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für Mathematik
in den Naturwissenschaften
Leipzig

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by

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Preprint no.: 64 2005

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1 Introduction

Let $\Omega \subset I\!R^n$ be a smooth bounded domain. We are interested in finding solutions to the following weighted eigenvalue problem:

$$(\triangle_p)^m u = \lambda g(x)|u|^{p-2}u \text{ in } \Omega \qquad (i)$$

with the Navier's boundary conditions

$$u = \triangle_p u = \cdots = \triangle_p^{\frac{n}{2}} u = 0 \text{ if } n = \text{even}$$

$$u = \triangle_p u = \cdots = \nabla(\triangle_p^{\frac{n-1}{2}} u) = 0 \text{ if } n = \text{odd}$$

where, $\triangle_p u = \text{div}(|\nabla u|^{p-2}\nabla u)$ is the p

Laplace operator and $p = \frac{n}{m} > 2$ and $g(x)$ is an indefinite weight function and $\lambda > 0$ is a parameter.

The function space $D_0^{m,p}(\Omega)$ is defined to be the completion of $C_0^\infty(\Omega)$ with respect to the norm

$$||u||_{D_0^{m,p}(\Omega)} = ||\nabla^m u||_p$$

where,

$$\nabla^m u = \begin{cases} \triangle^{\frac{m}{2}} u & \text{if } m = \text{even;} \\ \nabla(\triangle^{\frac{m-1}{2}} u) & \text{if } m = \text{odd.} \end{cases}$$

When, $mp < n$, by the Sobolev imbedding theorem,$D_0^{m,p}(\Omega)$ is continuously embedded into $L^r(\Omega)$ for all $r \in [1, q]$ where, $\frac{1}{q} = \frac{1}{p} - \frac{m}{n}$.

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Since $mp = n$, the limit case of sobolev impedding occurs i.e., $D_0^{m,p}(\Omega)$ is continously embedded in $L^q(\Omega)\ \forall\ q \in [1,\infty)$ but not in $L^\infty(\Omega)$.

When $m = 2$, we get our usual 4th order partial differential equation namely

$$\begin{cases} (\triangle_p)^2 u = \lambda g(x)|u|^{p-2}u & \text{in } \Omega; \\ u = \triangle_p u = 0 & \text{on } \partial\Omega. \end{cases}$$

Recently there is a considerable interest on higher order nonlinear problems especially for Fourth order and polyharmonic operators.Elliptic problems with exponential nonlinearity has been studied for second order and higher order problem has been studied by several author in both analytic and geometric setup. We refer some of the paper as [2],[4],[3],[5],[13].Existence of solution of polyharmonic operators has been studied recently in the papers [6],[7],[8],[9],[10].. The main difficulty for higher order problem is to prove the positivity of the solution because of the lack of maximum principle. Therefore, one is interested to obtain only sign changing solution.

When Ω is a ball say B_1 and $p = 2$,it is known that

$$\triangle_p^2 u = \lambda_1 |u|^{p-2}u \text{ in } B_1$$

$$u = \frac{\partial u}{\partial \nu} = 0 \text{ on } \partial B_1$$

has smallest eigen value $\lambda_1 > 0$, simple and corresponding eigen funcion does not change sign.

Main results:- Through this section, we will state our main results and discuss some properties of convex function.

Let H be a convex function on $I\!R$ and the Legendre transform H^* is given by,

$$H^*(q) = \sup_{p\in\mathbb{R}} \{p.q - H(p)\} \tag{1.1}$$

Definition:- The function space $L^1(H^*)$ is defined by,

$$L^1(H^*) \quad = \quad \left\{g \in L^1(\Omega); L^*(|g|) \in L^1(\Omega)\right\} \tag{1.2}$$

Define, the convex function $H(t)$ as

$$H(t) = \begin{cases} e^{t^{\frac{1}{p-1}}} - \displaystyle\sum_{k=0}^{2n-3} \frac{t^{\frac{k}{p-1}}}{k!}; & t \geq 0; \\ 0 & t \leq 0. \end{cases}$$

Then,the derivative $H'(p)$ is given by

$$H'(t) = \begin{cases} \frac{1}{(p-1)} \displaystyle\sum_{k=2n-2}^{\infty} \frac{t^{\frac{k-p+1}{p-1}}}{(k-1)!}; t \geq 0 \\ 0 & t \leq 0 \end{cases}$$

H' is clearly a convex function.

Theorem 2.1:- Suppose,$mp < n$ and $g \in L^{\frac{n}{mp}}(\Omega)$;$g$ changes sign.Then there exists an eigen value $\lambda_1 > 0$ and the corresponding sign changing eigen function $\phi_{\lambda_1} \in D_0^{m,p}(\Omega)$.

Theorem 2.2:- Suppose $mp = n$ and g satisfies the hypothesis:$g \in L^1(H^*) \cap L^1((H')^*)$ and $g \in L^{p_0}(\Omega)$; for some $p_0 > 1$,g changes sign. Then there exists an eigen value $\lambda_1 > 0$ and the corresponding signchanging eigen function ϕ_{λ_1}belongs to $D_0^{m,\frac{n}{m}}(\Omega)$.

Proof of the theorems:- Before going to the proof of the theorems we need to deduce some useful estimate which we describe as lemmas.

Lemma-1:- Let $\gamma > 0$, then for any convex function H,

(i) $(\gamma H)^*(q) = \gamma H^*\left(\frac{q}{\gamma}\right), \forall q \in I\!R$

(ii) If $H(t) = e^{t^{\frac{1}{p-1}}} - \sum_{k=0}^{2n-3} \frac{t^{\frac{k}{p-1}}}{k!}; \quad t \geq 0;$

$= 0 t \leq 0.$

then,

$$H^*(q) = \begin{cases} q\rho(q) - H(\rho(q)), q \geq 0 \\ \infty, q < 0 \end{cases}$$

where, ρ is the inverse of H'.

Proof:-

(i) We have,

$$\begin{aligned} (\gamma H)^*(q) &= \sup_p \{p.q - \gamma H(p)\} \\ &= \gamma. \sup_{p \in R} \{p.\frac{q}{\gamma} - H(p)\} \\ &= \gamma H^*(\frac{q}{\gamma}). \end{aligned}$$

(ii) $H^*(\gamma) = \sup\{\sup_{t \geq 0}(t.q - e^{t^{\frac{1}{p-1}}} + \sum_{k=0}^{2n-3} \frac{t^{\frac{k}{p-1}}}{k!}); \sup_{t<0} t.q\}$

Now for $q < 0, \sup_{t<0} t.q = \infty$.

Therefore, $H^*(q) = \infty$ if $q < 0$.

For, $q \geq 0$, suppose, $H^*(q) = t_0 q - H(t_0)$, then, $q = H'(t_0) = \sum_{k=2n-2}^{\infty} \frac{1}{(p-1)} . \frac{t^{\frac{k-p+1}{p-1}}}{(k-1)!}.$

Let, ρ be the inverse of H', then $t_0 = \rho(q)$. Hence,

$$H^*(q) = \begin{cases} q\rho(q) - H(\rho(q)); q \geq 0 \\ \infty; q < 0 \end{cases}$$

Lemma - II:- Let $0 < \gamma \leq 1$ and $g \in L^1(H^*) \cap L^1(H')^*)$, then for any set $A \subset \Omega$ we have

(i). $\gamma H^*(\frac{|g|}{\gamma}) \in L^1(A)$ and

(ii). $\int_A \gamma H^*(\frac{|g|}{\gamma}) \leq (1-\gamma)||(H')^*(|g|)||_{L^1(A)} + (\frac{1}{\gamma} - 1)||g||_{L^1(A)} + \gamma||H^*(|g|)||_{L^1(A)}$.

Proof:- By lemma 1, we have,

$$\begin{aligned} \int_A \gamma H^*\left(\frac{|g|}{\gamma}\right) &= \int_A \left[\frac{|g|}{\gamma}.\rho\left(\frac{|g|}{\gamma}\right) - H\left(\rho\frac{|g|}{\gamma}\right)\right].\gamma dx \\ &= \int_A |g|\rho\left(\frac{|g|}{\gamma}\right) - \gamma H\left(\rho\left(\frac{|g|}{\gamma}\right)\right) dx \end{aligned}$$

Since, H is convex so

$$\begin{aligned} H\left(\rho\left(\frac{|g|}{\gamma}\right)\right) &\geq H\left(\rho(|g|)\right) + H'(\rho(|g|))\left(\rho\left(\frac{|g|}{\gamma}\right)\right) - \rho(|g|) \\ &= H(\rho(|g|) + |g|\left(\rho\left(\frac{|g|}{\gamma}\right)\right) - \rho(|g|) \end{aligned}$$

Hence,

$$\begin{aligned} \int_A \gamma H^*\left(\frac{|g|}{\gamma}\right) &\leq \int_A |g|\rho\left(\frac{|g|}{\gamma}\right) dx - \gamma \int_A H\left(\rho(|g|)\right. - \gamma \int_A |g|\left(\rho\left(\frac{|g|}{\gamma} - \rho(|g|\right)\right) dx \\ &= (1-\gamma)\int_A |g|\rho(\frac{|g|}{\gamma})dx + \gamma\int_A [|g|\rho(|g|) - H(\rho(|g|)]dx \\ &= (1-\gamma)\int_A |g|\rho\left(\frac{|g|}{\gamma}\right) + \gamma\int_A H^*(|g|)dx \end{aligned}$$

Since, H' is a convex function, we have

$$pq \leq H'(p) + (H')^*(q)\ \forall\ p, q \in I\!R$$

and so

$$|g|\rho\left(\frac{|g|}{\gamma}\right) \leq H'\left(\rho\left(\frac{|g|}{\gamma}\right)\right) + (H')^*(|g|)$$

$$= \frac{|g|}{\gamma} + (H')^*(|g|)$$

Hence,

$$\int_A \gamma H^* \left(\frac{|g|}{\gamma}\right) \leq \left(\frac{1}{\gamma} - 1\right) \int_A (|g|)dx + (1-\gamma) \int_A (H')^*(|g|)dx + \gamma \int_A H^*(|g|)dx$$

This proves the lemma.

Trudinger - Moser Inequality:- Recall the Moser Trudinger inequality in [11],[12]. Following as in Adams [1] :

Theorem:-

If m is a positive integer less than n, then exists a constant $c = c(m,n)$ such that $\forall u \in D_0^{m,p}(\Omega)$ with the normalisation $||\nabla^m u||p \leq 1, p = n/m$ we have,

$$\int_\Omega \exp(\alpha |u(x)|^q)dx \leq c$$

$\forall\ 0 \leq \alpha \leq \alpha_0$ and $\dfrac{1}{p} + \dfrac{1}{q} = 1$, where

α_0 is given by

$$\alpha_0 = \begin{cases} \dfrac{n}{\omega_{n-1}} \left[\dfrac{\pi^{n/2} 2^m \Gamma\left(\frac{m+1}{2}\right)}{\Gamma\left(\frac{n-m+1}{2}\right)} \right]^q ; m = \text{ odd} \\ \dfrac{n}{\omega_{n-1}} \left[\dfrac{\pi^{n/2} 2^m \Gamma\left(\frac{m}{2}\right)}{\Gamma\left(\frac{n-m}{2}\right)} \right]^q ; m = \text{ even} \end{cases}$$

Moreover if $\alpha > \alpha_0$; then,

$$\sup_{||\nabla^m u||_{p \leq 1}} \int_\Omega \exp(\alpha |u(x)|^q)dx = \infty$$

Proof of theorem 2.1:- If λ_1 denote the principal eigenvalue, then $\lambda_1 = \inf_{u \in M} I[u]$, where, the total energy $I[u]$ is given by

$$I[u] = \int_\Omega |\nabla^m u|^p dx$$

and $M = \left\{ u \in D_0^{m,p}(\Omega); \int_\Omega g|u|^p dx = 1 \right\}$ is a C^1-mainfold.

We claim that

(i) $\lambda_1 > 0$ and

(ii) M is weakly closed.

Suppose,$\lambda_1 = 0$ and u_k be a minimising sequence.Then,

$$\begin{aligned} 1 &= \int_\Omega g|u_k|^p dx && (1.3) \\ &\le ||g||_{L(\frac{n}{mp})(\Omega)} ||u_k||^p_{L(\frac{np}{n-mp})(\Omega)} && (1.4) \\ &\le C||\nabla^m u_k||^p_{L^p(\Omega)} && (1.5) \\ &\to 0 && (1.6) \end{aligned}$$

which is a contradiction. So $\lambda_1 > 0$.

Suppose, u_m converges weakly to u in M.Then u_m is bounded in $D_0^{m,p}(\Omega)$..Since $\frac{n}{mp} > 1$ and $g \in L^{\frac{n}{mp}}(\Omega)$,so for any ball $B(R)$ of radius R arround origin, we have,

$$\int_{B(R)\cap\Omega} g|u_k|^p dx \to \int_{B(R)\cap\Omega} g|u|^p dx \qquad (1.7)$$

as $k \to \infty$. Let $\Omega^R = \Omega \cap B(R)^c$.Then

$$\int_{\Omega^R} g|u_k|^p dx \le ||g||_{L(\frac{n}{mp})(\Omega^R)} ||u_k||^p_{L(\frac{np}{n-mp})(\Omega^R)} \qquad (1.8)$$

Given $\epsilon > 0$ small choose $R > 0$ large enough such that $||g||_{L(\frac{n}{mp})(\Omega^R)} < \epsilon$.
Then,$\int_{\Omega^R} g|u_k|^p dx \le C\epsilon$.
Similarly, $\int_{\Omega^R} g|u|^p dx \le C\epsilon$
Therefore,

$$\begin{aligned} \int_\Omega g|u_k|^p dx &\to \int_\Omega g|u|^p dx && (1.9) \\ &= 1 && (1.10) \end{aligned}$$

Now M is weakly closed and I being weakly lower semi continous the infimum is achieved by some function ϕ_1.

Proof of theorem 2.2:-

We claim that

(i) $\lambda_1 > 0$ and

(ii) M is weakly closed.

(i) Suppose $\lambda_1 = 0$ and (u_m) be a minimizing sequence for λ_1. Then as

$$H^*(q) = \sup\left\{ \sup_{t\ge 0} \left(t.q - e^{t^{\frac{1}{p-1}}} + \sum_{k=0}^{2m-3} \frac{t^{\frac{k}{p-1}}}{k!} \right); \sup_{t<0} t.q \right\} \ge (t.q - e^{t^{\frac{1}{p-1}}})$$

so, $t.q \leq e^{t^{\frac{1}{p-1}}} + H^*(q)\ \forall t, q \in I\!R$.

Therefore,

$$\begin{aligned}
1 &= \int_\Omega g|u_m|^p dx && (1.11)\\
&\leq \frac{||u_m||^p}{\alpha_0(p-1)}\int_\Omega g(\frac{u_m}{||u||_m})^p(\alpha_0)^(p-1)dx && (1.12)\\
&\leq \frac{||u_m||^p}{\alpha_0(p-1)}[\int_\Omega exp(\alpha_0(\frac{u_m}{||u_m||})^{\frac{p}{p-1}}) + \int_\Omega H^*(|g|)] && (1.13)\\
&\leq \frac{||u_m||^p}{\alpha_0(p-1)}[C|\Omega| + \int_\Omega H^*(|g|)] && (1.14)\\
& && (1.15)
\end{aligned}$$

where, $C = \sup_{||\nabla^m u||_p \leq 1} \int_\Omega \exp(\alpha_0|u(x)|^q)dx$

Therefore, $1 \leq K||\nabla^m u||^p_{L^p(\Omega)} \to 0$ as $m \to \infty$,which leads to a contradiction.

So, $\lambda_1 > 0$ which proves our first claim.

(ii). To prove that M is weakly closed i.e., if $u_n \to u$ in M then $\int_\Omega g|u|^p dx = 1$.

Let B_R be the ball of radius R centered at origin. Let, (u_k) be a minimizing sequence in M.

Then, as (u_k) is bounded in $D_0^{m,p}(\Omega)$, let u_k converges weakly to$u$ in $D_0^{m,p}(\Omega)$.

Since, $g \in L^{p_0}_{loc}$ and $u_m \to u$ in $L^q(BR)$ for any $q \in [1, \infty)$ so,

$$\int_{B(R)\cap\Omega} g|u_k|^p dx \to \int_{B(R)\cap\Omega} g|u|^p dx$$

Let, $\Omega^R = \Omega \cap B(R)^c$.Then as ϵH is a convex function whenever H is convex, we have

$$\begin{aligned}
tq &\leq \epsilon H(t) + (\epsilon H)^*(q) && (1.16)\\
&\leq \epsilon e^{t^{\frac{1}{p-1}}} + \epsilon H^*(\frac{q}{\epsilon}) && (1.17)
\end{aligned}$$

By theorem A and lemma (2) we have,

$$\begin{aligned}
\int_{\Omega^R} g|u_k|^p dx &\leq \frac{||u_m||^p}{\alpha_0(p-1)}\int_\Omega g(\frac{u_m}{||u||_m})^p dx\\
&\leq \frac{||u_m||^p}{\alpha_0(p-1)}[\epsilon\int_\Omega exp(\alpha_0(\frac{u_m}{||u_m||})^{\frac{p}{p-1}}) + \int_{\Omega^R} \epsilon H^*(\frac{|g|}{\epsilon})]\\
&\leq \frac{||u_m||^p}{\alpha_0(p-1)}[\epsilon C|\Omega| + \int_\Omega \epsilon H^*(\frac{|g|}{\epsilon})]\\
&\leq \frac{||u_m||^p}{\alpha_0(p-1)}[\epsilon|\Omega|C + (1-\epsilon)||(H')^*(|g|)||_{L^1(\Omega^R)} + (\frac{1}{\epsilon}-1)||g||_{L^1(\Omega^R)}\\
&+ \epsilon||H^*(|g|)||_{L^1(\Omega^R)}].
\end{aligned}$$

First choose $\epsilon > 0$ small to make the 1st term small and then let $R \to \infty$ to make the other term small.
Hence, $\int_{\Omega^R} g|u_k|^p dx = O(\epsilon)$ as $R \to \infty$. Similarly, $\int_{\Omega^R} g|u|^p dx = O(\epsilon)$ as $R \to \infty$.
so, $\lim_{k\to\infty} \int_\Omega g|u_k|^p dx = \int_\Omega g|u|^p dx = 1$

This proves $M$is weakly closed. I being weakly lower semicontinous the inf is achieved and by Lagrange multiplier is an eigen function.

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