i \rangle_{L^2(M)}\}^2 \\ &\leq e^{-t\lambda_1} \sum_i \langle 1, \phi_i \rangle_{L^2(M)}^2 = \text{vol}(M) e^{-t\lambda_1}. \end{aligned}$$

It now follows that the series in Equation (1.d) converges for all smooth ϕ and ϕ^* and for all $t > 0$.

We return to the general setting. Adopt the notation established above.

Theorem 1.1. *Let $\phi \in C^\infty(M; V)$ and let $\phi^* \in C^\infty(M; V^*)$. There exists a complete asymptotic expansion $\beta(\phi, \phi^*, D, \mathcal{B})(t) \sim \sum_{n \geq 0} \beta_n(\phi, \phi^*, D, \mathcal{B}) t^{n/2}$ where the β_n are locally computable in terms of integrals over M , C_D , $C_{\mathcal{R}}$, and Σ .*

We use the dual connection $\tilde{\nabla}$ to covariantly differentiate sections of V^* . Let the dual endomorphism \tilde{S} on V^* define the dual boundary condition $\tilde{\mathcal{B}}$ for the dual operator \tilde{D} on $C_{\mathcal{R}}$. Near the boundary, we choose a local orthonormal frame $\{e_i\}$ for the tangent bundle of M so that e_m is the inward unit geodesic normal vector field; on Σ , we assume e_{m-1} is the inward unit normal of $\Sigma \subset C_D$. Let indices a, b range from 1 through $m-1$ and index the induced orthonormal frame for the

tangent bundle of the boundary; let indices u, v range from 1 through $m - 2$ and index the induced orthonormal frame for Σ . Let $L_{ab} := \Gamma_{abm}$ and $\tilde{L}_{uv} := \Gamma_{uv(m-1)}$ be the components of the second fundamental forms of $\partial M \subset M$ and $\Sigma \subset C_{\mathcal{D}}$, respectively. Let R be the Riemann curvature tensor with the sign convention that $R_{1221} = +1$ for the standard sphere in \mathbb{R}^3 .

Theorem 1.2. *There exist universal constants c_i so that:*

- (1) $\beta_0(\phi, \phi^*, D, \mathcal{B}) = \int_M \langle \phi, \phi^* \rangle dx$.
- (2) $\beta_1(\phi, \phi^*, D, \mathcal{B}) = -\frac{2}{\sqrt{\pi}} \int_{C_{\mathcal{D}}} \langle \phi, \phi^* \rangle dx'$.
- (3) $\beta_2(\phi, \phi^*, D, \mathcal{B}) = -\int_M \langle D\phi, \phi^* \rangle dx + \int_{C_{\mathcal{R}}} \{ \langle \phi_{;m} + S\phi, \phi^* \rangle \} dx'$
 $+ \int_{C_{\mathcal{D}}} \{ \frac{1}{2} L_{aa} \langle \phi, \phi^* \rangle - \langle \phi, \phi_{;m}^* \rangle \} dx' + c_0 \int_{\Sigma} \langle \phi, \phi^* \rangle dz$.
- (4) $\beta_3(\phi, \phi^*, D, \mathcal{B}) = \frac{4}{3\sqrt{\pi}} \int_{C_{\mathcal{R}}} \langle \phi_{;m} + S\phi, \phi_{;m}^* + \tilde{S}\phi^* \rangle dx'$
 $-\frac{2}{\sqrt{\pi}} \int_{C_{\mathcal{D}}} \{ \frac{2}{3} \langle \phi_{;mm}, \phi^* \rangle + \frac{2}{3} \langle \phi, \phi_{;mm}^* \rangle - \langle \phi_{;a}, \phi_{;a}^* \rangle + \langle E\phi, \phi^* \rangle$
 $-\frac{2}{3} L_{aa} \langle \phi, \phi^* \rangle_{;m} + \langle (\frac{1}{12} L_{aa} L_{bb} - \frac{1}{6} L_{ab} L_{ab} + \frac{1}{6} R_{amam}) \phi, \phi^* \rangle \} dx'$
 $+ \int_{\Sigma} \{ \langle (c_1 L_{m-1, m-1} + c_2 L_{uu} + c_3 \tilde{L}_{uu} + c_4 S) \phi, \phi^* \rangle$
 $+ c_5 \langle \phi, \phi^* \rangle_{;m-1} + c_6 \langle \phi, \phi^* \rangle_{;m} \} dz$.
- (5) *We have $c_0 = -\frac{1}{2}$, $c_3 = \frac{1}{2\sqrt{\pi}}$, $c_5 = \frac{1}{2\sqrt{\pi}}$, $c_6 = -\frac{2}{3\sqrt{\pi}}$.*
- (6) *We have $c_2 - \frac{1}{2}c_4 = \frac{2}{3\sqrt{\pi}}$.*

Remark 1.3. Our methods did not yield c_1 . They also did not permit us to complete the computation of $\{c_2, c_4\}$.

Theorem 1.2 follows if Σ is empty from results of [2, 4]; the new feature here is the additional integrands over Σ present in β_2 and β_3 and the partial information we have obtained concerning these terms.

Here is a brief guide to the paper. In Section 2, we use invariance theory to establish Assertions (1-4) of Theorem 1.2. In Section 3, we use product formulas and make a special case computation to show

$$(1.e) \quad c_0 = -\frac{1}{2}, \quad \frac{1}{\sqrt{\pi}} - c_3 - c_5 = 0, \quad \text{and} \quad c_2 - \frac{1}{2}c_4 + c_6 = 0.$$

In Section 4 we use special cases on the half-plane to complete the proof by showing

$$(1.f) \quad c_3 = \frac{1}{2\sqrt{\pi}}, \quad c_5 = \frac{1}{2\sqrt{\pi}}, \quad \text{and} \quad c_6 = -\frac{2}{3\sqrt{\pi}}.$$

The remainder of the paper is devoted to the proof of Theorem 1.1 using results of [13, 14, 15]; we believe the methods of Seeley [17, 18] could also be used. We switch focus completely at this stage. Instead of working invariantly and globally in the context of Riemannian manifolds using methods of invariance theory, we work locally in Euclidean space in a system of local coordinates. In Section 5, we introduce the function spaces which we shall need. In Section 6, we discuss various ‘model’ problems and in Section 7, we formulate a basic theorem on asymptotics. We obtain a number of estimates and conclude the proof of Theorem 1.1 in Section 8 by establishing a slightly more general result (see Theorem 8.2).

2. UNIVERSAL EXPRESSIONS FOR THE INVARIANTS β_n^Σ

Let β_n^Σ denote the additional invariant defined by integration over Σ . Dimensional analysis then shows that β_n^Σ can be computed by integrating invariants which are homogeneous of weight $n - 2$ in the jets of the symbol; see the discussion in [2, 4] on this point where a similar analysis was performed which studied the interior invariants and the boundary invariants for Dirichlet or Neumann problems. Thus trivially, $\beta_0^\Sigma = \beta_1^\Sigma = 0$, and therefore the first interesting contribution arises at the

β_2 level; this must be a constant multiple of $\langle \phi, \phi^* \rangle$. Assertions (1-3) of Theorem 1.2 now follow.

To study the form of the additional boundary integral over Σ appearing in β_3 , we investigate the local geometry near Σ . Fix a point $z_0 \in \Sigma$. Let $z = (z_1, \dots, z_{m-2})$ be local coordinates on Σ so

$$g(\partial_u^z, \partial_v^z)(z_0) = \delta_{uv} \quad \text{and} \quad \partial_u^z g(\partial_v^z, \partial_w^z)(z_0) = 0 \quad \text{for} \quad 1 \leq u, v, w \leq m-2.$$

By considering the geodesic flow from $\Sigma \subset C_{\mathcal{D}}$, we introduce coordinates

$$(z, y_1) \rightarrow \exp_z \{y_1 e_{m-1}(z)\}$$

so that y_1 is the signed geodesic distance from Σ to ∂M ; $C_{\mathcal{D}}$ corresponds to $y_1 \geq 0$ and $C_{\mathcal{R}}$ corresponds to $y_1 < 0$. The metric then satisfies:

$$g(\partial_u^z, \partial_1^y) = 0 \quad \text{and} \quad g(\partial_1^y, \partial_1^y) = 1.$$

Now use the geodesic flow of ∂M in M to introduce coordinates

$$(z, y_1, y_2) \rightarrow \exp_{(z, y_1)} \{y_2 e_m(z, y_1)\}$$

where y_2 is the geodesic distance to ∂M . We then have

$$g(\partial_u^z, \partial_2^y) = g(\partial_1^y, \partial_2^y) = 0 \quad \text{and} \quad g(\partial_2^y, \partial_2^y) = 1.$$

The only non-zero derivatives of the metric at z_0 are then given by the second fundamental forms:

$$\begin{aligned} \tilde{L}_{uv}(z_0) &= -\frac{1}{2} \partial_1^y g(\partial_u^z, \partial_v^z)(z_0), \\ L_{uv}(z_0) &:= -\frac{1}{2} \partial_2^y g(\partial_u^z, \partial_v^z)(z_0), \quad \text{and} \\ L_{m-1, m-1}(z_0) &:= -\frac{1}{2} \partial_2^y g(\partial_1^y, \partial_1^y)(z_0). \end{aligned}$$

The structure group is the orthogonal group $O(m-2)$ and we apply H. Weyl's Theorem [19] on the invariants of the orthogonal group. Assertion (2) now follows by writing down a basis for the set of invariants of weight 1 and applying the symmetry

$$\beta_n(\phi, \phi^*, D, \mathcal{B}) = \beta_n(\phi^*, \phi, \tilde{D}, \tilde{\mathcal{B}})$$

where \tilde{D} and $\tilde{\mathcal{B}}$ are the dual operator and dual boundary condition on the dual bundle V^* , respectively. The usual product and addition formulas then show the constants are dimension free and universal. This completes the proof of Theorem 1.2 (4).

3. RELATIONS AMONG THE UNIVERSAL COEFFICIENTS

The universal coefficient c_0 of Theorem 1.2 (3) can be determined by a special case calculation. Let M_+ be a compact convex subset of \mathbb{R}^2 with non-empty interior and smooth boundary ∂M_+ . We suppose that ∂M_+ contains a closed line segment Λ of positive length. Let M_- be the reflection of M_+ with respect to the line determined by Λ . We assume that $\Lambda = \partial M_+ \cap \partial M_-$. We set $N = M_+ \cup M_-$.

Let $\Delta = -\partial_1^2 - \partial_2^2$ be the usual flat Laplacian. Let

$$\begin{aligned} C_{\mathcal{R}}(M_{\pm}) &:= \Lambda, \quad C_{\mathcal{D}}(M_{\pm}) := \partial M_{\pm} - \Lambda, \\ C_{\mathcal{R}}(N) &:= \emptyset, \quad C_{\mathcal{D}}(N) := \partial N = C_{\mathcal{D}}(M_+) \cup C_{\mathcal{D}}(M_-). \end{aligned}$$

We take $\phi = \phi^* = 1$ and $S = 0$ to define u_{\pm} on M_{\pm} and u_N on N . Let

$$(3.a) \quad c(\gamma) := 4 \int_0^{\infty} \frac{\sinh((\pi - \gamma)s)}{\sinh(\pi s) \cdot \cosh(\gamma s)} ds.$$

The manifold N has two cusps of angle 2π at $\partial\Lambda$. The results of [3, 5] can be used to compute β_n , while Theorem 1.2 (3) can be used to see:

$$\begin{aligned}
(3.b) \quad \beta_N(\phi, \phi^*, \Delta, \mathcal{B})(t) &= \int_N dx - \sqrt{t} \left\{ \frac{2}{\sqrt{\pi}} \int_{\partial N} dx' \right\} \\
&+ t \left\{ \frac{1}{2} \int_{\partial N} L_{aa} dx' + 2c(2\pi) \right\} + O(t^{\frac{3}{2}}), \\
\beta_{M_+}(\phi, \phi^*, \Delta, \mathcal{B})(t) &= \int_{M_+} dx - \sqrt{t} \left\{ \frac{2}{\sqrt{\pi}} \int_{\partial M_+} dx' \right\} \\
&+ t \left\{ \frac{1}{2} \int_{C_{\mathcal{D}}(M_+)} L_{aa} dx' + 2c_0 \right\} + O(t^{\frac{3}{2}}).
\end{aligned}$$

We have by symmetry that $u_N(\phi, \phi^*, \Delta, \mathcal{B}) = u_{\pm}(\phi, \phi^*, \Delta, \mathcal{B})$ if $x \in M_{\pm}$ and $t > 0$ and trivially the normal derivative of N vanishes on Λ . Thus:

$$\begin{aligned}
(3.c) \quad \beta_N(\phi, \phi^*, \Delta, \mathcal{B})(t) &= \beta_{M_+}(\phi, \phi^*, \Delta, \mathcal{B})(t) + \beta_{M_-}(\phi, \phi^*, \Delta, \mathcal{B})(t) \\
&= 2\beta_{M_+}(\phi, \phi^*, \Delta, \mathcal{B})(t).
\end{aligned}$$

Since $\partial N = C_{\mathcal{D}}(M_+) \cup C_{\mathcal{D}}(M_-)$, we may use equations (3.b) and (3.c) to see that $4c_0 = 2c(2\pi)$. By Equation (3.a), $c(2\pi) = -1$. We may therefore conclude

$$c_0 = -\frac{1}{2}.$$

We use warped product formulae to obtain the two relationships between the coefficients given in Equation (1.e). Let $M_1 := [0, 1] \times S^1$ be the cylinder with the usual parameters (r, θ) . Set:

$$\begin{aligned}
ds_1^2 &:= dr^2 + d\theta^2, & C_{\mathcal{D}} &:= \{0, 1\} \times [0, \pi], \\
C_{\mathcal{R}} &:= \{0, 1\} \times [\pi, 2\pi], & \Sigma &:= \{0, 1\} \times \{0, \pi\}, \\
\Delta_1 &:= -\partial_r^2 - \partial_{\theta}^2, & S_1 &:= 0, \\
\phi_1 &:= 1, & \phi_1^* &:= 1.
\end{aligned}$$

Since Σ is discrete, dz is counting measure. Since all the structures are flat,

$$\beta_n(\phi_1, \phi_1^*, \Delta_1, \mathcal{B}_1) = 0 \quad \text{for } n \geq 3.$$

Let ε be a small real parameter. Let $M_2 := M_1 \times S^1$ and let Θ be the usual periodic parameter on the second circle. Let $f = f(r, \theta)$ be a smooth warping function and define

$$\begin{aligned}
ds_2^2 &= ds_1^2 + e^{2\varepsilon f(r, \theta)} d\Theta^2, & D_2 &:= \Delta_1 - e^{-2\varepsilon f(r, \theta)} \partial_{\Theta}^2, \\
d\text{vol}_2 &= e^{\varepsilon f} dr d\theta d\Theta.
\end{aligned}$$

We take the warping function to vanish identically near $r = 1$ and focus attention on $r = 0$. Note that D_2 is not self-adjoint. We let $\mathcal{B}_2 = \mathcal{B}_1$ induce the same boundary conditions; we must adjust S_2 appropriately to once again take pure Neumann boundary conditions on $C_{\mathcal{R}} \times S^1$ as the connection induced by D_2 having a non-trivial connection 1 form. We let $\phi_2 = 1$, but we set $\phi_2^* = e^{-\varepsilon f(r, \theta)}$ to compensate for the change in the volume element. Set $u_2 = u_1$. We verify that $u_2 = e^{-t\Delta_2, \mathcal{B}_2} \phi_2$ by computing:

$$\begin{aligned}
(\partial_t + D_2)u_2 &= (\partial_t + \Delta_1)u_1 = 0, \\
u_2(r, \theta, \Theta; 0) &= u_1(r, \theta; 0) = 1, \\
\mathcal{B}_2 u_2 &= 0.
\end{aligned}$$

Consequently we may compute:

$$\begin{aligned} \beta(1, \phi_2^*, \Delta_2, \mathcal{B}_2)(t) &= \int_{M_2} u_2(r, \theta, \Theta; t) \phi_2^*(r, \theta, \Theta) e^{\varepsilon f(r, \theta)} dr d\theta d\Theta \\ &= 2\pi \int_{M_1} u_1(r, \theta, \Theta; t) dr d\theta = 2\pi \beta(1, 1, \Delta_1, \mathcal{B}_1)(t), \quad \text{so} \\ (3.d) \quad \beta_3(1, \phi_2^*, D_2, \mathcal{B}_2) &= 2\pi \beta_3(1, 1, \Delta_1, \mathcal{B}_1) = 0. \end{aligned}$$

First take $f(r, \theta) = f(\theta)$ to be independent of the radial parameter near $r = 0$. We use equation (1.a) to see:

$$\begin{aligned} \omega_r &= \omega_\Theta = 0, & \omega_\theta &= -\frac{\varepsilon}{2} f_\theta, \\ \nabla_\theta \phi_2 &= (\partial_\theta + \omega_\theta) \phi_2 = -\frac{\varepsilon}{2} f_\theta, & \tilde{\nabla}_\theta \phi_2^* &= (\partial_\theta - \omega_\theta) \phi_2^* = -\frac{\varepsilon}{2} f_\theta, \\ -\phi_{2;a} \phi_{2;a}^* &= -\frac{\varepsilon^2}{4} f_\theta f_\theta e^{-\varepsilon f(\theta)}, & E &= \frac{\varepsilon}{2} f_{\theta\theta} + \frac{\varepsilon^2}{4} f_\theta f_\theta e^{-\varepsilon f(\theta)}. \end{aligned}$$

Consequently we have

$$(3.e) \quad \beta_3^{C_D}(\phi_2, \phi_2^*, D_2, \mathcal{B}) = -\frac{2}{\sqrt{\pi}} \int_{C_D} \frac{\varepsilon}{2} f_{\theta\theta} d\theta d\Theta = \frac{\varepsilon}{\sqrt{\pi}} \int_\Sigma f_\theta d\Theta.$$

Note that $c_3 \tilde{L}_{uu} \langle \phi_2, \phi_2^* \rangle = -c_3 f_\theta \varepsilon e^{-\varepsilon f}$ and $c_5 \langle \phi_2, \phi_2^* \rangle_{;m-1} = -c_5 f_\theta \varepsilon e^{-\varepsilon f}$. Because

$$\beta_3^\Sigma(\phi_2, \phi_2^*, D_2, \mathcal{B}) = -\varepsilon \int_\Sigma (c_3 + c_5) f_\theta d\Theta,$$

we have the desired relationship

$$\frac{1}{\sqrt{\pi}} - c_3 - c_5 = 0.$$

We now take $f(r, \theta) = f(r)$ where

$$f(0) = 0, \quad \partial_r f(0) = 1, \quad \text{and } \partial_r^k f(0) = 0 \text{ for } k > 1.$$

Since $\mathcal{B}\phi_2 = 0$ on $C_{\mathcal{R}}$, only C_D and Σ are relevant. We follow the discussion in Section 3 of [4] to show $\beta_3^{C_D} = 0$. We have

$$\begin{aligned} \phi &= 1, & \phi^* &= e^{-\varepsilon f(r)}, & \Gamma_{\Theta\Theta r} &= -\varepsilon e^{\varepsilon f}, \\ \omega_r &= -\frac{\varepsilon}{2}, & \tilde{\omega}_r &= \frac{\varepsilon}{2}, & S_2 &= \frac{1}{2} \varepsilon. \end{aligned}$$

Consequently, we may compute on C_D that:

$$\begin{aligned} \frac{2}{3} \langle \phi; mm, \phi^* \rangle + \frac{2}{3} \langle \phi, \phi^*; mm \rangle &= \frac{\varepsilon^2}{3}, \\ E &= -\omega_r \omega_r + \omega_r \Gamma_{\theta\theta r} = \frac{\varepsilon^2}{4}, \\ -\frac{2}{3} L_{aa} \langle \phi, \phi^* \rangle_{;m} &= -\frac{2\varepsilon^2}{3}, \\ \left(\frac{1}{12} L_{aa} L_{bb} - \frac{1}{6} L_{ab} L_{ab} + \frac{1}{6} R_{amam} \right) &= \left(\frac{1}{12} - \frac{1}{6} + \frac{1}{6} \right) \varepsilon^2, \\ \beta_3^{C_D} &= \left(\frac{4}{12} + \frac{3}{12} - \frac{8}{12} + \frac{1}{12} \right) \varepsilon^2 = 0. \end{aligned}$$

This implies that

$$\begin{aligned} 0 &= \beta_3^\Sigma(1, \phi_2^*, \Delta_2, \mathcal{B}_2) \\ &= \int_\Sigma \{ (c_2 L_{uu} + c_4 S) \phi_2, \phi_2^* \} + c_6 \langle \phi_2, \phi_2^* \rangle_{;m} \} dz \\ &= \left(-c_2 + \frac{1}{2} c_4 - c_6 \right) \varepsilon \text{vol}(\Sigma). \end{aligned}$$

This establishes Equation (1.e) by showing that

$$c_2 - \frac{1}{2} c_4 + c_6 = 0.$$

4. AN EXAMPLE ON THE HALF-PLANE

We now consider an example on the half-plane, where the classical Zaremba boundary value problem that we are considering has a simple spectral resolution. The two-dimensional Laplacian is given in polar coordinates by

$$\Delta = \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2}{\partial \varphi^2}.$$

We let $\phi = 0$ define the Dirichlet component and $\phi = \pi$ define the Neumann boundary component. The spectral resolution is then given by:

$$(4.a) \quad \psi_{\lambda,k}(\varphi, r) = \sqrt{\frac{2}{\pi}} \sin(\varphi[k + 1/2]) J_{k+1/2}(\lambda r), \quad k \in \mathbb{N}_0.$$

We can use this spectral resolution to write down the Fourier decomposition of the heat content where we assume a suitable decay of ϕ and ϕ^* at infinity to ensure this is well defined:

$$\begin{aligned} \beta(\phi, \phi^*, \mathcal{D}, \mathcal{B})(t) &= \sum_{k=0}^{\infty} \int_0^{\infty} d\lambda \lambda e^{-t\lambda^2} \gamma_{k,\lambda}(\phi) \gamma_{k,\lambda}(\phi^*) \quad \text{where} \\ \gamma_{k,\lambda}(f) &= \int_0^{\pi} d\varphi \int_0^{\infty} dr r f(\varphi, r) \psi_{\lambda,k}(\varphi, r). \end{aligned}$$

We first perform the λ integration using [10], Equation 6.633,

$$\int_0^{\infty} dx x e^{-tx^2} J_p(\alpha x) J_p(\beta x) = \frac{1}{2t} e^{-\frac{\alpha^2 + \beta^2}{4t}} I_p \left(\frac{\alpha\beta}{2t} \right).$$

This leads to the following representation of the heat content,

$$(4.b) \quad \begin{aligned} \beta(\phi, \phi^*, \mathcal{D}, \mathcal{B})(t) &= \frac{1}{\pi t} \sum_{k=0}^{\infty} \int_0^{\pi} d\varphi \int_0^{\infty} dr r \int_0^{\pi} d\varphi' \int_0^{\infty} dr' r' \phi(\varphi, r) \phi^*(\varphi', r') \\ &\quad \sin(\varphi[k + 1/2]) \sin(\varphi'[k + 1/2]) e^{-\frac{r^2 + r'^2}{4t}} I_{k+1/2} \left(\frac{rr'}{2t} \right). \end{aligned}$$

We will choose suitable angular parts for the localizing functions ϕ, ϕ^* to ensure that the angular integrals can be obtained in closed form. As we will see, an arbitrary r -dependence can be dealt with.

The basis for the forthcoming calculation is the integral representation of the Bessel function $I_{k+1/2}$ where k is an integer (we refer to [10], Equation 8.431.5 for details):

$$(4.c) \quad \begin{aligned} I_{k+1/2}(z) &= \frac{1}{\pi} \int_0^{\pi} d\theta e^{z \cos \theta} \cos([k + 1/2]\theta) \\ &\quad - \frac{\sin([k + 1/2]\pi)}{\pi} \int_0^{\infty} d\tau e^{-z \cosh \tau - (k + \frac{1}{2})\tau} \\ &= \frac{1}{\pi} \int_0^{\pi} d\theta e^{z \cos \theta} \cos([k + 1/2]\theta) \\ &\quad - \frac{1}{\pi} (-1)^k \int_0^{\infty} d\tau e^{-z \cosh \tau - (k + \frac{1}{2})\tau}. \end{aligned}$$

Remark 4.1. If we had studied pure Dirichlet or pure Neumann boundary conditions, then the relevant Bessel functions would be indexed by an integer rather than by the half integer $k + 1/2$. The second term in the representation given by Equation (4.c) would be absent in such a case.

Substituting the identity

$$(4.d) \quad \exp \left\{ -\frac{r^2 + r'^2}{4t} + \frac{rr'}{2t} \cos \theta \right\} = \exp \left\{ -\frac{(r - r')^2}{4t} + \frac{rr'}{2t} (\cos \theta - 1) \right\}$$

into (4.b) and using a saddle point argument, we can verify that the first term in (4.c) is ‘responsible’ for producing the volume and boundary contributions. We will show that the second term is ‘responsible’ for the contributions concentrated on Σ . As we are interested in the contributions concentrated on Σ , which we will denote by β^Σ , we only study the second term in (4.c).

We give another derivation of the identity $c_0 = -1/2$ to illustrate the general idea behind the calculation. We assume ϕ and ϕ^* to have the product form

$$\phi = \Omega_1(\varphi)R_1(r), \quad \phi^* = \Omega_2(\varphi)R_2(r).$$

We first study a constant angular part $\Omega_i(\varphi) = 1$, $i = 1, 2$, and perform the angular integrations,

$$\int_0^\pi d\varphi \sin\left(\varphi \left[k + \frac{1}{2}\right]\right) = \frac{1}{k + 1/2}.$$

This yields the identity

$$\begin{aligned} \beta^\Sigma(\phi, \phi^*, \mathcal{D}, \mathcal{B})(t) &= -\frac{1}{\pi^2 t} \int_0^\infty dr r \int_0^\infty dr' r' R_1(r) R_2(r') e^{-\frac{r^2+r'^2}{4t}} \\ &\quad \int_0^\infty d\tau e^{-\frac{rr'}{2t} \cosh \tau} \sum_{k=0}^\infty (-1)^k \frac{e^{-(k+1/2)\tau}}{(k+1/2)^2}. \end{aligned}$$

Thus only $r \sim 0$ and $r' \sim 0$ contribute to the asymptotic small t expansion of the heat content.

We substitute $y = r/\sqrt{t}$, $y' = r'/\sqrt{t}$, and expand around $r = 0$, $r' = 0$. To leading order this produces

$$\begin{aligned} \beta^\Sigma(\phi, \phi^*, \mathcal{D}, \mathcal{B})(t) &\sim -\frac{t}{\pi^2} R_1(0) R_2(0) \int_0^\infty d\tau \sum_{k=0}^\infty (-1)^k \frac{e^{-(k+1/2)\tau}}{(k+1/2)^2} \\ &\quad \int_0^\infty dy \int_0^\infty dy' y y' e^{-\frac{y^2+y'^2}{4} - \frac{yy'}{2} \cosh \tau}. \end{aligned}$$

The constant c_0 is determined by this triple integral and the sum over k . We perform the y' -integral using [10], Equation 3.322.2; this involves a complementary error function which, together with [10], Equation 6.286.1, yields

$$(4.e) \quad \begin{aligned} \beta^\Sigma(\phi, \phi^*, \mathcal{D}, \mathcal{B})(t) &\sim -\frac{t}{\pi^2} R_1(0) R_2(0) \sum_{k=0}^\infty \frac{(-1)^k}{(k+1/2)^2} \\ &\quad \int_0^\infty d\tau e^{-(k+1/2)\tau} \left\{ -\frac{4}{\sinh^2 \tau} + \frac{4\tau \cosh \tau}{\sinh^3 \tau} \right\}. \end{aligned}$$

The integration of the single terms in (4.e) is not possible as is seen from the $\tau \rightarrow 0$ behaviour. In order to use [10], Equation 3.541.1,

$$(4.f) \quad \int_0^\infty d\tau e^{-\mu\tau} \sinh^\alpha(\beta\tau) = \frac{1}{2^{\alpha+1}\beta} B\left(\frac{\mu}{2\beta} - \frac{\alpha}{2}, \alpha + 1\right),$$

with the beta-function $B(x, y)$, we need to introduce a regularizing factor $\sinh^\nu \tau$ in (4.e), integrate the single terms and perform the limit $\nu \rightarrow 0^+$ at the end of the calculation. We obtain

$$\begin{aligned} &\beta^\Sigma(\phi, \phi^*, \mathcal{D}, \mathcal{B})(t) \\ &\sim -\frac{t}{\pi^2} R_1(0) R_2(0) \sum_{k=0}^\infty \frac{(-1)^k}{(k+1/2)^2} \left[-2 \left(k + \frac{3}{2}\right) + \left(k + \frac{1}{2}\right)^2 \psi' \left(\frac{1}{2} \left[k + \frac{1}{2}\right]\right) \right] \end{aligned}$$

with the psi-function $\psi(x) = (d/dx) \ln \Gamma(x)$. The first term can be summed with the aid of the Hurwitz zeta function,

$$\sum_{k=0}^{\infty} (-1)^k \frac{k+3/2}{(k+1/2)^2} = \frac{\pi}{2} + 4C,$$

C being the Catalan constant. The summation over the derivatives of the ψ -function is performed using [10], Equation 8.363.8,

$$\psi'(x) = \zeta_H(2; x).$$

We use this to write

$$\begin{aligned} \sum_{k=0}^{\infty} (-1)^k \psi' \left(\frac{1}{2} \left[k + \frac{1}{2} \right] \right) &= \sum_{k=0}^{\infty} (-1)^k \sum_{l=0}^{\infty} \frac{1}{\left(l + \frac{1}{2} \left[k + \frac{1}{2} \right] \right)^2} \\ &= \lim_{\nu \rightarrow 0^+} \sum_{k=0}^{\infty} (-1)^k \sum_{l=0}^{\infty} \frac{1}{\left(l + \frac{1}{2} \left[k + \frac{1}{2} \right] \right)^{2+\nu}} \\ &= \lim_{\nu \rightarrow 0^+} \sum_{l=0}^{\infty} \sum_{k=0}^{\infty} \left\{ \frac{1}{\left(l + k + 1/4 \right)^{2+\nu}} - \frac{1}{\left(l + k + 3/4 \right)^{2+\nu}} \right\} \\ &= \lim_{\nu \rightarrow 0^+} \sum_{m=0}^{\infty} \sum_{l=0}^m \left\{ \frac{1}{\left(m + 1/4 \right)^{2+\nu}} - \frac{1}{\left(m + 3/4 \right)^{2+\nu}} \right\} \\ &= \lim_{\nu \rightarrow 0^+} \sum_{m=0}^{\infty} \left\{ \frac{m+1}{\left(m + 1/4 \right)^{2+\nu}} - \frac{m+1}{\left(m + 3/4 \right)^{2+\nu}} \right\} \\ &= \lim_{\nu \rightarrow 0^+} \left(\zeta_H(1+\nu; 1/4) - \zeta_H(1+\nu; 3/4) \right) + \frac{3}{4} \zeta_H(2; 1/4) - \frac{1}{4} \zeta(2; 3/4) \\ &= -\psi \left(\frac{1}{4} \right) + \psi \left(\frac{3}{4} \right) + \frac{3}{4} \zeta_H(2; 1/4) - \frac{1}{4} \zeta_H(2; 3/4) = \pi + 8C + \frac{\pi^2}{2}. \end{aligned}$$

This allows us to conclude

$$\beta^{\Sigma}(\phi, \phi^*, \mathcal{D}, \mathcal{B})(t) \sim -\frac{1}{2} t R_1(0) R_2(0),$$

which gives us another derivation of the result that $c_0 = -1/2$. Since we have chosen a constant angular dependence, we do not have $\langle \phi, \phi^* \rangle_{;m}$ terms. Also, given the localizing functions are assumed to be $C^\infty(M; V)$, we need $R_i(r) = R_i(-r)$, which implies $(\partial/\partial r) R_i(r)|_{r=0} = 0$; so we do not have $\langle \phi, \phi^* \rangle_{;m-1}$ terms either and we need not consider other terms in the asymptotic expansion for this example.

In order to obtain information about constants c_5 and c_6 , we need to study nontrivial angular dependences. The constant c_6 is studied by looking at

$$\Omega_1(\varphi) = 1, \quad \Omega_2(\varphi) = \sin \varphi.$$

Assuming $R_2(r) = r R_2'(0) + \mathcal{O}(r^2)$ as $r \rightarrow 0$, we have that $\phi_{;m}^*|_{r=0} = R_2'(0)$.

As in the previous calculation, we start by observing that

$$\int_0^\pi d\varphi \sin \left(\varphi \left[k + \frac{1}{2} \right] \right) \sin \varphi = (-1)^{k+1} \left(\frac{1}{2k-1} - \frac{1}{2k+3} \right).$$

This allows us to write the leading term of β^{Σ} as $t \rightarrow 0$ in the form

$$\begin{aligned} &\beta^{\Sigma}(\phi, \phi^*, \mathcal{D}, \mathcal{B})(t) \\ &\sim \frac{2t^{3/2}}{\pi^2} R_1(0) R_2'(0) \sum_{k=0}^{\infty} \left\{ \frac{1}{(2k-1)(2k+1)} - \frac{1}{(2k+1)(2k+3)} \right\} \\ &\quad \int_0^\infty d\tau e^{-(k+\frac{1}{2})\tau} \int_0^\infty dy \int_0^\infty dy' y y'^2 e^{-\frac{y^2+y'^2}{4} - \frac{yy'}{2} \cosh \tau}. \end{aligned}$$

To proceed as before with [10], Equations 3.322.2 and 6.286.1, we observe

$$\begin{aligned} I &:= \int_0^\infty dy' \int_0^\infty dy yy'^2 e^{-\frac{y^2+y'^2}{4} - \frac{yy'}{2} \cosh \tau} \\ &= -\frac{2}{\sinh \tau} \frac{d}{d\tau} \int_0^\infty dy' y' e^{-\frac{y'^2}{4}} \int_0^\infty dy e^{-\frac{y^2}{4} - \frac{yy'}{2} \cosh \tau}. \end{aligned}$$

The y -integral is evaluated using [10], Equation 3.322.2, the resulting y' -integral with [10], Equation 6.286.1. This shows that

$$I = -\frac{2\sqrt{\pi}}{\sinh \tau} \frac{d}{d\tau} \frac{1}{\cosh^2 \tau} {}_2F_1\left(1, \frac{3}{2}; 2; \tanh^2 \tau\right),$$

with the hypergeometric function ${}_2F_1(a, b; c; x)$. For the particular parameters involved, the hypergeometric function is

$${}_2F_1\left(1, \frac{3}{2}; 2; x\right) = -\frac{2(-1 + \sqrt{1-x})}{x\sqrt{1-x}},$$

which for $x = \tanh^2 \tau$ yields the identity

$${}_2F_1\left(1, \frac{3}{2}; 2; \tanh^2 \tau\right) = 2 \frac{\cosh^2 \tau}{\sinh^2 \tau} (-1 + \cosh \tau).$$

This shows that

$$I = -4\sqrt{\pi} \left\{ \frac{2 \cosh \tau}{\sinh^4 \tau} - \frac{2}{\sinh^4 \tau} - \frac{1}{\sinh^2 \tau} \right\} = \frac{\sqrt{\pi}}{\cosh^4\left(\frac{\tau}{2}\right)},$$

and consequently we have that

$$\begin{aligned} &\beta^\Sigma(\phi, \phi^*, \mathcal{D}, \mathcal{B})(t) \\ &\sim \frac{2t^{\frac{3}{2}}}{\pi^{\frac{3}{2}}} R_1(0) R_2'(0) \sum_{k=0}^{\infty} \left\{ \frac{1}{(2k-1)(2k+1)} - \frac{1}{(2k+1)(2k+3)} \right\} \\ (4.g) \quad &\times \int_0^\infty d\tau e^{-(k+\frac{1}{2})\tau} \frac{1}{\cosh^4\left(\frac{\tau}{2}\right)}. \end{aligned}$$

We evaluate the integral using [10], Equation 3.541.8. With $\mu = k + 1/2$ and with the standard notation [10]

$$\beta(x) = \frac{1}{2} \left[\Psi\left(\frac{x+1}{2}\right) - \Psi\left(\frac{x}{2}\right) \right] = \sum_{k=0}^{\infty} \frac{(-1)^k}{x+k}.$$

This shows that

$$\int_0^\infty \frac{e^{-\mu\tau}}{\cosh^4\left(\frac{\tau}{2}\right)} d\tau = \frac{2}{3}\mu \{1 + 2(\mu^2 - 1)[\beta(\mu+1) - \beta(\mu)]\}.$$

We note that

$$\frac{1}{(2k-1)(2k+1)} - \frac{1}{(2k+1)(2k+3)} = \frac{1}{(\mu-1)\mu(\mu+1)},$$

which allows us to write

$$(4.h) \quad \begin{aligned} &\beta^\Sigma(\phi, \phi^*, \mathcal{D}, \mathcal{B})(t) \\ &\sim \frac{2t^{\frac{3}{2}}}{3\pi^{\frac{3}{2}}} R_1(0) R_2'(0) \sum_{k=0}^{\infty} \left\{ \frac{1}{(\mu-1)(\mu+1)} + 2[\beta(\mu+1) - \beta(\mu)] \right\}. \end{aligned}$$

Our final task is the evaluation of the sum over k . To this end, we note that

$$\sum_{k=0}^{\infty} \frac{1}{(k - \frac{1}{2})(k + \frac{3}{2})} = 0, \quad \text{and}$$

$$\sum_{k=0}^{\infty} (\beta(k + 3/2) - \beta(k + 1/2)) = -\beta(1/2) = -\frac{\pi}{2}.$$

We may then conclude that

$$\beta^{\Sigma}(\phi, \phi^*, \mathcal{D}, \mathcal{B})(t) \sim -\frac{2}{3\sqrt{\pi}} R_1(0) R_2'(0) t^{3/2}.$$

This shows, as desired, that

$$c_6 = -\frac{2}{3\sqrt{\pi}}.$$

In order to determine c_5 , we choose

$$\Omega_1(\varphi) = 1 \quad \text{and} \quad \Omega_2(\varphi) = \cos \varphi.$$

We again suppose that $R_2(r) = rR_2'(0) + \mathcal{O}(r^2)$. Then as $r \rightarrow 0$,

$$\phi_{;m-1}^*|_{r=0} = R_2'(0).$$

The relevant angular part integration is

$$\int_0^{\pi} d\varphi \sin\left(\varphi \left[k + \frac{1}{2}\right]\right) \cos \varphi = \frac{1}{2k-1} + \frac{1}{2k+3}.$$

The y and y' integration, as well as the resulting τ -integration are the same as before, and the equation corresponding to (4.h) for this example is

$$(4.i) \quad \beta^{\Sigma}(\phi, \phi^*, \mathcal{D}, \mathcal{B})(t) \sim -\frac{t^{\frac{3}{2}}}{3\pi^{\frac{3}{2}}} R_1(0) R_2'(0) \times \sum_{k=0}^{\infty} (-1)^k \left\{ \frac{1}{\mu-1} + \frac{1}{\mu+1} + 4\mu(\beta(\mu+1) - \beta(\mu)) \right\}.$$

The remaining sum over k may be performed with the help of the identities

$$\sum_{k=0}^{\infty} \frac{(-1)^k}{k - \frac{1}{2}} = -\frac{1}{2}(\pi + 4), \quad \sum_{k=0}^{\infty} \frac{(-1)^k}{k + \frac{3}{2}} = \frac{1}{2}(4 - \pi), \quad \text{and}$$

$$\sum_{k=0}^{\infty} (-1)^k (k + 1/2)(\beta(k + 3/2) - \beta(k + 1/2)) = -\frac{\pi}{8}.$$

We add these relations to conclude that

$$\beta^{\Sigma}(\phi, \phi^*, \mathcal{D}, \mathcal{B})(t) \sim \frac{1}{2\sqrt{\pi}} R_1(0) R_2'(0) t^{3/2}.$$

This shows that

$$c_5 = \frac{1}{2\sqrt{\pi}}.$$

This establishes Equation (1.f) and thereby completes the proof of Theorem 1.2.

5. FUNCTION SPACES

The analysis in question is local so we shall suppose M is an open domain in \mathbb{R}^m with compact closure and with smooth boundary ∂M . For the sake of simplicity, we shall assume that the vector bundle in question is trivial; the analysis is similar in the bundle valued case. We write the Robin boundary operator in local coordinates in the form

$$Ru = \sum a^{ij}(x)\sigma_i\partial_{x_j} + d(x),$$

where $\sigma = (\sigma_1, \dots, \sigma_m)$ is the outward unit normal to ∂M and d is a smooth function on ∂M .

Let ν be the distance to ∂M and let x' denote a point on ∂M . We introduce coordinates $(x', \nu) \rightarrow x' + \nu\sigma$ on a collared neighborhood of the boundary to express

$$(5.a) \quad D(x, \partial_x) = -a(x')\partial_\nu^2 + \nu b(x', \nu)\partial_\nu^2 + L_1(x', \nu, \partial_{x'})\partial_\nu + L_2(x', \nu, \partial_{x'}).$$

In Equation (5.a), a is a smooth positive function on ∂M , b is smooth in a neighborhood of the boundary, and L_1 and L_2 are differential operators in x' of orders 1, 2, respectively, with smooth coefficients near ∂M .

Near Σ , M is diffeomorphic to $\Sigma \times B_+(\varepsilon)$ for some $\varepsilon > 0$, where

$$B_+(\varepsilon) := \{y = (y_1, y_2) : y_2 \geq 0, y_1^2 + y_2^2 < \varepsilon^2\}.$$

We may choose y_2 to be the normal parameter ν and use coordinates (z, y) near Σ . With these normalizations,

$$C_{\mathcal{D}} = \{(z, y) : y_2 = 0, y_1 \geq 0\} \quad \text{and} \quad C_{\mathcal{R}} = \{(z, y) : y_2 = 0, y_1 < 0\}.$$

We can express the operator D as:

$$(5.b) \quad \begin{aligned} D(x, \partial_x) = & -L(z, \partial_y) + A_2(z, y, \partial_z) \\ & + \sum_{i=1,2} (y_i B_{i2}(z, y, \partial_y) + A_{i1}(z, y, \partial_z)\partial_{y_i}). \end{aligned}$$

In this formulation, A_{i1} and B_{i2} , A_2 are differential operators with smooth coefficients of orders 1 and 2, respectively. Furthermore, L can be expressed as

$$(5.c) \quad L(z, \partial_y) = \sum_{i,j=1,2} A^{ij}(z)\partial_{y_i}\partial_{y_j},$$

where A^{ij} are smooth real valued functions on Σ such that the matrix $\{A^{ij}\}$ is symmetric positive definite and

$$(5.d) \quad L(z, \xi) \leq \varkappa_0^{-1}|\xi|^2 \quad \text{for all } z \in \Sigma \text{ and } \xi \in \mathbb{R}^{m-2}$$

with some positive \varkappa_0 . The boundary operator R can be represented in these coordinates as

$$(5.e) \quad R(x', \partial_x) = R_0(z, \partial_y) + y_1 B_1(z, y_1, \partial_y) + A_1(z, y, \partial_z),$$

where B_1 and A_1 are differential operators with respect to y and z of orders 1 with smooth coefficients, and

$$(5.f) \quad R_0(z, \partial_y) = \sum_{j=1,2} A^{j2}(z)\partial_{y_j}.$$

We blowup Σ and introduce polar coordinates near Σ by setting:

$$(z, \rho, \theta) \rightarrow (z, y) = (z, \rho \cos \theta, \rho \sin \theta).$$

Note that $\theta = 0$ defines $C_{\mathcal{D}}$ while $\theta = \pi$ defines $C_{\mathcal{R}}$.

Let $C^\infty(M_\Sigma)$ be the class of smooth functions on $C^\infty(M - \Sigma)$ which extend smoothly to the blowup. In what follows we shall suppose that a function ρ , which defined only locally, has been extended smoothly as a $C^\infty(M_\Sigma)$ function which is positive outside the original neighborhood.

Definition 5.1. For $\kappa > 0$, let $\mathcal{E}(\kappa)$ be the set of functions $U \in C^\infty([0, \infty))$ which satisfy estimates of the form

$$|\partial_\tau^k U(\tau)| \leq C_{k,\kappa'} \exp(-\kappa' \tau^2) \quad \text{for } k = 0, 1, 2, \dots \quad \text{and } \kappa' \in (0, \kappa).$$

The optimal constants $C_{k,\kappa'}$ define semi-norms

$$p_{k,\kappa'}(U) := \inf C_{k,\kappa'}$$

giving a Frechet space topology on $\mathcal{E}(\kappa)$. We use this topology to define subspaces of $\mathcal{E}(\kappa)$ of smooth functions on $C_{\mathcal{D}}$ and on $C_{\mathcal{R}}$:

$$\mathcal{E}(C_{\mathcal{D}}, \kappa) = C^\infty(C_{\mathcal{D}}, \mathcal{E}(\kappa)) \quad \text{and} \quad \mathcal{E}(C_{\mathcal{R}}, \kappa) = C^\infty(C_{\mathcal{R}}, \mathcal{E}(\kappa)).$$

Introduce the halfspace $\mathbb{R}_+^2 = \{y = (y_1, y_2) : y_2 \geq 0\}$ with polar coordinates (ρ, θ) for $\rho \in [0, \infty)$ and $\theta \in [0, \pi]$.

Definition 5.2. Let \mathcal{U} be a smooth function on $\mathbb{R}_+^2 - \{0\}$. We say that $\mathcal{U} \in \Lambda_\kappa^\mu$ if

- (1) $|\partial_\rho^k \partial_\theta^j \mathcal{U}(y)| \leq C_{k,j} \rho^{\mu-k}$ for $\rho \leq 1$ and for all k, j .
- (2) For large values of ρ , the function \mathcal{U} admits an asymptotic expansion

$$(5.g) \quad \mathcal{U}(y) \sim \sum_{j=0}^{\infty} \rho^{-j} \left\{ \frac{v_j(\theta)}{\rho} + \sum_{\pm} U_j^\pm(y_2) \chi\left(\frac{y_2}{\rho}\right) \right\},$$

where $v_j \in C^\infty([0, \pi])$, where χ is a smooth cutoff function on \mathbb{R}_+ which equals 1 for small τ and 0 for large τ , and where $U_j^\pm \in \mathcal{E}(\kappa)$ with “+” corresponding to $y_1 > 0$ and “-” corresponding to $y_1 < 0$.

The asymptotic expansion in Definition 5.2 is to be understood in the following sense. For any $N = 1, 2, \dots$ and for any multi-indices α and γ with $|\gamma| \leq |\alpha|$, we have a constant C , which is independent of θ and of ρ , so that:

$$(5.h) \quad \left| y^\gamma \partial_y^\alpha \left(U(\rho, \theta) - \sum_{j=0}^{N-1} \rho^{-j} \left\{ \frac{v_j(\theta)}{\rho} + \sum_{\pm} U_j^\pm(y_2) \chi\left(\frac{y_2}{\rho}\right) \right\} \right) \right| \leq C \rho^{-N}, \rho \geq 1.$$

One verifies that the class given in Definition 5.2 is independent of the particular χ chosen.

Definition 5.3. Let $\Lambda_\kappa^\mu(\Sigma) = C^\infty(\Sigma; \Lambda_\kappa^\mu)$ be the set of all functions $\mathcal{U} = \mathcal{U}(z, y)$ from $C^\infty(\Sigma \times (\mathbb{R}_+^2 - \{0\}))$ belonging to Λ_κ^μ for every $z \in \Sigma$. The coefficients v_j and U_j^\pm in the asymptotic expansion (5.g) and in the inequality (5.h) may depend on $z \in \Sigma$. We assume that these coefficients belong to $C^\infty(\Sigma, \mathcal{E}(\kappa))$ and that (5.g) and (5.h) can be differentiated with respect to z .

Definition 5.4. Let R_μ be the set of smooth functions on \mathbb{R}_+ so that:

- (1) $|\partial_\rho^k \mathcal{V}(\rho)| \leq C_k \rho^{\mu-k}$ for $\rho \leq 1$,
- (2) $\mathcal{V}(\rho) \sim \sum_{j=0}^{\infty} a_j \rho^{-j}$ as $\rho \rightarrow \infty$.

Remark 5.5. Condition (2) of Definition 5.4 means that we have estimates

$$|\partial_\rho^k (\mathcal{V}(\rho) - \sum_{j=0}^N a_j \rho^{-j})| \leq C_{N,k} \rho^{-N-k-1}$$

for all N and k and for $\rho \geq 1$.

We put

$$\mathcal{R}_\mu(\Sigma) := C^\infty(\Sigma, R_\mu).$$

6. MODEL PROBLEMS

We shall first consider boundary value problems on a half-line and then subsequently consider boundary value problems on the half-plane. We begin with the *Dirichlet problem*:

$$(6.a) \quad (\partial_t - \partial_\nu^2)t^{k/2}U\left(\frac{\nu}{2\sqrt{t}}\right) = t^{k/2-1}F\left(\frac{\nu}{2\sqrt{t}}\right), \quad U(0) = G,$$

where $\nu \geq 0$, $t > 0$ and $k = 0, 1, \dots$. The function $U = U(\nu)$ then satisfies

$$U''(\nu) + 2\nu U'(\nu) - 2kU(\nu) = -4F(\nu) \quad \text{for } \nu \geq 0, \quad U(0) = G.$$

The homogeneous equation for U (with $F = 0$) has two solutions

$$\psi_k(\nu) = \int_\nu^\infty (s - \nu)^k e^{-s^2} ds \quad \text{and} \quad \phi_k(\nu) = e^{-\nu^2} \partial_\nu^k e^{\nu^2}.$$

Therefore if $F = 0$ the only solution to (6.a) decaying for large ν is the function

$$U(\nu) = 2b\psi_k(\nu)/\Gamma((k+1)/2).$$

Let $G = 0$. We impose suitable decay properties on F to ensure the following integrals converge and set:

$$U(\nu) = \begin{cases} \frac{4}{k!}(\psi_k(\nu) \int_0^\nu e^{s^2} \phi_k(s) F(s) ds + \phi_k(\nu) \int_\nu^\infty e^{s^2} \psi_k(s) F(s) ds) & , k \text{ odd} , \\ \frac{4}{k!}(\psi_k(\nu) \int_0^\nu e^{s^2} \phi_k(s) F(s) ds + \phi_k(\nu) \int_\nu^\infty e^{s^2} \psi_k(s) F(s) ds) \\ \quad - \frac{8\psi_k(\nu)}{\Gamma((k+1)/2)(k/2)!} \int_0^\infty e^{s^2} \psi_k(s) F(s) ds & , k \text{ even} . \end{cases}$$

Proposition 6.1. *Let $a \in C^\infty(C_{\mathcal{D}})$ be a positive function, let $F \in \mathcal{E}(C_{\mathcal{D}}, \kappa)$, let $G \in C^\infty(C_{\mathcal{D}})$, and let $\kappa \leq \min_{x' \in C_{\mathcal{D}}} a^{-1}(x')$. Then there exists $U \in \mathcal{E}(C_{\mathcal{D}}, \kappa)$ so*

$$(\partial_t - a\partial_\nu^2)t^{k/2}U\left(\frac{\nu}{2\sqrt{t}}\right) = t^{k/2-1}F\left(\frac{\nu}{2\sqrt{t}}\right) \quad \text{and} \quad U(0) = G.$$

Proof. We make the change of variable $\tau = at$ to reduce the problem of Proposition 6.1 to that given in Equation (6.a) with $F = F(x')$ and $G = G(x')$ for $x' \in C_{\mathcal{D}}$. One can then use the formulae given above to see that $U \in \mathcal{E}(C_{\mathcal{D}}, \kappa)$ as claimed. \square

A similar argument can be given to deal with the Neumann problem:

Proposition 6.2. *Let $a \in C^\infty(C_{\mathcal{R}})$ be a positive function, let $F \in \mathcal{E}(C_{\mathcal{R}}, \kappa)$, let $H \in C^\infty(C_{\mathcal{R}})$, and let $\kappa \leq \min_{x' \in C_{\mathcal{R}}} a^{-1}(x')$. Then there exists $U \in \mathcal{E}(C_{\mathcal{R}}, \kappa)$ so*

$$(\partial_t - a\partial_\nu^2)t^{k/2}U\left(x', \frac{\nu}{2\sqrt{t}}\right) = t^{k/2-1}F\left(x', \frac{\nu}{2\sqrt{t}}\right) \quad \text{and} \quad U'(0) = H.$$

Next we study a model problem in the half-space \mathbb{R}_+^2 :

$$(6.b) \quad (\partial_t - \Delta_y)(t^{k/2}\mathcal{U}\left(\frac{y}{2\sqrt{t}}\right)) = t^{k/2-1}\mathcal{F}\left(\frac{y}{2\sqrt{t}}\right) \quad \text{for } y \in \mathbb{R}_+^2 \text{ and } t > 0,$$

$$(6.c) \quad \mathcal{U}(y_1, 0) = \mathcal{G}(\rho) \quad \text{for } y_1 > 0 \quad \text{and} \quad \rho\partial_{y_2}\mathcal{U}(y_1, 0) = \mathcal{H}(\rho) \quad \text{for } y_1 < 0,$$

where k is a nonnegative integer, where $\mathcal{F} \in \Lambda_\kappa^\mu$, and where $\mathcal{G}, \mathcal{H} \in R_\mu$. Equation (6.b) can be rewritten as

$$(6.d) \quad (\partial_\rho^2 + \frac{1}{\rho}\partial_\rho + \frac{1}{\rho^2}\partial_\theta^2)\mathcal{U} + (2\rho\partial_\rho + -2k)\mathcal{U} = -4\mathcal{F} \quad \text{on } \mathbb{R}_+^2.$$

We adopt the notation of Definition 5.4. We omit details of the proof of the following Theorem as it is analogous to the proof given for Proposition 2 [13].

Theorem 6.3. *Let $\kappa \in (0, 1)$ and $\mu \in (-1/2, 1/2)$. Let $\mathcal{F} \in \Lambda_\kappa^{\mu-2}$ and $\mathcal{G}, \mathcal{H} \in R_\mu$. Then there exists a unique solution to Equations (6.c) and (6.d) belonging to Λ_κ^μ .*

Next we consider the case when the operator and the right-hand sides in Equations (6.c) and (6.d) depend on a parameter. Let R_0 be the operator of Equation (5.f), let $L(z, \partial_y)$ be the operator of Equation (5.c), and let \varkappa_0 be the constant of Equation (5.d).

Theorem 6.4. *Let $\kappa \in (0, \kappa_0]$ and $\mu \in (-1/2, 1/2)$. If $\mathcal{F} \in \Lambda_\kappa^{\mu-2}(\Sigma)$ and if $\mathcal{G}, \mathcal{H} \in R_\mu(\Sigma)$, then there exists a unique element $\mathcal{U} \in \Lambda_\kappa^\mu(\Sigma)$ such that*

$$\begin{aligned} (\partial_t - L(z, \partial_y))(t^{k/2}\mathcal{U}(z, \frac{y}{2\sqrt{t}})) &= t^{k/2-1}\mathcal{F}(z, \frac{y}{2\sqrt{t}}) \text{ for } y \in \mathbb{R}_+^2 \text{ and } t > 0, \\ \mathcal{U}(z, y_1, 0) &= \mathcal{G}(z, y_1) \text{ for } y_1 > 0, \\ \rho(R_0(z, \partial_y)\mathcal{U})(z, y_1, 0) &= \mathcal{H}(z, -y_1) \text{ for } y_1 < 0. \end{aligned}$$

Proof. Let $B = B_{ij} = A^{-1/2}$ where $A = A^{ij}$. Set $Y_k = \sum_{j=1,2} B_{kj}y_j$. Then the equations given in Theorem 6.4 become Equations (6.b) and (6.c) where the right-hand side depends on the parameter $z \in \Sigma$. The desired result now follows from Theorem 6.3. \square

7. A THEOREM ON ASYMPTOTICS

We can now establish the result from which Theorem 1.1 will follow. Let

$$\kappa = \min_{x' \in \partial M} \varkappa(x')$$

where $\varkappa(x')$ is the best constant in the inequality

$$\sum a^{ij}(x')\xi_i\xi_j \leq \varkappa^{-1}(x')|\xi|^2 \text{ for } \xi \in \mathbb{R}^m.$$

Theorem 7.1. *Let $\Phi \in C^\infty(M_\Sigma)$. Then the weak solution of Equation (1.c) has the asymptotic representation:*

$$\begin{aligned} u(x, t) \sim \sum_{k=0}^{\infty} t^{k/2} \left\{ \Xi\left(\frac{\rho}{2\sqrt{t}}\right) \left(u_k(x) + \rho\eta(\nu/\rho)U_k^D\left(x', \frac{\nu}{2\sqrt{t}}\right) \right. \right. \\ \left. \left. + \rho\eta(\nu/\rho)U_k^R\left(x', \frac{\nu}{2\sqrt{t}}\right) \right) + \zeta(\rho)\mathcal{U}_k\left(z, \frac{y}{2\sqrt{t}}\right) \right\}. \end{aligned}$$

In the above, $\xi, \zeta,$ and η are smooth cutoff functions on \mathbb{R}_+ which vanish when the argument is greater than δ and which are equal to 1 if the argument is less than $\frac{1}{2}\delta$ where $\delta > 0$ is suitably chosen. We set $\Xi = 1 - \xi$. The functions u_k belong to $C^\infty(M_\Sigma)$, $U_k^D \in \mathcal{E}(C_{\mathcal{D}}, \kappa)$, $U_k^R \in \mathcal{E}(C_{\mathcal{R}}, \kappa)$ and $\mathcal{U}_k \in \Lambda_\kappa^\mu(\Sigma)$ with arbitrary $\mu \in [0, 1/2)$. Moreover the coefficient U_0^R equals zero.

The asymptotic expansion in this theorem is to be understood in the sense that the difference

$$\begin{aligned} r_N(x, t) &= u(x, t) - \sum_{k=0}^N t^{k/2} \left\{ \Xi\left(\frac{\rho}{2\sqrt{t}}\right) \left(u_k(x) + \rho\eta(\nu/\rho)U_k^D\left(x', \frac{\nu}{2\sqrt{t}}\right) \right. \right. \\ (7.a) \quad & \left. \left. + \rho\eta(\nu/\rho)U_k^R\left(x', \frac{\nu}{2\sqrt{t}}\right) \right) + \zeta(\rho)\mathcal{U}_k\left(z, \frac{y}{2\sqrt{t}}\right) \right\} \end{aligned}$$

satisfies the estimate

$$(7.b) \quad |\partial_t^{\alpha_0} \partial_x^\alpha r_N(x, t)| \leq C \begin{cases} t^{(N+1)/2 - \alpha_0 - |\alpha|/2} & \text{for } t \leq \rho^2 \\ t^{(N+1-\mu)/2 - \alpha_0} \rho^{\mu - |\alpha|} & \text{for } t \geq \rho^2. \end{cases}$$

Moreover, the same estimate (in a neighborhood of Σ) is valid for all the derivatives of r_N with respect to z where the constant involved may depend on the number of derivatives.

7.1. The main term. We are looking for u in the form

$$u(x, t) = \Xi\left(\frac{\rho}{2\sqrt{t}}\right)u_0(x) + v(x, t),$$

where $u_0 = \Phi$ and v satisfies the equations:

$$(7.c) \quad \begin{cases} (\partial_t + D(x, \partial_x))v = f & \text{in } M \times (0, T), \\ v = g & \text{on } C_{\mathcal{D}} \times (0, T), \\ Rv = h & \text{on } C_{\mathcal{R}} \times (0, T), \end{cases}$$

with initial condition

$$(7.d) \quad v = 0 \quad \text{on } M \text{ for } t = 0.$$

Here

$$(7.e) \quad \begin{aligned} f(x, t) &= -\Xi\left(\frac{\rho}{2\sqrt{t}}\right)Du_0(x) + [\partial_t + D, \xi\left(\frac{\rho}{2\sqrt{t}}\right)]u_0(x), \\ g &= -\Xi\left(\frac{\rho}{2\sqrt{t}}\right)u_0(x), \quad \text{and} \quad h = -R\Xi\left(\frac{\rho}{2\sqrt{t}}\right)u_0(x), \end{aligned}$$

where here and elsewhere $[\cdot, \cdot]$ denotes the commutator of two operators. Decompose the function u near Σ in an asymptotic series with respect to ρ :

$$f(x, t) \sim \Xi\left(\frac{\rho}{2\sqrt{t}}\right)f_{01}(x) + \zeta(\rho) \sum_{k=0}^{\infty} t^{-1+k/2} \mathcal{F}_{0k}\left(z, \frac{y}{2\sqrt{t}}\right),$$

where $f_{01} \in C^\infty(M_\Sigma)$ and $\mathcal{F}_{0k}(z, Y)$ belongs to $\Lambda_\kappa^\mu(\Sigma)$ and is equal to zero for small $|Y|$. The asymptotic expansion given above means that the remainder

$$q_N(x, t) = f(x, t) - \Xi\left(\frac{\rho}{2\sqrt{t}}\right)f_{01}(x) - \sum_{k=0}^N t^{-1+k/2} \mathcal{F}_{0k}\left(z, \frac{y}{2\sqrt{t}}\right)$$

satisfies the estimate

$$|\partial_t^{\alpha_0} \partial_x^\alpha \partial_z^\gamma q_N(x, t)| \leq |Ct^{(N-1)/2 - \alpha_0 - |\alpha|/2}|,$$

where $q_N(x, t) = 0$ for $\rho \leq \varepsilon\sqrt{t}$ and for some small positive ε .

Analogously, one can represent g and h as

$$g(x', t) = \Xi\left(\frac{\rho}{2\sqrt{t}}\right)\rho g_{01}(x') + \zeta(\rho)\Xi\left(\frac{\rho}{2\sqrt{t}}\right)g_{00}(z),$$

where $g_{00} \in C^\infty(\Sigma)$ and $g_{01} \in C^\infty(C_{\mathcal{D}})$, and

$$h(x', t) \sim \Xi\left(\frac{\rho}{2\sqrt{t}}\right)h_{01}(x') + \zeta(\rho)\rho^{-1} \sum_{k=0}^{\infty} t^{k/2} \mathcal{H}_{0k}\left(z, \frac{\rho}{2\sqrt{t}}\right),$$

where $h_{01} \in C^\infty(C_{\mathcal{R}})$ and $\mathcal{H}_{0k}(z, Y)$ are smooth functions from $R_\mu(\Sigma)$ which are equal to 0 for $|Y| \leq \varepsilon$.

Now, the function \mathcal{U}_0 can be found by solving the problem

$$\begin{aligned} (\partial_t + L(z, \partial_y))\mathcal{U}_0(z, \frac{y}{2\sqrt{t}}) &= t^{-1}(\Xi\left(\frac{\rho}{2\sqrt{t}}\right)\left(\frac{\rho}{\sqrt{t}}\right)^{-2}u_{00}(z, \theta) + \mathcal{F}_{00}(z, \frac{y}{2\sqrt{t}})), \\ \mathcal{U}_0(z, y_1, 0) &= \Xi\left(\frac{\rho}{2\sqrt{t}}\right)g_{00}(z) \quad \text{for } y_1 > 0, \\ \rho(R_0\mathcal{U}_0)(z, y_1, 0) &= \mathcal{H}_{00}(z, \frac{\rho}{2\sqrt{t}}) \quad \text{for } y_1 < 0. \end{aligned}$$

By Theorem 6.4, this boundary value problem has a solution from $\Lambda_\kappa^\mu(\Sigma)$. Similarly, the function U_0^D satisfies the relations

$$(\partial_t - a\partial_\nu^2)U_0^D\left(\frac{\nu}{2\sqrt{t}}\right) = 0, \quad U_0^D(0) = g_{01}(x')$$

and, by Proposition 6.1, has a solution $U \in \mathcal{E}(C_{\mathcal{D}}, \kappa)$. The remainder

$$w(x, t) = u(x, t) - \Xi\left(\frac{\rho}{2\sqrt{t}}\right)(u_0(x) + \eta(\nu/\rho)\rho U_0^D(x', \frac{\nu}{2\sqrt{t}})) - \zeta(\rho)\mathcal{U}_0\left(\frac{y}{2\sqrt{t}}\right)$$

satisfies the Equations

$$(7.f) \quad \begin{cases} (\partial_t + D(x, \partial_x))w = f & \text{in } M \times (0, T), \\ w = g & \text{on } C_{\mathcal{D}} \times (0, T), \\ Rw = h & \text{on } C_{\mathcal{R}} \times (0, T), \end{cases}$$

and the initial condition

$$(7.g) \quad w = 0 \quad \text{on } M \text{ for } t = 0.$$

One can verify that the right-hand sides in (7.f) admit the following asymptotic expansions as $t \rightarrow 0$:

$$(7.h) \quad f(x, t) \sim \sum_{k=1}^{\infty} t^{k/2-1} \left\{ \Xi\left(\frac{\rho}{2\sqrt{t}}\right) \left(f_k(x) + \rho\eta(\nu/\rho) F_k^D\left(x', \frac{\nu}{2\sqrt{t}}\right) \right. \right. \\ \left. \left. + \rho\eta(\nu/\rho) F_k^R\left(x', \frac{\nu}{2\sqrt{t}}\right) \right) + \zeta(\rho) \mathcal{F}_k\left(z, \frac{y}{2\sqrt{t}}\right) \right\}$$

with $f_k \in C^\infty(M_\Sigma)$, $F_k^D \in \mathcal{E}(C_{\mathcal{D}}, \varkappa)$, $F_k^R \in \mathcal{E}(C_{\mathcal{R}}, \varkappa)$ and $\mathcal{F}_k \in \Lambda_{\varkappa}^{\mu-1}(\Sigma)$,

$$(7.i) \quad g(x', t) \sim \sum_{k=1}^{\infty} t^{k/2} \left(\Xi\left(\frac{\rho}{2\sqrt{t}}\right) \rho g_k(x') + \zeta(\rho) \mathcal{G}_k\left(z, \frac{\rho}{2\sqrt{t}}\right) \right),$$

with $g_k \in C^\infty(C_{\mathcal{D}})$, $\mathcal{G}_k \in \mathcal{R}_{\mu+1}(\Sigma)$, and

$$(7.j) \quad h(x', t) \sim \sum_{k=1}^{\infty} t^{(k-1)/2} \left(\Xi\left(\frac{\rho}{2\sqrt{t}}\right) h_k(x') + \zeta(\rho) \frac{\sqrt{t}}{\rho} \mathcal{H}_k\left(z, \frac{\rho}{2\sqrt{t}}\right) \right)$$

with $h_k \in C^\infty(C_{\mathcal{R}})$, $\mathcal{H}_k \in \mathcal{R}_{\mu+1}(\Sigma)$. The asymptotic expansion of Equation (7.h) is to be understood in the following sense. Let

$$f^{(N)}(x, t) = \sum_{k=1}^N t^{k/2-1} \left\{ \Xi\left(\frac{\rho}{2\sqrt{t}}\right) \left(f_k(x) + \rho\eta(\nu/\rho) F_k^D\left(x', \frac{\nu}{2\sqrt{t}}\right) \right. \right. \\ \left. \left. + \rho\eta(\nu/\rho) F_k^R\left(x', \frac{\nu}{2\sqrt{t}}\right) \right) + \zeta(\rho) \mathcal{F}_k\left(z, \frac{y}{2\sqrt{t}}\right) \right\}$$

then the remainder $R_N = f - f^{(N)}$ satisfies the estimate

$$(7.k) \quad |\partial_t^{\alpha_0} \partial_x^\alpha \partial_z^\gamma R_N(x, t)| \leq C \begin{cases} t^{(N-1)/2-\alpha_0-|\alpha|/2} & \text{for } t \leq \rho^2 \\ t^{(N-1)/2-\alpha_0+(1-\mu)/2} \rho^{\mu-1-|\alpha|} & \text{for } t \geq \rho^2, \end{cases}$$

for all $\alpha_0 = 0, 1, \dots$ and multi-indices $\alpha = (\alpha_1, \dots, \alpha_m)$ and $\gamma = (\gamma_1, \dots, \gamma_{m-2})$ with nonnegative integer components. The derivatives with respect to z are defined and should be taken into account only in a neighborhood of Σ , outside of this neighborhood γ is zero.

If we denote by $g^{(N)}(x', t)$ and $h^{(N)}(x', t)$ the partial sums in (7.i) and (7.j) from 1 to N and introduce the remainder terms $R_{gN} = g - g^{(N)}$ and $R_{hN} = h - h^{(N)}$ then the asymptotic representations (7.i) and (7.j) mean that

$$(7.l) \quad |\partial_t^{\alpha_0} \partial_{x'}^{\alpha'} \partial_z^\gamma R_{gN}(x, t)| \leq C \begin{cases} t^{(N+1)/2-\alpha_0-|\alpha|/2} & \text{for } t \leq \rho^2 \\ t^{N/2-\alpha_0-\mu/2} \rho^{\mu+1-|\alpha|} & \text{for } t \geq \rho^2, \end{cases}$$

and

$$(7.m) \quad |\partial_t^{\alpha_0} \partial_{x'}^{\alpha'} \partial_z^\gamma R_{hN}(x, t)| \leq C \begin{cases} t^{N/2-\alpha_0-|\alpha|/2} & \text{for } t \leq \rho^2 \\ t^{N/2-\alpha_0-\mu/2} \rho^{\mu-|\alpha|} & \text{for } t \geq \rho^2, \end{cases}$$

for all $\alpha_0 = 0, 1, \dots$ and multi-indices $\alpha' = (\alpha_1, \dots, \alpha_{m-1})$ and $\gamma = (\gamma_1, \dots, \gamma_{m-2})$. Roughly speaking the estimates for the remainder terms are the same as the estimate for the next terms in the asymptotic expansions (7.h)–(7.j).

The form of the right-hand sides (7.h)–(7.j) are more general than we need but it is convenient to consider this more general form in order to unify the construction of other terms in the asymptotic expansion for u .

7.2. Higher order terms. We first describe the construction of the terms u_1 , U_1^D , U_1^R and \mathcal{U}_1 .

The term u_1 . We take $u_1 = 2f_1$. Then

$$\begin{aligned} & (\partial_t + D(x, \partial_x)) \left(t^{1/2} \Xi \left(\frac{\rho}{2\sqrt{t}} \right) u_1(x) \right) - t^{-1/2} \Xi \left(\frac{\rho}{2\sqrt{t}} \right) f_1(x) \\ &= \frac{\rho}{4t} \xi' \left(\frac{\rho}{2\sqrt{t}} \right) u_1(x) - t^{1/2} \left[D(x, \partial_x), \xi \left(\frac{\rho}{2\sqrt{t}} \right) \right] u_1(x) \\ &+ t^{1/2} \Xi \left(\frac{\rho}{2\sqrt{t}} \right) D(x, \partial_x) u_1(x). \end{aligned}$$

Decomposing the function u_1 near Σ in asymptotic series with respect to ρ we obtain that the right-hand side is asymptotically equal to

$$t^{1/2} \Xi \left(\frac{\rho}{2\sqrt{t}} \right) f_{11}(x) + \sum_{k=1}^{\infty} t^{k/2-1} \zeta(\rho) \mathcal{F}_{1k} \left(z, \frac{y}{2\sqrt{t}} \right)$$

with $f_{11} \in C^\infty(M_\Sigma)$ and $\mathcal{F}_{1k} \in \Lambda_\kappa^\mu(\Sigma)$. Moreover $\mathcal{F}_{1k}(z, Y) = 0$ for $|Y| \leq \varepsilon$ for some positive ε . So, we have compensated the term containing f_1 in the right-hand side of (7.h) and the discrepancy, which came, can be included in the remaining terms in the right-hand side in (7.h). We shall denote the new right-hand sides by the same letters.

The term \mathcal{U}_1 . We find this function from the equation

$$(\partial_t - L(z, \partial_y)) \left(t^{1/2} \mathcal{U}_1 \left(z, \frac{y}{2\sqrt{t}} \right) \right) = t^{-1/2} \mathcal{F}_1 \left(z, \frac{y}{2\sqrt{t}} \right)$$

supplied with boundary conditions

$$\mathcal{U}_1(z, y_1, 0) = \mathcal{G}_1(z, y_1) \quad \text{for } y_1 > 0$$

and

$$\rho \sum_{j=1}^2 A^{j2}(z) (\partial_{y_j} \mathcal{U}_1)(z, y_1, 0) = \mathcal{H}_1(z, -y_1) \quad \text{for } y_1 < 0.$$

The discrepancy in the equation brought by this term is equal to

$$\begin{aligned} & (\partial_t + D) \left(t^{1/2} \zeta(\rho) \mathcal{U}_k \left(z, \frac{y}{2\sqrt{t}} \right) \right) - t^{-1/2} \zeta(\rho) \mathcal{F}_k \left(z, \frac{y}{2\sqrt{t}} \right) \\ &= t^{1/2} \zeta(\rho) (D + L) \mathcal{U}_k \left(z, \frac{y}{2\sqrt{t}} \right) + t^{1/2} [D, \zeta(\rho)] \mathcal{U}_k \left(z, \frac{y}{2\sqrt{t}} \right). \end{aligned}$$

Using (5.b) and the asymptotic expansion at infinity for functions from the class Λ one can show that the right-hand side of the equation above has asymptotics

$$\begin{aligned} & \sum_{k=2}^{\infty} t^{k/2-1} \left\{ \Xi \left(\frac{\rho}{2\sqrt{t}} \right) \left(f_{1k}(x) + \eta(\nu/r) F_{1k}^D \left(x', \frac{\nu}{2\sqrt{t}} \right) \right. \right. \\ & \left. \left. + \eta(\nu/r) F_{1k}^R \left(x', \frac{\nu}{2\sqrt{t}} \right) \right) + \zeta(\rho) \mathcal{F}_{2k} \left(z, \frac{y}{2\sqrt{t}} \right) \right\}, \end{aligned}$$

where $f_{1k} \in C^\infty(M_\Sigma)$, $F_{1k}^D \in \mathcal{E}(C_{\mathcal{D}}, \kappa)$, $F_{1k}^R \in \mathcal{E}(C_{\mathcal{R}}, \kappa)$ and $\mathcal{F}_{2k} \in \Lambda_\kappa^{\mu-1}(\Sigma)$.

The discrepancy in the Dirichlet boundary condition is zero and in the Robin boundary condition is

$$\begin{aligned} & R(t^{1/2} \zeta \mathcal{U}_1) - \zeta(\rho) \rho^{-1} \mathcal{H}_k \\ &= \sum_{k=2}^{\infty} t^{(k-1)/2} \left(\Xi \left(\frac{\rho}{2\sqrt{t}} \right) h_{1k}(x') + \zeta(\rho) \frac{\sqrt{t}}{\rho} \mathcal{H}_{1k} \left(z, \frac{\rho}{2\sqrt{t}} \right) \right), \end{aligned}$$

where $h_{1k} \in C^\infty(C_{\mathcal{R}})$, $\mathcal{H}_{1k} \in \mathcal{R}_{\mu+1}(\Sigma)$. So, one can see that \mathcal{U}_1 compensates the terms \mathcal{F}_1 , \mathcal{G}_1 and \mathcal{H}_1 in the right-hand sides of (7.h)–(7.j). The discrepancies brought by \mathcal{U}_1 have lower order and can be included in terms in the asymptotic expansions (7.h)–(7.j) with $k \geq 2$.

The terms U_1^D and U_1^R . We define the function U_1^D , $x' \in C_D$, as a solution of the boundary value problem

$$\begin{aligned} (\partial_t - a(x')\partial_\nu^2)(t^{1/2}U_1^D(x', \nu)) &= t^{-1/2}F_1^D(x', \nu), \\ U_1^D(x', 0) &= g_1(x') \end{aligned}$$

with x' considered as a parameter. We have

$$\begin{aligned} &(\partial_t + D(x, \partial_x))\left(t^{1/2}\eta\left(\frac{\nu}{\rho}\right)\rho U_1^D\left(x', \frac{\nu}{2\sqrt{t}}\right)\right) \\ &\quad - t^{-1/2}\Xi\left(\frac{\rho}{2\sqrt{t}}\right)\eta\left(\frac{\nu}{\rho}\right)\rho F_1^D\left(x', \frac{\nu}{2\sqrt{t}}\right) \\ &= t^{1/2}\Xi\left(\frac{\rho}{2\sqrt{t}}\right)\eta\left(\frac{\nu}{\rho}\right)(D(x, \partial_x) + a(x')\partial_\nu^2)\rho U_1^D\left(x', \frac{\nu}{2\sqrt{t}}\right) \\ &\quad + t^{1/2}\left[D(x, \partial_x), \Xi\left(\frac{\rho}{2\sqrt{t}}\right)\eta\left(\frac{\nu}{\rho}\right)\right]\rho U_1^D\left(x', \frac{\nu}{2\sqrt{t}}\right). \end{aligned}$$

By Equation (5.a), the right-hand side in this equation asymptotically equals

$$\sum_{j=2}^{\infty} t^{j/2-1}\zeta(\rho)\mathcal{F}_{1j}\left(z, \frac{y}{2\sqrt{t}}\right) + \rho \sum_{j=2}^{\infty} t^{j/2-1}\Xi\left(\frac{\rho}{2\sqrt{t}}\right)\eta(\nu/\rho)F_{1j}^D\left(x', \frac{\nu}{2\sqrt{t}}\right)$$

with $\mathcal{F}_{1j} \in \Lambda_\kappa^{\mu-1}(\Sigma)$ and $F_{1j}^D \in \mathcal{E}(C_D, \kappa)$.

The term U_1^R is then constructed analogously. Thus, we have constructed the approximation of the solution which contains all terms in the asymptotic expansion of Theorem 7.1 with $k = 1$ and which compensates all terms in the asymptotic representations (7.h)–(7.j) of the right-hand sides of (7.f) with $k = 1$. Continuing this procedure we can compensate terms with $k = 2, 3, \dots$. Therefore, if we put

$$\begin{aligned} u_N(x, t) &= \sum_{k=0}^N t^{k/2} \left\{ \Xi\left(\frac{\rho}{2\sqrt{t}}\right) \left(u_k(x) + \rho \eta\left(\frac{\nu}{\rho}\right) U_k^D\left(x', \frac{\nu}{2\sqrt{t}}\right) \right. \right. \\ &\quad \left. \left. + \rho \eta\left(\frac{\nu}{\rho}\right) U_k^R\left(x', \frac{\nu}{2\sqrt{t}}\right) \right) + \zeta(\rho) \mathcal{U}_k\left(z, \frac{y}{2\sqrt{t}}\right) \right\} \end{aligned}$$

and $r_N = u - u_N$. The function r_N then satisfies the equations:

$$\begin{cases} (\partial_t + D(x, \partial_x))r_N = f_N & \text{in } M \times (0, T) \\ r_N = g_N & \text{on } C_D \times (0, T), \\ Ru_N = h_N & \text{on } C_{\mathcal{R}} \times (0, T), \end{cases}$$

with the initial condition $r_N = 0$ on M for $t = 0$. The right-hand sides in these relations admit the asymptotic expansions (7.h)–(7.j), respectively, where summation is started from $k = N + 1$.

8. ESTIMATE OF THE REMAINDER TERM IN THE ASYMPTOTICS

In this section, we complete the proof of Theorem 7.1 by establishing the remainder estimate given in Equation 7.b. The proof rests on a result (Theorem 8.1 below) obtained by Johansson [12]. First we introduce some additional notation. Let T be a positive number and $Q_T = M \times (0, T)$, $\Gamma_T^D = C_D \times (0, T)$ and $\Gamma_T^R = C_{\mathcal{R}} \times (0, T)$. We introduce also weighted Sobolev spaces. Let $\ell = 0, 1, \dots$, $\beta \in \mathbb{R}$. The space $W_\beta^{2\ell, \ell}(Q_T)$ consists of functions on Q_T with the finite norm

$$\|u\|_{W_\beta^{2\ell, \ell}(Q_T)} = \left(\int_{Q_T} \rho^{2(\beta-2\ell)} \sum_{|\bar{\alpha}| \leq 2\ell} \rho^{2|\bar{\alpha}|} |\partial_t^{\alpha_0} \partial_x^\alpha u|^2 dx dt \right)^{1/2}$$

where we set $\bar{\alpha} := (\alpha_0, \alpha)$ and $|\bar{\alpha}| := 2\alpha_0 + |\alpha|$. For $s = 1/4, 3/4, \dots$, introduce the trace spaces $W_\beta^{2s, s}(\Gamma_T^D)$ with the norm

$$\|u\|_{W_\beta^{2s, s}(\Gamma_T^D)} = \left(\int_0^T \|u(\cdot, t)\|_{V_\beta^{2s}(C_D)}^2 dt + \int_{C_D} \rho^{2\beta} \|u(x, \cdot)\|_{H^s(0, T)}^2 dx \right)^{1/2}.$$

Here H^s stands for the standard Sobolev space on the interval $(0, T)$. If k is a positive integer, then $V_\beta^{k-1/2}(C_D)$ is the space of traces on C_D of functions from the space $V_\beta^k(M)$ with the norm

$$\|v\|_{V_\beta^k(M)} = \left(\int_M \sum_{|\alpha| \leq k} \rho^{2(\beta-k+|\alpha|)} |\partial_x^\alpha v(x)|^2 dx \right)^{1/2}.$$

The norm in $V_\beta^{k-1/2}(C_D)$ is defined by

$$\|w\|_{V_\beta^{k-1/2}(C_D)} = \inf \{ \|v\|_{V_\beta^k(M)} : v \in V_\beta^k(M), v|_{C_D} = w \}.$$

Analogously, one can define the space $W_\beta^{2s,s}(\Gamma_T^D)$.

The closure of functions from the space $W_\beta^{2\ell,\ell}(Q_T)$ equal to 0 for small t will be denoted by $W_{\beta,0}^{2\ell,\ell}(Q_T)$. Analogously one defines the spaces $W_{\beta,0}^{2s,s}(\Gamma_T^D)$ and $W_{\beta,0}^{2s,s}(\Gamma_T^R)$.

If $|\bar{\alpha}| < 2\ell - n/2 - 1$ then functions $u \in W_\beta^{2\ell,\ell}(Q_T)$ have continuous derivatives of order $\bar{\alpha}$ in Q_T and

$$|\partial_t^{\alpha_0} \partial_x^\alpha u(x, t)| \leq C \rho^{2\ell - |\bar{\alpha}| - n/2 - 1 - \beta} \|u\|_{W_\beta^{2\ell,\ell}(Q_T)}.$$

This estimate can be obtained from the analogous estimate for functions from non-weighted spaces (see [6], Chapter 3) and homogeneity arguments. If $u \in W_\beta^{2\ell,\ell}(Q_T)$ and $|\bar{\alpha}| + 2m < 2\ell - n/2 - 1$ with a nonnegative m then, clearly,

$$(8.a) \quad |\partial_t^{\alpha_0} \partial_x^\alpha u(x, t)| \leq C t^m \rho^{2\ell - |\bar{\alpha}| - n/2 - 1 - \beta} \|u\|_{W_\beta^{2\ell,\ell}(Q_T)}.$$

The proof of the following result is contained in [12].

Theorem 8.1. *Let $\ell \geq 1$ be an integer and let β satisfy $1/2 < -\beta + 2\ell < 3/2$. If $f \in W_{\beta,0}^{2\ell-2,\ell-1}(Q_T)$, $g \in W_{\beta,0}^{2\ell-1/2,\ell-1/4}(\Gamma_T^D)$ and $h \in W_{\beta,0}^{2\ell-3/2,\ell-3/4}(\Gamma_T^R)$ then there exists a unique solution $u \in W_{\beta,0}^{2\ell,\ell}(Q_T)$ to problem (7.f). This solution satisfies the estimate*

$$(8.b) \quad \|u\|_{W_\beta^{2\ell,\ell}(Q_T)} \leq C \left(\|f\|_{W_\beta^{2\ell-2,\ell-1}(Q_T)} + \|g\|_{W_\beta^{2\ell-1/2,\ell-1/4}(\Gamma_T^D)} + \|h\|_{W_\beta^{2\ell-3/2,\ell-3/4}(\Gamma_T^R)} \right).$$

Now we are in a position to prove the remainder estimate (7.b). According to the construction of the terms in the asymptotic expansion given in Theorem 7.1 (see the end of Sect. 7.2) the remainder (7.a) satisfies the boundary value problem (7.f), (7.g), where the right-hand sides admit the asymptotic representations (7.h)–(7.j) with summation starting with $k = N + 1$. Therefore these right-hand sides are estimated by the right-hand sides in (7.k)–(7.m). This implies that the derivative of order k with respect to t and all derivatives with respect to z (in a neighborhood of Σ) belong to

$$W_{\beta,0}^{2\ell-2,\ell-1}(Q_T), \quad W_{\beta,0}^{2\ell-1/2,\ell-1/4}(\Gamma_T^D) \quad \text{and} \quad W_{\beta,0}^{2\ell-3/2,\ell-3/4}(\Gamma_T^R),$$

for $\ell < (N - 2k - 1)/2$ and $2\ell - \beta < 2 + \mu$, respectively. We suppose here that μ is an arbitrary number from the interval $(0, 1/2)$. Now applying Theorem 8.1 we obtain that $\partial_t^k r_N$ together with all derivatives with respect to z (in a neighborhood of Σ) belongs to $W_{\beta,0}^{2\ell,\ell}(Q_T)$ for $1/2 < -\beta + 2\ell < 1 + \mu$ and $\ell < (N - 2k - 1)/2$. This implies that in a neighborhood of Σ the integral

$$\int \rho^{2(\beta-2\ell)} \sum_{|\alpha| \leq 2\ell} \rho^{2|\alpha|} |\partial_y^\alpha \partial_t^{k-1} \partial_z^\gamma u|^2 dy$$

is bounded uniformly with respect to t and z . By the usual imbedding theorem we obtain

$$|\partial_y^\alpha \partial_t^{k-1} \partial_z^\gamma u| \leq C \rho^{2\ell - |\alpha| - 1 - \beta}$$

for $|\alpha| < 2\ell - 1$. Choosing $\sigma = 2\ell - 1 - \beta$ close to $1/2$ and then taking $\mu \in (\sigma, 1/2)$ we can rewrite the above estimate as

$$(8.c) \quad |\partial_y^\alpha \partial_t^{k-1} \partial_z^\gamma u| \leq Ct^m \rho^{\sigma-|\alpha|}$$

which is valid for $|\alpha| + 2k + 2m < N - 2$ and for arbitrary multi-index γ .

In order to obtain a remainder estimate outside a neighborhood of Σ one can use (8.a) which gives

$$(8.d) \quad |\partial_t^{\alpha_0} \partial_x^\alpha r_N(x, t)| \leq Ct^m$$

for $|\bar{\alpha}| + 2m < 2\ell - n/2 - 1$ and for $\rho \geq \varepsilon$ where ε is a small positive number. Estimates (8.c) and (8.d) imply

$$(8.e) \quad |\partial_t^{\alpha_0} \partial_x^\alpha r_N(x, t)| \leq Ct^m \rho^{\sigma-|\alpha|}$$

for $|\bar{\alpha}| + 2m < N - n/2 - 2$ and for arbitrary $\sigma \in (0, 1/2)$.

In order to obtain estimate (7.b) for r_N we proceed as follows. We choose an integer $M > N$ and represent the remainder term r_N as

$$(8.f) \quad r_N = r_M + \sum_{k=N+1}^M t^{k/2} \left\{ \Xi\left(\frac{\rho}{2\sqrt{t}}\right) \left(u_k(x) + \rho\eta(\nu/\rho) U_k^D\left(x', \frac{\nu}{2\sqrt{t}}\right) \right. \right. \\ \left. \left. + \rho\eta(\nu/\rho) U_k^R\left(x', \frac{\nu}{2\sqrt{t}}\right) \right) + \zeta(\rho) \mathcal{U}_k\left(z, \frac{y}{2\sqrt{t}}\right) \right\}.$$

One can check that all the terms in the summation satisfy estimate (7.b). By choosing M sufficiently large, we obtain estimate (7.b) for r_M from (8.e). The proof of Theorem 7.1 is complete. \square

Let Σ_D and Σ_R denote the boundaries of C_D and C_R , respectively. Clearly, functions from $C^\infty(M_\Sigma)$ may take different values on Σ_D and Σ_R . The existence of the asymptotic series given in Theorem 1.1 is a special case of the following more general result:

Theorem 8.2. *Let u be the solution to problem (1.c) and let $\sigma \in C^\infty(M_\Sigma)$. Then the following asymptotic expansion for u is valid:*

$$(8.g) \quad \int_M \sigma(x) u(x, t) dx \sim \sum_{k=0}^{\infty} t^k \int_M \sigma(x) u_k(x) dx + \sum_{k=0}^{\infty} t^{k/2} (ta_k + t^{1/2}b_k + t^{3/2}c_k),$$

where

$$a_k = \int_\Sigma \int_{-\pi}^{\pi} v_k(z, \theta) dz d\theta, \quad b_k = \int_{C_D} w_k^D(x') dx' + \int_{C_R} w_k^R(x') dx'$$

and

$$c_k = \int_{\Sigma_D} h_k^D(z) dz + \int_{\Sigma_R} h_k^R(z) dz.$$

Here u_k , v_k , w_k^D , w_k^R , h_k^D and h_k^R are smooth functions whose values at a given point depend only on values of φ and its derivatives at this point.

Proof. We have

$$(8.h) \quad \int_M \sigma(x) \Xi\left(\frac{\rho}{2\sqrt{t}}\right) u_k(x) dx = \int_M \sigma(x) u_k(x) dx - \int_M \xi\left(\frac{\rho}{2\sqrt{t}}\right) \sigma(x) u_k(x) dx.$$

From $\sigma, u_k \in C^\infty(M_\Sigma)$ it follows that

$$\sigma(x) u_k(x) = \sum_{j=0}^N u_{kj}(z, \theta) \rho^j + O(\rho^{N+1})$$

for each N , which implies that

$$\int_M \xi\left(\frac{\rho}{2\sqrt{t}}\right) \sigma(x) u_k(x) dx = t \sum_{j=0}^{\infty} c_j \int_{\Sigma} \int_{-\pi}^{\pi} u_{kj}(z, \theta) dz d\theta t^{j/2},$$

where c_j are constants independent of the initial data φ . Therefore, the integrals in (8.h) give the first sum in the right-hand side of (8.g) in Theorem 8.2 and terms of the form ta_k in (8.g).

Next, consider the integral

$$(8.i) \quad \int_M \sigma(x) \Xi\left(\frac{\rho}{2\sqrt{t}}\right) \rho \eta\left(\frac{\nu}{\rho}\right) U_k^D\left(x', \frac{\nu}{2\sqrt{t}}\right) dx.$$

Let us introduce a cutoff function $\zeta(y_1)$ which is equal to 1 for $|y_1| \leq \varepsilon/2$ and 0 for $|y_1| \geq \varepsilon$, where ε is a small positive number. Then we represent (8.i) for small t as

$$(8.j) \quad \begin{aligned} & \int_M (1 - \zeta(y_1)) \rho \eta\left(\frac{\nu}{\rho}\right) \sigma(x) U_k^D\left(x', \frac{\nu}{2\sqrt{t}}\right) dx \\ & + \int_{\Sigma} \int_0^{\infty} \int_0^{\infty} \zeta(y_1) \rho \sigma(z, y) U_k^D\left(z, y_1, \frac{y_2}{2\sqrt{t}}\right) dy dz \\ & + \int_{\Sigma} \int_0^{\infty} \int_0^{\infty} \zeta(y_1) \rho \left(\eta\left(\frac{y_2}{\rho}\right) - 1\right) \sigma(z, y) U_k^D\left(z, y_1, \frac{y_2}{2\sqrt{t}}\right) dy dz \\ & - \int_{\Sigma} \int_0^{\infty} \int_0^{\infty} \xi\left(\frac{\rho}{2\sqrt{t}}\right) \rho \eta\left(\frac{y_2}{\rho}\right) \sigma(z, y) U_k^D\left(z, y_1, \frac{y_2}{2\sqrt{t}}\right) dy dz \end{aligned}$$

where ρ is equal to $\sqrt{y_1^2 + y_2^2}$ in coordinates $y = (y_1, y_2)$. One can check directly that the first two integrals in (8.j) have asymptotics

$$\sum_{j=0}^{\infty} t^{(1+j)/2} \int_{C_D} q_{kj}(x') dx'$$

and the third integral has an expansion of the form

$$\sum_{j=0}^{\infty} t^{1+j/2} \int_{\Sigma_D} h_{kj}^{(1)}(z) dz.$$

Making change of variables $y_1 = 2\sqrt{t}Y_1$ and $y_2 = 2\sqrt{t}Y_2$ we can rewrite the last integral in (8.j) as

$$4t^{3/2} \int_{\Sigma} \int_0^{\infty} \int_0^{\infty} \xi(\rho) \rho \eta(Y_2/\rho) \sigma(z, \sqrt{t}Y) U_k^D(z, 2\sqrt{t}Y_1, Y_2) dY_1 dY_2 dz.$$

Since $U_k^D \in \mathcal{E}(C_D, \kappa)$, the last integral has the asymptotics

$$\sum_{j=0}^{\infty} t^{(3+j)/2} \int_{\Sigma_D} h_{kj}^{(2)}(z) dz,$$

where $h_{kj}^{(2)}$ are integrals with respect to Y_2 of linear combinations of functions $\partial_{Y_1}^s U_k^D(z, Y_1, Y_2)|_{Y_1=0}$ multiplied by explicit weights. The term

$$\int_M \sigma(x) \Xi\left(\frac{\rho}{2\sqrt{t}}\right) \rho \eta(\nu/\rho) U_k^R\left(x', \frac{\nu}{2\sqrt{t}}\right) dx$$

is considered analogously.

It remains to obtain an asymptotic expansion of the term

$$(8.k) \quad \int_{\Sigma} \int_0^{\infty} \int_0^{\infty} \sigma(x) \zeta(\rho) \mathcal{U}_k\left(z, \frac{y}{2\sqrt{t}}\right) dz dy_1 dy_2.$$

Using the asymptotic expansion for the function \mathcal{U}_k for large second argument:

$$\mathcal{U}_k(z, Y) \sim \sum_{j=0}^{\infty} |Y|^{-j} \left\{ \frac{v_{kj}(z, \theta)}{|Y|} + \sum_{\pm} U_{kj}^{\pm}(z, Y_2) \chi\left(\frac{Y_2}{|Y|}\right) \right\},$$

we obtain integrals similar to the ones just considered. Moreover, one can show that the coefficient v_{00} is equal to zero because of vanishing of the analogous coefficient in the asymptotics of the right-hand side in the equation for the function \mathcal{U}_0 . Reasoning as above we arrive at the required asymptotic representation for these integrals. This completes the proof of Theorem 8.2 and thereby of Theorem 1.1. \square

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REFERENCES

- [1] I. G. Avramidi, *Heat kernel asymptotics of Zaremba boundary value problem*, Mathematical Physics, Analysis and Geometry **7** (2004), 9-46.
- [2] M. van den Berg, S. Desjardins, and P. Gilkey, *Functoriality and heat content asymptotics for operators of Laplace type*, Topological Methods in Nonlinear Analysis **2** (1993), 147–162.
- [3] M. van den Berg and J.-F. Le Gall, *Mean curvature and the heat equation*, Math. Zeit. **215** (1994), 437–464.
- [4] M. van den Berg and P. Gilkey, *Heat content asymptotics of a Riemannian manifold with boundary*, J. Funct. Anal. **120** (1994), 48–71.
- [5] M. van den Berg and S. Srisatkunarajah, *Heat flow and Brownian motion for a region in \mathbb{R}^2 with a polygonal boundary*, Probab. Th. Rel. Fields **86** (1990), 41–52.
- [6] O. V. Besov, V. P. Il'in, and S. M. Nikol'skii, **Integral representations of functions and imbedding theorems**. Vol. I. Translated from the Russian. Scripta Series in Mathematics. Edited by Mitchell H. Taibleson. V. H. Winston and Sons, Washington, D.C.; Halsted Press [John Wiley and Sons], New York-Toronto, Ont.-London, 1978.
- [7] J. S. Dowker, *The $N \cup D$ problem*, hep-th/0007129.
- [8] J. S. Dowker, *The hybrid spectral problem and Robin boundary conditions*, math.sp/0409442.
- [9] J. S. Dowker, P. Gilkey, and K. Kirsten, *On properties of the asymptotic expansion of the heat trace for the N/D problem*, Int. J. Math **12** (2001), 505–517.
- [10] I.S. Gradshteyn and I.M. Ryzhik, **Table of Integrals, Series and Products**, Academic Press, New York, 1965.
- [11] D. Jakobson, M. Levitin, N. Nadirashvili, and I. Polterovich, *Spectral problems with mixed Dirichlet-Neumann boundary conditions: isospectrality and beyond*, math.SP/0409154.
- [12] T. Johansson, *Solvability of a boundary value problem for the heat equation in weighted Sobolev spaces*, to appear.
- [13] V. A. Kozlov, *Asymptotics as $t \rightarrow 0$ of solutions of the heat equation in a domain with a conical point*, (Russian). Matem. Sbornik. **136** (1988), 384–395; Math. USSR Sbornik **64** (1989), 383–395.
- [14] V. Kozlov and V. Maz'ya, *On singularities of solutions to the first boundary value problem for the heat equation in domains with conic points 1*. (Russian). Izv. VUZov. Matem. **2** (1987), 38–47; translation in J. Sov. Math. **31** (1987), 61–74.
- [15] V. Kozlov and V. Maz'ya, *On singularities of solutions to the first boundary value problem for the heat equation in domains with conic points 2*. (Russian). Izv. VUZov. Matem. **3** (1987), 37–44; translation in J. Sov. Math. **31** (1987), 49–57.
- [16] H. P. McKean and I. M. Singer, *Curvature and the eigenvalues of the Laplacian*, J. Differential Geometry **1** (1967), 43–69.
- [17] R. Seeley, *Trace expansions for the Zaremba problem*, Comm. Partial Differential Equations **27** (2002), 2403–2421.
- [18] R. Seeley, *Trace expansions for the Zaremba problem*, Comm. Partial Differential Equations **28** (2003), 601–616.
- [19] H. Weyl, **The Classical Groups**, Princeton Univ. Press, Princeton, 1946.

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