

Max-Planck-Institut
für Mathematik
in den Naturwissenschaften
Leipzig

A new spectral cancellation in quantum gravity

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Preprint no.: 67

2005



A NEW SPECTRAL CANCELLATION IN QUANTUM GRAVITY

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ABSTRACT. A general method exists for studying Abelian and non-Abelian gauge theories, as well as Euclidean quantum gravity, at one-loop level on manifolds with boundary. In the latter case, boundary conditions on metric perturbations h can be chosen to be completely invariant under infinitesimal diffeomorphisms, to preserve the invariance group of the theory and BRST symmetry. In the de Donder gauge, however, the resulting boundary-value problem for the Laplace type operator acting on h is known to be self-adjoint but not strongly elliptic. The present paper shows that, on the Euclidean four-ball, only the scalar part of perturbative modes for quantum gravity is affected by the lack of strong ellipticity. Interestingly, three sectors of the scalar-perturbation problem remain elliptic, while lack of strong ellipticity is “confined” to the remaining fourth sector. The integral representation of the resulting ζ -function asymptotics on the Euclidean four-ball is also obtained; this remains regular at the origin by virtue of a peculiar spectral identity obtained by the authors. There is therefore encouraging evidence in favour of the functional determinant with fully diff-invariant boundary conditions remaining well defined, at least on the four-ball, although severe technical obstructions remain in general.

1. INTRODUCTION

This paper is motivated by the authors’ struggle over many years with an important problem in quantum field theory and spectral geometry, i.e. the functional determinant in Euclidean quantum gravity on manifolds with non-empty boundary. The related open issues are not yet settled, but there is a sufficient amount of new calculations to justify further efforts, as we are going to see shortly.

The subject of boundary effects in quantum field theory (Deutsch and Candelas in Ref. 1) has always received a careful consideration in the literature by virtue of very important physical and mathematical motivations, that can be summarized as follows.

(i) Boundary data play a crucial role in the functional-integral approach (DeWitt in Ref. 2), in the quantum theory of the early universe (Hartle-Hawking and Hawking in Ref. 3) in supergravity (Hawking in Ref. 4) and even in string theory (Abouelsaoud et al. in Ref. 5).

(ii) The way in which quantum fields react to the presence of boundaries is responsible for remarkable physical effects, e.g. the attractive Casimir force among perfectly conducting parallel plates (Bordag et al., Milton, Nesterenko et al. in Ref. 6), which can be viewed as arising from differences of zero-point energies of the quantized electromagnetic field.

(iii) The spectral geometry of a Riemannian manifold (Gilkey in Ref. 7) with boundary is a fascinating problem where many new results have been derived over the last few years (Kirsten in Ref. 8, Vassilevich in Ref. 9).

(iv) Boundary terms (Moss in Ref. 10) in heat-kernel expansions have become a major subject of investigation in quantum gravity (Avramidi in Ref. 11), since they shed new light on one-loop conformal anomalies (Esposito et al., Moss and Poletti in Ref. 12, Tsoupros in Ref. 13) and one-loop divergences (Esposito in Ref. 14, Esposito et al. in Ref. 15).

In our paper we are interested in boundary conditions for metric perturbations that are completely invariant under infinitesimal diffeomorphisms, since they are part of the general scheme according to which the boundary conditions are preserved under the action of the symmetry group of the theory (Barvinsky in Ref. 16, Moss and Silva in Ref. 17, Avramidi and Esposito in Ref. 18). In field-theoretical language, this means setting to zero at the boundary that part πA of the gauge field A that lives on the boundary \mathcal{B} (π being a projection operator):

$$(1) \quad \left[\pi A \right]_{\mathcal{B}} = 0,$$

as well as the gauge-fixing functional,

$$(2) \quad \left[\Phi(A) \right]_{\mathcal{B}} = 0,$$

and the whole ghost field

$$(3) \quad [\varphi]_{\mathcal{B}} = 0.$$

For Euclidean quantum gravity, Eq. (1) reads as

$$(4) \quad [h_{ij}]_{\mathcal{B}} = 0,$$

where h_{ij} are perturbations of the induced three-metric. To arrive at the gravitational counterpart of Eqs. (2) and (3), note first that, under infinitesimal diffeomorphisms, metric perturbations $h_{\mu\nu}$ transform according to

$$(5) \quad \widehat{h}_{\mu\nu} \equiv h_{\mu\nu} + \nabla_{(\mu} \varphi_{\nu)},$$

where ∇ is the Levi-Civita connection on the background four-geometry with metric g , and $\varphi_{\nu} dx^{\nu}$ is the ghost one-form (strictly, our presentation is simplified: there are two independent ghost fields obeying Fermi statistics, and we will eventually multiply by -2 the effect of φ_{ν} to take this into account). In geometric language, the infinitesimal variation $\delta h_{\mu\nu} \equiv \widehat{h}_{\mu\nu} - h_{\mu\nu}$ is given by the Lie derivative along φ of the four-metric g . For manifolds with boundary, Eq. (5) implies that (Esposito et al. in Ref. 19, Avramidi et al. in Ref. 20)

$$(6) \quad \widehat{h}_{ij} = h_{ij} + \varphi_{(i;j)} + K_{ij}\varphi_0,$$

where the stroke denotes three-dimensional covariant differentiation tangentially with respect to the intrinsic Levi-Civita connection of the boundary, while K_{ij} is the extrinsic-curvature tensor of the boundary. Of course, φ_0 and φ_i are the normal and tangential components of the ghost, respectively. By virtue of Eq. (6), the boundary conditions (4) are “gauge invariant”, i.e.

$$(7) \quad \left[\widehat{h}_{ij} \right]_{\mathcal{B}} = 0,$$

if and only if the whole ghost field obeys homogeneous Dirichlet conditions, so that

$$(8) \quad [\varphi_0]_{\mathcal{B}} = 0,$$

$$(9) \quad [\varphi_i]_{\mathcal{B}} = 0.$$

The conditions (8) and (9) are necessary and sufficient since φ_0 and φ_i are independent, and three-dimensional covariant differentiation commutes with the operation of restriction to the boundary. We are indeed assuming that the boundary \mathcal{B} is smooth and not totally geodesic, i.e. $K_{ij} \neq 0$. However, for totally geodesic boundaries, having $K_{ij} = 0$, the condition (8) is no longer necessary.

On imposing boundary conditions on the remaining set of metric perturbations, the key point is to make sure that *the invariance of such boundary conditions under the infinitesimal transformations (5) is again guaranteed by (8) and (9)*, since otherwise one would obtain incompatible sets of boundary conditions on the ghost field. Indeed, on using the DeWitt–Faddeev–Popov formalism for the ⟨out|in⟩ amplitudes of quantum gravity, it is necessary to use a gauge-averaging term in the Euclidean action, of the form²

$$(10) \quad I_{g.a.} = \frac{1}{16\pi G} \int_{\mathcal{M}} \frac{\Phi_\nu \Phi^\nu}{2\alpha} \sqrt{\det g} \, d^4x,$$

where Φ_ν is any functional which leads to self-adjoint (elliptic) operators on metric and ghost perturbations. One then finds that

$$(11) \quad \delta\Phi_\mu(h) \equiv \Phi_\mu(h) - \Phi_\mu(\widehat{h}) = \mathcal{F}_\mu^\nu \varphi_\nu,$$

where \mathcal{F}_μ^ν is an elliptic operator that acts linearly on the ghost field. Thus, if one imposes the boundary conditions

$$(12) \quad \left[\Phi_\mu(h) \right]_{\mathcal{B}} = 0,$$

and if one assumes that the ghost field can be expanded in a complete orthonormal set of eigenfunctions $u_\nu^{(\lambda)}$ of \mathcal{F}_μ^ν which vanish at the boundary, i.e.

$$(13) \quad \mathcal{F}_\mu^\nu u_\nu^{(\lambda)} = \lambda u_\mu^{(\lambda)},$$

$$(14) \quad \varphi_\nu = \sum_\lambda C_\lambda u_\nu^{(\lambda)},$$

$$(15) \quad \left[u_\mu^{(\lambda)} \right]_{\mathcal{B}} = 0,$$

the boundary conditions (12) are automatically gauge-invariant under the Dirichlet conditions (8) and (9) on the ghost.

Having obtained the general recipe expressed by Eqs. (4) and (12), we can recall what they imply on the Euclidean four-ball. This background is relevant for one-loop quantum cosmology in the limit of small three-geometry on the one hand (Schleich in Ref. 21), and for spectral geometry and spectral asymptotics on the other hand.^{8,9} As shown in Ref. 19, if one chooses the de Donder gauge-fixing functional

$$(16) \quad \Phi_\mu(h) = \nabla^\nu \left(h_{\mu\nu} - \frac{1}{2} g_{\mu\nu} g^{\rho\sigma} h_{\rho\sigma} \right),$$

which has the virtue of leading to an operator of Laplace type on $h_{\mu\nu}$ in the one-loop functional integral, Eq. (12) yields the mixed boundary conditions

$$(17) \quad \left[\frac{\partial h_{00}}{\partial \tau} + \frac{6}{\tau} h_{00} - \frac{\partial}{\partial \tau} (g^{ij} h_{ij}) + \frac{2}{\tau^2} h_{0i}^{|i} \right]_{\mathcal{B}} = 0,$$

$$(18) \quad \left[\frac{\partial h_{0i}}{\partial \tau} + \frac{3}{\tau} h_{0i} - \frac{1}{2} \frac{\partial h_{00}}{\partial x^i} \right]_{\mathcal{B}} = 0.$$

In Refs. 15, 19, the boundary conditions (4), (17) and (18) were used to evaluate the full one-loop divergence of quantized general relativity on the Euclidean four-ball, including all $h_{\mu\nu}$ and all ghost modes. However, the meaning of such a calculation became unclear after the discovery in Ref. 18 that the boundary-value problem for the Laplacian P acting on metric perturbations is not strongly elliptic by virtue of tangential derivatives in the boundary conditions (17) and (18). Moreover, the work by Dowker and Kirsten in Ref. 22 had proved even earlier, in a simpler case, that the boundary-value problem with tangential derivatives is, in general, not strongly elliptic. Strong ellipticity^{8,18} is a technical requirement ensuring that a unique smooth solution of the boundary-value problem exists which vanishes at infinite geodesic distance from the boundary. If it is fulfilled, this ensures that the L^2 trace of the heat semigroup e^{-tP} exists, with the associated global heat-kernel asymptotics that yields one-loop divergence and one-loop effective action. However, when strong ellipticity does not hold, the L^2 trace of e^{-tP} acquires a singular part,¹⁸ and hence ζ -function calculations may become ill-defined.

All of this has motivated our analysis, which therefore derives in Sec. 2 the eigenvalue conditions for scalar modes. Section 3 obtains the first pair of resulting scalar-mode ζ -functions and Sec. 4 studies the remaining elliptic and non-elliptic parts of spectral asymptotics. Results and open problems are described in Sec. 5.

2. EIGENVALUE CONDITIONS FOR SCALAR MODES ON THE FOUR-BALL

On the Euclidean four-ball, which can be viewed as the portion of flat Euclidean four-space bounded by a three-sphere of radius q , metric perturbations $h_{\mu\nu}$ can be expanded in terms of hyperspherical harmonics as (Lifshitz and Khalatnikov in Ref. 23, Esposito et al. in Ref. 24)

$$(19) \quad h_{00}(x, \tau) = \sum_{n=1}^{\infty} a_n(\tau) Q^{(n)}(x),$$

$$(20) \quad h_{0i}(x, \tau) = \sum_{n=2}^{\infty} \left[b_n(\tau) \frac{Q_{|i}^{(n)}(x)}{(n^2 - 1)} + c_n(\tau) S_i^{(n)}(x) \right],$$

$$(21) \quad \begin{aligned} h_{ij}(x, \tau) &= \sum_{n=3}^{\infty} d_n(\tau) \left[\frac{Q_{|ij}^{(n)}(x)}{(n^2 - 1)} + \frac{c_{ij}}{3} Q^{(n)}(x) \right] + \sum_{n=1}^{\infty} \frac{e_n(\tau)}{3} c_{ij} Q^{(n)}(x) \\ &+ \sum_{n=3}^{\infty} \left[f_n(\tau) \left(S_{i|j}^{(n)}(x) + S_{j|i}^{(n)}(x) \right) + k_n(\tau) G_{ij}^{(n)}(x) \right], \end{aligned}$$

where $\tau \in [0, q]$ and $Q^{(n)}(x)$, $S_i^{(n)}(x)$ and $G_{ij}^{(n)}(x)$ are scalar, transverse vector and transverse-traceless tensor hyperspherical harmonics, respectively, on a unit three-sphere with metric c_{ij} . By insertion of the expansions (19)-(21) into the eigenvalue equation for the Laplacian acting on $h_{\mu\nu}$, and by setting $\sqrt{E} \rightarrow iM$, which corresponds to a rotation of contour in the ζ -function analysis (Barvinsky et al. in Ref. 25) one finds the modes as linear combinations of modified Bessel functions of first kind. Modified Bessel functions of the second kind are not included

to ensure regularity at the origin $\tau = 0$. For details, we refer the reader to the work by Esposito et al. in Ref. 26.

The boundary conditions (4), (17), (18), (8), (9), jointly with the mode-expansions on the four-ball, can be used to obtain homogeneous linear systems that yield, implicitly, the eigenvalues of our problem. The conditions for finding non-trivial solutions of such linear systems are given by the vanishing of the associated determinants; these yield the eigenvalue conditions $\delta(E) = 0$, i.e. the equations obeyed by the eigenvalues by virtue of the boundary conditions. For the purpose of a rigorous analysis, we need the full expression of such eigenvalue conditions for each set of coupled modes. Upon setting $\sqrt{E} \rightarrow iM$, we denote by $D(Mq)$ the counterpart of $\delta(E)$, bearing in mind that, strictly, only $\delta(E)$ yields implicitly the eigenvalues, while $D(Mq)$ is more convenient for ζ -function calculations.²⁵

In particular, we here focus on scalar modes (for the whole set of modes, see again the work in Ref. 26). For all $n \geq 3$, coupled scalar modes a_n, b_n, d_n, e_n are ruled by a determinant reading as

$$(22) \quad D_n(Mq) = \det \rho_{ij}(Mq),$$

with degeneracy n^2 , where ρ_{ij} is a 4×4 matrix with entries

$$(23) \quad \rho_{11} = I_n(Mq) - MqI'_n(Mq), \quad \rho_{12} = MqI'_n(Mq),$$

$$(24) \quad \rho_{13} = (2-n)I_{n-2}(Mq) + MqI'_{n-2}(Mq), \quad \rho_{14} = (2+n)I_{n+2}(Mq) + MqI'_{n+2}(Mq),$$

$$(25) \quad \rho_{21} = -(n^2-1)I_n(Mq), \quad \rho_{22} = 2MqI'_n(Mq) + 6I_n(Mq),$$

$$(26) \quad \rho_{23} = 2(n+1)MqI'_{n-2}(Mq) - (n^2-6n-7)I_{n-2}(Mq),$$

$$(27) \quad \rho_{24} = -2(n-1)MqI'_{n+2}(Mq) - (n^2+6n-7)I_{n+2}(Mq),$$

$$(28) \quad \rho_{31} = 0, \quad \rho_{32} = -I_n(Mq),$$

$$(29) \quad \rho_{33} = \frac{(n+1)}{(n-2)}I_{n-2}(Mq), \quad \rho_{34} = \frac{(n-1)}{(n+2)}I_{n+2}(Mq),$$

$$(30) \quad \rho_{41} = 3I_n(Mq), \quad \rho_{42} = -2I_n(Mq), \quad \rho_{43} = -I_{n-2}(Mq), \quad \rho_{44} = -I_{n+2}(Mq).$$

The hardest part of our analysis is the investigation of the equation obtained by setting to zero the determinant (22). For this purpose, we first exploit the recurrence relations among I_n, I_{n+1} and I'_n to find

$$(31) \quad \rho_{11} = I_n(w) - wI'_n(w), \quad \rho_{12} = wI'_n(w), \quad \rho_{13} = wI'_n(w) + nI_n(w), \quad \rho_{14} = wI'_n(w) - nI_n(w),$$

$$(32) \quad \rho_{21} = -(n^2-1)I_n(w), \quad \rho_{22} = 2(wI'_n(w) + 3I_n(w)),$$

$$(33) \quad \rho_{23} = (n+1) \left\{ \left[3(n+1) + \frac{2n(n-1)(n+3)}{w^2} \right] I_n(w) + 2 \left[w + \frac{(n-1)(n+3)}{w} \right] I'_n(w) \right\},$$

$$(34) \quad \rho_{24} = (n-1) \left\{ \left[3(n-1) + \frac{2n(n+1)(n-3)}{w^2} \right] I_n(w) - 2 \left[w + \frac{(n+1)(n-3)}{w} \right] I'_n(w) \right\},$$

$$(35) \quad \rho_{31} = 0, \quad \rho_{32} = -I_n(w),$$

$$(36) \quad \rho_{33} = \frac{(n+1)}{(n-2)} \left[\left(1 + \frac{2n(n-1)}{w^2} \right) I_n(w) + \frac{2(n-1)}{w} I_n'(w) \right],$$

$$(37) \quad \rho_{34} = \frac{(n-1)}{(n+2)} \left[\left(1 + \frac{2n(n+1)}{w^2} \right) I_n(w) - \frac{2(n+1)}{w} I_n'(w) \right],$$

$$(38) \quad \rho_{41} = 3I_n(w), \quad \rho_{42} = -2I_n(w),$$

$$(39) \quad \rho_{43} = - \left(1 + \frac{2n(n-1)}{w^2} \right) I_n(w) - \frac{2(n-1)}{w} I_n'(w),$$

$$(40) \quad \rho_{44} = - \left(1 + \frac{2n(n+1)}{w^2} \right) I_n(w) + \frac{2(n+1)}{w} I_n'(w).$$

The resulting determinant, despite its cumbersome expression, can be studied by introducing the variable

$$(41) \quad y \equiv \frac{I_n'(w)}{I_n(w)},$$

which leads to

$$(42) \quad D_n(w) = \frac{48n(1-n^2)}{(n^2-4)} I_n^4(w) (y-y_1)(y-y_2)(y-y_3)(y-y_4),$$

where

$$(43) \quad y_1 \equiv -\frac{n}{w}, \quad y_2 \equiv \frac{n}{w}, \quad y_3 \equiv -\frac{n}{w} - \frac{w}{2}, \quad y_4 \equiv \frac{n}{w} - \frac{w}{2},$$

and hence

$$(44) \quad \frac{(n^2-4)}{48n(1-n^2)} D_n(w) = \left(I_n'(w) + \frac{n}{w} I_n(w) \right) \left(I_n'(w) - \frac{n}{w} I_n(w) \right) \\ \cdot \left(I_n'(w) + \left(\frac{w}{2} + \frac{n}{w} \right) I_n(w) \right) \left(I_n'(w) + \left(\frac{w}{2} - \frac{n}{w} \right) I_n(w) \right).$$

3. FIRST PAIR OF SCALAR-MODE ζ -FUNCTIONS

In our problem, the differential operator under investigation is the Laplacian on the Euclidean four-ball acting on metric perturbations. The boundary conditions for vector, tensor and ghost modes correspond to a familiar mixture of Dirichlet and Robin boundary conditions for which integral representation of the ζ -function and heat-kernel coefficients are immediately obtained. New features arise instead from Eq. (44), that gives rise to four different ζ -functions. On studying the first line of Eq. (44), we can exploit the work by Dowker and Kirsten in Ref. 22 and the uniform asymptotic expansion of Bessel functions and their first derivatives (see Appendix) to say that the integral representation of the resulting ζ -function reads as

$$(45) \quad \zeta_A^\pm(s) \equiv \frac{(\sin \pi s)}{\pi} \sum_{n=3}^{\infty} n^{-(2s-2)} \int_0^\infty dz z^{-2s} \frac{\partial}{\partial z} \log \left[z^{-\beta_\pm(n)} \left(zn I_n'(zn) \pm n I_n(zn) \right) \right].$$

With our notation, $\beta_+(n) = n, \beta_-(n) = n+2$, where these factors are fixed by the leading behaviour of the eigenvalue condition as $z \rightarrow 0$ (Bordag et al. in Ref.

27); the uniform asymptotic expansion of modified Bessel functions and their first derivatives can be used to find (hereafter $\tau = \tau(z) \equiv (1 + z^2)^{-\frac{1}{2}}$)

$$(46) \quad znI'_n(zn) \pm nI_n(zn) \sim \frac{n}{\sqrt{2\pi n}} \frac{e^{n\eta}}{\sqrt{\tau}} (1 \pm \tau) \left(1 + \sum_{k=1}^{\infty} \frac{p_{k,\pm}(\tau)}{n^k} \right),$$

where

$$(47) \quad p_{k,\pm}(\tau) \equiv (1 \pm \tau)^{-1} \left(v_k(\tau) \pm \tau u_k(\tau) \right),$$

for all $k \geq 1$, and

$$(48) \quad \log \left(1 + \sum_{k=1}^{\infty} \frac{p_{k,\pm}(\tau)}{n^k} \right) \sim \sum_{k=1}^{\infty} \frac{T_{k,\pm}(\tau)}{n^k}.$$

Thus, the ζ -functions (45) obtain, from the first pair of round brackets in Eq. (46), the contributions (cf. Ref. 22)

$$(49) \quad A_+(s) \equiv \sum_{n=3}^{\infty} n^{-(2s-2)} \frac{(\sin \pi s)}{\pi} \int_0^{\infty} dz z^{-2s} \frac{\partial}{\partial z} \log \left(1 + (1 + z^2)^{-\frac{1}{2}} \right),$$

$$(50) \quad A_-(s) \equiv \sum_{n=3}^{\infty} n^{-(2s-2)} \frac{(\sin \pi s)}{\pi} \int_0^{\infty} dz z^{-2s} \frac{\partial}{\partial z} \log \left(\frac{1 - (1 + z^2)^{-\frac{1}{2}}}{z^2} \right),$$

where z^2 in the denominator of the argument of the log arises, in Eq. (50), from the extra z^{-2} in the prefactor $z^{-\beta-(n)}$ in the definition (45). Moreover, the second pair of round brackets in Eq. (46) contributes $\sum_{j=1}^{\infty} A_{j,\pm}(s)$, having defined

$$(51) \quad A_{j,\pm}(s) \equiv \sum_{n=3}^{\infty} n^{-(2s+j-2)} \frac{(\sin \pi s)}{\pi} \int_0^{\infty} dz z^{-2s} \frac{\partial}{\partial z} T_{j,\pm}(\tau(z)),$$

where, from the formulae

$$(52) \quad T_{1,\pm} = p_{1,\pm},$$

$$(53) \quad T_{2,\pm} = p_{2,\pm} - \frac{1}{2} p_{1,\pm}^2,$$

$$(54) \quad T_{3,\pm} = p_{3,\pm} - p_{1,\pm} p_{2,\pm} + \frac{1}{3} p_{1,\pm}^3,$$

we find

$$(55) \quad T_{1,\pm} = -\frac{3}{8} \tau \pm \frac{1}{2} \tau^2 - \frac{5}{24} \tau^3,$$

$$(56) \quad T_{2,\pm} = -\frac{3}{16} \tau^2 \pm \frac{3}{8} \tau^3 + \frac{1}{8} \tau^4 \mp \frac{5}{8} \tau^5 + \frac{5}{16} \tau^6,$$

$$(57) \quad T_{3,\pm} = -\frac{21}{128} \tau^3 \pm \frac{3}{8} \tau^4 + \frac{509}{640} \tau^5 \mp \frac{25}{12} \tau^6 + \frac{21}{128} \tau^7 \pm \frac{15}{8} \tau^8 - \frac{1105}{1152} \tau^9,$$

and hence, in general,

$$(58) \quad T_{j,\pm}(\tau) = \sum_{a=j}^{3j} f_a^{(j,\pm)} \tau^a.$$

We therefore find, from the first line of Eq. (44), contributions to the generalized ζ -function, from terms in round brackets in Eq. (46), equal to

$$(59) \quad \chi_A^\pm(s) = \omega_0(s)F_0^\pm(s) + \sum_{j=1}^{\infty} \omega_j(s)F_j^\pm(s),$$

where, for all $\lambda = 0, j$,

$$(60) \quad \omega_\lambda(s) \equiv \sum_{n=3}^{\infty} n^{-(2s+\lambda-2)} = \zeta_H(2s+\lambda-2; 3) = \zeta_R(2s+\lambda-2) - 1 - 2^{-(2s+\lambda-2)},$$

while, from Eqs. (49)–(51),

$$(61) \quad F_0^+(s) \equiv \frac{(\sin \pi s)}{\pi} \int_0^\infty dz z^{-2s} \frac{\partial}{\partial z} \log\left(1 + (1+z^2)^{-\frac{1}{2}}\right),$$

$$(62) \quad F_0^-(s) \equiv -2 \frac{(\sin \pi s)}{\pi} \int_0^\infty dz \frac{z^{-(2s-1)}}{(1+z^2)} - F_0^+(s) = -1 - F_0^+(s),$$

$$(63) \quad F_j^\pm(s) \equiv \frac{(\sin \pi s)}{\pi} \sum_{a=j}^{3j} L^\pm(s, a, 0) f_a^{(j, \pm)},$$

having set (this general definition will prove useful later, and arises from a more general case, where τ^a is divided by the b -th power of $(1 \pm \tau)$ in Eq. (58))

$$(64) \quad L^\pm(s, a, b) \equiv \int_0^1 \tau^{2s+a} (1-\tau)^{-s} (1+\tau)^{-s} \left(\pm b(1 \pm \tau)^{-b-1} - a\tau^{-1}(1 \pm \tau)^{-b} \right) d\tau.$$

Moreover, on considering

$$(65) \quad L_0^+(s) \equiv \frac{\pi}{\sin \pi s} F_0^+(s),$$

and changing variable from z to τ therein, all L -type integrals above can be obtained from

$$(66) \quad Q(\alpha, \beta, \gamma) \equiv \int_0^1 \tau^\alpha (1-\tau)^\beta (1+\tau)^\gamma d\tau.$$

In particular, we will need

$$(67) \quad L_0^+(s) = -Q(2s, -s, -s-1),$$

$$(68) \quad L^+(s, a, b) = bQ(2s+a, -s, -s-b-1) - aQ(2s+a-1, -s, -s-b),$$

where, from the integral representation of the hypergeometric function, one has (Gradshteyn and Ryzhik in Ref. 28)

$$(69) \quad Q(\alpha, \beta, \gamma) = \frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{\Gamma(\alpha+\beta+2)} F(-\gamma, \alpha+1; \alpha+\beta+2; -1).$$

For example, explicitly,

$$(70) \quad L_0^+(s) = -\frac{\Gamma(2s+1)\Gamma(1-s)}{\Gamma(s+2)} F(s+1, 2s+1; s+2; -1).$$

Now we exploit Eqs. (45), (46) and (59) to write

$$(71) \quad \zeta_A^+(s) = \chi_A^+(s) + \frac{(\sin \pi s)}{\pi} \sum_{n=3}^{\infty} n^{-(2s-2)} \int_0^\infty dz \left[\frac{z^{-(2s-1)}}{2(1+z^2)} + nz^{-(2s+1)} \left(\sqrt{1+z^2} - 1 \right) \right].$$

Hence we find

$$(72) \quad \zeta_A^+(0) = \lim_{s \rightarrow 0} \left[\omega_0(s) F_0^+(s) + \sum_{j=1}^{\infty} \omega_j(s) F_j^+(s) + (\zeta_A^+(s) - \chi_A^+(s)) \right].$$

The first limit in Eq. (72) is immediately obtained by noting that

$$(73) \quad \lim_{s \rightarrow 0} L_0^+(s) = -\log(2),$$

and hence

$$(74) \quad \lim_{s \rightarrow 0} \omega_0(s) F_0^+(s) = \lim_{s \rightarrow 0} \left[\zeta_H(2s-2; 3) \frac{(\sin \pi s)}{\pi} L_0^+(s) \right] = 0.$$

To evaluate the second limit in Eq. (72), we use

$$(75) \quad \lim_{s \rightarrow 0} L^+(s, a, 0) = -1,$$

and bear in mind that $\omega_j(s)$ is a meromorphic function with first-order pole, as $s \rightarrow 0$, only at $j = 3$ by virtue of the limit

$$(76) \quad \lim_{y \rightarrow 1} \left[\zeta_R(y) - \frac{1}{(y-1)} \right] = \gamma.$$

Hence we find (see coefficients in Eq. (57))

$$(77) \quad \begin{aligned} \lim_{s \rightarrow 0} \sum_{j=1}^{\infty} \omega_j(s) F_j^+(s) &= \lim_{s \rightarrow 0} \frac{(\sin \pi s)}{\pi} \sum_{j=1}^{\infty} \omega_j(s) \left[\sum_{a=j}^{3j} L^+(s, a, 0) f_a^{(j,+)} \right] \\ &= -\frac{1}{2} \sum_{a=3}^9 f_a^{(3,+)} = -\frac{1}{720}, \end{aligned}$$

while, from Eqs. (71) and (69),

$$(78) \quad \begin{aligned} \lim_{s \rightarrow 0} (\zeta_A^+(s) - \chi_A^+(s)) &= \lim_{s \rightarrow 0} \left(\frac{1}{4} \zeta_H(2s-2; 3) + \frac{1}{4\sqrt{\pi}} \frac{\Gamma(s-\frac{1}{2})}{\Gamma(s+1)} \zeta_H(2s-3; 3) \right) \\ &= -\frac{5}{4} + \frac{1079}{240}. \end{aligned}$$

We therefore find, with the same algorithms as in Ref. 27,

$$(79) \quad \zeta_A^+(0) = -\frac{5}{4} + \frac{1079}{240} - \frac{1}{2} \sum_{a=3}^9 f_a^{(3,+)} = \frac{146}{45},$$

$$(80) \quad \zeta_A^-(0) = -\frac{5}{4} + \frac{1079}{240} + 5 - \frac{1}{2} \sum_{a=3}^9 f_a^{(3,-)} = \frac{757}{90}.$$

These results have been double-checked by using also the powerful analytic technique in Ref. 25.

4. FURTHER SPECTRAL ASYMPTOTICS: ELLIPTIC AND NON-ELLIPTIC PARTS

As a next step, the second line of Eq. (44) suggests considering ζ -functions having the integral representation (cf. Eq. (45) and see further comments in Sec. 5)

$$(81) \quad \zeta_B^\pm(s) \equiv \frac{(\sin \pi s)}{\pi} \sum_{n=3}^{\infty} n^{-(2s-2)} \int_0^\infty dz z^{-2s} \frac{\partial}{\partial z} \log \left[z^{-\beta_\pm(n)} \left(z n I'_n(zn) + \left(\frac{z^2 n^2}{2} \pm n \right) I_n(zn) \right) \right].$$

To begin, we exploit again the uniform asymptotic expansion of modified Bessel functions and their first derivatives to find (cf. Eq. (46))

$$(82) \quad z n I'_n(zn) + \left(\frac{z^2 n^2}{2} \pm n \right) I_n(zn) \sim \frac{n^2}{2\sqrt{2\pi n}} \frac{e^{n\eta}}{\sqrt{\tau}} \left(\frac{1}{\tau} - \tau \right) \left(1 + \sum_{k=1}^{\infty} \frac{r_{k,\pm}(\tau)}{n^k} \right),$$

where we have (bearing in mind that $u_0 = v_0 = 1$)

$$(83) \quad r_{k,\pm}(\tau) \equiv u_k(\tau) + \frac{2\tau}{(1-\tau^2)} \left((v_{k-1}(\tau) \pm \tau u_{k-1}(\tau)) \right),$$

for all $k \geq 1$. Hereafter we set

$$(84) \quad \Omega \equiv \sum_{k=1}^{\infty} \frac{r_{k,\pm}(\tau(z))}{n^k},$$

and rely upon the formula

$$(85) \quad \log(1 + \Omega) \sim \sum_{k=1}^{\infty} (-1)^{k+1} \frac{\Omega^k}{k}$$

to evaluate the uniform asymptotic expansion (cf. Eq. (48))

$$(86) \quad \log \left(1 + \sum_{k=1}^{\infty} \frac{r_{k,\pm}(\tau(z))}{n^k} \right) \sim \sum_{k=1}^{\infty} \frac{R_{k,\pm}(\tau(z))}{n^k}.$$

The formulae yielding $R_{k,\pm}$ from $r_{k,\pm}$ are exactly as in Eqs. (52)–(54), with T replaced by R and p replaced by r (see, however, comments below Eq. (90)). Hence we find, bearing in mind Eq. (83),

$$(87) \quad R_{1,\pm} = (1 \mp \tau)^{-1} \left(\frac{17}{8} \tau \mp \frac{1}{8} \tau^2 - \frac{5}{24} \tau^3 \pm \frac{5}{24} \tau^4 \right),$$

$$(88) \quad R_{2,\pm} = (1 \mp \tau)^{-2} \left(-\frac{47}{16} \tau^2 \pm \frac{15}{8} \tau^3 - \frac{21}{16} \tau^4 \pm \frac{3}{4} \tau^5 - \frac{1}{16} \tau^6 \mp \frac{5}{8} \tau^7 + \frac{5}{16} \tau^8 \right),$$

$$(89) \quad \begin{aligned} R_{3,\pm} &= (1 \mp \tau)^{-3} \left(\frac{1721}{384} \tau^3 \mp \frac{441}{128} \tau^4 + \frac{597}{320} \tau^5 \mp \frac{1033}{960} \tau^6 + \frac{239}{80} \tau^7 \right. \\ &\mp \frac{28}{5} \tau^8 + \frac{2431}{576} \tau^9 \pm \frac{221}{192} \tau^{10} - \frac{1105}{384} \tau^{11} \pm \frac{1105}{1152} \tau^{12} \left. \right), \end{aligned}$$

and therefore

$$(90) \quad R_{j,\pm}(\tau(z)) = (1 \mp \tau)^{-j} \sum_{a=j}^{4j} C_a^{(j,\pm)} \tau^a,$$

where, unlike what happens for the $T_{j,\pm}$ polynomials, the exponent of $(1 \mp \tau)$ never vanishes. Note that, at $\tau = 1$ (i.e. $z = 0$), our $r_{k,+}(\tau)$ and $R_{k,+}(\tau)$ are singular. Such a behaviour is not seen for any of the strongly elliptic boundary-value problems.⁸ This technical difficulty motivates our efforts below and is interpreted by us as a clear indication of the lack of strong ellipticity proved, on general ground, in Ref. 18.

The $\zeta_B^-(s)$ function is more easily dealt with. It indeed receives contributions from terms in round brackets in Eq. (82) equal to (cf. Eq. (50) and bear in mind that $\beta_- - \beta_+ = 2$ in Eq. (81))

$$\begin{aligned} B_-(s) &\equiv \sum_{n=3}^{\infty} n^{-(2s-2)} \frac{(\sin \pi s)}{\pi} \int_0^{\infty} dz z^{-2s} \frac{\partial}{\partial z} \log \left(\frac{\frac{1}{\tau(z)} - \tau(z)}{z^2} \right) \\ (91) \quad &= \omega_0(s) \frac{(\sin \pi s)}{\pi} \int_0^{\infty} dz z^{-2s} \frac{\partial}{\partial z} \log \frac{1}{\sqrt{1+z^2}} = -\frac{1}{2} \omega_0(s), \end{aligned}$$

and $\sum_{j=1}^{\infty} B_{j,-}(s)$, having defined, with $\lambda = 0, j$ (cf. Eq. (51))

$$(92) \quad \omega_{\lambda}(s) \equiv \sum_{n=3}^{\infty} n^{-(2s+\lambda-2)} = \zeta_H(2s + \lambda - 2; 3),$$

$$(93) \quad B_{j,-}(s) \equiv \omega_j(s) \frac{(\sin \pi s)}{\pi} \int_0^{\infty} dz z^{-2s} \frac{\partial}{\partial z} R_{j,-}(\tau(z)).$$

On using the same method as in Sec. 3, the formulae (81)–(93) lead to

$$(94) \quad \zeta_B^-(0) = -\frac{5}{4} + \frac{1079}{240} + \frac{5}{2} - \frac{1}{16} \sum_{a=3}^{12} C_a^{(3,-)} = \frac{206}{45},$$

a result which agrees with a derivation of $\zeta_B^-(0)$ relying upon the method of Ref. 25.

Although we have stressed after Eq. (90) the problems with the $\zeta_B^+(s)$ part, for the moment let us proceed formally in the same way as above. Thus we define, in analogy to Eq. (91),

$$(95) \quad B_+(s) \equiv \omega_0(s) \frac{(\sin \pi s)}{\pi} \int_0^{\infty} dz z^{-2s} \frac{\partial}{\partial z} \log \left(\frac{1}{\tau(z)} - \tau(z) \right),$$

and, in analogy to Eq. (93),

$$(96) \quad B_{j,+}(s) \equiv \omega_j(s) \frac{(\sin \pi s)}{\pi} \int_0^{\infty} dz z^{-2s} \frac{\partial}{\partial z} R_{j,+}(\tau(z)).$$

In order to make the presentation as transparent as possible, we write out the derivatives of $R_{j,+}$. On changing integration variable from z to τ we define

$$(97) \quad C_j(\tau) \equiv \frac{\partial}{\partial \tau} R_{j,+}(\tau),$$

and we find the following results:

$$(98) \quad C_1(\tau) = (1-\tau)^{-2} \left(\frac{17}{8} - \frac{1}{4}\tau - \frac{1}{2}\tau^2 + \frac{5}{4}\tau^3 - \frac{5}{8}\tau^4 \right),$$

$$(99) \quad C_2(\tau) = (1-\tau)^{-3} \left(-\frac{47}{8}\tau + \frac{45}{8}\tau^2 - \frac{57}{8}\tau^3 + \frac{51}{8}\tau^4 - \frac{21}{8}\tau^5 - \frac{33}{8}\tau^6 + \frac{45}{8}\tau^7 - \frac{15}{8}\tau^8 \right),$$

$$(100) \quad C_3(\tau) = (1-\tau)^{-4} \left(\frac{1721}{128} \tau^2 - \frac{441}{32} \tau^3 + \frac{1635}{128} \tau^4 - \frac{163}{16} \tau^5 + \frac{1545}{64} \tau^6 - \frac{227}{4} \tau^7 \right. \\ \left. + \frac{4223}{64} \tau^8 - \frac{221}{16} \tau^9 - \frac{5083}{128} \tau^{10} + \frac{1105}{32} \tau^{11} - \frac{1105}{128} \tau^{12} \right),$$

so that the general expression of $C_j(\tau)$ reads as

$$(101) \quad C_j(\tau) = (1-\tau)^{-j-1} \sum_{a=j-1}^{4j} K_a^{(j)} \tau^a, \quad \forall j = 1, \dots, \infty.$$

These formulae engender a $\zeta_B^+(0)$ which can be defined, after change of variable from z to τ , by splitting the integral with respect to τ , in the integral representation of $\zeta_B^+(s)$, according to the identity

$$\int_0^1 d\tau = \int_0^\mu d\tau + \int_\mu^1 d\tau,$$

and taking the limit as $\mu \rightarrow 1$ *after having evaluated the integral*. More precisely, since the integral on the left-hand side is independent of μ , we can choose μ small on the right-hand side so that, in the interval $[0, \mu]$ (and only there!), we can use the uniform asymptotic expansion of the integrand where the negative powers of $(1-\tau)$ are harmless. Moreover, independence of μ also implies that, after having evaluated the integrals on the right-hand side, we can take the $\mu \rightarrow 1$ limit. Within this framework, the limit as $\mu \rightarrow 1$ of the second integral on the right-hand side yields vanishing contribution to the asymptotic expansion of $\zeta_B^+(s)$.

With this *caveat*, on defining (cf. (66))

$$(102) \quad Q_\mu(\alpha, \beta, \gamma) \equiv \int_0^\mu \tau^\alpha (1-\tau)^\beta (1+\tau)^\gamma d\tau,$$

we obtain the representations

$$(103) \quad B_+(s) = -\omega_0(s) \frac{(\sin \pi s)}{\pi} \left[-Q_\mu(2s, -s-1, -s) + Q_\mu(2s, -s, -s-1) - Q_\mu(2s-1, -s, -s) \right],$$

$$(104) \quad B_{j,+}(s) = -\omega_j(s) \frac{(\sin \pi s)}{\pi} \sum_{a=j-1}^{4j} K_a^{(j)} Q_\mu(2s+a, -s-j-1, -s).$$

The relevant properties of $Q_\mu(\alpha, \beta, \gamma)$ can be obtained by observing that this function is nothing but a hypergeometric function of two variables,²⁸ i.e.

$$(105) \quad Q_\mu(\alpha, \beta, \gamma) = \frac{\mu^{\alpha+1}}{\alpha+1} F_1(\alpha+1, -\beta, -\gamma, \alpha+2; \mu, -\mu).$$

In detail, a summary of results needed to consider the limiting behaviour of $\zeta_B^+(s)$ as $s \rightarrow 0$ is

$$(106) \quad \omega_0(s) \frac{(\sin \pi s)}{\pi} \sim -5s + \mathcal{O}(s^2),$$

$$(107) \quad \omega_j(s) \frac{(\sin \pi s)}{\pi} \sim \frac{1}{2} \delta_{j,3} + \tilde{b}_{j,1} s + \mathcal{O}(s^2),$$

$$(108) \quad \lim_{\mu \rightarrow 1} Q_\mu(2s, -s-1, -s) \sim -\frac{1}{s} + \mathcal{O}(s^0),$$

$$(109) \quad \lim_{\mu \rightarrow 1} Q_\mu(2s, -s, -s-1) \sim \log(2) + O(s),$$

$$(110) \quad \lim_{\mu \rightarrow 1} Q_\mu(2s-1, -s, -s) \sim \frac{1}{2s} + O(s),$$

$$(111) \quad \begin{aligned} & \lim_{\mu \rightarrow 1} Q_\mu(2s+a, -s-j-1, -s) \\ &= \frac{\Gamma(-j-s)\Gamma(a+2s+1)}{\Gamma(a-j+s+1)} F(a+2s+1, s, a-j+s+1; -1) \\ &\sim \frac{b_{j,-1}(a)}{s} + b_{j,0}(a) + O(s), \end{aligned}$$

where

$$(112) \quad \tilde{b}_{j,1} = -1 - 2^{2-j} + \zeta_R(j-2)(1 - \delta_{j,3}) + \gamma\delta_{j,3},$$

$$(113) \quad b_{j,-1}(a) = \frac{(-1)^{j+1}}{j!} \frac{\Gamma(a+1)}{\Gamma(a-j+1)} (1 - \delta_{a,j-1}),$$

and we only strictly need $b_{3,0}(a)$ which, unlike the elliptic cases studied earlier, now depends explicitly on a and is given by (ψ being the standard notation for the logarithmic derivative of the Γ -function)

$$(114) \quad b_{3,0}(a) = \frac{1}{6} \frac{\Gamma(a+1)}{\Gamma(a-2)} \left[-\log(2) - \frac{1}{4}(6a^2 - 9a + 1) \frac{\Gamma(a-2)}{\Gamma(a+1)} + 2\psi(a+1) - \psi(a-2) - \psi(4) \right].$$

Remarkably, the coefficient of $\frac{1}{s}$ in the small- s behaviour of the generalized ζ -function $\zeta_B^+(s)$ is zero because it is equal to

$$(115) \quad \lim_{s \rightarrow 0} s \zeta_B^+(s) = \sum_{a=2}^{12} b_{3,-1}(a) K_a^{(3)} = \frac{1}{6} \sum_{a=3}^{12} a(a-1)(a-2) K_a^{(3)},$$

which vanishes by virtue of the rather peculiar general property

$$(116) \quad \sum_{a=j}^{4j} \frac{\Gamma(a+1)}{\Gamma(a-j+1)} K_a^{(j)} = \sum_{a=j}^{4j} \prod_{l=0}^{j-1} (a-l) K_a^{(j)} = 0, \quad \forall j = 1, \dots, \infty,$$

and hence we find eventually

$$(117) \quad \begin{aligned} \zeta_B^+(0) &= -\frac{5}{4} + \frac{1079}{240} + \frac{5}{2} - \frac{1}{2} \sum_{a=2}^{12} b_{3,0}(a) K_a^{(3)} - \sum_{j=1}^{\infty} \tilde{b}_{j,1} \sum_{a=j-1}^{4j} b_{j,-1}(a) K_a^{(j)} \\ &= \frac{5}{4} + \frac{1079}{240} + \frac{599}{720} = \frac{296}{45}, \end{aligned}$$

because the infinite sum on the first line of Eq. (117) vanishes by virtue of Eqs. (113) and (116), and exact cancellation of $\log(2)$ terms is found to occur by virtue of Eq. (116).

To cross-check our analysis we remark that, on applying the technique of Ref. 25, one finds

$$(118) \quad \zeta_B^+(0) = -\frac{15}{4} + \frac{1079}{240} - \frac{1}{720} = \frac{67}{90},$$

where $-\frac{1}{720}$ results from working in the $n \rightarrow \infty$ and $w \rightarrow 0$ limit in

$$\left(I'_n(w) + \left(\frac{w}{2} + \frac{n}{w} \right) I_n(w) \right)$$

on the second line of Eq. (44); such a term then reduces to $\left(I'_n(w) + \frac{n}{w} I_n(w) \right)$, *not identically but only in the region where $n \rightarrow \infty$ and $w \rightarrow 0$* . Such a discrepancy still needs to be understood.

Remaining contributions to $\zeta(0)$, being obtained from strongly elliptic sectors of the boundary-value problem, are instead found to agree with the results in Ref. 19, i.e.

$$(119) \quad \zeta(0)[\text{transverse traceless modes}] = -\frac{278}{45},$$

$$(120) \quad \zeta(0)[\text{coupled vector modes}] = \frac{494}{45},$$

$$(121) \quad \zeta(0)[\text{decoupled vector mode}] = -\frac{15}{2},$$

$$(122) \quad \zeta(0)[\text{scalar modes}(a_1, e_1; a_2, b_2, e_2)] = -17,$$

$$(123) \quad \zeta(0)[\text{scalar ghost modes}] = -\frac{149}{45},$$

$$(124) \quad \zeta(0)[\text{vector ghost modes}] = \frac{77}{90},$$

$$(125) \quad \zeta(0)[\text{decoupled ghost mode}] = \frac{5}{2}.$$

5. CONCLUDING REMARKS

We have studied the analytically continued eigenvalue conditions for metric perturbations on the Euclidean four-ball, in the presence of boundary conditions completely invariant under infinitesimal diffeomorphisms in the de Donder gauge and with the α parameter set to 1 in Eq. (10). This has made it possible to prove that only one sector of the scalar-mode determinant is responsible for lack of strong ellipticity of the boundary-value problem (see second line of Eq. (44) and the analysis in Secs. 3 and 4). The first novelty with respect to the work in Ref. 18 is a clear separation of the elliptic and non-elliptic sectors of spectral asymptotics for Euclidean quantum gravity. We have also shown that one can indeed obtain a regular ζ -function asymptotics at small s in the non-elliptic case by virtue of the remarkable identity (116). Our prescription for the $\zeta(0)$ value differs from the result first obtained in Ref. 19, where, however, neither the strong ellipticity issue¹⁸ nor the non-standard spectral asymptotics of our Sec. 4 had been considered.

As far as we can see, the issues raised by our results are as follows.

(i) The integral representation (81) is legitimate because the second line of Eq. (44) corresponds to the eigenvalue conditions, for $n \geq 3$,

$$(126) \quad F_B^\pm(n, x) \equiv J'_n(x) + \left(-\frac{x}{2} \pm \frac{n}{x} \right) J_n(x) = 0.$$

For both choices of sign in front of $\frac{n}{x}$, if x_i is a root, then so is $-x_i$, with positive eigenvalue $E_i = x_i^2$ (having set the 3-sphere radius $q = 1$ for simplicity). For

any fixed n , there is a countable infinity of roots x_i and they grow approximately linearly with the integer i counting such roots. The function F_B^\pm admits therefore a canonical-product representation (Ahlfors in Ref. 29) which ensures that the integral representation (81) reproduces the standard definition of generalized ζ -function, i.e.

$$\zeta(s) \equiv \sum_{E_k > 0} d(E_k) E_k^{-s},$$

where $d(E_k)$ is the degeneracy of the eigenvalue E_k .

(ii) Even though the lack of strong ellipticity implies that the functional trace of the heat semigroup no longer exists, and hence the Mellin transform relating ζ -function to integrated heat kernel cannot be exploited, it remains possible to define the functional determinant of the operator P acting on metric perturbations. For this purpose, a weaker assumption provides a sufficient condition, i.e. the existence of a sector in the complex plane free of eigenvalues of the leading symbol of P (Seeley in Ref. 30). Note also that, if one looks at the A_1 heat-kernel coefficient for boundary conditions involving tangential derivatives,⁸ it is exactly for the ball that the potentially divergent pieces involving the extrinsic curvature in A_1 cancel. Thus, on the Euclidean ball cancellations take place that maybe could explain why $\zeta(0)$ is finite. This might be therefore *a very particular result for the ball*.

(iii) By virtue of standard recurrence relations among Bessel functions, the eigenvalue conditions (126) are equivalent to studying the eigenvalue conditions

$$(127) \quad \tilde{F}_B^\pm(n, x) = J_n(x) \mp \frac{2}{x} J_{n-1}(x) = 0,$$

where the eigenvalues $E(i, n, \pm)$ are obtained by squaring up the roots $x(i, n, \pm)$. The equation for $\tilde{F}_B^-(n, x)$ can be further re-expressed in the form

$$(128) \quad \left(1 + \frac{4n}{x^2}\right) J_n(x) - \frac{2}{x} J_{n-1}(x) = 0.$$

The functions \tilde{F}_B^\pm differ therefore by one term only, and this term gets small as x gets larger. The numerical analysis confirms indeed that a $\rho(i, n)$ positive and much smaller than 1 exists such that one can write

$$(129) \quad E(i, n, +) = E(1, n, +)\delta_{i,1} + E(i-1, n, -)(1 + \rho(i, n))(1 - \delta_{i,1}),$$

for all $n \geq 3$ and for all $i \geq 1$. One therefore finds a “decomposition” of the desired $\zeta_B^+(0)$ according to (Esposito et al. in Ref. 31)

$$(130) \quad \zeta_B^+(0) = \zeta_B^-(0) + \lim_{l \rightarrow 0} \sum_{n=3}^{\infty} n^2 E^{-l}(1, n, +) + \sum_{k=1}^{\infty} (-1)^k \frac{r_k(0)}{k},$$

where $E(1, n, +)$ is obtained by squaring up³¹

$$(131) \quad x(1, n, +) \sim \pm 2\sqrt{n-1} \left[1 + \frac{1}{2}(n-1)^{-2} - \frac{17}{8}(n-1)^{-4} + O((n-1)^{-6}) \right],$$

while³¹

$$(132) \quad r_k(0) \equiv \lim_{l \rightarrow 0} \left[l \sum_{n=3}^{\infty} n^2 \sum_{i=1}^{\infty} E^{-l}(i, n, -) \rho^k(i+1, n) \right].$$

Unfortunately, Eq. (132) cannot be used with (130) and (131) to check our result (117), because neither the eigenvalues $E(i, n, -)$ nor the correction factor

$$(133) \quad \rho(i+1, n) \equiv \frac{E(i+1, n, +)}{E(i, n, -)} - 1$$

are explicitly known.

(iv) The remarkable factorization of eigenvalue conditions, with resulting isolation of elliptic part of spectral asymptotics (transverse-traceless, vector and ghost modes, all modes in finite-dimensional sub-spaces and three of the four equations for scalar modes), suggests trying to re-assess functional integrals on manifolds with boundary, with the hope of being able to obtain unique results from the non-elliptic contribution. If this cannot be achieved, the two alternatives below should be considered again.

(v) Luckock boundary conditions (Luckock in Ref. 32), which engender BRST-invariant amplitudes but are not diffeomorphism invariant.¹⁵ They have already been applied by Moss and Poletti in Refs. 12 and 33.

(vi) Non-local boundary conditions that lead to surface states in quantum cosmology and pseudo-differential operators on metric and ghost modes (Marachevsky and Vassilevich, Esposito in Ref. 34). Surface states are particularly interesting since they describe a transition from quantum to classical regime in cosmology entirely ruled by the strong ellipticity requirement, while pseudo-differential operators are a source of technical complications.

There is therefore encouraging evidence in favour of Euclidean quantum gravity being able to drive further developments in quantum field theory, quantum cosmology and spectral asymptotics (see early mathematical papers by Grubb in Ref. 35, Gilkey and Smith in Ref. 36) in the years to come.

APPENDIX: OLVER EXPANSIONS

In Secs. 3 and 4 we use the uniform asymptotic expansion of modified Bessel functions I_ν first found by Olver in Ref. 37:

$$(134) \quad I_\nu(z\nu) \sim \frac{e^{\nu\eta}}{\sqrt{2\pi\nu}(1+z^2)^{\frac{1}{4}}} \left(1 + \sum_{k=1}^{\infty} \frac{u_k(\tau)}{\nu^k} \right),$$

where

$$(135) \quad \tau \equiv (1+z^2)^{-\frac{1}{2}}, \quad \eta \equiv (1+z^2)^{\frac{1}{2}} + \log \left(\frac{z}{1+\sqrt{1+z^2}} \right).$$

This holds for $\nu \rightarrow \infty$ at fixed z . The polynomials $u_k(\tau)$ can be found from the recurrence relation²⁷

$$(136) \quad u_{k+1}(\tau) = \frac{1}{2}\tau^2(1-\tau^2)u'_k(\tau) + \frac{1}{8} \int_0^\tau d\rho (1-5\rho^2)u_k(\rho),$$

starting with $u_0(\tau) = 1$. Moreover, the first derivative of I_ν has the following uniform asymptotic expansion at large ν and fixed z :

$$(137) \quad I'_\nu(z\nu) \sim \frac{e^{\nu\eta}}{\sqrt{2\pi\nu}} \frac{(1+z^2)^{\frac{1}{4}}}{z} \left(1 + \sum_{k=1}^{\infty} \frac{v_k(\tau)}{\nu^k} \right),$$

with the v_k polynomials determined from the u_k according to²⁷

$$(138) \quad v_k(\tau) = u_k(\tau) + \tau(\tau^2 - 1) \left[\frac{1}{2}u_{k-1}(\tau) + \tau u'_{k-1}(\tau) \right],$$

starting with $v_0(\tau) = u_0(\tau) = 1$.

ACKNOWLEDGMENTS

We are indebted to Ivan Avramidi for inspiration provided by previous collaboration with some of us and by continuous correspondence, and to Gerd Grubb for enlightening correspondence. K. Kirsten is grateful to the Baylor University Research Committee, to the Max-Planck-Institute for Mathematics in the Sciences (Leipzig, Germany) and to the INFN for financial support. The work of G. Esposito and K. Kirsten has been partially supported also by PRIN *SINTESI*. A.K. is grateful to CARIPO Science Foundation for financial support. His work was also partially supported by the Russian Foundation for Basic Research under the Grant No. 02-02-16817 and by the Scientific School Grant No. 2338.2003.2

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