Max-Planck-Institut
für Mathematik
in den Naturwissenschaften
Leipzig

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by

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Preprint no.: 68 2005

STABILITY THEOREMS FOR CHIRAL BAG BOUNDARY CONDITIONS

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ABSTRACT. We study asymptotic expansions of the smeared L^2-traces Fe^{-tP^2} and FPe^{-tP^2}, where P is an operator of Dirac type and F is an auxiliary smooth endomorphism. We impose chiral bag boundary conditions depending on an angle θ. Studying the θ-dependence of the above trace invariants, θ-independent pieces are identified. The associated stability theorems allow one to show the regularity of the eta function for the problem and to determine the most important heat kernel coefficient on a four dimensional manifold.

1. Introduction

Let M be a compact connected m-dimensional Riemannian manifold with smooth boundary ∂M. Let D be an operator of Laplace type on a vector bundle V over M. Let $D_\mathcal{B}$ be the realization of D which is defined by a strongly elliptic boundary condition $\mathcal{B}$. Work of Greiner [1] and Seeley [2, 3] shows that the fundamental solution of the heat equation $e^{-tD_\mathcal{B}}$ is an infinitely smoothing operator which is of trace class. Let $F \in C^\infty(\mathrm{End}(V))$ be an auxiliary endomorphism of V which is used for localization. As $t \downarrow 0$, there is a complete asymptotic expansion

$$\mathrm{Tr}_{L^2}\left\{Fe^{-tD_\mathcal{B}}\right\} \sim \sum_{n=0}^{\infty} a_n(F,D,\mathcal{B})t^{(m-n)/2}\,.$$

The *heat asymptotics* a_n are locally computable. There exist suitable local endomorphisms $e_n(x,D)$ and $e_{n,\nu}(y,D,\mathcal{B})$ of V so that

$$a_n(F,D,\mathcal{B}) = \int_M \mathrm{Tr}_V\left\{F(x)e_n(x,D)\right\}dx + \sum_{\nu=0}^{n-1}\int_{\partial M}\mathrm{Tr}_V\left\{F^{(\nu)}e_{n,\nu}(y,D,\mathcal{B})\right\}dy\,.$$

In this equation, dx and dy denote the Riemannian measures on M and on ∂M, respectively, and $F^{(\nu)}$ denotes the ν^{th} covariant derivative of F with respect to the inward unit normal using the canonical connection determined by D.

These asymptotics can be computed quite explicitly; the principle of "not feeling the boundary" shows that the interior invariants $e_n(x,D)$ do not depend on the boundary condition chosen. The invariants $e_n(x,D)$ vanish for n odd and are known for $n = 0, 2, 4, 6, 8$, see for example [4, 5, 6]. The boundary invariants $e_{n,\nu}(y,D,\mathcal{B})$ are considerably more subtle. The invariants for n odd do not vanish. For mixed boundary conditions they are known for $n = 0, 1, 2, 3, 4, 5$, and for many other boundary conditions varying amounts of information are known. We refer to [7, 8, 9] for further details concerning these formulas.

There are various stability theorems for these invariants that play a crucial role in their evaluation. In Section 2, we present results in the context of chiral bag boundary conditions [10, 11]. To motivate these stability results, we first give examples in the standard setting which have proved to be crucial in past analysis.

Key words and phrases. bag boundary conditions, operator of Dirac type, zeta and eta invariants, variational formulas

2000 *Mathematics Subject Classification.* 58J50.

The following stability formulae arise from conformal variations of the operator. Assertion (1) follows from work of Branson and Ørsted [12, 13]; Assertion (2) was first observed in [14].

Theorem 1.1. *Let $\mathcal{B}$ define an elliptic boundary condition for an operator D of Laplace type on a compact m-dimensional Riemannian manifold M with smooth boundary. Let $h \in C^\infty(M)$, let $D_\varepsilon := e^{-2\varepsilon h}D$, let $F \in C^\infty(\text{End}(V))$, and let $F_\varepsilon := e^{-2\varepsilon h}F$. Then*

1. $\partial_\varepsilon a_m(1, D_\varepsilon, \mathcal{B}) = 0$.
2. $\partial_\varepsilon a_{m-2}(F_\varepsilon, D_\varepsilon, \mathcal{B}) = 0$.

Proof. We proceed formally, the necessary analytic justification can be found in [15]:

$$\begin{aligned} &\sum_{n=0}^{\infty} \partial_\varepsilon a_n(1, D_\varepsilon, \mathcal{B}) t^{(n-m)/2} \sim \partial_\varepsilon \operatorname{Tr}_{L^2}\left\{e^{-tD_{\varepsilon,\mathcal{B}}}\right\} \\ = \quad & -t \operatorname{Tr}_{L^2}\left\{(\partial_\varepsilon D_{\varepsilon,\mathcal{B}}) e^{-tD_{\varepsilon,\mathcal{B}}}\right\} = 2t \operatorname{Tr}_{L^2}\left\{h D_{\varepsilon,\mathcal{B}} e^{-tD_{\varepsilon,\mathcal{B}}}\right\} \\ = \quad & -2t\partial_t \operatorname{Tr}_{L^2}\left\{h e^{-tD_{\varepsilon,\mathcal{B}}}\right\} \sim -2t\partial_t \sum_{n=0}^{\infty} a_n(h, D_\varepsilon, \mathcal{B}) t^{(n-m)/2} \\ \sim \quad & \sum_{n=0}^{\infty} (m-n) a_n(h, D_\varepsilon, \mathcal{B}) t^{(n-m)/2}\,. \end{aligned}$$

Equating powers in the relevant asymptotic expansions yields

$$\partial_\varepsilon a_n(1, D_\varepsilon, \mathcal{B}) = (m-n) a_n(h, D_\varepsilon, \mathcal{B})\,.$$

We set $n = m$ to complete the proof of Assertion (1).

Let $D_{\delta,\mathcal{B}} := D_{\mathcal{B}} - \delta F$. We compute

$$\begin{aligned} &\sum_{n=0}^{\infty} \partial_\delta a_n(1, D_\delta, \mathcal{B}) t^{(n-m)/2} \sim \partial_\delta \operatorname{Tr}_{L^2}\left\{e^{-tD_{\delta,\mathcal{B}}}\right\} \\ = \quad & -t \operatorname{Tr}_{L^2}\left\{(\partial_\delta D_{\delta,\mathcal{B}}) e^{-tD_{\delta,\mathcal{B}}}\right\} = t \operatorname{Tr}_{L^2}\left\{F e^{-tD_{\delta,\mathcal{B}}}\right\} \\ \sim \quad & t \sum_{n=0}^{\infty} a_n(F, D_\delta, \mathcal{B}) t^{(n-m)/2}\,. \end{aligned}$$

Equating terms in the asymptotic expansions yields

$$\partial_\delta a_n(1, D - \delta F, \mathcal{B}) = a_{n-2}(F, D - \delta F, \mathcal{B})\,. \tag{1.a}$$

We now consider the joint variation $D_{\delta,\varepsilon,\mathcal{B}} := e^{-2\varepsilon h}(D_{\mathcal{B}} - \delta F)$. We use Assertion (1) and Equation (1.a) to see

$$\begin{aligned} &\partial_\varepsilon a_m(1, e^{-2\varepsilon h}(D - \delta F), \mathcal{B}) = 0, \\ &\partial_\delta a_m(1, e^{-2\varepsilon h}(D - \delta F), \mathcal{B}) = a_{m-2}(e^{-2\varepsilon h}F, e^{-2\varepsilon h}(D - \delta F), \mathcal{B})\,. \end{aligned}$$

Consequently:

$$\begin{aligned} 0 \quad = \quad & \partial_\delta \partial_\varepsilon a_m(1, e^{-2\varepsilon h}(D - \delta F), \mathcal{B}) = \partial_\varepsilon \partial_\delta a_m(1, e^{-2\varepsilon h}(D - \delta F), \mathcal{B}) \\ = \quad & \partial_\varepsilon a_{m-2}(e^{-2\varepsilon h}F, e^{-2\varepsilon h}(D - \delta F), \mathcal{B})\,. \end{aligned}$$

Assertion (2) follows by setting $\delta = 0$. □

Here is a brief guide to the remainder of this paper. In Section 2, we will use a similar strategy to establish stability results for the zeta and eta invariants, when chiral bag boundary conditions [10, 11] are imposed. In Section 3, we will use these arguments to establish the regularity at $s = 0$ of the eta invariant. Although this result can be derived from Theorem 2.3.5 [16], it seemed worth giving a straightforward and elementary argument adapted to the setting at hand which is more conceptual in nature and which is of interest in its own right; the discussion in [16] dealt with a very general setting and was, perhaps, somewhat opaque. The evaluation of invariants in arbitrary dimensions, even at the level of a_2, has turned out to be extremely involved [17]. As an application of the stability results, we will see in Section 4 that it is relatively easy to find the coefficient a_4 for the case $F = 1$ in the most relevant case of a four dimensional manifold. In Section 5, we give a perturbative result showing that the a_4 coefficient does exhibit θ dependence in the general setting. In Section 6, we remove the assumption that M is orientable and in the final section, we make some concluding remarks.

2. Chiral bag boundary conditions

We start this section by describing the chiral bag boundary conditions. Let γ give the vector bundle V a $\mathrm{Clif}(TM)$ module structure. If $\{e_i\}$ is a local orthonormal frame for the tangent bundle TM, then $\gamma_i := \gamma(e_i)$ forms a collection of skew-adjoint matrices satisfying the Clifford commutation relations:

$$\gamma_i\gamma_j + \gamma_j\gamma_i = -2\delta_{ij}\,\mathrm{Id}_V\ .$$

Let ∇ be a connection on V and let ψ be an auxiliary endomorphism of V. We form the associated operator of Dirac type

$$P := \gamma_i\nabla_{e_i} + \psi\,.$$

Assume that $m = 2\bar m$ is even and that M is oriented. Let

$$\tilde\gamma := (\sqrt{-1})^{\bar m}\gamma_1...\gamma_m$$

be the normalized orientation; $\tilde\gamma$ is self-adjoint and $\tilde\gamma^2 = \mathrm{Id}_V$. It is the generalization of γ_5 to arbitrary even dimension.

We assume as a compatibility condition that

$$\tilde\gamma P + P\tilde\gamma = 0\,.$$

This means that if we decompose $V = V_+ \oplus V_-$ into the ±1 eigenbundles of $\tilde\gamma$, then we may also decompose

$$P = P_+ + P_- \quad\text{where}\quad P_\pm : C^\infty(V_\pm) \to V_\mp\,.$$

Choosing suitable boundary conditions for P is crucial. Whereas spectral questions for many boundary conditions have been analyzed in great detail [7, 8], an understanding of so-called chiral bag boundary conditions is still in its infancy; see, however, [17, 18, 19]. These boundary conditions are defined as follows. Set

$$\chi_\theta := -\tilde\gamma e^{\theta\tilde\gamma}\gamma_m \quad\text{for}\quad \theta\in\mathbb{R}\,.$$

Since γ_m anti-commutes with $\tilde\gamma$, one has

$$\chi_\theta^2 = \tilde\gamma e^{\theta\tilde\gamma}\gamma_m\tilde\gamma e^{\theta\tilde\gamma}\gamma_m = \tilde\gamma\gamma_m\tilde\gamma\gamma_m e^{-\theta\tilde\gamma}e^{\theta\tilde\gamma} = \mathrm{Id}_V\ .$$

We consider the projection operator on the -1 spectrum of χ_θ, which is given by

$$\mathcal{B}_\theta := \tfrac12(1-\chi_\theta)\,.$$

This defines suitable boundary conditions for P; let $P_{\mathcal{B}_\theta}$ be the realization of P. We set $D_{\mathcal{B}_\theta} = P^2_{\mathcal{B}_\theta}$ and let the associated boundary operator be

$$\mathcal{B}^1_\theta\phi := \mathcal{B}_\theta\phi \oplus \mathcal{B}_\theta P\phi\,.$$

This is an elliptic boundary condition so the heat trace asymptotics are well defined [20]. To simplify the notation, we set

$$a_n(F,P,\mathcal{B}_\theta) := a_n(F,D,\mathcal{B}^1_\theta)\,.$$

The study of the coefficient a_m is crucial to understanding the chiral anomaly in the zeta function regularization; see, for example, the discussion in [18]. The angle θ occurring in these boundary conditions is a substitute for introducing small quark masses to drive the breaking of chiral symmetry [19, 21, 22]. One of the first papers where the chiral boundary conditions were introduced is the work by Hrasko and Balog [10], with first applications to chiral bag models being presented in [11].

The stability results we are going to prove naturally relate heat equation invariants and eta invariants. The eta invariants measure the spectral asymmetry of $P_{\mathcal{B}_\theta}$; they are defined as follows. If $F \in C^\infty(\mathrm{End}(V))$, then we may expand

$$\mathrm{Tr}_{L^2}\left\{FP_{\mathcal{B}_\theta}e^{-tP^2_{\mathcal{B}_\theta}}\right\} = \sum_{n=0}^{\infty} a_n^\eta(F,P,\mathcal{B}_\theta)t^{(n-m-1)/2}\,.$$

The eta invariants $a_n^\eta(F,P,\mathcal{B}_\theta)$ are again locally determined. We refer to [23] for a further discussion of the eta invariant in this setting.

We now come to one of the main results of this article.

Theorem 2.1. *Let $P = \gamma_i\nabla_{e_i} + \psi$ be an operator of Dirac type on a compact oriented smooth manifold of even dimension m with boundary. Let $F \in C^\infty(\mathrm{End}(V))$. Assume that $P\tilde\gamma + \tilde\gamma P = 0$ and that $F\tilde\gamma + \tilde\gamma F = 0$. Then*

(1) $\partial_\theta a_m(1,P,\mathcal{B}_\theta) = 0$.
(2) $\partial_\theta a_m^\eta(1,P,\mathcal{B}_\theta) = 0$.
(3) $\partial_\theta a_{m-1}^\eta(F,P,\mathcal{B}_\theta) = 0$.

Proof. The central technical point is to replace the given variation by one where the boundary condition is held fixed, an idea that first occurred in [24]. We consider the gauge transformation defined by $e^{\frac12\theta\tilde\gamma}$. Since $\tilde\gamma$ anti-commutes with P and with γ_m, we may compute:

$$P_\theta := e^{-\frac12\theta\tilde\gamma}Pe^{\frac12\theta\tilde\gamma} = e^{-\theta\tilde\gamma}P,$$
$$e^{-\frac12\theta\tilde\gamma}\chi_\theta e^{\frac12\theta\tilde\gamma} = -e^{-\frac12\theta\tilde\gamma}\tilde\gamma e^{\theta\tilde\gamma}\gamma_m e^{\frac12\theta\tilde\gamma} = -e^{-\frac{\theta}{2}\tilde\gamma}\tilde\gamma e^{\frac{\theta}{2}\tilde\gamma}\gamma_m = \chi_0\,.$$

Thus by gauge invariance,

$$a_n(1,P,\mathcal{B}_\theta) = a_n(1,P_\theta,\mathcal{B}_0)\,.$$

We have $\partial_\theta P_\theta = -\tilde\gamma P_\theta$; since we are multiplying on the left by $\tilde\gamma$, the domain defined by the boundary condition $\mathcal{B}_0$ is not perturbed. We can now argue exactly as in the proof of Theorem 1.1 to see:

$$\begin{aligned}\partial_\theta\,\mathrm{Tr}_{L^2}\left\{e^{-tP^2_{\theta,\mathcal{B}_0}}\right\} &= -2t\,\mathrm{Tr}_{L^2}\left\{\partial_\theta(P_{\theta,\mathcal{B}_0})P_{\theta,\mathcal{B}_0}e^{-tP^2_{\theta,\mathcal{B}_0}}\right\}\\ = \quad 2t\,\mathrm{Tr}_{L^2}\left\{\tilde\gamma P^2_{\theta,\mathcal{B}_0}e^{-tP^2_{\theta,\mathcal{B}_0}}\right\} &= -2t\partial_t\,\mathrm{Tr}_{L^2}\left\{\tilde\gamma e^{-tP^2_{\theta,\mathcal{B}_0}}\right\}\,.\end{aligned}$$

Expanding in an asymptotic series then yields:

$$\begin{aligned}&\sum_{n=0}^{\infty}\partial_\theta a_n(1,P_\theta,\mathcal{B}_0)t^{(n-m)/2} \sim \sum_{n=0}^{\infty} -2t\partial_t\left\{a_n(\tilde\gamma,P_\theta,B_0)t^{(n-m)/2}\right\}\\ \sim\quad &\sum_{n=0}^{\infty}(m-n)a_n(\tilde\gamma,P_\theta,\mathcal{B}_0)t^{(n-m)/2}\,.\end{aligned}$$

Equatating terms in the relevant asymptotic expansions yields:

$$\partial_\theta a_n(1,P,\mathcal{B}_\theta) = \partial_\theta a_n(1,P_\theta,\mathcal{B}_0) = (m-n)a_n(\tilde\gamma,P_\theta,\mathcal{B}_0)\,. \tag{2.a}$$

Assertion (1) then follows by setting $n = m$ in Equation (2.a).

The argument is similar to prove Assertion (2). We compute:

$$\begin{aligned}
&\sum_{n=0}^{\infty}\partial_\theta a_n^\eta(1,P,\mathcal{B}_\theta)=\sum_{n=0}^{\infty}\partial_\theta a_n^\eta(1,P_\theta,\mathcal{B}_0)\\
\sim\quad&\partial_\theta\operatorname{Tr}_{L^2}\left\{P_{\theta,\mathcal{B}_0}e^{-tP^2_{\theta,\mathcal{B}_0}}\right\}=\operatorname{Tr}_{L^2}\left\{\partial_\theta(P_{\theta,\mathcal{B}_0})(1-2tP^2_{\theta,\mathcal{B}_0})e^{-tP^2_{\theta,\mathcal{B}_0}}\right\}\\
=\quad&(1+2t\partial_t)\operatorname{Tr}_{L^2}\left\{-\tilde\gamma P_{\theta,\mathcal{B}_0}e^{-tP^2_{\theta,\mathcal{B}_0}}\right\}\\
\sim\quad&-(1+2t\partial_t)\sum_{n=0}^{\infty}a_n^\eta(\tilde\gamma,P_\theta,\mathcal{B}_0)^{(n-m-1)/2}\\
\sim\quad&\sum_{n=0}^{\infty}(m-n)a_n^\eta(\tilde\gamma,P_\theta,\mathcal{B}_0)t^{(n-m-1)/2}.
\end{aligned}$$

This yields the relation

$$\partial_\theta a_n^\eta(1,P,\mathcal{B}_\theta)=(m-n)a_n^\eta(\tilde\gamma,P_\theta,\mathcal{B}_0)\,.$$

Assertion (2) follows by setting $n=m$.

To prove Assertion (3), consider a variation of the form $P_\delta:=P-\delta F$. We argue as in the proof of Theorem 1.1 (2) to compute:

$$\begin{aligned}
&\sum_{n=0}^{\infty}\partial_\delta a_n(1,P_\delta,\mathcal{B}_\theta)t^{(n-m)/2}\sim\partial_\delta\operatorname{Tr}_{L^2}\left\{e^{-tP^2_{\delta,\mathcal{B}_\theta}}\right\}\\
=\quad&-2t\operatorname{Tr}_{L^2}\left\{(\partial_\delta P_{\delta,\mathcal{B}_\theta})P_{\delta,\mathcal{B}_\theta}e^{-tP^2_{\delta,\mathcal{B}_\theta}}\right\}=2t\operatorname{Tr}_{L^2}\left\{FP_{\delta,\mathcal{B}_\theta}e^{-tP^2_{\delta,\mathcal{B}_\theta}}\right\}\\
\sim\quad&2\sum_{k=0}^{\infty}a_k^\eta(F,P,\mathcal{B}_\theta)t^{(k-m+1)/2}\,.
\end{aligned}$$

Equating terms in the asymptotic expansion then yields

$$2a_k^\eta(F,P,\mathcal{B}_\theta)=\partial_\delta a_{k+1}(1,P_\delta,\mathcal{B}_\theta)\,. \tag{2.b}$$

Since $P_\delta\tilde\gamma+\tilde\gamma P_\delta=0$, we can apply Theorem 2.1 (1). Differentiating Equation (2.b) with respect to θ, setting $k+1=m$ and $\delta=0$, then yields the desired result. □

Remark 2.2. It is natural to try a similar approach to study $\partial_\theta a_{m-1}(F,P,\mathcal{B}_\theta)$. In fact, however, this fails. One would compute:

$$\begin{aligned}
&\sum_{n=0}^{\infty}\partial_\delta a_n^\eta(1,P_\delta,\mathcal{B}_\theta)t^{(n-m-1)/2}\sim\partial_\delta\operatorname{Tr}_{L^2}\left\{P_{\delta,\mathcal{B}_\theta}e^{-tP^2_{\delta,\mathcal{B}_\theta}}\right\}\\
=\quad&\operatorname{Tr}_{L^2}\left\{(\partial_\delta P_{\delta,\mathcal{B}_\theta})(1-2tP^2_{\delta,\mathcal{B}_\theta})e^{-tP^2_{\delta,\mathcal{B}_\theta}}\right\}\\
=\quad&-(1+2t\partial_t)\operatorname{Tr}_{L^2}\left\{Fe^{-tP^2_{\delta,\mathcal{B}_\theta}}\right\}\\
\sim\quad&-(1+2t\partial_t)\sum_{k=0}^{\infty}a_k(F,P_\delta,\mathcal{B}_\theta)t^{(k-m)/2}\\
\sim\quad&\sum_{k=0}^{\infty}(m-k-1)a_k(F,P_\delta,\mathcal{B}_\theta)t^{(k-m)/2}\,.
\end{aligned}$$

Equating terms in the asymptotic expansions then yields

$$\partial_\delta a_{k+1}^\eta(1,P_\delta,\mathcal{B}_\theta)=(m-k-1)a_k(F,P_\delta,\mathcal{B}_\theta)\,. \tag{2.c}$$

Setting $k+1=m$ yields no information about $a_{m-1}(F,P_\delta,\mathcal{B}_\theta)$.

Remark 2.3. Assertion (1) of Theorem 2.1 explains certain observations made in the literature. For example, the calculation of the heat kernel coefficients on the ball with $F=1$ showed, in general dimensions, a strong dependence on θ. It was noticed that in $m = 2$, respectively $m = 4$, this dependence disappeared when $a_2(1, P, \mathcal{B}_\theta)$, respectively $a_4(1, P, \mathcal{B}_\theta)$, was considered [25]. The above result shows why this must be the case.

The same pattern could be observed in the coefficient $a_2(1, P, \mathcal{B}_\theta)$ found in [17]. Whereas in general the θ-dependence enters in terms of hypergeometric functions, these terms are multiplied by $(m - 2)$ and they thus vanish at $m = 2$; see Equation (2d) in [17].

3. Regularity of the eta invariant at s=0

In this section we apply Theorem 2.1 (2) to discuss the regularity of the eta invariant at $s = 0$.

The eta invariant was introduced by Atiyah et. al. [26] for closed manifolds. We extend those definitions to the setting at hand as follows. Let

$$\eta(s, P, \mathcal{B}_\theta) := \mathrm{Tr}_{L^2}\left\{P_{\mathcal{B}_\theta}(P^2_{\mathcal{B}_\theta})^{-(s+1)/2}\right\}.$$

(There is a bit of fuss with the 0-spectrum that can be dealt with appropriately). The standard pseudo-differential calculus, see, for example, the discussion in [24], can be used to show $\eta(s, P, \mathcal{B}_\theta)$ has a meromorphic extension to $\mathbb{C}$ with isolated simple poles with locally computable residues. If $P_{\mathcal{B}_\theta}$ is self-adjoint with respect to some Hermitian metric on V and if $\{\lambda_\nu\}$ is the spectrum of $P_{\mathcal{B}_\theta}$, each eigenvalue being repeated according to multiplicity, then

$$\eta(s, P, \mathcal{B}_\theta) := \sum_{\lambda_\nu \neq 0} \mathrm{sign}(\lambda_\nu)|\lambda_\nu|^{-s}.$$

A-priori, $s = 0$ need not be a regular value of η. However, the pole of the eta function at $s = 0$ can be related to the trace invariant $a^\eta_m(1, P, \mathcal{B}_\theta)$:

$$\mathrm{Res}_{s=0}\, \eta(s, P, \mathcal{B}_\theta) = 2\Gamma(\tfrac{1}{2})^{-1} a^\eta_m(1, P, \mathcal{B}_\theta).$$

We will show that $a^\eta_m(1, P, \mathcal{B}_\theta) = 0$ and thus η is regular at $s = 0$. One then sets

$$\eta(P, \mathcal{B}_\theta) := \tfrac{1}{2}\{\eta(s, P, \mathcal{B}_\theta) + \dim\ker\{P_{\mathcal{B}_\theta}\}\}\big|_{s=0} \in \mathbb{C}/\mathbb{Z}$$

as a global measure of the spectral asymmetry of P. For closed manifolds, this invariant plays a central role in the index theorem for manifolds with boundary [26]. It can also be used to study K-theory groups and equivariant bordism groups [27] and to study metrics of positive scalar curvature [28]. Thus it is important to understand this invariant in the context of bag boundary conditions.

Theorem 3.1. *Let P be an operator of Dirac type on a compact oriented smooth manifold of even dimension m with boundary. Then $a^\eta_m(1, P, \mathcal{B}_\theta) = 0$.*

Proof. Let Q be an auxilary first order partial differential operator on V. There is a complete asymptotic expansion, see for example Lemma 2.6 [24],

$$\mathrm{Tr}_{L^2}\left\{Qe^{-tP^2_{\mathcal{B}_\theta}}\right\} = \sum_{n=0}^{\infty} a^\eta_n(Q, P, \mathcal{B}_\theta)t^{(n-m-1)/2}.$$

Let P_ε be a smooth 1-parameter family of operators of Dirac type. We assume the leading symbol of P_ε agrees with the leading symbol of P on ∂M. Thus $\mathcal{B}_\theta$ defines an elliptic boundary condition for this family as well. Using the notation

$\dot{P}_\varepsilon = \partial_\varepsilon(P_{\varepsilon,\mathcal{B}_\theta})$, we compute:

$$\begin{aligned}
&\sum_{n=0}^\infty \partial_\varepsilon a_n^\eta(1, P_\varepsilon, \mathcal{B}_\theta) t^{(n-m-1)/2} \sim \partial_\varepsilon \operatorname{Tr}_{L^2}\left\{P_{\varepsilon,\mathcal{B}_\theta} e^{-tP^2_{\varepsilon,\mathcal{B}_\theta}}\right\}\\
= \;& \operatorname{Tr}_{L^2}\left\{\dot{P}_\varepsilon(1-2tP^2_{\varepsilon,\mathcal{B}_\theta}) e^{-tP^2_{\varepsilon,\mathcal{B}_\theta}}\right\}\\
= \;& (1+2t\partial_t)\operatorname{Tr}_{L^2}\left\{\dot{P}_\varepsilon e^{-tP^2_{\varepsilon,\mathcal{B}_\theta}}\right\}\\
\sim \;& \sum_{n=0}^\infty (1+2t\partial_t) a_n^\eta(\dot{P}_\varepsilon, P_\varepsilon, \mathcal{B}_\theta) t^{(n-m-1)/2}\\
\sim \;& \sum_{n=0}^\infty (n-m) a_n^\eta(\dot{P}_\varepsilon, P_\varepsilon, \mathcal{B}_\theta) t^{(n-m-1)/2}\,.
\end{aligned}$$

Equating coefficients in the asymptotic expansion then yields

$$\partial_\varepsilon a_n^\eta(1, P_\varepsilon, \mathcal{B}_\theta) = (n-m) a_n^\eta(\dot{P}_\varepsilon, P_\varepsilon, \mathcal{B}_\theta)\,.$$

Setting $n = m$ then yields the well known variational principle:

$$\partial_\varepsilon a_m^\eta(1, P_\varepsilon, \mathcal{B}_\theta) = 0\,.$$

Equation (2.c) is a special case of this construction.

Using a partition of unity, we can construct a smooth 1-parameter family of operators P_ε which are of Dirac type with respect to Riemannian metrics g_ε so that $P_0 = P$, so that $P_\varepsilon = P$ on ∂M, and so that the metric defined by P_1 is product near the boundary. Since a_m^η is unchanged by this variation, we may assume without loss of generality that the Riemannian metric in question is product near the boundary in proving Theorem 3.1.

We say that a connection ∇ on V is compatible if $\nabla\gamma = 0$; such connections always exist [29]. Choose such a connection on V and expand $P = \gamma_i\nabla_{e_i} + \psi$. Let

$$P_\varepsilon := \gamma_i\nabla_{e_i} + \varepsilon\psi\,.$$

Since $a_m^\eta(1, P_\varepsilon, \mathcal{B}_\theta)$ is independent of ε, setting $\varepsilon = 0$ shows we may assume $\psi = 0$ in proving Theorem 3.1. Since $\psi = 0$, $\tilde{\gamma}$ anti-commutes with P. Thus Theorem 2.1 (2) permits to restrict to the case $\theta = 0$.

By results in [24], there are local invariants $a_n^\eta(x, P)$ and $a_n^\eta(y, P, \mathcal{B}_0)$ so

$$a_n^\eta(1, P, \mathcal{B}_\theta) = \int_M a_n^\eta(x, P)dx + \int_{\partial M} a_n^\eta(y, P, \mathcal{B}_0)dy\,.$$

The interior invariant $a_n^\eta(x, P)$ is homogeneous of order n in the jets of the symbol of P and the boundary invariant $a_n^\eta(y, P, \mathcal{B}_0)$ is homogeneous of order $n-1$ in the jets of the symbol of P. There is a parity constraint, by which the interior invariants $a_n^\eta(x, P)$ vanish if n is *even*. In particular, this invariant plays no role in the study of $a_m^\eta(1, P, \mathcal{B}_0)$.

Let U be a small contractible open neighborhood of $y \in \partial M$. Since ∇ is a compatible connection, Theorem 1.4 [29] shows one may decompose

$$V|_U = \Delta \otimes V_1 \quad \text{and} \quad \nabla|_U = \nabla^s \otimes \operatorname{Id} + \operatorname{Id} \otimes \nabla^1 \tag{3.a}$$

where ∇^s is the spin connection on the spin bundle Δ and where ∇^1 is an auxiliary connection on an auxiliary vector bundle V_1. Since the structures are product near the boundary, the boundary invariant $a_m^\eta(y, P, \mathcal{B}_0)$ is a universal polynomial in the covariant derivatives of the curvature tensors of (∇^s, ∇^1); the invariants in question are local so the possible failure of the decomposition given in Equation (3.a) to exist globally plays no role. Since $\tilde{\gamma}$ depends on the orientation, the structure group is $SO(m-1)$. The normal vector plays no role since the structures are product near the boundary; these invariants are defined by the structures on ∂M.

The invariant $a^\eta_m(y,P,B_0)$ is homogeneous of odd degree $m-1$ in the derivatives of the metric on the boundary and in the derivatives of the connection 1-form of ∇^1. We can decompose $a^\eta_m = a^\eta_{m,+} + a^\eta_{m,-}$ where $a^\eta_{m,+}$ is an $O(m-1)$ invariant and where $a^\eta_{m,-}$ changes sign if the orientation is reversed. By H. Weyl's theorem, all local scalar $O(m-1)$ invariants are constructed by taking traces and contracting tangential indices in pairs. Since $a^\eta_{m,+}(y,P,\mathcal{B}_0)$ is homogeneous of order $m-1$ and since $m-1$ is odd, this local invariant must vanish; there are no odd order invariants in the jets of the curvature of the metric g and of the auxiliary connection ∇^1 since the structures are product near the boundary.

The invariant $a^\eta_{m,-}$ changes sign if the orientation is reversed. The invariance theory used in the heat equation proof of the local index theorem shows that this invariant must vanish as well, see for example the discussion in [15, 30]. □

Remark 3.2. It is crucial in the analysis performed above that invariants involving the endomorphism ψ or the second fundamental form L are eliminated by performing a relevant homotopy. For example, if $m=4$, the invariants $L_{aa}R_{ijij}$ and $\operatorname{Tr}(\psi_{;m}\psi)$ are $O(3)$ invariants on the boundary which could a-priori enter. Let ε be the totally anti-symmetric tensor. The invariant $\{\varepsilon_{abc}L_{ad}R_{bcdm}\}$ is an $SO(3)$ invariant which changes sign if the orientation is reversed; this invariant plays a crucial role studying the axial anomaly for a Euclidean Taub-NUT metric, see, for example, the discussion in [31]. In our setting, this invariant is eliminated from consideration since we perturb the problem so that the second fundamental form vanishes.

Remark 3.3. We have shown that $\eta(s,P,\mathcal{B}_\theta)$ is regular at $s=0$. We defined

$$\eta(P,\mathcal{B}_\theta) := \tfrac{1}{2}\big\{\eta(s,P,\mathcal{B}_\theta) + \dim\ker\{P_{\mathcal{B}_\theta}\}\big\}\big|_{s=0} \in \mathbb{C}/\mathbb{Z}$$

as a measure of the global spectral asymmetry of P. Although this invariant is not locally computable, the derivative is locally computable. If $P_\varepsilon := P + \varepsilon\Xi$ where Ξ is a 0^{th} order operator, then one has, see for example [15], that

$$\partial_\varepsilon \eta(P_\varepsilon,\mathcal{B}_\theta)|_{\epsilon=0} = 2\Gamma(\tfrac{1}{2})^{-1} a_{m-1}(\Xi,P,\mathcal{B}_\theta)\,.$$

If $\partial M = \emptyset$, then $a_{m-1}(\Xi,P,\mathcal{B}_\theta)=0$ and $\eta(P_\varepsilon,\mathcal{B}_\theta)$ is independent of ε. This is not, however, the case if ∂M is non-empty. Take $\Xi := -f_o\tilde\gamma\gamma_m$ where f_o vanishes to second order on the boundary and where f_o is supported near the boundary. Set $\theta=0$, and set $m=4$. By Theorem IV.1 [32],

$$a_3(\Xi,P,\mathcal{B}_0) = (4\pi)^{-3/2}(384)^{-1}\int_{\partial M} 30\dim(V) f_{o;mmm}dy\,.$$

Thus the eta invariant is **not** a homotopy invariant on an even dimensional manifold with these boundary conditions if the boundary is non-empty.

4. Evaluating the coefficient $a_4(1,P,\mathcal{B}_\theta)$

As a further immediate application of Theorem 2.1 we are able to evaluate $a_4(1,P,\mathcal{B}_\theta)$ for $P=\gamma_i\nabla_{e_i}+\psi$ when $m=4$ and when $P\tilde\gamma+\tilde\gamma P=0$; this is the most relevant dimension. Given that $\partial_\theta a_4(1,P,\mathcal{B}_\theta)=0$ for $m=4$, we have $a_4(1,P,\mathcal{B}_\theta)=a_4(1,P,\mathcal{B}_0)$. However, for $\theta=0$ the boundary conditions reduce to standard boundary conditions of mixed type. To use the known results for mixed boundary conditions [7, 8] we introduce some notation. We adopt the Bochner formalism. If D is an operator of Laplace type, there is a unique connection ∇^D and a unique endomorphism E so that

$$D = -g^{ij}\nabla^D_i\nabla^D_j - E\,.$$

Let χ be an auxiliary Hermitian endomorphism used to define the splitting of the bundle V. With the projectors

$$\Pi_\pm = \tfrac{1}{2}(1 \pm \chi),$$

the mixed boundary operator is defined as

$$\mathcal{B}_0^1 \phi = \Pi_- \phi\,|_{\partial M} \oplus (\nabla_m^D + S)\Pi_+\phi\big|_{\partial M} = 0.$$

Let Ω_{ij}^D be the curvature of the connection ∇^D. Furthermore, let $\tau := R_{ijji}$ be the scalar curvature, with the convention that the components R_{ijkl} of the curvature of the Levi-Civita connection are such that $R_{1212} = -1$ for the standard metric on S^2. Finally, let $\rho_{ij} = R_{ikkj}$ be the Ricci tensor and L_{ab} the second fundamental form and let ';' (resp. ':') denote the covariant derivative (resp. tangential covariant derivative) with respect to the connection ∇^D and the Levi-Civita connection of M (resp. of ∂M). Using this notation, furthermore $\rho^2 = \rho_{ij}\rho_{ij}$ and $R^2 = R_{ijkl}R_{ijkl}$, the following result has been shown in [14, 33].

Theorem 4.1.

$$\begin{aligned}
& a_4(1, D, \mathcal{B}_0^1) \\
= \; & \frac{1}{360(4\pi)^{m/2}} \int_M \mathrm{Tr}_V[60E_{;kk} + 60\tau E + 180E^2 + 30\Omega_{ij}^D\Omega_{ij}^D + 12\tau_{;kk} \\
& +5\tau^2 - 2\rho^2 + 2R^2]dx \\
& + \int_{\partial M} \mathrm{Tr}_V\,[(240\Pi_+ - 120\Pi_-)\,E_{;m} + (42\Pi_+ - 18\Pi_-)\,\tau_{;m} \\
& +120EL_{aa} + 20\tau L_{aa} - 4\rho_{mm}L_{aa} - 12R_{ambm}L_{ab} + 4R_{abcb}L_{ac} \\
& +\frac{1}{21}(280\Pi_+ + 40\Pi_-)\,L_{aa}L_{bb}L_{cc} + \frac{1}{21}(168\Pi_+ - 264\Pi_-)\,L_{ab}L_{ab}L_{cc} \\
& +\frac{1}{21}(224\Pi_+ + 320\Pi_-)\,L_{ab}L_{cb}L_{ac} + 720SE + 120S\tau \\
& +144SL_{aa}L_{bb} + 48SL_{ab}L_{ab} + 480S^2L_{aa} + 480S^3 \\
& +60\chi\chi_{:a}\Omega_{am}^D - 12\chi_{:a}\chi_{:a}L_{bb} - 24\chi_{:a}\chi_{:b}L_{ab} - 120\chi_{:a}\chi_{:a}S\big]\,dy\,.
\end{aligned}$$

For the present case under investigation, we have by [29, 34] that writing

$$P^2 = (\gamma_i\nabla_i + \psi)^2 := -g_{ij}\nabla_i^D\nabla_j^D - E,$$

the induced connection ∇_i^D satisfies

$$\nabla_i^D = \nabla_i + \omega_i \quad \text{where} \quad \omega_i = -\tfrac{1}{2}(\psi\gamma_i + \gamma_i\psi).$$

Furthermore, in terms of the induced connection ∇_i^D, P can be written as

$$P = \gamma_i\nabla_i^D + \phi \quad \text{where} \quad \phi = \psi - \gamma_i\omega_i.$$

To state all ingredients needed for the actual evaluation of Theorem 4.1 for the chiral bag boundary condition with $\theta = 0$, let Ω_{ij} be the curvature of the compatible connection. Then, for any concrete example, the formula in Theorem 4.1 can be evaluated from [29, 34]

$$\begin{aligned}
\Omega_{ij}^D &= \Omega_{ij} + \nabla_i\omega_j - \nabla_j\omega_i + \omega_i\omega_j - \omega_j\omega_i, \\
E &= -\tfrac{1}{2}\gamma_i\gamma_j\Omega_{ij}^D - \gamma_i\nabla_i\phi - \phi^2, \\
S &= \tfrac{1}{2}\Pi_+(-\gamma_m\psi + \psi\gamma_m - L_{aa})\Pi_+, \\
\chi &= -\tilde\gamma\gamma_m, \qquad \chi_{:a} = \tilde\gamma\gamma_b L_{ab} + \omega_a\chi - \chi\omega_a\,.
\end{aligned}$$

5. A PERTURBATIVE RESULT – $\partial_\theta a_m(1, P, \mathcal{B}_\theta)|_{\theta=0}$

In this section, we drop the assumption $P\tilde\gamma + \tilde\gamma P = 0$ and examine, at least in part, the θ dependence.

Theorem 5.1. *Expand $P = \gamma_i \nabla_{e_i} + \psi$ where ∇ is compatible. Let $\psi = \psi_o + \psi_e$ where ψ_o anti-commutes with $\tilde\gamma$ and ψ_e commutes with $\tilde\gamma$. Then:*

1. $\partial_\theta\{a_m(1, P, \mathcal{B}_\theta)\}|_{\theta=0} = -2a^\eta_{m-1}(\tilde\gamma\psi_e, P, \mathcal{B}_0)$.
2. *Let $m = 4$, let $d = \dim(V)$, let $\psi_o = f_0\tilde\gamma\gamma_m$, and let $\psi_e = f_e\tilde\gamma$ where f_o and f_e are smooth functions on M and where f_o is supported near ∂M. Then:*

$$\partial_\theta\{a_m(1, P, \mathcal{B}_\theta)\}|_{\theta=0} = \frac{d}{16\pi^2}\int_{\partial M}\{2f_ef_oL_{aa} + 2f_e^2 f_o - 2f_{e;m}f_o - f_ef_{o;m}\}dy\,.$$

Remark 5.2. Assertion (2) shows that $a_m(1, P, \mathcal{B}_\theta)$ exhibits non-trivial θ dependence in general.

Proof. Consider the gauge transformation defined by $e^{\frac12\theta\tilde\gamma}$:

$$\begin{aligned}P_\theta &:= e^{-\frac12\theta\tilde\gamma}(P - \psi_e + \psi_e)e^{\frac12\theta\tilde\gamma} = e^{-\theta\tilde\gamma}(P - \psi_e) + \psi_e\\ &= e^{-\theta\tilde\gamma}P + (1 - e^{-\theta\tilde\gamma})\psi_e,\\ e^{-\frac12\theta\tilde\gamma}\chi_\theta e^{\frac12\theta\tilde\gamma} &= -e^{-\frac12\theta\tilde\gamma}\tilde\gamma e^{\theta\tilde\gamma}\gamma_m e^{\frac12\theta\tilde\gamma} = -e^{-\frac{\theta}{2}\tilde\gamma}\tilde\gamma e^{\frac{\theta}{2}\tilde\gamma}\gamma_m = \chi_0\,.\end{aligned}$$

By gauge invariance and a direct calculation, we have

$$a_n(1, P, \mathcal{B}_\theta) = a_n(1, P_\theta, \mathcal{B}_0) \quad\text{and}\quad \partial_\theta P_\theta|_{\theta=0} = -\tilde\gamma P + \tilde\gamma\psi_e\,.$$

Since we are multiplying on the left by $\tilde\gamma$, the domain of P_θ defined by the boundary condition $\mathcal{B}_0$ is not perturbed. We compute:

$$\begin{aligned}&\partial_\theta \operatorname{Tr}_{L^2}\left\{e^{-tP^2_{\theta,\mathcal{B}_0}}\right\}\Big|_{\theta=0} = -2t\operatorname{Tr}_{L^2}\left\{\partial_\theta(P_{\theta,\mathcal{B}_0})P_{\mathcal{B}_0}e^{-tP^2_{\mathcal{B}_0}}\right\}\Big|_{\theta=0}\\ =\;& -2t\operatorname{Tr}_{L^2}\left\{(-\tilde\gamma P^2_{\mathcal{B}_0} + \tilde\gamma\psi_eP_{\mathcal{B}_0})e^{-tP^2_{\mathcal{B}_0}}\right\}\\ =\;& -2t\partial_t\operatorname{Tr}_{L^2}\left\{\tilde\gamma e^{-tP^2_{\mathcal{B}_0}}\right\} - 2t\operatorname{Tr}_{L^2}\left\{\tilde\gamma\psi_eP_{\mathcal{B}_0}e^{-tP^2_{\mathcal{B}_0}}\right\}\,.\end{aligned}$$

Expanding in an asymptotic series then yields:

$$\begin{aligned}&\sum_{n=0}^\infty \partial_\theta a_n(1, P_\theta, \mathcal{B}_0)t^{(n-m)/2}\Big|_{\theta=0}\\ \sim\;& -2t\partial_t\sum_{j=0}^\infty a_j(\tilde\gamma, P, \mathcal{B}_0)t^{(j-m)/2} - 2t\sum_{k=0}^\infty a_k^\eta(\tilde\gamma\psi_e, P, \mathcal{B}_0)t^{(k-m-1)/2}\,.\end{aligned}$$

Equating terms in the relevant asymptotic expansions yields:

$$\begin{aligned}&\partial_\theta a_n(1, P, \mathcal{B}_\theta)\big|_{\theta=0} = \partial_\theta a_n(1, P_\theta, \mathcal{B}_0)\big|_{\theta=0}\\ =\;& (m-n)a_n(\tilde\gamma, P, \mathcal{B}_0) - 2a^\eta_{n-1}(\tilde\gamma\psi_e, P, \mathcal{B}_0)\,.\end{aligned}$$

Assertion (1) then follows by setting $n = m$.

We specialize this to the case $m = 4$. We apply Theorem 12 [34]. Set

$$\begin{aligned}&\psi := f_o\tilde\gamma\gamma_m + f_e\tilde\gamma, \quad \psi_e := f_e\tilde\gamma, \quad \psi_o := f_o\tilde\gamma\gamma_m,\\ &\chi := -\tilde\gamma\gamma_m, \qquad\qquad\quad \Psi := \gamma_i\psi\gamma_i, \quad \Psi_T := \gamma_a\psi\gamma_a\,.\end{aligned}$$

After adjusting for the different sign convention employed, setting $m = 4$, and using $f = \tilde\gamma\psi_e = f_e$ scalar, we find

$$\begin{aligned}
a_3^\eta(f,P,\mathcal{B}_0) = -\frac{1}{192\pi^2}\bigg\{ &\int_M f_e \operatorname{Tr}_V \{\{6\psi_{;i} + 3\Psi\gamma_i\psi\}_{;i} \\
&+\{-\tau\psi - 6\gamma_i\gamma_j W_{ij}\psi + 6\psi\psi_{;i}\gamma_i - 3\psi\psi\Psi\}\}dx \\
+\int_{\partial M} &\{ \operatorname{Tr}_V\{12 f_{e;m}\chi\psi\} + f_e \operatorname{Tr}_V\{6\chi\psi_{;m} + 6\psi_{;m} + 6\chi\gamma_m\gamma_a\psi_{;a} \\
&-12\chi\psi L_{aa} - 2\psi L_{aa} - 6\chi\gamma_m\psi\psi + 3\gamma_m\psi\Psi_T \\
&-3\chi\gamma_m\psi\chi\psi + 6\chi\gamma_a W_{am}\}\}dy\bigg\}.
\end{aligned}$$

Let $d := \dim(V)$. We compute:

$$\begin{aligned}
&\operatorname{Tr}_V\{6\psi_{;ii} - \tau\psi - 6\gamma_i\gamma_j W_{ij}\psi - 3\psi\psi\Psi\} = 0,\\
&\operatorname{Tr}_V(3\Psi_e\gamma_i\psi_o)_{;i} = 12\operatorname{Tr}_V(\tilde\gamma\gamma_m\tilde\gamma\gamma_m)(f_e f_o)_{;m} = 12d(f_{e;m}f_o + f_{o;m}f_e),\\
&\operatorname{Tr}_V(3\Psi_o\gamma_i\psi_e)_{;i} = -6\operatorname{Tr}_V(\tilde\gamma\gamma_m\gamma_m\tilde\gamma)(f_e f_o)_{;m} = 6d(f_{e;m}f_o + f_{o;m}f_e),\\
&\operatorname{Tr}_V(6\psi_o\psi_{e;m}\gamma_m) = 6\operatorname{Tr}_V(\tilde\gamma\gamma_m\tilde\gamma\gamma_m)f_o f_{e;m} = 6df_{e;m}f_o,\\
&\operatorname{Tr}_V(6\psi_e\psi_{o;m}\gamma_m) = 6\operatorname{Tr}_V(\tilde\gamma\tilde\gamma\gamma_m\gamma_m)f_e f_{o;m} = -6df_e f_{o;m},\\
&f_e\operatorname{Tr}_V\{(3\Psi\gamma_i\psi)_{;i} + 6\psi\psi_{;}\gamma_i\} = 12d(f_e^2 f_0)_{;m}.
\end{aligned}$$

Thus interior integral is in divergence form. Next we study the boundary term:

$$\begin{aligned}
&f_e\operatorname{Tr}_V\{6\psi_{;m} + 6\chi\gamma_m\gamma_a\psi_{;a} - 2\psi L_{aa} - 6\chi\gamma_m\psi\psi + 6\chi\gamma_a W_{am}\} = 0,\\
&f_{e;m}\operatorname{Tr}_V\{12\chi\psi\} = -12df_{e;m}f_o,\\
&f_e\operatorname{Tr}_V\{6\chi\psi_{;m}\} = -6df_e f_{o;m},\\
&f_e\operatorname{Tr}_V\{-12\chi\psi L_{aa}\} = 12df_e f_o L_{aa},\\
&f_e\operatorname{Tr}_V\{3\gamma_m\psi\Psi_T\} = 3f_e^2 f_o\operatorname{Tr}_V\{\gamma_m\tilde\gamma\gamma_m\gamma_a\tilde\gamma\gamma_a + \gamma_m\tilde\gamma\gamma_a\tilde\gamma\gamma_m\gamma_a\} = 18df_e^2 f_o,\\
&f_e\operatorname{Tr}_V\{-3\chi\gamma_m\psi\chi\psi\} = -3f_e^2 f_o\operatorname{Tr}_V\{\tilde\gamma\gamma_m\gamma_m\tilde\gamma\tilde\gamma\gamma_m\tilde\gamma\gamma_m + \tilde\gamma\gamma_m\gamma_m\tilde\gamma\gamma_m\tilde\gamma\gamma_m\tilde\gamma\}\\
&\qquad = 6df_e^2 f_o.
\end{aligned}$$

Assertion (2) now follows after combining terms. □

6. Orientability

We have assumed in previous sections that M is orientable. In fact, it is possible to relax this restriction. We decompose $\partial M = N_1 \cup ... \cup N_\ell$ into connected components. Assume that each component N_i is orientable but that M need not be orientable. On each component N_i, choose an orientation to define $\tilde\gamma$ and thereby define chiral bag boundary conditions. Let $\check M$ be the orientable double cover; $\partial\check M = \check N_1 \cup ... \cup \check N_{2\ell}$ where each N_i lifts to two distinct components in the double cover. Let $\check\gamma$ be defined by an orientation of $\check M$.

Let P be an operator of Dirac type on M. We can define $\tilde\gamma$ locally on M; we say that P is *admissible* if $\tilde\gamma P + P\tilde\gamma = 0$ near any point P of M; replacing $\tilde\gamma$ by $-\tilde\gamma$ does not change this condition so this is an invariantly defined notion. Equivalently, if $\check P$ is the associated operator on $\check M$, this condition simply means that $\check P\check\gamma + \check\gamma\check P = 0$. Similarly, we say that F is *admissible* if $F\tilde\gamma + \tilde\gamma F = 0$ for any local orientation of M or equivalently if $\check F\check\gamma + \check\gamma\check F = 0$. Since the heat trace invariants are multiplicative under finite coverings, applying Theorem 2.1 for $\{\check M, \check P, \check F\}$ (replacing if necessary χ_θ by $-\chi_\theta$ on certain components if necessary), then yields the following result for $\{M, P, F\}$.

Theorem 6.1. *Let P be an operator of Dirac type on a compact non-orientable smooth manifold of even dimension m with orientable boundary. Let $\mathcal{B}_\theta$ be chiral bag boundary conditions defined by orientations of the components of ∂M. Let $F \in C^\infty(\operatorname{End}(V))$. Assume that P and F are admissible. Then:*

(1) $\partial_\theta a_m(1,P,\mathcal{B}_\theta)=0$.
(2) $\partial_\theta a^\eta_m(1,P,\mathcal{B}_\theta)=0$.
(3) $\partial_\theta a^\eta_{m-1}(F,P,\mathcal{B}_\theta)=0$.

Similarly Theorem 3.1 extends immediately to yield:

Theorem 6.2. *Let P be an operator of Dirac type on a compact smooth manifold of even dimension m with orientable boundary ∂M. Let $\mathcal{B}_\theta$ be chiral bag boundary conditions defined by orientations of the components of ∂M. Then $a^\eta_m(1,P,\mathcal{B}_\theta)=0$.*

7. Conclusions

The crucial point of our arguments in Section 2 is that we work with the associated first order operator. A chiral gauge transformation is used to relate the original problem with θ-dependent boundary condition to a problem where the operator is transformed to a θ-dependent one and the boundary conditions to a θ-independent one. In a formula, the chiral gauge transformation amounts to expressing $P_{\theta,\mathcal{B}_0}=e^{-\theta\tilde\gamma}P_{\mathcal{B}_0}$, the domains of the appropriate operators being crucial to understanding the arguments involved. To emphasize: $\tilde\gamma P_{\mathcal{B}_0}\neq -P_{\mathcal{B}_0}\tilde\gamma$. From the above stated relation the variation of invariants with θ can be evaluated and is given in Theorem 2.1. As an application of this theorem, we have given a very elegant proof of the regularity of the eta invariant at $s=0$ in $m=4$ dimensions. Furthermore, the coefficient $a_4(1,\gamma_j\nabla_{e_j},\mathcal{B}_\theta)$ in $m=4$ dimensions could be found using results from standard mixed boundary conditions, as it does not depend on θ. Note, however, that in general $a_4(1,P,\mathcal{B}_\theta)$ shows non-trivial θ-dependence, see Theorem 5.1. The eta invariant was computed [35, 36] for certain closed even dimensional pin^c manifolds; their K-theory groups and some equivariant bordism groups were then determined. The results of Section 6 show the eta invariant is well defined with chiral bag boundary conditions even if $\partial M\neq\emptyset$. However, the observation made in Remark 3.3 shows that the eta invariant is no longer a homotopy invariant and thus a more careful study of the boundary contribution needs to be made before the results cited above can be extended to the case $\partial M\neq\emptyset$.

Acknowledgements

Research of PG was partially supported by the MPI (Leipzig, Germany). KK acknowledges support by the Baylor University Research Committee and by the MPI (Leipzig, Germany).

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