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by

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Abstract

Tunnelling is studied here as a variational problem formulated in terms of a functional which approximates the rate function for large deviations in Ising systems with Glauber dynamics and Kac potentials, [8]. The spatial domain is a two-dimensional square of side L with reflecting boundary conditions. For L large enough the penalty for tunnelling from the minus to the plus equilibrium states is determined. Minimizing sequences are fully characterized and shown to have approximately a planar symmetry at all times, thus departing from the Wulff shape in the initial and final stages of the tunnelling.

AMS (MOS) subject classification: 82C05

1 Introduction

Tunnelling in the $d = 2$ ferromagnetic Ising model at low temperatures has been object of many studies, mainly focused on metastability, namely the analysis of the Glauber dynamics when an external magnetic field $h > 0$ is present and the initial state is close to the minus Gibbs state at $h = 0$. We are instead interested here in studying a bistable equilibrium with "oscillations" between the two minimizers. Such a case has been considered by Martinelli, [25], in the n.n. ferromagnetic Ising model in a $d = 2$ square of side L, proving upper and lower bounds for the [random] transition time from the "plus" to the "minus" state (and viceversa) in the limit as $L \to \infty$. Much earlier Comets had attacked the problem in the context of Ising systems with Kac interactions. Supposing the side L of the square to be proportional to the range γ^{-1} of the Kac interaction, Comets [8] derived the large deviations rate function in the asymptotics of small γ. A "sharp" analysis of the path followed during the tunnelling is however still an open problem in both models.

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Tunnelling is usually studied in two steps: the first one is based on a loss of memory property, namely that configurations close to one of the two stable states can be successfully coupled with large probability before leaving the neighborhood. Such estimates seem within the reach of the present techniques, as in [14] very strong properties of Glauber dynamics have been established. The second step for tunnelling requires to solve a variational problem involving the large deviations rate function. In this paper we concentrate on the latter aspect and study tunnelling in a purely variational setting. For simplicity we replace the Comets rate function by an "easier functional", already considered in [3] in the $d = 1$ version of the model. The extension to the true Comets functional and then to the Ising system may still require a non trivial work, but we believe that the main physical features of the actual tunnelling excursion are already captured by our results.
The extension from $d = 1$ to $d > 1$ is in general far from trivial. Large deviations and tunnelling have been studied by Jona-Lasinio and Mitter, [21], for stochastic perturbations of the Allen-Cahn equation, partially extending the $d = 1$ work by Faris and Jona-Lasinio, [17] (see also [9]), but, as far as we know, a full analysis in $d = 2$ is open also for the Ginzburg-Landau action functional associated with the Allen-Cahn equation. Even more subtle is the analysis of tunnelling under time constraints, namely when the excursion between the two stable states is required to occur within a given time interval. The picture in such a case may be dramatically different if time is short, and the optimal pattern may involve multiple nucleations. Results of this type are proved in $d = 1$ for the Ginzburg-Landau functional and Allen-Cahn equation, [22],[23], and for the non local interaction considered here, [10]; most of the proofs are still missing in the multi-dimensional case, but a clear picture of the phenomenon can at least be outlined, [23].
Geometric patterns are the main issues in a multi-dimensional analysis. In the sharp interface limit (i.e. when the spatial domain, a square of side L in our case, is observed in rescaled variables so that it always appears as a unit square as $L \to \infty$) the tunnelling orbits are moving surfaces which describe the boundaries of the set where the plus phase is located. In $d = 1$ this is simply a point which moves from an endpoint of the unit interval to the other one (Neumann boundary conditions are responsible for the nucleation to start from the boundaries of the domain). To see geometrical effects we thus need to go to $d > 1$.
An important factor is then played by the Wulff shape. As it is well known (and briefly discussed in Section 2) in $d = 2$ dimensions the set with minimal perimeter for a given area θ is a quarter of a circle around a vertex of the unit square Q_1, or a rectangle with three sides lying on ∂Q_1 (again this is due to Neumann boundary conditions). Rectangles appear if their area and the area of the complement (in Q_1) are both larger than a critical value θ_{crit}, otherwise we observe a quarter of a circle. As Wulff shapes describe states with minimal free energies under the area constraint, one usually expects that if the process is "slow" and the transformation "adiabatic" then the tunnelling patterns are determined by sequences of "equilibrium" Wulff shapes. It is however evident from the above description that tunnelling orbits cannot always be close to Wulff shapes as there is a discontinuity at θ_{crit}. One possible scenario is depicted in (a) of Figure 1 where the Wulff shape is deformed to interpolate around θ_{crit} between the two different regimes. We will prove instead that the optimal tunnelling in our diffused interface model is all the way planar as in pattern (b) of Figure 1, namely that it is convenient to nucleate initially in a less efficient way, the cost being recovered in the end. More discussions on this point can be found in Section 2.

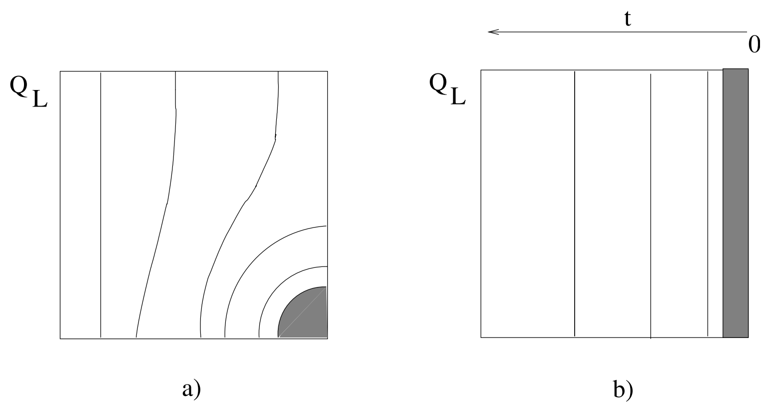

Figure 1: In a) and b) we depict two possible tunnelling paths in the sharp interface regime. In figure a) a small droplet (Wulff shape) of the $-$phase (dark region) nucleates at a vertex of the square Q_L. It then invades Q_L as time increases, gradually changing its interface, and eventually becomes a rectangle. Our results, valid for the diffused interface model, show that a) is *not* minimizing, and that the minimizing path is the one corresponding to figure b). In this path we have initially a nucleation of a flat interface (dark rectangular region), which smoothly invades Q_L.

The content of the paper is outlined in Section 2, see Subsection 2.9, after defining the model and stating the main results. We just mention here that the restriction to $d = 2$ can be hopefully lifted, an extension to $d = 3$ is presently under way (we still miss the case $d > 3$ due to a poor control of Wulff shapes). For reasons of brevity imposed by the journal we have written in a separate paper the analysis of the invariant manifolds for a non local version of the Allen-Cahn equation, which is used here to characterize the optimal tunnelling orbits.

2 Definitions and results

We consider a continuum model of a two-dimensional magnet where states are functions $m \in L^\infty(Q_L, [-1, 1])$, $Q_L = \{r \in \mathbb{R}^2 : |r \cdot e_1| \le L/2, |r \cdot e_2| \le L/2\}$, $r \cdot e_1$ and $r \cdot e_2$ the x and y components of r. $m(r)$ is interpreted as a magnetization density which may be related, by a coarse graining procedure, to an underlying Ising spin configuration, hence the restriction to $[-1, 1]$. Time evolution is described by orbits which are smooth functions $u = u(r, t)$, $r \in Q_L$, t in $\mathbb{R}$ or in an interval of $\mathbb{R}$, $|u| \le 1$.

2.1 The penalty functional

The "action" of an orbit $u(\cdot)$ restricted to an interval $[t_0, t_1]$ of its domain of definition is

$$A_{L;t_0,t_1}(u) = F_L(u(\cdot, t_0)) + I_{L;t_0,t_1}(u)$$

where $F_L(m)$, the free energy of the state m, is

$$F_L(m) = \int_{Q_L} \phi_\beta(m)\, dr + \frac{1}{4}\int_{Q_L\times Q_L} J^{\mathrm{neum}}(r,r')[m(r)-m(r')]^2 dr\, dr'. \tag{2.1}$$

$J^{\mathrm{neum}}(r,r')$ is the interaction coupling constant (with Neumann boundary conditions), namely $J^{\mathrm{neum}}(r,r') = \sum_{r''\simeq r'} J(r,r'')$, where $r'' \simeq r'$ means that r'' is equal to r' modulo reflections along the lines $\{y = \pm(2n+1)L/2\}$ and $\{x = \pm(2n+1)L/2\}$, $n \in \mathbb{Z}$. We suppose $J(r,r') = J(0,r'-r)$, $J(0,r)$ depends only on $|r|$ and is a smooth non negative function supported in the unit ball and $\int J(0,r) = 1$. We finally suppose that

$$j(0,x) = \int J((0,0),(x,y))\, dy \tag{2.2}$$

is a non increasing function of x when $x > 0$. We take $\beta > 1$ and

$$\phi_\beta(m) = \tilde\phi_\beta(m) - \min_{|s|\le 1}\tilde\phi_\beta(s), \qquad \tilde\phi_\beta(m) = -\frac{m^2}{2} - \frac{1}{\beta}\mathcal{S}(m)$$

$$\mathcal{S}(m) = -\frac{1-m}{2}\log\frac{1-m}{2} - \frac{1+m}{2}\log\frac{1+m}{2}.$$

Finally,

$$I_{L;t_0,t_1}(u) = \frac{1}{4}\int_{t_0}^{t_1}\int_{Q_L} [u_t - f_L(u)]^2\, dr\, dt,$$

where u_t is the time derivative of u and

$$f_L(u) = -\frac{\delta F_L(u)}{\delta u} = J^{\mathrm{neum}} * u - a_\beta(u), \qquad a_\beta(u) = \frac{1}{\beta}\,\mathrm{arctanh}(u),$$

$$J^{\mathrm{neum}} * u(r) = \int_{Q_L} J^{\mathrm{neum}}(r,r')u(r')\, dr'. \tag{2.3}$$

As mentioned in the Introduction, $A_{L;t_0,t_1}(u)$ is a simplified version of the Comets large deviations rate function for Glauber dynamics in a ferromagnetic Ising system with Kac potential $J_\gamma(r,r') = \gamma^2 J(\gamma r, \gamma r')$.
Later on, in the course of the proofs, we will consider rectangles $Q_{L',L} = \{(x,y) : |x| \le L', |y| \le L/2\}$ with $L' \in (0,+\infty]$ and call channel the set $Q_{\infty,L}$. The definition of F_L in (2.1) naturally extends to domains $Q_{L',L}$ in which cases it will be denoted by $F_{Q_{L',L}}$, as a functional on $L^\infty(Q_{L',L},[-1,1])$.

2.2 Dynamics: the semigroups S_t and T_t.

We denote by S_t the semi-group generated by the L^2-gradient dynamics, namely $S_t(u_0) = u(\cdot, t)$ is the solution to the non local evolution equation

$$u_t = f_L(u) = -\frac{\delta F_L(u)}{\delta u}, \qquad u(\cdot, 0) = u_0. \tag{2.4}$$

The velocity field $f_L(u)$ is Lipschitz when restricted to sets of the form $\{\|u\|_\infty \leq b\}$, $b < 1$. Then it is not difficult to prove global existence of $S_t(u_0)$ if $\|u_0\|_\infty < 1$, see [2] for details.

In order to exploit results already existing in the literature we will also consider the semigroup $T_t(u_0)$ generated by the equation

$$u_t = -u + \tanh\{\beta J^{\text{neum}} * u\}, \qquad u(\cdot, 0) = u_0, \tag{2.5}$$

which we will set either in Q_L or in $Q_{\infty,L}$. Observe that T_t decreases the energy F_L and that its fixed points are the same as for S_t and they are critical points of F_L. The evolution defined by (2.5) is derived from Glauber dynamics with Kac potentials in a scaling limit, see [14].

2.3 The cost of tunnelling

The action $A_{L;t_0,t_1}(u)$ is always non negative, as the integrands in F_L and $I_{L;t_0,t_1}$ are non negative. Actually $A_{L;t_0,t_1}(u) > 0$ unless $u(r,t) \equiv \pm m_\beta$, where $m_\beta > 0$ is such that $m_\beta = \tanh\{\beta m_\beta\}$ (recall the assumption $\beta > 1$). Therefore $m^{(\pm)}(r) \equiv \pm m_\beta$ have the interpretation of the [only] two equilibrium states of the system and tunnelling describes orbits which connect such states. Thus the space of tunnelling orbits in a time $T > 0$ is

$$\mathcal{U}_{L,T} = \{u \in C^\infty(Q_L \times [0,T]) : u(r,0) = -m_\beta,\ u(r,T) = m_\beta \ \text{ for all } r \in Q_L\}$$

and, calling $I_{L,T}(u) = I_{L;0,T}(u)$, we define the cost of tunnelling as

$$P_L := \inf_{T>0} \inf_{u \in \mathcal{U}_{L,T}} I_{L,T}(u) \tag{2.6}$$

noticing that since $F_L(m^{(-)}) = 0$, $A_{L,T}(u) = A_{L;0,T}(u) = I_{L,T}(u)$ when $u \in \mathcal{U}_{L,T}$.
As mentioned in the Introduction the problem is completely different if restrictions on T are imposed, but in this paper we will only study problem (2.6). To motivate our results let us first describe some properties of $A_{L;t_0,t_1}$.

2.4 Reversibility

First notice that $I_{L;t_0,t_1}(u) = 0$ if $u(\cdot, t) = S_{t-t_0}(u(\cdot, t_0))$, S_t being defined in Subsection 2.2. Given $u(\cdot, t)$, $t \in [t_0, t_1]$, call $u^{\text{rev}}(\cdot, t_0 + s) = u(\cdot, t_1 - s)$, $s \in [0, t_1 - t_0]$. Then

$$A_{L;t_0,t_1}(u) = A_{L;t_0,t_1}(u^{\text{rev}}). \tag{2.7}$$

To show (2.7), which is proved in [3], it suffices to expand the square in the integral defining $I_{L;t_0,t_1}$ and recall that $f_L(u) = -\delta F_L(u)/\delta u$. As a consequence of (2.7),

$$I_{L;t_0,t_1}(u) \geq F_L(u(\cdot,t_1)) - F_L(u(\cdot,t_0)) \tag{2.8}$$

$$I_{L;t_0,t_1}(u) = F_L(u(\cdot,t_1)) - F_L(u(\cdot,t_0)) \quad \text{if } u(\cdot,t_0) = S_{t_1-t_0}(u(\cdot,t_1)) \tag{2.9}$$

Remark 2.1. Note that if v then $I_{L,T}(u) \geq F_L(u(\cdot,t))$ for any $t \in [0,T]$.

2.5 The Wulff shape

Given any tunnelling orbit $u \in \mathcal{U}_{L,T}$ and $\alpha \in (-m_\beta, m_\beta)$, by continuity there must be a time $t \in (0,T)$ when $u(\cdot,t) \in \Sigma_\alpha$,

$$\Sigma_\alpha = \big\{ m \in L^\infty(Q_L,[-1,1]) : \fint_{Q_L} m = \alpha \big\}.$$

Thus from (2.8)

$$I_{L,T}(u) \geq \inf \big\{ F_L(m) : m \in \Sigma_\alpha \big\} \quad \text{for any } \alpha \in (-m_\beta, m_\beta), \tag{2.10}$$

hence the intuition that optimality in tunnelling requires closeness to the Wulff shape, namely the minimizer on the r.h.s. of (2.10). The Wulff problem is well understood in the limit $L \to \infty$. As the infimum on the r.h.s. of (2.10) scales proportionally to L, (L^{d-1} in d dimensions)

$$\lim_{L\to\infty} \inf \Big\{ \frac{F_L(m)}{L} : m \in \Sigma_\alpha \Big\} = c_\beta \inf \Big\{ P(E, \mathrm{int}(Q_1)) : E \subseteq Q_1, |E| = \frac{1}{2} - \vartheta_\alpha \Big\}$$

where $P(E, \mathrm{int}(Q_1))$ denotes the perimeter of the BV set E in the interior of Q_1, $|E|$ is the Lebesgue measure of E, $\vartheta_\alpha \in (-1/2, 1/2)$ is defined by

$$(\frac{1}{2} - \vartheta_\alpha) m_\beta - [1 - (\frac{1}{2} - \vartheta_\alpha)] m_\beta = \alpha \tag{2.11}$$

ϑ_α has a clear geometrical interpretation: the magnetization α can in fact be realized by putting m_β in the rectangle $\{(x,y) \in Q_1 : x \geq \vartheta_\alpha\}$ and $-m_\beta$ in its complement. c_β has the meaning of a surface tension, which in the present model is equal to $c_\beta = F^{(1)}(\bar{m})$. Namely c_β is the one-dimensional free energy $F^{(1)}$ of the one-dimensional instanton $\bar{m}(x), x \in \mathbb{R}$, where

$$F^{(1)}(m) = \int_{\mathbb{R}} \phi_\beta(m)\, dx + \frac{1}{4} \int_{\mathbb{R}} \int_{\mathbb{R}} j(x,x')[m(x) - m(x')]^2 \, dx dx', \tag{2.12}$$

with $j(x,x')$ as in (2.2) and $\bar{m}$ is the non-zero, antisymmetric solution of

$$\bar{m} = \tanh\{j * \bar{m}), \tag{2.13}$$

see [14], [15], [16].

The limit Wulff problem

$$\inf \left\{ P(E, \mathrm{int}(Q_1)) : E \subseteq Q_1,\ |E| = \frac{1}{2} - \theta \right\} \tag{2.14}$$

of minimizing the perimeter functional $P(E, \mathrm{int}(Q_1))$ is explicitly solved. Indeed (2.14) admits a solution and any solution E_θ is such that $Q_1 \cap \partial E_\theta$ is smooth [24]. Moreover $Q_1 \cap \partial E_\theta$ is connected and has constant curvature. Hence it is contained either in a circle or in a line. In addition the contact between ∂E_θ and ∂Q_1 is orthogonal. Let θ_{crit} be defined by

$$\frac{1}{2} - \theta_{\mathrm{crit}} = \frac{\pi R^2}{4}, \quad \text{where } \frac{2\pi R}{4} = 1.$$

Then the following result holds.

Proposition 2.1. *If $|\theta| \le \theta_{\mathrm{crit}}$ then $Q_1 \cap \partial E_\theta$ is a segment parallel to one of the coordinate axes and intersecting two of the opposite sides of ∂Q_1. If $|\theta| \ge \theta_{\mathrm{crit}}$ then $Q_1 \cap \partial E_\theta$ is a quarter of circle of radius $\frac{2}{\sqrt{\pi}}(\frac{1}{2} - \theta)^{1/2}$ centered at one of the four corners of Q_1.*

Remark 2.2. As already remarked in the Introduction, for L large enough a tunnelling orbit cannot always be close to the Wulff shape, as the Wulff shape varies discontinuously when α crosses the critical value at which $\vartheta_\alpha = \theta_{\mathrm{crit}}$. When $\alpha = 0$ the Wulff shape is planar and this may suggest that optimal orbits become eventually (approximately) planar. Two scenarios are then conceivable: (a) the plus phase grows initially as a quarter of circle around a corner and then progressively deforms to end up into a planar wave as $\alpha \to 0$; (b) the plus phase starts from the very beginning planar, so that in the limit picture the perimeter is discontinuous at time 0, jumping from 0 to its maximal value. In any case, both scenarios evidently contradict the intuitive idea that optimal orbits follow Wulff shapes. A discussion on this issue can be found in [26] in the context of statistical mechanics.
Planar symmetry suggests relevance of $d = 1$ tunnelling.

2.6 Tunnelling in one dimension

With $F^{(1)}$ defined in (2.12), let $m \in L^\infty([-L/2, L/2], [-1, 1])$ and set

$$m^e(r) = m(r \cdot e_1), \qquad r \in Q_L.$$

Then

$$F_L(m^e) = LF^{(1)}(m). \tag{2.15}$$

Let $\mathcal{U}^{(1)}_{L,T}$ be the $d = 1$ tunnelling orbits in a time T and $P^{(1)}_L$ the $d = 1$ tunnelling cost associated with the functional $F^{(1)}$. We then have from (2.15)

$$P_L \le LP^{(1)}_L. \tag{2.16}$$

In [3],[4] it is proved that

$$P_L^{(1)} = F_L^{(1)}(\hat{m}_L) \tag{2.17}$$

where $\hat{m}_L$ is the unique non zero, strictly monotone antisymmetric function of x which solves the equation

$$\hat{m}_L(x) = \tanh\{j^{\text{neum}} * \hat{m}_L(x)\}, \quad |x| \le L/2, \tag{2.18}$$

with j^{neum} obtained from j (see (2.2)) by reflections at $\pm L/2$.

2.7 S_t-invariant manifolds

It is proved in [2] that $\hat{m}_L^e := (\hat{m}_L)^e$ is "dynamically connected" to $m^{(\pm)}$ in the sense that there are two S_t-invariant, one-dimensional manifolds, $\mathcal{W}_\pm = \{v_L^{(\pm)}(\cdot, s), s \in \mathbb{R}\}$, which connect $\hat{m}_L^e$ to $m^{(-)}$ and, respectively, to $m^{(+)}$. $v_L^{(\pm)}(\cdot, s)$ are planar functions (i.e. constant in the vertical direction) which satisfy the following two properties:

$$\lim_{s\to-\infty} \|v_L^{(\pm)}(\cdot, s) - \hat{m}_L^e\|_2 = 0, \quad \lim_{s\to\infty} \|v_L^{(\pm)}(\cdot, s) - m^{(\pm)}\|_2 = 0, \tag{2.19}$$

where $\|\cdot\|_2$ is the L^2 norm in Q_L, and

$$S_t(v_L^{(\pm)}(\cdot, s)) = v_L^{(\pm)}(\cdot, s+t) \quad \text{for all } s \in \mathbb{R} \text{ and all } t \ge 0.$$

Moreover $F_L(v_L^{(\pm)}(\cdot, s)) < LF_L^{(1)}(\hat{m}_L)$ for any $s \in \mathbb{R}$.

2.8 Main results

Theorem 2.3. *For L large enough*

$$P_L = LF_L^{(1)}(\hat{m}_L) \tag{2.20}$$

Theorem 2.3 will be proved starting from Section 5. It suggests that the best strategy for tunnelling is to use orbits with planar symmetry, a statement made precise in Theorem 2.4 below which will be proved in Section 4 using heavily results from [2].

Theorem 2.4. *For all L large enough, if $\{T_n, u_n\}$ is a minimizing sequence for (2.6), then $\lim_{n\to+\infty} T_n = +\infty$ and, given any $\epsilon > 0$ there exists a positive integer n_0 such that for any $n \ge n_0$, u_n (or its image under a rotation by an integer multiple of $\pi/2$) has the following properties. There is $s \in (0, T_n)$ so that $\|u_n(\cdot, s) - \hat{m}_L^e\|_2 \le \epsilon$ and there are τ' and τ'' positive so that*

$$\|u_n(\cdot, t) - v_L^{(-)}(\cdot, \tau' - t)\|_2 \le \epsilon, \quad t \in [0, s] \tag{2.21}$$

$$\|u_n(\cdot, t) - v_L^{(+)}(\cdot, -\tau'' + (t-s))\|_2 \le \epsilon, \quad t \in [s, T_n]. \tag{2.22}$$

Theorem 2.4 proves that the best tunnelling is obtained by orbits which have (approximately) a planar symmetry and which (approximately) follow the one-dimensional manifolds connecting saddle and stable points, first in the time reverse direction and then, after crossing the saddle, along the forward time direction. Initially the orbits look far from optimal, in the sense that it would be cheaper to gain the same value of total magnetization by following a different pattern, closer to the corresponding Wulff shape; but overall such an initial cost is recovered by smaller costs afterwards. In the limit $L \to \infty$ and rescaling penalties by dividing by L, we see that in optimal orbits the free energy jumps at time 0 to a value which then remains constant: in the limit the whole penalty is paid at time 0^+ . Thus the pattern b) in Figure 1 rather than a) is what we actually observe in tunnelling events.

2.9 Content of the paper

In Section 3 we reduce the proof of Theorem 2.3 to the proof that • when $u(\cdot,t) \in \Sigma_\alpha$ with $|\alpha|$ small then $u(\cdot,t)$ is very close to a planar instanton, Theorem 3.1; • calling $m = u(\cdot,t)$, t as above, then either $T_s(m) \to \hat{m}^e_L$ as $s \to \infty$, or else $T_s(m)$ at some time s is close to a planar instanton suitably shifted away from the origin, Theorem 3.2; • if m is close to a planar instanton suitably shifted away to the right or to the left of the origin, then $T_s(m)$ is attracted by $m^{(-)}$ or respectively by $m^{(+)}$, Theorem 3.3. We conclude Section 3 by showing that indeed Theorem 2.3 follows from Theorems 3.1–3.3.
In Section 4 we prove Theorem 2.4 as a consequence of Theorem 2.3 and of existence and stability of the invariant manifolds $\mathcal{W}_\pm$, properties which are proved in a companion paper, [2].
In Sections 5, 6, 7 we prove Theorem 3.1: in Section 5 we quote from the literature lower bounds on the free energy cost of deviations from equilibrium (Peierls estimates). In Section 6 we prove that the distance from an instanton can be controlled in terms of the free energy, Theorem 6.1, and in Section 7 we conclude the proof of Theorem 3.1.
In Section 8 we prove Theorems 3.2 and 3.3, relying again on the companion paper [2], thus concluding the proof of Theorem 2.3.
In the Appendix, Sections 9 and 10, we prove the spectral properties of some operators obtained by linearizing the flows T_t and S_t which have been used in the proofs of Theorems 3.1–3.3.

3 Scheme of proof of Theorem 2.3

By (2.16) and (2.17), $P_L \le LF^{(1)}_L(\hat{m}_L)$, so that Theorem 2.3 will be proved once we show that for L large enough

$$P_L \ge LF^{(1)}_L(\hat{m}_L) \tag{3.1}$$

Thus we may take arbitrarily $\epsilon > 0$, restrict to $T > 0$ and $u \in \mathcal{U}_{L,T}$ such that

$$I_{L,T}(u) \le P_L + \epsilon \le LF^{(1)}_L(\hat{m}_L) + \epsilon \tag{3.2}$$

and show that if L is large enough for any such u

$$I_{L,T}(u) \geq L F_L^{(1)}(\hat{m}_L) \tag{3.3}$$

The main point is an a-priori characterization of the tunnelling orbits which satisfy (3.2) at times t when $u(\cdot,t) \in \Sigma_\alpha$ with $|\vartheta_\alpha| < \theta_0$ (ϑ_α as in (2.11)) where θ_0 is fixed arbitrarily with the only requirement that

$$0 < \theta_0 < \theta_{\text{crit}} \tag{3.4}$$

(how large is L in our analysis will depend also on the value of θ_0). As we will see in Section 7, the proof of convergence to the Wulff shape as $L \to \infty$, see Proposition 7.1, essentially contains closeness to the instanton in the following sense:
For any $\delta > 0$ there are $\epsilon(\delta) > 0$ and $L(\delta)$ so that if $0 < \epsilon < \epsilon(\delta)$, $L > L(\delta)$, $m \in \Sigma_\alpha$ with $|\vartheta_\alpha| \leq \theta_0$ and $F_L(m) < L F_L^{(1)}(\hat{m}_L) + \epsilon$, then, modulo a rotation of an integer multiple of $\pi/2$, there is $\xi \in (-L/2, L/2)$ so that $\|m - \bar{m}_{\xi,L}\|_1 \leq \delta L^2$, where

$$\bar{m}_{\xi,L}(r) = \bar{m}(r \cdot e_1 - \xi), \quad r \in Q_L \tag{3.5}$$

The bound $\|m - \bar{m}_{\xi,L}\|_1 \leq \delta L^2$ is however far from what needed in the proof of (3.1), but it is an important ingredient in the proof of a much sharper estimate, where "the error" $\|m - \bar{m}_{\xi,L}\|_2$ vanishes instead of growing as $L \to \infty$. This is the main technical point in the paper, its precise statement is the content of:

Theorem 3.1. *There are L_0 and $\epsilon_0(L) \in (0, L^{-100})$, so that for any $L \geq L_0$ if*

$$m \in \Sigma_\alpha \text{ with } |\vartheta_\alpha| \leq \theta_0 \text{ and } F_L(m) < L F_L^{(1)}(\hat{m}_L) + \epsilon,\ \epsilon \in (0, \epsilon_0(L)) \tag{3.6}$$

then there exists $\xi \in (-\theta_0 L - 1, \theta_0 L + 1)$ such that, modulo a rotation of an integer multiple of $\pi/2$,

$$\|m - \bar{m}_{\xi,L}\|_2 < L^{-100}. \tag{3.7}$$

Remarks. Theorem 3.1 as well as Theorems 3.2 and 3.3 below, are proved in the next sections and in an appendix. The bound L^{-100} is not optimal. It can be proved, analogously to (8.9), that $|\vartheta_\alpha + \xi/L| < L^{-100}$. Our proof of Theorem 3.1 uses in an essential way two dimensions but it can be hopefully extended to $d = 3$ with an argument introduced by Bodineau and Ioffe and by extending the theory of Wulff shapes to $d = 3$, work in preparation.

The proof of (3.1) proceeds with a characterization of the critical points of $F_L(\cdot)$. For this purpose we use the dynamics with semigroup T_t defined by equation (2.5).

Theorem 3.2. *There exists $L_1 \geq L_0$ such that for any $L \geq L_1$ the following holds. If m satisfies* (3.6) *then either there is a time t when $T_t(m) \in \Sigma_{\alpha'}$, α' such that $|\vartheta_{\alpha'}| = \theta_0$, or else $\lim_{t\to\infty} T_t(m) = \hat{m}_L^e$ in $L^2(Q_L)$ (modulo a rotation of an integer multiple of $\pi/2$).*

Theorem 3.3. *There exists $L_2 \geq L_1$ such that for any $L \geq L_2$ the following holds. If $m \in \Sigma_{\alpha'}$ for some α' such that $\vartheta_{\alpha'} = \pm\theta_0$ and if there exists ξ such that $\|m - \bar{m}_{\xi,L}\|_2 < L^{-100}$, then*

$$\lim_{t\to\infty} T_t(m) = m^{(\mp)} \qquad \text{in } L^2(Q_L). \tag{3.8}$$

The proof of (3.1), giving Theorems 3.1, 3.2 and 3.3 for proved, is then concluded using the following corollary:

Corollary 3.4. *Let L_2 be as in Theorem 3.3 and $L > L_2$. Then for any $u \in \mathcal{U}_{L,T}$ which satisfies* (3.2) *with ϵ as in* (3.6)*, there exists $t^* \in (0,T)$ so that*

$$F_L(u(\cdot,t^*)) \geq LF^{(1)}(\hat{m}_L). \tag{3.9}$$

and

$$\lim_{t\to\infty} T_t(u(\cdot,t^*)) = \hat{m}^e_L \qquad \text{in } L^2(Q_L). \tag{3.10}$$

Proof. Let $u \in \mathcal{U}_{L,T}$ be as in the statement, $\alpha(t)$ such that $u(\cdot,t) \in \Sigma_{\alpha(t)}$ and $I = \{t \in [0,T] : |\vartheta_{\alpha(t)}| \leq \theta_0\}$. Since $\vartheta_{\alpha(0)} = 1/2$, $\vartheta_{\alpha(T)} = -1/2$ and $\theta_0 < 1/2$, by continuity there is an interval $[t_0,t_1] \subset I$, where $\vartheta_{\alpha(t_0)} = \theta_0$ and $\vartheta_{\alpha(t_1)} = -\theta_0$. By Theorems 3.1, 3.2 and 3.3, $[t_0,t_1]$ is the disjoint union of the intervals I_+, I_- and $\hat{I}$, respectively where $u(\cdot,t)$ is attracted by $m^{(+)}$, $m^{(-)}$ and $\hat{m}^e_L$. By Theorem 3.3 $I_+ \ni t_1$ and $I_- \ni t_0$, thus $I_\pm$ are both non empty. Moreover, since the equilibria $\pm m_\beta$ are stable, by the continuity of motion I_+ and I_- are open. Then necessarily also $\hat{I} \neq \emptyset$ and hence there is a time $t^* \in (t_0,t_1)$ so that (3.10) holds. Since T_t decreases the energy F_L, see (6.2), (3.9) follows from (3.10), (6.2) and (2.15).

□

Conclusion of the proof of Theorem 2.3.

From (2.8) and (3.9) it follows that, if L is large enough,

$$I_{L,T}(u) = A_{L,T}(u) \geq A_{L;t^*}(u) = F_L(u(\cdot,0)) + I_{L,t^*}(u) \geq F_L(u(\cdot,t^*)) \geq LF^{(1)}(\hat{m}_L).$$

hence (3.3). Theorem 2.3 is proved.

□

In $d = 1$, see [3], [4] (and [17] for Allen-Cahn), it is proved that for L large enough if $F^{(1)}_L(m) \leq F^{(1)}_L(\hat{m}_L) + \epsilon$ and m is a critical point, then $m \in \{m^+, m^-, \hat{m}_L\}$. (The statement becomes evident in Allen-Cahn once it is formulated in terms of a one dimensional point particle in a conservative field). It then follows that if $P^{(1)}_L(u) \leq F^{(1)}_L(\hat{m}_L) + \epsilon$ then at all t, $u(\cdot,t)$ is attracted by $\{m^+, m^-, \hat{m}_L\}$. In $d = 2$ we know that such a property is valid only at times t when $u(\cdot,t) \in \Sigma_\alpha$ with α such that $|\vartheta_\alpha| \leq \theta_0$. The proof of Theorem 2.3 can be worked out also with such a weaker statement, but the extension to Glauber dynamics

in Ising models with Kac potentials seems to require the stronger property (or a suitable substitute).
We have shown that the proof of Theorem 2.3 is reduced to the proof of Theorems 3.1, 3.2 and 3.3, which will be given in the next sections. While the proof of Theorems 3.2 and 3.3 is an extension of the proof of analogous statements in the $d = 1$ case, see [3], the proof of Theorem 3.1 requires really new considerations, due to geometrical complexities of the higher dimension and will take most of the paper.

4 Proof of Theorem 2.4

In this section we prove Theorem 2.4 using Theorem 2.3 which is thus given for proved. Let $\{u_n, T_n\}$ be a minimizing sequence for (2.6), i.e., $u_n \in \mathcal{U}_{L,T_n}$ and $\lim_{n\to\infty} I_{L,T_n}(u_n) = P_L = LF^{(1)}(\hat{m}_L)$, where the last equality follows from Theorem 2.3. Then for any $\epsilon > 0$ there exists n_ϵ so that for any $n \ge n_\epsilon$

$$I_{L,T_n}(u_n) \le LF^{(1)}(\hat{m}_L) + \epsilon. \tag{4.1}$$

By Corollary 3.4 if $L > L_2$ and ϵ is as in (3.6), then for any $n \ge n_\epsilon$ there is a time $s_n \in (0, T_n)$ (s_n will be the time s in Theorem 2.4) so that

$$\lim_{t\to\infty} T_t\big(u(\cdot, s_n)\big) = \hat{m}^e_L, \quad F_L(u_n(\cdot, s_n)) \ge LF^{(1)}(\hat{m}_L) = F_L(\hat{m}^e_L) \tag{4.2}$$

By (2.8), $I_{L,T_n}(u_n) \ge I_{L,s_n}(u_n) \ge F_L(u_n(\cdot, s_n))$, then, using (4.1),

$$0 \le F_L(u_n(\cdot, s_n)) - F_L(\hat{m}^e_L) \le \epsilon \tag{4.3}$$

The function

$$w_n(\cdot, t) = u_n(\cdot, s_n - t), \qquad t \in (0, s_n). \tag{4.4}$$

satisfies the identity

$$\frac{dw_n}{dt} = J^{\text{neum}} * w_n - \frac{1}{\beta}\,\text{arctanh}(w_n) + K_n \tag{4.5}$$

where K_n is defined by (4.5) itself. We consider (4.5) as an equation in w_n, regarding K_n as a "known term". In the next lemma we will prove that K_n is "small" and then as a consequence and relying heavily on [2] that w_n follows closely the S_t-invariant manifold $\mathcal{W}_-$.

Lemma 4.1. *Let $\epsilon > 0$, u_n satisfy (4.1), s_n as in (4.2), w_n as in (4.4) and K_n as in (4.5). Then for n sufficiently large*

$$\|K_n\|^2 := \int_0^{s_n} \int_{Q_L} K_n(r,t)^2 \; dr dt < \epsilon. \tag{4.6}$$

Furthermore there exists $c > 0$ independent of n so that

$$\|w_n(\cdot, 0) - \hat{m}^e_L\|_2^2 \le c\epsilon. \tag{4.7}$$

Proof. From (4.1) and (2.15) it follows that

$$F_L(\hat{m}_L^e) + \epsilon \geq \int_0^{s_n} \int_{Q_L} [(u_n)_t - f_L(u_n)]^2 = I_{L,s_n}(u_n). \tag{4.8}$$

From (2.7) and (2.9), recalling that $F_L(u_n(\cdot, 0)) = 0$, it follows that

$$I_{L,s_n}(u_n) = A_{L;s_n}(u_n) = A_{L,s_n}(w_n) = I_{L,s_n}(w_n) + F_L(u_n(\cdot, s_n)) = \|K_n\|^2 + F_L(u_n(\cdot, s_n)).$$

which, together with (4.8) and (4.2), implies (4.6).
Let $\Sigma := \{m \in L^\infty(Q_L, (-1,1)) : \lim_{t\to\infty} \|T_t(m) - \hat{m}_L^e\|_2 = 0\}$. In [2, Theorem 7.2] it is proved that there is c so that

$$\|m - \hat{m}_L^e\|_2^2 \leq c[F_L(m) - F_L(\hat{m}_L^e)] \qquad \text{for all } m \in \Sigma. \tag{4.9}$$

By (4.2) $u_n(\cdot, s_n) = w_n(\cdot, 0) \in \Sigma$, therefore (4.7) follows from (4.9) and (4.3). ∎

We will prove the properties of w_n stated in Theorem 2.4 by investigating the evolution equation (4.5) and exploiting that K_n is small. Smallness of K_n is however not enough: if we only knew the bounds on K_n from Lemma 4.1 we could not predict (even approximately) the evolution of w_n. Recall in fact that $\hat{m}_L^e$ is a stationary solution of the unperturbed evolution so that, no matter how small is K_n, it would nonetheless be larger than the unperturbed force in a correspondingly small neighborhood of $\hat{m}_L^e$. In other words, when close to $\hat{m}_L^e$ the evolution is essentially ruled by K_n. Besides this, the initial datum $w_n(\cdot, 0)$ is in the domain of attraction of $\hat{m}_L^e$ with "the wrong dynamics" T_t, under the "right evolution" S_t it may no longer converge to $\hat{m}_L^e$ but rather to m^- or even m^+. In conclusion the evolution of $w_n(\cdot, 0)$ may have completely different behavior if we only had the information of Lemma 4.1 concerning smallness of K_n and closeness of $w_n(\cdot, 0)$ to $\hat{m}_L^e$.
Let us now remind what proved in [2], in particular Theorem 7.3 of [2]. Call $S_t^K(m)$ the flow generated by the equation $u_t = J^{\text{neum}} * u - \beta^{-1}\text{arctanh}(u) + K$, $u(\cdot, 0) = m$, where $K = K(r, t)$, $(r, t) \in Q_L \times \mathbb{R}_+$, is a smooth space-time dependent force.
Then for any $\zeta > 0$ there is $\epsilon' > 0$ so that if $\|K\| < \epsilon'$ and $\|m - \hat{m}_L^e\|_2 < \epsilon'$ only the following two alternatives hold:

- For all times $t \geq 0$, $\|S_t^K(m) - \hat{m}_L^e\|_2 < \zeta$
- There are $t^* > 0$ and $\sigma \in \{-, +\}$ so that $\|S_t^K(m) - \hat{m}_L^e\|_2 < 2\|v_L^{(\sigma)}(\cdot, -\tau) - \hat{m}_L^e\|_2$ for all $t \leq t^*$ while $\|S_t^K(m) - v_L^{(\sigma)}(\cdot, -\tau + (t - t^*)\|_2 < \zeta$ for all $t \geq t^*$.

Let us now prove the statements in Theorem 2.4 referring to $\mathcal{W}_-$, calling ϵ^* the parameter ϵ in Theorem 2.4 to avoid confusion with the ϵ of (4.1) and identifying $s = s_n$. Recall that $u_n(s_n - t) = S_t^{K_n}(w_n(0))$, $t \in [0, s_n]$, we are only writing the time variable in the argument of the functions.

We choose: τ such that $\sup_{s\le -\tau} \|v_L^{(-)}(s) - \hat m_L^e\|_2 \le \epsilon^*/10$; $\zeta < \epsilon^*/10$; ϵ' is determined by τ and ζ as above; ϵ in (4.1) so that $\epsilon < \epsilon'$ and $c\epsilon < \epsilon^*$, $c\epsilon$ as in (4.7), so that the inequality $\|u_n(\cdot, s_n) - \hat m_L^e\|_2 \le \epsilon^*$ in Theorem 2.4 follows from (4.7).
Since $u_n(0) = m^{(-)}$ the first alternative above is excluded and in the second alternative $\sigma = -$. Let t^* be as in the second alternative. We then have $\|u_n(s_n - t) - v_L^{(-)}(-\tau + t - t^*)\|_2 < \epsilon^*$ for $t \in [t^*, s_n]$. For $t \in [0, t^*]$ we write

$$\|u_n(s_n - t) - v_L^{(-)}(-\tau + t - t^*)\|_2 \le \|u_n(s_n - t) - \hat m_L^e\|_2 + \|v_L^{(-)}(-\tau + t - t^*) - \hat m_L^e\|_2$$

which is $\le 3 \sup_{s\le -\tau} \|v_L^{(-)}(s) - \hat m_L^e\|_2 \le 3\epsilon^*/10$. (2.21) is thus proved with $\tau' = -\tau + (s_n - t^*)$.

The proof of (2.22) is analogous. We now take

$$w_n^+(\cdot, t) = u_n(\cdot, s_n + t), \qquad t \in [0, T_n - s_n],$$

so that w_n^+ satisfies the "equation"

$$\frac{dw_n^+}{dt} = J^{\text{neum}} * w_n^+ - \frac{1}{\beta}\,\text{arctanh}(w_n^+) + K_n^+ \tag{4.10}$$

with K_n^+ defined by (4.10). Analogously to Lemma 4.1, $\|K_n^+\| < \epsilon$ for n large enough. Since $S^{K^+}_{T_n - t_n}(w_n^+) = m^{(+)}$, the first alternative is again excluded and the second one is followed with $\sigma = +$. Again we require τ so large and ζ so small (and ϵ correspondingly small) that $\|v_L^{(+)}(\cdot, -\tau) - \hat m_L^e\|_2 \le \epsilon^*/10$ and $\zeta < \epsilon^*$. Then (2.22) follows with $\tau'' = \tau' + t^*$ (t^* the time appearing in the second alternative applied to the present case). Notice finally that if $\epsilon^* \to 0$ the time τ in the above construction diverges and then we need also $T_n \to \infty$ as stated in Theorem 2.4.

5 Local equilibrium and Peierls estimates

The heuristics behind the proof of Theorem 3.1 goes as follows. The Wulff theorem and the limit Wulff shape suggest that if $u(\cdot,\cdot)$ satisfies (3.2), at times t when $u(\cdot, t) \in \Sigma_\alpha$ with $|\vartheta_\alpha| \le \theta_0$, to "zero order" $u(\cdot, t)$ looks like

$$W_{\alpha,L} := m_\beta 1_{\{(x,y):x\ge L\vartheta_\alpha\}} - m_\beta 1_{\{(x,y):x<L\vartheta_\alpha\}} \tag{5.1}$$

To a next approximation we expect $u(\cdot, t)$ close to $\bar m_{\xi,L}$ with ξ such that $\bar m_{\xi,L} \in \Sigma_\alpha$. Behind this picture is the intuition that it does not pay to have deviations from $+m_\beta$ and $-m_\beta$ away from the interface and that the actual profile at the interface is not exactly as sharp as in $W_{\alpha,L}$ but rather the diffuse interface defined by the $d = 1$ instanton $\bar m$ shifted by ξ.
In this section we quote from the literature lower bounds on the free energy due to deviations from equilibrium [Peierls estimates], in the next one we prove lower bounds due to deviations from the instanton shape and in Section 7 we use all that to prove Theorem 3.1.
Local equilibrium and deviations from equilibrium as usual in statistical mechanics are defined in terms of "averages" and of "coarse grained" variables. We briefly recall the main notion adapted to the present context.

Definition 5.1. *(Coarse graining). We denote by $\mathcal{D}^{(\ell)}$, $\ell > 0$, the partition of $\mathbb{R}^2$ into the squares $\{(x,y) : x \in [n\ell,(n+1)\ell), y \in [n'\ell,(n'+1)\ell)\}$, n, n' integers, and by $C_r^{(\ell)}$ the square of $\mathcal{D}^{(\ell)}$ which contains r. Then the ℓ-coarse grained image $m^{(\ell)}$ of a function $m \in L^\infty(\mathbb{R}^2)$ is*

$$m^{(\ell)}(r) := \fint_{C_r^{(\ell)}} m(r')$$

Definition 5.2. *(Geometrical notions). A set is $\mathcal{D}^{(\ell)}$-measurable if it is union of squares in $\mathcal{D}^{(\ell)}$, two sets are connected if their closures have non empty intersection and B is a vertical connection if it is a $\mathcal{D}^{(\ell_+)}$-measurable, connected set which is connected to both lines $\{y = \pm L/2\}$. Given a $\mathcal{D}^{(\ell_+)}$-measurable region $\Lambda \subset Q_L$ we call $\delta_{\text{out}}^{\ell_+}[\Lambda]$ the union of all squares of $\mathcal{D}^{(\ell_+)}$ in $Q_L \setminus \Lambda$ which are connected to Λ.*

Definition 5.3. *(Phase indicators). Given an "accuracy parameter" $\zeta > 0$ and $m \in L^\infty(\mathbb{R}^2,[-1,1])$, we define the "local phase indicator"*

$$\eta^{(\zeta,\ell)}(m;r) = \begin{cases} \pm 1 & \text{if } |m^{(\ell)}(r) \mp m_\beta| \le \zeta, \\ 0 & \text{otherwise.} \end{cases}$$

Given $\ell_- > 0$, ℓ_+ an integer multiple of ℓ_-, $\mathcal{D}^{(\ell_+)}$ a coarser partition of $\mathcal{D}^{(\ell_-)}$, we define the "global phase indicator"

$$\Theta^{(\zeta,\ell_-,\ell_+)}(m;r) = \begin{cases} \pm 1 & \text{if } \eta^{(\zeta,\ell_-)}(m;\cdot) = \pm 1 \text{ in } C_r^{(\ell_+)} \cup \delta_{\text{out}}^{\ell_+}[C_r^{(\ell_+)}], \\ 0 & \text{otherwise.} \end{cases}$$

$\eta^{(\zeta,\ell)}(m;r)$ and $\Theta^{(\zeta,\ell_-,\ell_+)}(m;r)$ are defined also for functions $m \in L^\infty(Q_L,[-1,1])$ by simply extending m to $\mathbb{R}^2$ by reflections along the lines $\{y = (2n+1)L/2\}$ and $\{x = (2n+1)L/2\}$, $n \in \mathbb{Z}$.

Definition 5.3 introduces the notion of "local equilibrium": a point r is attributed to the plus phase if $\Theta^{(\zeta,\ell_-,\ell_+)}(m;r) = 1$, to the minus phase if $\Theta^{(\zeta,\ell_-,\ell_+)}(m;r) = -1$ while, if $\Theta^{(\zeta,\ell_-,\ell_+)}(m;r) = 0$, r belongs to a contour, contours being the maximal connected components of $\{r : \Theta^{(\zeta,\ell_-,\ell_+)}(m;r) = 0\}$. Local equilibrium in r requires closeness to m_β in a large region, the 9 squares in Fig. 2. By choosing ℓ_- small we try to approximate point-wise closeness (which would be too strong a request as the energy is defined by integrals) while taking ℓ_+ large we try to approximate global equilibrium. Very little is needed for local equilibrium to fail as exemplified in Fig. 2.

Definition 5.4. *(Choice of parameters). We choose ℓ_- and ℓ_+ as functions of ζ and L. The definition is used only when ζ is small and L much larger than ℓ_+, and the dependence on L is only through the requirement that $L\ell_\pm^{-1}$ is an integer. We require that for ζ small enough: $\ell_- \in [\zeta^2/2,\zeta^2]$; $\ell_+ \in [\zeta^{-4}/2,\zeta^{-4}]$ with ℓ_+ an integer multiple of ℓ_-; Q_L to be the closure of union of squares of $\mathcal{D}^{(\ell_+)}$; each square of $\mathcal{D}^{(\ell_+)}$ to be union of squares of $\mathcal{D}^{(\ell_-)}$.*

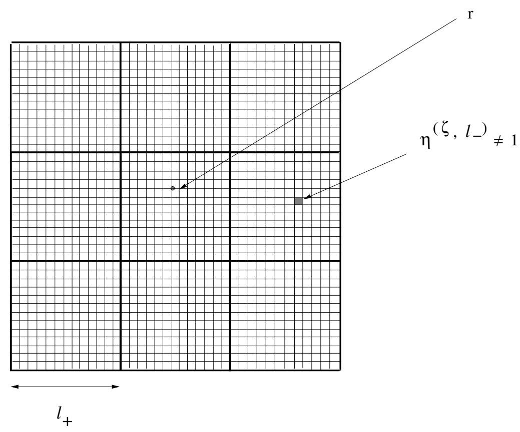

Figure 2: Nine large squares belonging to $\mathcal{D}^{(\ell_+)}$. The small squares are instead elements of $\mathcal{D}^{(\ell_-)}$. Even if $\eta^{(\zeta,\ell_-)}(m;\cdot)=1$ in all small squares except the one in grey, nonetheless $\Theta^{(\zeta,\ell_-,\ell_+)}(m;r)=0$

We have the following two theorems, whose proof is (essentially) contained in [27]:

Theorem 5.5. *There exist $c>0$ and $\omega>0$ such that if $\zeta>0$ is small enough and ℓ_-, ℓ_+ and L are as above, the following assertions hold. Let $\Lambda\subset Q_L$ be $\mathcal{D}^{(\ell_-)}$-measurable, and let m be such that $\eta^{(\zeta,\ell_-)}(m;r)=1$ for all $r\in Q_L$ at distance ≤ 1 from Λ. Then there exists a function ψ satisfying*

$$
\begin{aligned}
&\psi = m \text{ outside } \Lambda,\\
&\eta^{(\zeta,\ell_-)}(\psi;\cdot)=1 \text{ in } \Lambda,\\
&\psi(r)=\tanh\{J^{\mathrm{neum}}*\psi(r)\}, \qquad r\in\Lambda,\\
&|\psi(r)-m_\beta|\le ce^{-\omega\,\mathrm{dist}(r,Q_L\setminus\Lambda)}, \qquad r\in\Lambda,\\
&F_L(\psi)\le F_L(m).
\end{aligned}
$$

The analogous statement holds if $\eta^{(\zeta,\ell_-)}(m;\cdot)=-1$, provided m_β is replaced by $-m_\beta$.

Theorem 5.6. *There exists $c_1>0$ such that if ζ is small enough and ℓ_-, ℓ_+ and L are as above, the following assertions hold. Let $\Lambda\subset Q_L$ be $\mathcal{D}^{(\ell_+)}$-measurable and let m be such that $\Theta^{(\zeta,\ell_-,\ell_+)}(m;\cdot)=1$ in $\delta^{\ell_+}_{\mathrm{out}}[\Lambda]$. Then there exists a function ψ satisfying*

$$
\begin{aligned}
&\psi = m \text{ outside } \Lambda,\\
&\eta^{(\zeta,\ell_-)}(\psi;\cdot)=1 \text{ in } \Lambda,\\
&F_L(m)\ge F_L(\psi)+c_1\zeta^2(\ell_-)^2N_0,
\end{aligned}
\tag{5.2}
$$

where N_0 denotes the number of squares of $\mathcal{D}^{(\ell_+)}\cap\Lambda$ where $\Theta^{(\zeta,\ell_-,\ell_+)}(m;\cdot)=0$. The analogous statement holds if $\Theta^{(\zeta,\ell_-,\ell_+)}(m;\cdot)=-1$ in $\delta^{\ell_+}_{\mathrm{out}}[\Lambda]$.

6 Free energy bounds in the channel

This section continues the "preparation" to the proof of Theorem 3.1. We will estimate here the cost of deviations from the instanton shape. The natural setup for the problem is the channel $Q_{\infty,L}$, in the next section we will in fact eventually reduce from Q_L to $Q_{\infty,L}$. Our main result is an extension to $Q_{\infty,L}$ of a $d=1$ result in [27]:

Theorem 6.1. *There is c so that for any L large enough and for any $m \in L^\infty(Q_{\infty,L},[-1,1])$ such that $\liminf_{x\to\pm\infty} m(x,y) \gtrless 0$ uniformly in y and such that for some $\xi \in \mathbb{R}$, $\|m-\bar m^e_\xi\|_2^2 < \infty$,*

$$F_{Q_{\infty,L}}(m) - F_{Q_{\infty,L}}(\bar m^e) \geq \begin{cases} cL^{-[22+36\beta]}, & \text{if } \inf_{\xi\in\mathbb{R}} \|m-\bar m^e_\xi\|_2^2 > L^{-24\beta-8} \\ \\ cL^{-[2+12(\beta+1)]} \inf_{\xi\in\mathbb{R}} \|m-\bar m^e_\xi\|_2^2, & \text{if } \inf_{\xi\in\mathbb{R}} \|m-\bar m^e_\xi\|_2^2 \leq L^{-24\beta-8} \end{cases} \tag{6.1}$$

The dependence on L in (6.1) is not optimal. Theorem 6.1 will be proved in the rest of the section, the proof is essentially perturbative, it is in fact obtained by expanding $F_{Q_{\infty,L}}(m)$ around $F_{Q_{\infty,L}}(\bar m^e)$. The linear term disappears as the instanton is a critical point; the second order term in the expansion becomes then the leading one. Its analysis requires the study of the spectral properties of a linear operator, which is the second derivative of the functional and hence also the operator obtained by linearizing the time flow around $\bar m^e$. The spectral properties of such an operator are interesting in their own right, the analysis far from trivial and rather long, we have thus decided to just use in this section the outcome of the theory leaving details and proofs to an appendix, where the issue is presented in a self contained fashion.
Thus a spectral gap estimate will allow us to prove the desired lower bounds to a second order approximation, the analysis of the energy landscape away from the instanton shape where non linear effects are dominant requires a different set of ideas. Both close and away from the instanton shape, dynamical properties of the flow $T_t(m)$ play a dominant role, as well as in the proofs of Theorems 5.5-5.6. We thus begin our analysis by quoting from the literature some basic properties of the time flow.

6.1 Monotonicity of energy

The semigroup T_t generated by (2.5) (either in $\mathbb{R}^d$ or else in Q_L or in $Q_{\infty,L}$ with $J \to J^{\text{neum}}$, at the moment our notation does not distinguish among them) has the following properties (which explains why they are useful in proving energy bounds):

(i) T_t decreases the energy F (respectively in $\mathbb{R}^d$ or else in Q_L and in $Q_{\infty,L}$): $F(T_t(m)) \leq F(T_s(m))$ for $s \leq t$ and if $\lim_{t\to\infty} T_t(m) \to m^*$ uniformly on the compacts then

$$\liminf_{t\to\infty} F(T_t(m)) \geq F(m^*) \tag{6.2}$$

(ii) As $t\to\infty$, $T_t(m)$ converges by subsequences uniformly on the compacts to a solution of the stationary equation $m = \tanh\{\beta J * m\}$ (with $J \to J^{\text{neum}}$ in Q_L or $Q_{\infty,L}$).

6.2 Properties of the instanton

In [16] it is proved that there exists $a > 0$ so that

$$\lim_{x\to\infty} e^{\alpha x}\bar m'(x) = a, \tag{6.3}$$

where $\alpha > 0$ is such that

$$p_- \int_{\mathbb{R}} j(0,x)e^{\alpha x} = 1, \quad p_- = \lim_{x\to\infty} p(x) = \beta(1-m_\beta^2) < 1.$$

The finite volume instanton $\hat m_L$ is close to $\bar m$ restricted to $[-L/2, L/2]$, we will just need here that their energies are exponentially close: there are $c > 0$ and $\omega > 0$ so that for all L,

$$|F_L^{(1)}(\hat m_L) - F^{(1)}(\bar m)| \le ce^{-\omega L} \tag{6.4}$$

A function m may be close in shape to the instanton without being close to $\bar m$, but rather to one of its translates. The choice of which one to choose among all $\bar m_\xi$ depends on the applications, particularly useful is the notion of "center of a function m" which is a value $\xi \in \mathbb{R}$ such that $\int_{\mathbb{R}} m\bar m'_\xi p_\xi^{-1} = 0$, $p_\xi = \beta(1-\bar m_\xi^2)$, see (6.10) below for a geometrical interpretation. The notion extends to the channel where a center ξ_m of m is such that

$$\int_{Q_{\infty,L}} m(r)\bar m'_{\xi_m}(r\cdot e_1)p_{\xi_m}^{-1}(r\cdot e_1) = 0, \qquad p_\xi(x) = \beta[1-\bar m_\xi^2(x)] \tag{6.5}$$

The center is related to a minimization problem:

Lemma 6.1. *Let $m \in L^\infty(Q_{\infty,L}, [-1,1])$ then if $\inf_{\xi\in\mathbb{R}} \|m - \bar m_\xi^e\|_2$ is sufficiently small there exists ξ_m such that* (6.5) *holds and*

$$\|m - \bar m_{\xi_m}^e\|_2^2 \le \frac{1}{1-m_\beta^2} \inf_{\xi\in\mathbb{R}} \|m - \bar m_\xi^e\|_2^2 \tag{6.6}$$

Proof. The proof of existence of a center in [27] for the $d = 1$ case extends straightforwardly to the present case showing that if $\inf_{\xi\in\mathbb{R}} \|m - \bar m_\xi^e\|_2 < \epsilon$ with $\epsilon > 0$ small enough, then there is a unique ξ_m which satisfies (6.5) and moreover $\|m - \bar m_{\xi_m}^e\|_2 \le c\epsilon$.
Let

$$f(\xi) := \int_{Q_{\infty,L}} \frac{[m(r) - \bar m_\xi(r\cdot e_1)]^2}{p_{\xi_m}(r\cdot e_1)} dr, \quad b := \sqrt{f(\xi_m)}$$

Since $p_{\xi_m}^{-1} \le (1-m_\beta^2)^{-1}$ and $\|m - \bar m_{\xi_m}^e\|_2 \le c\epsilon$ then $b^2 \le (1-m_\beta^2)^{-1}(c\epsilon)^2$ for all m such that $\inf_{\xi\in\mathbb{R}} \|m - \bar m_\xi^e\|_2 < \epsilon$.
We claim that $f(\xi)$ has a unique minimum at $\xi = \xi_m$ if ϵ is small enough. For notational simplicity we suppose $\xi_m = 0$. Call f' and f'' the first and second derivatives of f w.r.t. ξ. By an explicit computation, $f'(0) = 0$ and

$$f''(0) \ge 2\int \frac{(\bar m')^2}{p} dr - 2b\Big[\int \frac{(\bar m'')^2}{p} dr\Big]^{1/2}$$

Hence there is $\epsilon^* > 0$ so that

$$f''(\xi) \ge \int \frac{(\bar m')^2}{p} dr, \quad |\xi| \le \epsilon^*,\ b \le \epsilon^*$$

which proves that $f(\xi)$ has a unique minimum at $\xi = 0$ when $\xi \in [-\epsilon^*, \epsilon^*]$, (if $b \le \epsilon^*$). Recall that since $\xi_m = 0$, $f(0) = b^2$. Call

$$A(\xi)^2 = \int \frac{(\bar m - \bar m_\xi)^2}{p} dr, \quad A^2 = \inf_{|\xi| \ge \epsilon^*} A(\xi)^2 > 0$$

We write $f(\xi) = \int [\{m - \bar m\} - \{\bar m_\xi - \bar m\}]^2 p^{-1} dr = b^2 + A(\xi)^2 - 2 \int (m - \bar m)(\bar m_\xi - \bar m) p^{-1} dr$, hence $f(\xi) \ge A(\xi)^2 \big(1 - \dfrac{b}{A(\xi)}\big)^2 \ge A^2/2$ for b small enough and $|\xi| \ge \epsilon^*$. Then

$$f(0) = b^2 < A^2/2 \le \inf_{|\xi| \ge \epsilon^*} f(\xi)$$

for b small enough thus proving the claim that 0 is the unique minimizer of f.

Using that $p_{\xi_m} < \beta$ and that $1 - \bar m_\xi^2 > 1 - m_\beta^2$ we then have

$$\begin{aligned} \|m - \bar m^e_{\xi_m}\|_2^2 &\le \int_{Q_{\infty,L}} \frac{\beta}{p^e_{\xi_m}} [m - \bar m^e_{\xi_m}]^2 = \inf_{\xi \in \mathbb{R}} \beta \int_{Q_{\infty,L}} \frac{[m - \bar m^e_\xi]^2}{p^e_{\xi_m}} \\ &\le \frac{1}{1 - m_\beta^2} \inf_{\xi \in \mathbb{R}} \|m - \bar m^e_\xi\|_2^2 \end{aligned}$$

□

6.3 Spectral estimates

Linear stability of the instanton shape has been proved in $d = 1$ and its validity extends to $Q_{\infty,L}$. In an appendix, Sections 9 and 10, we will in fact prove the following statements. In order to simplify notation, we will drop the superscript "e"to denote the extension of a function on $\mathbb{R}$ to the channel $Q_{\infty,L}$. Recalling that $g_L(m) := -m + \tanh\{\beta J^{\rm neum} * m\}$, the first order term in f in the expansion of $g_L(\bar m_\xi + \psi)$, gives

$$\Omega_\xi \psi = -\psi + p_\xi\ J^{\rm neum} * \psi, \qquad p_\xi = \beta(1 - \bar m_\xi^2) \tag{6.7}$$

We regard Ω_ξ as an operator on L^∞ and/or L^2. Ω_ξ has eigenvalue 0 with eigenvector $\bar m'_\xi$. Ω_ξ is a self-adjoint operator on $L^2(Q_L, p_\xi^{-1})$ and denoting by $\langle \cdot, \cdot \rangle_\xi$ the scalar product in such a space, there is a positive number κ (called a in Theorem 10.1) so that

$$\langle \psi, \Omega_\xi \psi \rangle_\xi \le -\frac{\kappa}{L^2} \langle \psi, \psi \rangle_\xi, \quad \langle \psi, \bar m'_\xi \rangle_\xi = 0 \tag{6.8}$$

and $c > 0$ so that (see Theorem 9.4)

$$\|e^{\Omega_\xi t} \psi\|_\infty \le c e^{-(\kappa/L^2)t} \|\psi\|_\infty, \quad \langle \psi, \bar m'_\xi \rangle_\xi = 0 \tag{6.9}$$

The center of m defined in (6.5) is then a value ξ such that $m - \bar{m}_\xi$ has no component along the maximal eigenvector:

$$\langle (m - \bar{m}_\xi), \bar{m}'_\xi \rangle_\xi = 0 \tag{6.10}$$

Indeed, $\langle \bar{m}_\xi, \bar{m}'_\xi \rangle_\xi = 0$ as $\bar{m}$ is antisymmetric and $\bar{m}'$ symmetric.

6.4 Stability of the instanton

We start by proving a weaker version of (6.1), which follows from the stability of the instanton. In $d = 1$ the instanton $\bar{m}$ is "stable" in the sense that if $m \in L^\infty(\mathbb{R}, [-1, 1])$ is such that $\liminf_{x\to\pm\infty} m(x) \gtrless 0$ then there exists $\widehat{\xi}$ such that

$$\lim_{t\to\infty} T_t(m) = \bar{m}_{\widehat{\xi}}, \qquad \text{in } L^\infty(\mathbb{R}) \tag{6.11}$$

(6.11) extends to the channel $Q_{\infty,L}$. The proof in $d = 1$ starts by showing stability of the linearized evolution around $\bar{m}$, it then proceeds by proving first local and then global stability in $\liminf_{x\to\pm\infty} m(x) \gtrless 0$. Linear stability has been already discussed in the previous subsection and with the spectral gap estimate of (6.8) and (6.9) (as L is fixed the fact that the gap vanishes as $L \to \infty$ is inconsequential) the $d = 1$ proof extends straightforwardly and we will just outline it.

Theorem 6.2. *Let $m \in L^\infty(Q_{\infty,L}, [-1, 1])$ be such that $\liminf_{x\to\pm\infty} m(x, y) \gtrless 0$ uniformly in y. Then there exists $\widehat{\xi}$ such that*

$$\lim_{t\to\infty} T_t(m) = \bar{m}^e_{\widehat{\xi}} \qquad \text{in } L^\infty(Q_{\infty,L}). \tag{6.12}$$

and

$$F_{Q_{\infty,L}}(m) \geq F_{Q_{\infty,L}}(\bar{m}^e_{\widehat{\xi}}) = F_{Q_{\infty,L}}(\bar{m}^e) = c_\beta L \tag{6.13}$$

Proof. By standard arguments it follows from the linear stability estimates of Subsection 6.3 that if $\|m - \bar{m}^e_{\widehat{\xi}}\| \leq \epsilon$ with $\epsilon > 0$ small enough, then (6.12) is verified. The global stability statement in the theorem follows from the above local stability using the same argument as in $d = 1$, see [11] [15], [16] (details are omitted).
(6.13) follows from (6.12) and (i) of Subsection 6.1.

□

Lemma 6.3. *Let $m \in L^\infty(Q_{\infty,L}, [-1, 1])$. Then there exists a constant $c > 0$ such that*

$$\|\nabla(T_t(m) - e^{-t} m)\|_\infty \leq c\|\nabla J\|_\infty \tag{6.14}$$

Moreover, there exists $\tau > 0$ so that for any $t \geq \tau$ and any $m \in L^\infty(Q_{\infty,L}, [-1, 1])$

$$\|T_t(m)\|_\infty \leq m_\beta + \frac{1 - m_\beta}{2} \tag{6.15}$$

Proof. The integral version of (2.5) yields

$$T_t(m) - e^{-t}m = \int_0^t e^{-t-s}\tanh\{\beta J^{\rm neum} * T_s(m)\}$$

hence (6.14). A comparison theorem holds for (2.5) so that $T_t(-1) \le T_t(m) \le T_t(1)$ which then gives (6.15). □

Lemma 6.4. *There exists a constant $c > 0$ such that if $m \in L^\infty(Q_{\infty,L}, [-1,1])$ and $\xi \in \mathbb{R}$ then*

$$\|T_t(m) - \bar m^e_\xi\|_\infty \le 2e^{-t} + c\Big(\|T_t(m) - \bar m^e_\xi\|_2 + e^{-t}\|m - \bar m^e_\xi\|_2\Big)^{2/3}$$

Proof. We may assume $\xi = 0$ and write simply $\bar m^e$. The function $\psi = T_t(m) - \bar m^e - e^{-t}(m - \bar m^e)$ has bounded derivative hence (see for instance [19]) there exists $c > 0$ (which depends on the L^∞ norm of the derivative) so that $\|\psi\|_\infty \le c\|\psi\|_2^{2/3}$. Thus

$$\|T_t(m) - \bar m^e\|_\infty \le \|\psi\|_\infty + 2e^{-t} \le 2e^{-t} + c\Big(\|T_t(m) - \bar m^e - e^{-t}(m - \bar m^e)\|_2\Big)^{2/3}$$

□

Lemma 6.5. *If $m \in L^\infty(Q_{\infty,L}, [-1,1])$ and $m - \bar m^e_\xi \in L^2(Q_{\infty,L})$, then for all $t \ge 0$*

$$e^{-2(\beta+1)t}\|m - \bar m^e_\xi\|_2^2 \le \|T_t(m) - \bar m^e_\xi\|_2^2 \le e^{2(\beta-1)t}\|m - \bar m^e_\xi\|_2^2 \tag{6.16}$$

and

$$\Big|\frac{d}{dt}\|T_t(m) - \bar m^e_\xi\|_2^2\Big| \le 2(\beta-1)\|T_t(m) - \bar m^e_\xi\|_2^2 \tag{6.17}$$

Proof. Supposing for simplicity $\xi = 0$, we write $\bar m^e$ instead of $\bar m^e_\xi$. Define $v = T_t(m) - \bar m^e$. Then $v_t = -v + \tanh\{\beta J^{\rm neum} * T_t(m)\} - \tanh\{\beta J^{\rm neum} * \bar m^e\}$. Hence

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|v\|^2 + \|v\|^2 &\le \beta|(v, J^{\rm neum} * v)|\\ \frac{1}{2}\frac{d}{dt}\|v\|^2 + \|v\|^2 &\ge -\beta|(v, J^{\rm neum} * v)|\end{aligned}$$

which proves (6.17) which, by integration, yields (6.16).

□

Proof of Theorem 6.1

To simplify notation we omit in this proof the superscript "e" to denote extension to $Q_{\infty,L}$ and observe that from Lemma 6.5 it follows that at any time t and for any ξ, $\|T_t(m) - \bar m_\xi\|_2^2 < \infty$, as this holds at time 0 by assumption.

We consider first the case when $\inf_{\xi\in\mathbb{R}} \|m-\bar m_\xi\|_2^2 > L^{-24\beta-8}$. There are then two possible alternatives: (a) at all times $\inf_\xi \|T_t(m)-\bar m_\xi\|_2^2 > L^{-24\beta-8}$; (b) There is a time $t<\infty$ when $\inf_\xi \|T_t(m)-\bar m_\xi\|_2^2 \le L^{-24\beta-8}$.
Case (b). By (6.17) for any ξ, $\|T_t(m)-\bar m_\xi\|_2^2$ is a continuous function of t and for any t, $\|T_t(m)-\bar m_\xi\|_2^2$ is a continuous function of ξ which diverges as $|\xi|\to\infty$, (recall the properties of $\bar m$ in Subsection 6.2). It then follows that $\inf_\xi \|T_t(m)-\bar m_\xi\|_2^2$ is a continuous function of t so that there is a time t_0 when $\inf_\xi \|T_{t_0}(m)-\bar m_\xi\|_2^2 = L^{-24\beta-8}$. Since $F_{Q_{\infty,L}}(T_{t_0}(m)) \le F_{Q_{\infty,L}}(m)$, this case is contained in the case when $\inf_{\xi\in\mathbb{R}} \|m-\bar m_\xi\|_2^2 \le L^{-24\beta-8}$ which will be examined next (postponing the analysis of case (a)).
Suppose then that $\inf_{\xi\in\mathbb{R}} \|m-\bar m_\xi\|_2^2 \le L^{-24\beta-8}$ and let τ be such that $e^{-\tau}=L^{-6}$. Call $m^* = T_\tau(m)$. By Theorem 6.2, there is $\xi^*\equiv\xi_{m*}$ so that (6.5) and (6.6) holds. By definition we have

$$F_{Q_{\infty,L}}(m^*) - F_{Q_{\infty,L}}(\bar m_{\xi^*}) = -\frac{1}{\beta}\int_{Q_{\infty,L}} \mathcal{S}(m^*) - \mathcal{S}(\bar m_{\xi^*})$$
$$-\frac12 \int_{Q_{\infty,L}\times Q_{\infty,L}} J^{\rm neum}(r,r')\{m^*(r)m^*(r') - \bar m_{\xi^*}(r)\bar m_{\xi^*}(r')\}$$

Calling $v = m^* - \bar m_{\xi^*}$ and $\alpha = \max(\|m^*\|_\infty, \|\bar m_{\xi^*}\|_\infty)$,

$$-\Big(\mathcal{S}(m^*) - \mathcal{S}(\bar m_{\xi^*})\Big) \ge -\mathcal{S}'(\bar m_{\xi^*})v + \frac{1}{2(1-\bar m_{\xi^*}^2)}v^2 - \frac{\alpha}{3(1-\alpha^2)^2}|v|^3.$$

By (6.15) if L is large enough, $\alpha \le (1+m_\beta)/2 < 1$. Calling

$$\mathcal{L}_\xi = p_\xi^{-1}\Omega_\xi, \quad \mathcal{L}_\xi v = J^{\rm neum} * v - p_\xi^{-1} v, \quad p_\xi = \beta(1-\bar m_\xi^2)$$

where Ω_{ξ^*} is defined in (6.7). We denote by (v,w) the scalar product on $L^2(Q_{\infty,L})$ and regard $\mathcal{L}_\xi$ as an operator on $L^2(Q_{\infty,L})$. We have

$$F_{Q_{\infty,L}}(m^*) - F_{Q_{\infty,L}}(\bar m_{\xi^*}) \ge -\frac12 (v, \mathcal{L}_{\xi^*}v) - \frac{\alpha\|v\|_\infty}{3\beta(1-\alpha^2)^2}(v,v)$$

Since $(v,\mathcal{L}_{\xi^*}v) = \langle v, \Omega_{\xi^*}v\rangle_{\xi^*}$, recalling that $\langle v, \bar m'_{\xi^*}\rangle_{\xi^*} = 0$, by the L^2 spectral gap theorem, (6.8), and using that $p_{\xi^*} \ge \beta[1-m_\beta^2]$ we get

$$F_{Q_{\infty,L}}(m^*) - F_{Q_{\infty,L}}(\bar m_{\xi^*}) \ge \frac{\kappa}{2L^2}\,\langle v,v\rangle_{\xi^*} - (v,v)\frac{\alpha\|v\|_\infty}{3\beta(1-\alpha^2)^2}$$
$$\ge (v,v)\Big(\frac{\kappa}{2L^2\beta(1-m_\beta^2)} - \frac{\alpha\|v\|_\infty}{3\beta(1-\alpha^2)^2}\Big)$$

By Lemma 6.4 with $t=\tau$ and $\xi=\xi^*$ after recalling that $v = T_\tau(m) - \bar m_{\xi^*}$,

$$\|v\|_\infty \le 2L^{-6} + c\big(\|m^* - \bar m_{\xi^*}\|_2 + L^{-6}\|m - \bar m_{\xi^*}\|_2\big)^{2/3}.$$

There exists $\hat{\xi} \in \mathbb{R}$ such that

$$\|m - \bar{m}_{\hat{\xi}}\|_2 \le 2 \inf_{\xi \in \mathbb{R}} \|m - \bar{m}_\xi\|_2.$$

By Lemma 6.5

$$e^{-(\beta+1)\tau}\|m - \bar{m}_{\xi^*}\|_2 \le \|m^* - \bar{m}_{\xi^*}\|_2,$$

and again by Lemma 6.5 and (6.6)

$$\begin{aligned}\|m^* - \bar{m}_{\xi^*}\|_2 &\le \frac{1}{\sqrt{1-m_\beta^2}}\|m^* - \bar{m}_{\hat{\xi}}\|_2 \le \frac{1}{\sqrt{1-m_\beta^2}} e^{(\beta-1)\tau}\|m - \bar{m}_{\hat{\xi}}\|_2 \\ &\le \frac{2}{\sqrt{1-m_\beta^2}} L^{6(\beta-1)-12\beta-4}\end{aligned}$$

so that, for L large enough,

$$\|v\|_\infty \le 2L^{-6} + c\left(L^{6(\beta-1)-12\beta-4} + L^{-6+6(\beta+1)+6(\beta-1)-12\beta-4}\right)^{2/3} \le 3L^{-6},$$

and

$$F_{Q_{\infty,L}}(m^*) - F_{Q_{\infty,L}}(\bar{m}_{\xi^*}) \ge \frac{\kappa}{4L^2\beta(1-m_\beta^2)}\,(v,v)$$

By (6.16), $\|v\|_2^2 \ge \|m - \bar{m}_{\xi^*}\|_2^2 e^{-2(\beta+1)\tau} \ge \inf_\xi \|m - \bar{m}_\xi\|_2^2 e^{-2(\beta+1)\tau}$. Recalling that $e^{-\tau} = L^{-6}$,

$$F_{Q_{\infty,L}}(m) - F_{Q_{\infty,L}}(\bar{m}_{\xi^*}) \ge \frac{\kappa}{4\beta(1-m_\beta^2)}\, L^{-[2+12(\beta+1)]}\, \inf_\xi \|m - \bar{m}_\xi\|_2^2.$$

Case (a), namely when at all times t, $\inf_\xi \|T_t(m) - \bar{m}_\xi\|_2^2 > L^{-24\beta-8}$. By Theorem 6.2 for any $\epsilon > 0$ there are t and ξ so that

$$\|T_t(m) - \bar{m}_\xi\|_\infty < \epsilon$$

By (6.6), calling $m^* = T_t(m)$ and ξ^* the center of m^*,

$$\|m^* - \bar{m}_{\xi^*}\|_\infty < \frac{\epsilon}{1-m_\beta^2} \tag{6.18}$$

We can then proceed as above and, calling $v = m^* - \bar{m}_{\xi^*}$,

$$F_{Q_{\infty,L}}(m^*) - F_{Q_{\infty,L}}(\bar{m}_{\xi^*}) \ge (v,v)\Big(\frac{\kappa}{2L^2\beta(1-m_\beta^2)} - \frac{\alpha\epsilon}{3\beta(1-\alpha^2)^2(1-m_\beta^2)}\Big)$$

By choosing ϵ small enough,

$$F_{Q_{\infty,L}}(m^*) - F_{Q_{\infty,L}}(\bar{m}_{\xi^*}) \ge (v,v)\Big(\frac{\kappa}{4L^2\beta(1-m_\beta^2)}\Big) \ge \|m^* - \bar{m}_{\xi^*}\|^2\Big(\frac{\kappa}{4L^2\beta(1-m_\beta^2)}\Big)$$

Since we are in case (a), $\|m^* - \bar{m}_{\xi^*}\|^2 = \|T_t(m) - \bar{m}_{\xi^*}\|^2 > L^{-24\beta-8}$ and (6.1) follows.

□

7 Proof of Theorem 3.1

In this section we will prove properties of orbits whose penalty is close to optimal which in the end will lead to the proof of Theorem 3.1. Recalling that $\theta_0 \in (0, \theta_{\text{crit}})$, see (3.4), we fix once for all θ_1 and θ_2 so that

$$\frac{1}{2} > \theta_2 > \theta_1 > \theta_0 \tag{7.1}$$

The values of L for which our analysis applies will depend on the actual choice of such parameters. With $W_{\alpha,L}$ defined in (5.1) and $\bar{m}$ in (2.13), for any $\delta > 0$ we set

$$\mathcal{N}_{\delta,L} := \Big\{ m \in \Sigma_\alpha : |\vartheta_\alpha| \le \theta_0,\ \fint_{Q_L} |m - W_{\alpha,L}| < \delta \Big\} \tag{7.2}$$

and will study functions m which satisfy

$$m \in \mathcal{N}_{\delta,L}, \quad F_L(m) < c_\beta L + \epsilon L, \qquad c_\beta = F^{(1)}(\bar{m}) \tag{7.3}$$

By supposing $\epsilon > 0$ smaller and smaller we will prove that (7.3) forces m progressively closer to $\bar{m}^e_\xi$, for a suitable value of ξ. Before entering into the whole issue we remark:

Lemma 7.1. *For every $a > 0$ there is L_a so that for all $L \ge L_a$ the following holds. Let (u_n, T_n) be an optimizing orbit, namely such that $\liminf_{n\to\infty} I_{L,T_n}(u_n) \le F^{(1)}_L(\hat{m}_L)L$. Then for all n large enough and all $t \in [0, T_n]$*

$$F_L(u_n(\cdot,t)) < c_\beta L + (\frac{1}{L^a})\, L, \quad L \ge L_a \tag{7.4}$$

Proof. For any $\delta > 0$ if n is large enough, $F_L(u_n(\cdot,t)) < F^{(1)}_L(\hat{m}_L)L + \delta$. By (6.4)

$$F_L(u_n(\cdot,t)) < L(c_\beta + ce^{-\omega L}) + \delta$$

Choose $\delta = (2L^a)^{-1}$ and L_a so that $ce^{-\omega L_a} < (2L^a)^{-1}$ and (7.4) follows.

□

Thus we can take in (7.3) $\epsilon = L^{-a}$ with a as large as desired, provided $L \ge L_a$ and that we restrict to optimizing sequences. Our first result is that the energy bound in (7.3) already implies that $m \in \mathcal{N}_{\delta,L}$, which is a corollary of the convergence theorem to the Wulff shape of Subsection 2.5.

Proposition 7.1. *For any $\delta > 0$ there exist $\epsilon > 0$ and $\bar{L}$ such that if $L \ge \bar{L}$, $m \in \Sigma_\alpha$ with $|\vartheta_\alpha| \le \theta_0$ and $F_L(m) < c_\beta L + \epsilon L$, then $m \in \mathcal{N}_{\delta,L}$ (modulo a rotation of an integer multiple of $\pi/2$).*

Proof. In the course of the proof we use the following notation: given a set $A \subset Q_1$ we call f_A the function equal to m_β in A and to $-m_\beta$ in $Q_1 \setminus A$ and $f_{A,L}$ its image as a function on Q_L, i.e. $f_{A,L}(Lr) = f_A(r)$. If $L = 1$ we simply write f_A. Let $E_{\vartheta_\alpha} \subseteq Q_1$ be a solution of (2.14) with $|E_{\vartheta_\alpha}| = 1/2 - \vartheta_\alpha$.
We argue by contradiction. Thus we suppose that there is $\delta > 0$ such that for any $\epsilon > 0$ and any $\bar{L}$ positive the following holds. There exist α such that $|\vartheta_\alpha| \leq \theta_0$, $L > \bar{L}$ and $m \in \Sigma_\alpha$ such that $F_L(m) < c_\beta L + \epsilon L$ and

$$\min_{E_{\vartheta_\alpha}} \fint_{Q_L} |m - f_{E_{\vartheta_\alpha},L}\, dr| \geq \delta.$$

We can then find an increasing sequence $\{L_h\}$ converging to $+\infty$ as $h \to +\infty$, α_h such that $|\vartheta_{\alpha_h}| \leq \theta_0$, and functions $m_h \in \Sigma_{\alpha_h}$ satisfying

$$\frac{F_{L_h}(m_h)}{L_h} < c_\beta + \frac{1}{h}, \qquad \min_{E_{\vartheta_{\alpha_h}}} \fint_{Q_{L_h}} |m_h - f_{E_{\vartheta_\alpha},L_h}\, dr| \geq \delta. \tag{7.5}$$

Rescale the functions m_h by defining $v_h(r) := m_h(L_h r)$, $r \in Q_1$. Then there is a (not relabelled) subsequence so that $\alpha_h \to \alpha$ as $h \to +\infty$ with $|\vartheta_\alpha| \leq \theta_0$ while $\{v_h\}$ converges in $L^1(Q_1)$ to a function f_A (i.e. equal to m_β in A and to $-m_\beta$ in $Q_1 \setminus A$, $A \in BV$, and $\fint f_A = \alpha$, [2]. Using the Γ-convergence of the rescaled sequence of functionals,

$$c_\beta \geq \liminf \frac{F_{L_h}(m_h)}{L_h} \geq c_\beta P(A, \text{int}(Q_1)). \tag{7.6}$$

Since $|A| \in [\frac{1}{2} - \theta_0, \frac{1}{2} + \theta_0]$, $P(A, \text{int}(Q_1)) \geq 1$ (1 being the minimal perimeter when the area is in $[\frac{1}{2} - \theta_0, \frac{1}{2} + \theta_0]$) hence from (7.6) $P(A, \text{int}(Q_1)) = 1$ and A is a minimizer of the perimeter. By rescaling the second equation in (7.5)

$$\min_{E_{\vartheta_{\alpha_h}}} \fint_{Q_1} |v_h - f_{E_{\vartheta_{\alpha_h}}}| \geq \delta.$$

As $h \to \infty$ (along the converging subsequence)

$$\min_{E_{\vartheta_\alpha}} \fint_{Q_1} |f_A - f_{E_{\vartheta_\alpha}}| \geq \delta, \tag{7.7}$$

which gives the desired contradiction because the left hand side on (7.7) vanishes. □

All functions $W_{\alpha,L}$ in $\mathcal{N}_{\delta,L}$ have value $+m_\beta$ in $\{(x,y) : x \geq \theta_0 L\}$ and value $-m_\beta$ in $\{(x,y) : x \leq -\theta_0 L\}$. Such a property evidently fails for the generic element of $\mathcal{N}_{\delta,L}$, however a weaker property holds, namely there are two vertical strips (see below), one in $A_+ = \{(x,y) : x \in [\theta_0 L, \theta_1 L]\}$ (recall (7.1) for notation) and the other one in $A_- = -A_+$ where on a large fraction of points $\Theta^{(\zeta,\ell_-,\ell_+)} = 1$, respectively $\Theta^{(\zeta,\ell_-,\ell_+)} = -1$. Under the additional assumption that (7.3) holds with ϵ small enough (yet independent of L) there are "vertical connections" (recall Definition 5.2) where identically $\Theta^{(\zeta,\ell_-,\ell_+)} = 1$ and $\Theta^{(\zeta,\ell_-,\ell_+)} = -1$. If we further strengthen

the assumption by supposing $\epsilon = L^{-2}$ and L large, then we will prove that $\eta^{(\zeta,\ell_-)} = 1$ for $x \geq \theta_2 L$ and $\eta^{(\zeta,\ell_-)} = -1$ for $x \leq -\theta_2 L$.
We define the strips $S(n)$ by

$$S(n) := [n\ell_+, (n+1)\ell_+) \times [-L/2, L/2)$$

Let $Z_L^{\pm} \subset \mathbb{Z}$ be the set of all $n \in \mathbb{Z}$ such that $S(n) \subset A_{\pm}$ and $Z_L = Z_L^+ \cup Z_L^-$.

Proposition 7.2. *There exists a constant $c = c(\zeta, \ell_-, \ell_+)$ such that for any $m \in \mathcal{N}_{\delta,L}$ there are $n_{\pm} \in Z_L^{\pm}$ such that $\Theta^{(\zeta,\ell_-,\ell_+)}(m,\cdot) \neq \pm 1$ in at most $N_\delta := \dfrac{c\delta}{\theta_1 - \theta_0} L$ squares of $\mathcal{D}^{(\ell_+)}$ inside $S(n_{\pm})$.*

Proof. The value of $\Theta^{(\zeta,\ell_-,\ell_+)}(m,\cdot)$ on $S(n)$ is determined by the value of $\eta^{(\zeta,\ell_-)}$ on a strip which is three times larger than $S(n)$. With reference to Figure 2 in fact if the middle square is in $S(n)$ then all 9 squares are needed to determine the value of $\Theta^{(\zeta,\ell_-,\ell_+)}$ in the middle one. Set then $S^{(3)}(n) := [(n-1)\ell_+, (n+2)\ell_+) \times [-L/2, L/2)$. By definition of $\mathcal{N}_{\delta,L}$,

$$\sum_{n \in Z_L^-} \int_{S^{(3)}(n)} |m + m_\beta| + \sum_{n \in Z_L^+} \int_{S^{(3)}(n)} |m - m_\beta| \leq 3 \int_{Q_L} |m - W_{\alpha,L}| < 3\delta L^2.$$

Therefore there are two strips $S^{(3)}(n_{\pm})$ such that

$$\int_{S^{(3)}(n_{\pm})} |m \mp m_\beta| \leq 3\delta L^2 \left(\frac{\ell_+}{L(\theta_1 - \theta_0)} \right) = \frac{3\ell_+ \delta}{\theta_1 - \theta_0}\, L.$$

This implies that $\eta^{(\zeta,\ell^-)}(m,\cdot) \neq \pm 1$ on at most

$$N_\delta := \left(\frac{3\ell_+ \delta L}{\theta_1 - \theta_0} \right) \frac{1}{\zeta(\ell_-)^2}$$

squares of $\mathcal{D}^{(\ell_+)}$ inside $S^{(3)}(n_{\pm})$. Thus there are at most $9N_\delta$ squares in $S(n_+)$ (resp. in $S(n_-)$) where $\Theta^{(\zeta,\ell_-,\ell_+)} \neq 1$ (resp. $\Theta^{(\zeta,\ell_-,\ell_+)} \neq -1$). $\square$

Proposition 7.3. *There are δ, L^* and ϵ^* all positive so that if m satisfies (7.3) with $L \geq L^*$ and $\epsilon \in (0, \epsilon^*)$, then there are two vertical connections $B_{\mp}$, one in $\mathcal{B}_- = \{(x,y) : x \in [-L\theta_2, -L\theta_0]\}$ and the other one in $\mathcal{B}_+ = \{(x,y) : x \in [L\theta_0, L\theta_2]\}$ where $\Theta^{(\zeta,\ell_-,\ell_+)}(m,\cdot)$ is identically equal to -1 and respectively to $+1$.*

Proof. The proof is by contradiction and it is based on successive modifications of m into new functions which if the vertical connections were absent would lead to a final function which has from one side energy smaller than the initial one and, on the other side, larger than $c_\beta L + \epsilon L$, which is the desired contradiction. We first outline the main steps of the proof leaving out details which will later be filled in. By symmetry we may restrict to the case where the vertical connection is absent in $\mathcal{B}_-$ and it may or may not be absent in $\mathcal{B}_+$.

1. The absence of a vertical connection in $\mathcal{B}_-$ implies that the set

$$\{r \in Q_L : \Theta^{(\zeta,\ell_-,\ell_+)}(m;r) > -1\}$$

connects $S(n_-)$ to $\{(x,y) \in Q_L : x = -\theta_2 L\}$. From this it will follow that the number K_0 of $\mathcal{D}^{(\ell_+)}$-squares strictly to the left of $S(n_-)$ where $\Theta^{(\zeta,\ell_-,\ell_+)}(m;\cdot) = 0$ is $K_0 \geq c_0(\theta_2 - \theta_1)L$, c_0 a positive constant.

2. It is possible to modify m only in $S_3(n_-)$ in such a way that the new function $\tilde{m}$ verifies $\Theta^{(\zeta,\ell_-,\ell_+)}(\tilde{m};r) = -1$, $r \in S(n_-)$ and $F_L(\tilde{m}) \leq F_L(m) + c'\delta L$, c' a positive constant.

3. By Theorem 5.6 applied to $\tilde{m}$ with Λ the region strictly to the left of $S(n_-)$ there exists $m^* = \tilde{m}$ on Λ^c such that $\eta^{(\zeta,\ell_-)}(m^*;\cdot) = -1$ on Λ and $F_L(m^*) \leq F_L(\tilde{m}) - c_1\zeta^2\ell_-^2 K_0$.

4. By Theorem 5.5 we can further modify m^* into a new function ψ_- equal to m^* outside Λ in such a way that $\eta^{(\zeta,\ell_-)}(\psi_-;r) = -1$, $r \in \Lambda$, $\psi_-(x,y) = -m_\beta$, $x < -L\theta_2 - 1$ and $F_L(\psi_-) \leq F_L(m^*) + c''e^{-\omega(1/2-\theta_2)L}$, c'' a positive constant.

Conclusion of proof.

Call ψ the function where the analogous modifications are made to the right of the origin, namely by repeating steps 2-4 above (notice that a vertical connection in $\mathcal{B}_+$ may very well exist, in which case we do not have the lower bound for the corresponding K_0 as in item 1). The "previous errors" occur therefore twice while the gain term only once, in the worst case, then

$$F_L(m) \geq F_L(\psi) - 2c'\delta L\ell_- - 2c''e^{-\omega(1/2-\theta_2)L} + c_1\zeta^2\ell_-^2\{c_0(\theta_2-\theta_1)L\} \tag{7.8}$$

Since $\psi(x,y) = \pm m_\beta$ in $x \geq L/2 - 1$ and respectively $x \leq -L/2 + 1$, $F_L(\psi) = F_{Q_{\infty,L}}(\tilde{\psi})$ where $\tilde{\psi}(x,y) = -m_\beta$ for $x \leq -L/2$ and $= m_\beta$ for $x \geq L/2$. Then by (6.13),

$$F_L(m) \geq c_\beta L - 2c'\delta L\ell_- - 2c''e^{-\omega(1/2-\theta_2)L} + c_1\zeta^2\ell_-^2\{c_0(\theta_2-\theta_1)L\} \tag{7.9}$$

which for δ small enough yields for all $L \geq L^*$

$$F_L(m) \geq c_\beta L - 2c''e^{-\omega(1/2-\theta_2)L^*} + \frac{c_1}{2}\,\zeta^2\ell_-^2\{c_0(\theta_2-\theta_1)L^*\} \tag{7.10}$$

Choosing L^* large enough and setting $2\epsilon^* = \dfrac{c_1}{4}\,\zeta^2\ell_-^2\{c_0(\theta_2-\theta_1)L^*\}$,

$$F_L(m) \geq c_\beta L + 2\epsilon^* \tag{7.11}$$

which is the desired contradiction.

Remarks. The above argument is strictly two dimensional. Indeed the lower bound on K_0 grows like L in all dimensions (the "thin fingers effect"), while the error in item 2 grows as $c\delta L^{d-1}$ which in $d > 2$ wins against the "gain term". A different argument developed by Bodineau and Ioffe seems to apply in $d > 2$ and since the theory of Wulff shape can be partially extended to $d = 3$ the results in this paper seem to extend to $d = 3$.

While items 3 and 4 are self explanatory, items 1 and 2 do need a proof.:

Proof of item 1. We call $\pm$ or 0 a $\mathcal{D}^{(\ell_+)}$-square where $\Theta^{(\zeta,\ell_-,\ell_+)}(m,\cdot)$ is ± 1 or 0, respectively. We need the following definitions: given $C \in \mathcal{D}^{(\ell_+)}$ we set

$$\mathcal{S}_{\text{left}}(C) := \Big\{ \widehat{C} \in \mathcal{D}^{(\ell_+)} \cap Q_L : \text{ for any } (x,y) \in \widehat{C} \text{ there is } x' > x \text{ with } (x',y) \in C \Big\}$$

$$\mathcal{S}_{\text{vert}}(C) := \Big\{ \widehat{C} \in \mathcal{D}^{(\ell_+)} \cap Q_L : \text{ for any } (x,y) \in \widehat{C} \text{ there is } y' \in (-\frac{L}{2}, \frac{L}{2}) \text{ with } (x,y') \in C \Big\}$$

Denote by K the number of 0-squares to the left of $S(n_-)$ (included). Item 1 then follows from the two alternatives below:

Case (i). Assume that there exists a $-$ square $C_0 \in \mathcal{D}^{(\ell_+)} \cap S(n_-)$ such that the strip $\mathcal{S}_{\text{left}}(C_0) \cap \{(x,y) \in Q_L : -\theta_2 L \le x \le n_- \ell_+\}$ contains only $-$ squares. For each C' in the strip we have that $\mathcal{S}_{\text{vert}}(C')$ contains at least one $-$ square, because $C' \subset \mathcal{S}_{\text{vert}}(C')$. On the other hand $\mathcal{S}_{\text{vert}}(C')$ cannot consist entirely of $-$ squares by our assumption that there is no vertical connection. Since the sets $\Theta^{(\zeta,\ell_-,\ell_+)} = 1$ and $\Theta^{(\zeta,\ell_-,\ell_+)} = -1$ are not connected, there must be at least one 0-square in $\mathcal{S}_{\text{vert}}(C')$. Thus $K \ge \dfrac{(\theta_2-\theta_1)L}{\ell_+}$.

Case (ii). Any $-$ square $C_0 \in \mathcal{D}^{(\ell_+)} \cap S(n_-)$ is such that $\mathcal{S}_{\text{left}}(C_0)$ contains at least a 0-square. In this case $K \ge \dfrac{L}{\ell_+} - N_\delta$ by definition of $S(n_-)$.

Proof of item 2. Call $\tilde{m}$ the function obtained from m by putting $-m_\beta$ on all squares connected to those in $S(n_-)$ where $\eta^{(\zeta,\ell_-)}(m,\cdot)$ is not identically -1. Then

$$F_L(m) \ge F_L(\tilde{m}) - c_J \ell_+ N_\delta,$$

where $c_J > 0$ is a constant depending only on J hence by Proposition 7.2 item 2 is proved.

As already remarked items 3 and 4 are self explanatory and the Proposition is therefore proved.

□

Corollary 7.4. *In the same context as in Proposition 7.3 assume in addition that* (7.3) *is verified with* $\epsilon = L^{-2}$. *Then* $\eta^{(\zeta,\ell_-)}(m,(x,y)) = \pm 1$ *for all* $x \ge \theta_2 L$ *and respectively* $x \le -\theta_2 L$.

Proof. By Proposition 7.3 there are two vertical connections $B_\mp$ respectively to the right of $x = -\theta_2 L$ and to the left of $x = \theta_2 L$ where $\Theta^{(\zeta,\ell_-,\ell_+)}(m,\cdot) = \mp 1$. Arguing again by contradiction and referring for definiteness to what happens to the left of B_-, if $\Theta^{(\zeta,\ell_-,\ell_+)}(m,\cdot) \ne -1$ somewhere on the left of B_-, necessarily $\Theta^{(\zeta,\ell_-,\ell_+)}(m,\cdot) = 0$ somewhere to the left of

B_-. Then by Theorem 5.6 there is ψ equal to m to the right of B_- (included) with $\Theta^{(\zeta,\ell_-,\ell_+)}(\psi,\cdot)=-1$ on the left of B_- and such that

$$F_L(m) \geq F_L(\tilde{m}) + c_1\zeta^2\ell_-^2$$

The same argument used in the proof of Proposition 7.3 shows that

$$F_L(m) \geq c_\beta L - c'' e^{-\omega(1/2-\theta_2)L} + c_1\zeta^2\ell_-^2$$

which leads to a contradiction because $L\epsilon = L^{-1} < c_1\zeta^2(\ell_-)^2 - c''e^{-\omega(1/2-\theta_2)L}$ for L large enough.

□

Lemma 7.1 and Theorem 7.5 conclude the proof of Theorem 3.1.

Theorem 7.5. *Assume $m \in \Sigma_\alpha$, $|\vartheta_\alpha| \leq \theta_0$ and that*

$$F_L(m) \leq LF^{(1)}(\bar{m}) + \epsilon, \qquad \epsilon < L^{-600-[2+24(\beta-1)]}. \tag{7.12}$$

Then there exists ξ with $|\xi| \leq \theta_0 L + 1$ such that

$$\|\bar{m}^e_\xi - m\|_2^2 \leq L^{-100}.$$

Proof. To simplify notation we omit also in this proof the superscript "e" to denote extension to $Q_{\infty,L}$. We distinguish two cases, case 1 is when (7.13) below is satisfied and case 2 when it is not (we will see that the second case contradicts the assumptions of the theorem and thus it will not occur). Let θ_2 be as in Corollary 7.4.
Case 1: There exists $\xi \in \mathbb{R}$ such that,

$$\|\bar{m}_\xi - m\|^2_{L^2(Q_{\theta_2 L,L})} \leq L^{-300} \tag{7.13}$$

We split the free energy as

$$F_L(m) = F_{Q_{\theta_2 L-1,L}}\left(m_{Q_{\theta_2 L-1,L}} |\, m_{Q^c_{\theta_2 L-1,L}}\right) + F_{Q^c_{\theta_2 L-1,L}}\left(m_{Q^c_{\theta_2 L-1,L}}\right), \tag{7.14}$$

where for $f,g \in L^\infty(Q_L,(-1,1))$ and $A \subseteq Q_L$

$$\begin{aligned} F_A(f) &:= \int_A \phi_\beta(f)\,dr + \frac{1}{4}\int_{A\times A} J^{\text{neum}}(r,r')[f(r)-f(r')]^2 dr\,dr' \\ F_A(f|g) &:= F_A(f) + \frac{1}{2}\int_{A\times(Q_L\setminus A)} J^{\text{neum}}(r,r')[f(r)-g(r')]^2 dr\,dr'. \end{aligned}$$

The L^2-continuity of the energy implies that

$$F_{Q_{\theta_2 L-1,L}}\left(m_{Q_{\theta_2 L-1,L}} |\, m_{Q^c_{\theta_2 L-1,L}}\right) \geq LF^{(1)}(\bar{m}) - L^{-150}$$

for L sufficiently large. By (7.14) and (7.12)

$$LF^{(1)}(\bar m)+\epsilon-LF^{(1)}(\bar m)+L^{-150}\ge F_{Q^c_{\theta_2 L-1,L}}\left(m_{Q^c_{\theta_2 L-1,L}}\right) \tag{7.15}$$

Since $\bar m_\xi$ converges exponentially fast to $\pm m_\beta$ as x diverges, for suitable constants $\bar c$ and $\bar c'$ we have that

$$\|m-\bar m_\xi\|_{L^2(Q^c_{\theta_2 L-1,L})}\le \|m-\mathrm{sign}(x)m_\beta\|_{L^2(Q^c_{\theta_2 L-1,L})}+\bar c e^{-\bar c' L}$$

Then by (7.15), the analysis of Case 1 will be concluded by showing that there is a constant $c>0$ so that

$$F_{Q^c_{\theta_2 L-1,L}}\left(m_{Q^c_{\theta_2 L-1,L}}\right)\ge c\int_{|x|\ge L\theta_2}|m-\mathrm{sign}(x)m_\beta|^2 \tag{7.16}$$

By Corollary 7.4, for L sufficiently large, $\eta^{(\zeta,\ell_-)}(m,r)=\mp 1$ when $x<-\theta_2 L$ and $x>\theta_2 L$. Using this we are going to prove that for any $r=(x,y):|x|\ge\theta_2 L$,

$$\phi_\beta(m(r))+\frac14\int_{Q_L\setminus Q_{\theta_2 L}}J^{\mathrm{neum}}(r,r')[m(r)-m(r')]^2\,\mathrm{d}r'\ge c(m(r)-m_\beta)^2 \tag{7.17}$$

which yields (7.16).
We consider only $x>0$, as the case $x<0$ is proved in exactly the same way. To prove (7.17), we first observe that $\phi_\beta(\pm m_\beta)=0$, $\phi_\beta(m)>0$ for $m\notin\{\pm m_\beta\}$ and $\phi_\beta(m)$ is strictly convex in $\pm m_\beta$. Therefore there exists a constant $c>0$ such that

$$\phi_\beta(m)\ge c\min\{|m-m_\beta|^2,\ |m+m_\beta|^2\}.$$

Thus if $m(r)>0$ the first term on the l.h.s. of (7.17) already yields the bound.
If $m(r)\le 0$ we call $J^{(\ell_-)}(r,r')=\displaystyle\fint_{C^{(\ell_-)}_{r'}}J(r,r'')dr''$ and have by the Lipschitz continuity of J

$$\int J(r,r')\Big(m(r)-m(r')\Big)^2\ge\int J^{(\ell_-)}(r,r')\Big(m(r)-m(r')\Big)^2-c\ell_- \tag{7.18}$$

By Cauchy-Schwartz,

$$\fint_{C^{(\ell_-)}_{r'}}\Big(m(r)-m(r'')\Big)^2\ge\Big(m(r)-\fint_{C^{(\ell_-)}_{r'}}m(r'')\Big)^2\ge\Big(m_\beta-\zeta\Big)^2$$

which inserted in (7.18) gives

$$\int_{Q_L\setminus Q_{\theta_2 L}}J(r,r')\Big(m(r)-m(r')\Big)^2\ge\frac12(m_\beta-\zeta)^2-c\ell_-$$

for ζ small enough, recall $\ell_-\le\zeta^2/2$, the r.h.s. $\ge m_\beta^2/4$ and (7.17) follows.

Case 2: The complementary case is when (7.13) does not hold, we will prove that such a case cannot actually happen. Indeed by Corollary 7.4 and Theorem 5.5 there is a function ψ equal to m on $|x| \le \theta_2 L$, such that $\psi = \pm m_\beta$ on $x > L/2 - 1$ and $x < -L/2 + 1$ and

$$F_L(m) \ge F_L(\psi) - ce^{-\omega(1/2-\theta_2)L} \tag{7.19}$$

Calling ϕ the function on $Q_{\infty,L}$ equal to ψ on Q_L and to $\pm m_\beta$ on $x < -L/2$ and $x > L/2$, by Theorem 6.1 we have

$$F_L(\psi) = F_{Q_{\infty,L}}(\phi) \ge c_\beta L + \inf_\xi cL^{-[2+24(\beta-1)]} \int_{|x|<\theta_2 L} |m - \bar m_\xi|^2 \ge c_\beta L + cL^{-[2+24(\beta-1)]-300} \tag{7.20}$$

(7.19)–(7.20) contradict (7.12) for L large.

□

8 Proof of Theorems 3.2 and 3.3

The proof follows the one dimensional analysis in [5], see also [6, 7, 13], and uses some spectral properties proved in an appendix, Sections 9 and 10.

8.1 Spectral analysis

Recalling that $g_L(m) := -m + \tanh\{\beta J^{\text{neum}} * m\}$, the first order term in f in the expansion of $g_L(\bar m_{\xi,L} + f)$, $|\xi| < L/2$, gives

$$\Omega_{\xi,L} f = -f + p_{\xi,L}\, J^{\text{neum}} * f, \qquad p_{\xi,L} = \beta\, \cosh^{-2}\{\beta J^{\text{neum}} \star \bar m^e_{\xi,L}\} \tag{8.1}$$

(the zero order term is however not missing because $\bar m_{\xi,L}$ is not a critical point unless $L = \infty$). We will regard here $\Omega_{\xi,L}$ as an operator on $L^\infty(Q_L)$, fix in the sequel $r \in (0,1)$ and restrict to ξ such that $|\xi| \le rL/2$. $\Omega_{\xi,L}$ is a shorthand for $\Omega_{\bar m_{\xi,L}}$ which is among the operators Ω_m considered in Section 9. Due to the planar symmetry (invariance under vertical shifts) some of its spectral properties just follow from the $d = 1$ analysis and are valid for all L large enough. We refer to Subsection 9.1, here we just mention that the maximal eigenvalue is $\lambda_{\xi,L}$ with eigenvector a strictly positive, planar function $e_{\xi,L}(\cdot)$. In the notation of Subsection 9.1 eigenvalue and eigenfunction are denoted by λ_m and e_m respectively, where $m = \bar m_{\xi,L}$.
The crucial property of $\Omega_{\xi,L}$ for our applications here is invertibility: the inverse $\Omega^{-1}_{\xi,L}$ of $\Omega_{\xi,L}$ exists and it is a bounded operator on the orthogonal complement of the one dimensional subspace spanned by $e_{\xi,L}$, more precisely there is a constant $\kappa > 0$ so that

$$\|\Omega^{-1}_{\xi,L}\|_\infty \le \frac{\kappa}{L^2} \tag{8.2}$$

on the aforementioned orthogonal complement. Unlike the previous property, (8.2) is not a consequence of the analogous property in $d = 1$ (where the bound on the r.h.s. is proved to be independent of L). The extension to $d = 2$ is proved in the appendix as a direct consequence of Theorem 9.4, see (9.26).

8.2 Fibers

Following [5], we introduce fibers in the space $L^\infty(Q_L;[-1,1])$, defined as

$$\mathcal{B}_{\xi,L} := \{m \in L^\infty(Q_L;[-1,1]) : m = \bar{m}_{\xi,L} + \phi, \pi_{\xi,L}(\phi) = 0\} \tag{8.3}$$

where

$$\pi_{\xi,L}(\phi) = \frac{\langle e_{\xi,L}\phi\rangle_{\xi,L}}{\langle e_{\xi,L}e_{\xi,L}\rangle_{\xi,L}}, \quad \langle f,g\rangle_{\xi,L} := \int_{Q_L} fg\, p_{\xi,L}^{-1} \tag{8.4}$$

and call

$$\mathcal{B}_{\epsilon,\xi,L} := \{\bar{m}_{\xi,L} + \phi \in \mathcal{B}_{\xi,L} : \|\phi\|_\infty \le \epsilon\}, \quad \mathcal{B'}_{\epsilon,\xi,L} := \{m \in \mathcal{B}_{\epsilon,\xi,L} : m(x,y) = m(x,0)\} \tag{8.5}$$

Theorem 8.1. *For any L large enough, the only solution of $g_L(m) = 0$ with $m \in \mathcal{B}_{L^{-3},\xi,L}$, $|\xi| \le rL/2$ is $\hat{m}_L^e$.*

Proof. The analogous property in $d = 1$ has been proved in a stronger form in [5], thus the theorem will follow once we show that any solution of $g_L(m) = 0$ in $\{\mathcal{B}_{L^{-3},\xi,L}, |\xi| \le rL/2\}$ is necessarily in $\{\mathcal{B'}_{L^{-3},\xi,L}, |\xi| \le rL/2\}$.
Following Section 4 of [5], we consider the auxiliary equation

$$g_L(m) - \pi_{\xi,L}\big(g_L(m)\big)e_{\xi,L} = 0, \quad m \in \mathcal{B}_{L^{-3},\xi,L} \tag{8.6}$$

We will prove that any solution of (8.6) is in $\mathcal{B'}_{L^{-3},\xi,L}$. The theorem will then follow because if $g_L(m) = 0$ then m satisfies (8.6).
For ϕ as above we define

$$R_{\xi,L}(\phi) = g_L(\bar{m}_{\xi,L} + \phi) - g_L(\bar{m}_{\xi,L}) - \Omega_{\xi,L}\phi$$

By a Taylor expansion to second order, there is c so that

$$\|R_{\xi,L}(\phi)\|_\infty \le c\|\phi^2\|_\infty, \quad \|R_{\xi,L}(\phi_1) - R_{\xi,L}(\phi_2)\|_\infty \le c\{\|\phi_1\|_\infty + \|\phi_2\|_\infty\}\|\phi_1 - \phi_2\|_\infty \tag{8.7}$$

For L large enough, let $\mathcal{A}_{\xi,L}$ be the following operator on $\mathcal{B}_{\xi,L}$:

$$\mathcal{A}_{\xi,L}(\phi) := -\Omega_{\xi,L}^{-1}\Big\{[g_L(\bar{m}_{\xi,L}) - \pi_{\xi,L}\big(g_L(\bar{m}_{\xi,L})\big)e_{\xi,L}] + [R_{\xi,L}(\phi) - \pi_{\xi,L}(R_{\xi,L}(\phi))e_{\xi,L}]\Big\}$$

If ϕ is a fixed point of $\mathcal{A}_{\xi,L}(\cdot)$ and $\|\phi\| \le L^{-3}$, then $\bar{m}_{\xi,L} + \phi$ solves (8.6).
In [12] it has been proved that there is C so that, for α as in (6.3),

$$\|g_L(\bar{m}_{\xi,L})\|_\infty \le Ce^{-\alpha(L-2|\xi|)}$$

which implies that for L large enough

$$\|g_L(\bar{m}_{\xi,L}) - \pi_{\xi,L}\big(g_L(\bar{m}_{\xi,L})\big)e_{\xi,L}\|_\infty \le Ce^{-\bar{c}L}$$

From (8.2) and (8.7) it then follows that

$$\|\mathcal{A}_{\xi,L}(\phi)\|_\infty \le c_0 L^2[Ce^{-\bar{c}L} + L^{-6}]$$

Thus, for all L large enough $\mathcal{A}_{\xi,L}$ maps the set $\mathcal{B}_{L^{-3},\xi,L}$ into itself. Moreover $\mathcal{A}_{\xi,L}$ maps $\mathcal{B}'_{L^{-3},\xi,L}$ into itself. By (8.7) and (8.2) we have

$$\|\mathcal{A}_{\xi,L}(\phi_1) - \mathcal{A}_{\xi,L}(\phi_2)\|_\infty \le \delta\|\phi_1 - \phi_2\|_\infty, \qquad \delta < L_1^{-1},$$

so that $\mathcal{A}_{\xi,L}$ is a contraction on $\mathcal{B}_{L^{-3},\xi,L}$ and since $\mathcal{B}'_{L^{-3},\xi,L}$ is invariant, the unique fixed point ϕ_ξ is in $\mathcal{B}'_{L^{-3},\xi,L}$, namely it has planar symmetry. As already remarked, solutions of (8.6) are fixed points of $\mathcal{A}_{\xi,L}$. We have thus shown that solutions of (8.6) have planar symmetry, which, as argued before, proves the theorem.

□

8.3 Proof of Theorem 3.2

Let m, L and ϵ as in Theorem 3.2. Since the function $t \to \alpha(t)$, $\alpha(t) = \fint_{Q_L} T_t(m)$, $t \ge 0$, is continuous and since $|\vartheta_{\alpha(0)}| \le \theta_0$, either there is a time $t^* \ge 0$ when $|\vartheta_{\alpha(t^*)}| = \theta_0$ or else any limit point m^* (in L^∞) of $T_t(m)$ is in Σ_α with $|\vartheta_\alpha| \le \theta_0$.
Being a limit point, m^* is stationary and by lower semi-continuity, $F_L(m^*) \le F_L(m) < LF_L^{(1)}(\hat{m}_L) + \epsilon$. Since $\epsilon < \epsilon_0(L)$, by Theorem 3.1 there is ξ, $|\xi| \le \theta_0 L + 1$, so that $\|m^* - \bar{m}_{\xi,L}\|_2 < L^{-100}$. Since m^* is stationary its derivative is bounded hence there is a constant c so that

$$\|m^* - \bar{m}_{\xi,L}\|_\infty \le c\big(\|m^* - \bar{m}_{\xi,L}\|_2\big)^{2/3} < c(L^{-100})^{2/3} \tag{8.8}$$

We omit the proof that if $\|m - \bar{m}_{\xi,L}\|_\infty < \zeta$, ζ small enough, then m is in a fiber $\mathcal{B}_{\xi',L}$ with $|\xi' - \xi| \le c\zeta$, which is analogous to its $d = 1$ version proved in [5]. Using such a statement by (8.8) for L large enough $m \in \mathcal{B}_{L^{-3},\xi',L}$, $|\xi'| \le rL/2$, $r < 1$ and by Theorem 8.1 we then conclude that $m^* = \hat{m}^e_L$. Theorem 3.2 is proved.

8.4 Proof of Theorem 3.3

By symmetry we may restrict to $m \in \Sigma_\alpha$ with $\vartheta_\alpha = -\theta_0$. By assumption $\|m - \bar{m}_{\xi,L}\|_2 < L^{-100}$; we are going to show that for L large enough,

$$|-\theta_0 - \frac{\xi}{L}| \le L^{-100} \tag{8.9}$$

Indeed, $\|m - \bar{m}_{\xi,L}\|_1 \le 4\|m - \bar{m}_{\xi,L}\|_2 < 4L^{-100}$, so that $|\fint_{Q_L} m - \fint_{Q_L} \bar{m}_{\xi,L}| \le \dfrac{4L^{-100}}{L^2}$. and (8.9) follows for L large enough because $\fint_{Q_L} m = \alpha$, $\vartheta_\alpha = -\theta_0$ and using (6.3).

Theorem 8.2. *For any ϵ and $r \in (0,1)$ there is $L(\epsilon, r)$ so that for all $L > L(\epsilon, r)$ the following holds. Let $m \in L^\infty(Q_L)$ be such that there is $\xi_0 \in (-\frac{L}{2}, -\frac{rL}{2})$ so that $\|m - \bar{m}_{\xi_0,L}\|_\infty \le \epsilon$, then*

$$\lim_{t\to\infty} \|T_t(m) - m^+\|_\infty = 0 \tag{8.10}$$

Proof. By assumption

$$m(x,y) \ge \bar{m}_{\xi_0}(x) - 2\epsilon, \qquad \text{for all } (x,y) \in Q_L$$

In Proposition 8.2, Theorem 8.3 and Proposition 8.4 of [4] it has been proved that for L large (how large depending on ϵ and r) $T_t\big(\bar{m}_{\xi_0} - 2\epsilon\big)$ converges to m^+. Thus (8.10) follows from the comparison theorem.

□

By Lemmas 6.4 and 6.5,

$$\|T_t(m) - \bar{m}_{\xi,L}\|_\infty \le 2e^{-t} + c\Big([e^{-t} + e^{2(\beta-1)t}]\|m - \bar{m}_{\xi,L}\|_2\Big)^{2/3}$$

Choosing t suitably large (independently of L) the r.h.s. becomes $< \epsilon$ and by (8.9), $T_t(m)$ satisfies the assumption of Theorem 8.2 with $r < \theta_0$ and L large enough. Then Theorem 3.3 follows from Theorem 8.2, noticing that convergence in $L^\infty(Q_L)$ implies convergence in $L^2(Q_L)$, because Q_L is bounded.

APPENDIX

9 Spectral estimates, sup norms

The analysis in this appendix refers to functions on the square Q_L and on the channel $Q_{\infty,L}$, in the latter case we will consider only one function, the instanton $\bar{m}^e$. For brevity we call planar a function or a kernel where the dependence on the point r is only via its x coordinate $x = r \cdot e_1$.

Definition. The set $\mathcal{M}_L$ consists of the instanton $\bar{m}^e \in L^\infty(Q_{\infty,L}, (-1,1))$ and of the family of planar functions $m \in L^\infty(Q_L, [-1,1])$ which are in either one of the following two classes (r below a fixed number in $(0,1)$):

- $\bar{m}_{\xi,L}$, $|\xi| \le \dfrac{rL}{2}$
- $\|m - \hat{m}^e_L\|_\infty \le \epsilon(L)$, $\epsilon(L) > 0$ a small number which will be fixed later.

9.1 Maximal eigenvalue and eigenvector

We call A_m, $m \in \mathcal{M}_L$, the operator on $L^\infty(Q_L)$ or $L^\infty(Q_{\infty,L})$ if $m = \bar{m}^e$, whose kernel is

$$A_m(r, r') = p_m(r) J^{\mathrm{neum}}(r, r') \tag{9.1}$$

If $m = \bar{m}_{\xi,L}$, then $p_m = \cosh^{-2}\{\beta J^{\mathrm{neum}} * m\}$, otherwise $p_m = \beta(1 - m^2)$. If $m = \bar{m}^e$ or $m = \hat{m}^e_L$ the two expressions coincide. The different choices are due to different applications, e.g. if we linearize around the flow $T_t(m)$ or $S_t(m)$.
In [12] it is proved that given $r \in [0, 1)$ there are L_r and $\epsilon(L)$ so that for all $L \ge L_r$ and any $m \in \mathcal{M}_L$, there are $\lambda_m > 0$ and e_m so that, with $s_m = p_m^{-1} e_m$,

$$\int A_m(r, r') e_m(r') dr' = \lambda_m e_m(r), \qquad \int s_m(r) A_m(r, r') dr' = \lambda_m s_m(r') \tag{9.2}$$

e_m is a strictly positive, smooth planar function in $L^\infty(Q_L)$ that we normalize so that

$$\int s_m e_m = \int e_m^2 p_m^{-1} = \langle e_m, e_m \rangle_m = 1 \tag{9.3}$$

λ_m is an eigenvalue of A_m with strictly positive right and left eigenvectors, e_m and s_m, in agreement with the Perron-Frobenius theorem which is indeed behind the proof of the above statements. The function $e^{(1)}_m(x)$ on $[-L/2, L/2]$ or $\mathbb{R}$ for the instanton, defined by $x \to e_m(r), r \cdot e_1 = x$, is the eigenvector for the $d = 1$ problem with interaction j as in (2.2), however $\int e^{(1)}_m(x)^2 dx = L^{-1}$ due to (9.3). In the case $m = \bar{m}^e$, $\lambda_m = 1$ and $e_m(r) = c\bar{m}'(r \cdot e_1)/\sqrt{L}$, c a normalization independent of L.
The above statements are verified in a large class of functions m, those which follow are instead more restrictive. All bounds below are uniform in $\mathcal{M}_L$ but we keep reference to the specific $m \in \mathcal{M}_L$ for future applications.

- There are $c_\pm > 0$ and $\alpha'_m > 0$ so that

$$1 - c_+ e^{-2\alpha'_m L} \le \lambda_m \le 1 + c_+ e^{-2\alpha'_m L} \tag{9.4}$$

- For each $m \in \mathcal{M}_L$ define x_m as $x_m = 0$ if $m = \bar{m}^e$, $x_m = \xi$ if $m = \bar{m}_{\xi,L}$ and $x_m = 0$ for the remaining m. Then there are $s > 0$ and $\delta < 1$ so that

$$p_m(r) \le \delta, \quad |r \cdot e_1 - x_m| \ge s \tag{9.5}$$

and there are $\alpha_m > 0$, $\alpha''_m > 0$ and c so that

$$e_m(r) \le \frac{c}{\sqrt{L}}\, e^{-\alpha_m |r \cdot e_1 - x_m|}, \quad e_m(r)^{-1} \le c\sqrt{L}\, e^{\alpha''_m |r \cdot e_1 - x_m|} \tag{9.6}$$

- We will also use that there is a constant c so that

$$\|p_m^{-1}\|_\infty \le c \tag{9.7}$$

- As mentioned, all the previous bounds are uniform in $\mathcal{M}_L$, by suitably resetting the coefficients.

9.2 Reduction to Markov chains

Let K_m be the Markov operator whose transition probability kernel is

$$K_m(r, r') = \frac{A_m(r, r')e_m(r')}{\lambda_m e_m(r)} \tag{9.8}$$

Since $A_m^n(r, r') = e_m(r)\lambda_m^n K_m^n(r, r')e_m(r')^{-1}$ we will derive bounds on A_m^n and consequently on the spectrum of A_m and of $\Omega_m := A_m - 1$ from properties of K_m^n. The important point of the transformation (9.8) is that K_m is a Perron-Frobenius Markov kernel to which the high temperature Dobrushin techniques apply.
Calling $x = r \cdot e_1$ and $y = r \cdot e_2$ we can write $K_m(r, r')$ as

$$K_m(r, r') = P_m(x, x')q_{x,x'}(y, y') \tag{9.9}$$

where, relative to the measure $K_m(r, r')dr'$, $P_m(x, x')$ is the marginal distribution of x' and $q_{x,x'}(y, y')$ is the conditional distribution of y' given x' (to simplify notation we drop sometimes the suffix m). The explicit expression of $P_m(x, x')$ is

$$P_m(x, x') = \frac{p_m(x)j^{\text{neum}}(x, x')e_m(x')}{\lambda_m e_m(x)} \tag{9.10}$$

where $j(x, x')$ is defined in (2.2) and $e_m(x) \equiv e_m(x, y)$ (recall that $e_m(r)$ is planar). (9.10) is (9.8) in the $d = 1$ case with interaction $j(x, x')$. Notice that due to the planar symmetry assumption the marginal $P_m(x, x')$ does not depend on the y coordinate of r. In the sequel we will consider the probability density

$$q_{x,x'}(z) = \frac{J((x, 0), (x', z))}{j(x, x')} \tag{9.11}$$

on $\mathbb{R}$ noticing that the variable $y' := y + z$ modulo reflections at $\pm nL/2$ has the law $q_{x,x'}(y, y')$ and sometimes, by an abuse of notation, we will write $q_{x,x'}(z)$ for $q_{x,x'}(y, y')$.

9.3 One dimensional results

To study the dependence on the initial point r of the Markov chain with transition probability kernel $K_m(r, r')$ we will use couplings (for brevity we may shorthand $x = r \cdot e_1$ and $y = r \cdot e_2$). We first recall some one dimensional results proved in [12]. Call

$$W^{(1)}[x, x'] = \left[w_m(x) + w_m(x')\right] \mathbf{1}_{\{x \neq x'\}}, \quad w_m(x) := e_m(x)^{-1} \tag{9.12}$$

e_m and $m \in \mathcal{M}_L$ below regarded as functions of x.

Theorem 9.1. *There are c and $\omega^{(1)}$ positive and for any $(x_0, x_0') \in [-L/2, L/2]^2$ (or $\mathbb{R}^2$) a process on $([-L/2, L/2]^2)^{\mathbb{N}}$ (or $(\mathbb{R}^2)^{\mathbb{N}}$) whose expectation is denoted by $\mathcal{E}^{(1)}_{x_0,x_0'}$ so that its marginal distributions are the Markov chains with transition probability* (9.10) *and, for any L large enough and $n \geq 1$,*

$$\mathcal{E}^{(1)}_{x_0,x_0'}\Big(W^{(1)}[x(n), x'(n)]\Big) \leq cW^{(1)}[x(0), x'(0)]e^{-\omega^{(1)}n} \tag{9.13}$$

Moreover if for some n, $x(n) = x'(n)$ then $x(n + k) = x'(n + k)$ for all $k \geq 0$.

9.4 Couplings and Wasserstein distance

For any $(r_0, r'_0) \in Q_L \times Q_L$ (or $Q_{\infty,L} \times Q_{\infty,L}$ if $m = \bar{m}^e$) we define a process $\{r(n), r'(n), n \in \mathbb{N}\}$, $r(0) = r_0$, $r'(0) = r'_0$, with values on $Q_L \times Q_L$ (or $Q_{\infty,L} \times Q_{\infty,L}$) as follows.
The marginal distribution of $\{x(n), x'(n), n \in \mathbb{N}\}$ is set equal to the law $\mathcal{P}^{(1)}_{x_0,x'_0}$ of the process defined in Theorem 9.1.
To complete the definition we must give the law of $\{y(n), y'(n), n \in \mathbb{N}\}$ conditioned on the trajectory

$$(\underline{x}, \underline{x}') = \{(x(n), x'(n)), n \in \mathbb{N}\},$$

which we consider in the sequel as fixed. Define then $n_0,\ n_1 \in \mathbb{N} \cup \{+\infty\}$ as

$$n_0 := \inf\{n \in \mathbb{N} :\ x(n_0) = x'(n_0)\}, \quad n_1 := \inf\{n \in \mathbb{N} :\ n \geq n_0 \text{ and } |y(n) - y'(n)| \leq 1\},$$

where the infimum over the empty set is defined as $+\infty$. This means that n_0 is the first time when the x-coordinates couple, and n_1 is the first time at which the y-coordinates get close after the x-coordinates have coupled.
For $n \leq n_1$, $y(n)$ and $y'(n)$ are independent of each other and distributed with the law of the Markov chain with transition probability (9.11) which starts respectively from y_0 and y'_0.
If $n_1 < \infty$ the conditional law of $\{y(n), y'(n), n \in [n_1, n_1 + k_0]\}$, k_0 as in Lemma 9.2 below, given $y(n_1), y'(n_1)$ is Π, Π the probability in Lemma 9.2 below.
If $y(n_1 + k_0) = y'(n_1 + k_0)$, $y'(n) = y(n)$ for $n \geq n_1 + k_0$ with $y(n)$ having the law of the Markov chain with transition probability (9.11).
If instead $y(n_1 + k_0) \neq y'(n_1 + k_0)$ we repeat the previous procedure with n_0 replaced by $n_1 + k_0$ and so forth.

Lemma 9.2. *There are π_0 and k_0 positive and for any (y_0, y'_0, X), $|y_0 - y'_0| \leq 1$, $X = (x_0, .., x_{k_0})$, a probability $\Pi = \Pi_{(y_0,y'_0,X)}$ on $[-L/2, L/2]^{k_0+1} \times [-L/2, L/2]^{k_0+1}$ such that the marginal distributions of $y(\cdot)$ and $y'(\cdot)$ are the Markov chains with transition probability* (9.11) *starting from y_0 and y'_0 and ($\mathcal{E}^{(1)}_{x_0,x'_0}$ below as in Theorem 9.1),*

$$\mathcal{E}^{(1)}_{x_0,x'_0}\Big(\Pi_{(y_0,y'_0,X)}\big(\{y(k_0) = y'(k_0)\}\big)\Big) \geq \pi_0 \tag{9.14}$$

The lemma follows easily from the smoothness properties of the transition kernel, its proof is just as in its one dimensional version in [12] and it is omitted.

We call $\mathcal{P}_{r_0,r'_0}$ the joint law of $\{r(n), r'(n), n \in \mathbb{N}\}$ as defined above and denote by $\mathcal{E}_{r_0,r'_0}$ expectation w.r.t. $\mathcal{P}_{r_0,r'_0}$. $\mathcal{P}_{r_0,r'_0}$ is a coupling of the Markov chains starting from r_0 and r'_0 and with transition probability K_m. Indeed, for any $f \in L^\infty(Q_L)$ or $f \in L^\infty(Q_{\infty,L})$, and any $n \geq 1$,

$$\mathcal{E}_{r_0,r'_0}\big(f(r(n)\big) = \int_{Q_L} K^n_m(r_0, r) f(r), \quad \mathcal{E}_{r_0,r'_0}\big(f(r'(n)\big) = \int_{Q_L} K^n_m(r'_0, r) f(r) \tag{9.15}$$

Recalling (9.12) we define a distance $W[r, r']$, on $Q_L \times Q_L$ or on $Q_{\infty,L} \times Q_{\infty,L}$ as

$$W[r, r'] = \big[w_m(r) + w_m(r')\big]\ \mathbf{1}_{\{r \neq r'\}} = W^{(1)}[x, x'], \quad w_m(r) := e_m(r)^{-1} \tag{9.16}$$

($x = r \cdot e_1$ above) and call

$$R_{n,r_0,r_0'} = \mathcal{E}_{r_0,r_0'}\big(W[(r(n), r'(n)]\big) \tag{9.17}$$

$R_{n,r_0,r_0'}$ is an upper bound for the Wasserstein distance between $K_m^n(r_0, \cdot)$ and $K_m^n(r_0', \cdot)$ relative to the distance (9.16).

Theorem 9.3. *There are positive constants L^*, c and ω so that for any of the above chains and any $L > L^*$, $n \geq 1$:*

$$R_{n,r_0,r_0'} \leq c e^{-(\omega/L^2)n} W[r_0, r_0'] \tag{9.18}$$

The proof of Theorem 9.3, which is postponed, uses Theorem 9.1 to reduce to the case when $x(\cdot) = x'(\cdot)$. Then the y coordinates (regarded on the whole axis and then reduced to $[-L/2, L/2]$ by reflections) perform independent random walks with increments having the same law (which depends on $\underline{x}$) till when they get at distance ≤ 1. By Lemma 9.2 they couple after a time k_0 with probability $\pi_0 > 0$ and the proof of Theorem 9.3 will then be concluded with an estimate of the probability of the time when two independent walks get closer than 1. We will see that such a probability is positive independently of L and of the starting points provided the time is proportional to L^2 (recall that the y coordinates are defined modulo reflections at $\pm nL/2$).

9.5 L^∞ bounds

The Markov chain K_m has an invariant probability measure $\mu(r)dr$ (recall the normalization of e_m and s_m in Subsection 9.1)

$$\mu(r) := s_m(r) e_m(r) = e_m(r)^2 p_m(r)^{-1}, \qquad \int \mu(r) K_m(r, r') dr = \mu(r') \tag{9.19}$$

Let $\psi \in L^\infty(Q_L)$ and $u = \psi w_m$. By the invariance of μ,

$$\int K_m^n(r_0, r')[u(r') - \mu(u)]dr' = \int \mu(r_0') \mathcal{E}_{r_0,r_0'}\big(u(r(n)) - u(r'(n))\big) dr_0' \tag{9.20}$$

We write $u(r) - u(r') = \dfrac{\tilde{u}(r)}{w_m(r)} w_m(r) - \dfrac{\tilde{u}(r')}{w_m(r')} w_m(r')$, $\tilde{u} = u - \mu(u)$ where, by an abuse of notation, $\mu(u) = \int \mu(r) u(r) dr$. Thus

$$\big|u(r) - u(r')\big| \leq \|\frac{\tilde{u}}{w_m}\|_\infty W[r, r'] \tag{9.21}$$

Hence by (9.18)

$$|\int K_m^n(r_0, r')[u(r') - \mu(u)]| \leq \|\frac{\tilde{u}}{w_m}\|_\infty c e^{-(\omega/L^2)n} \big(w_m(r_0) + C'\big) \tag{9.22}$$

The term with C' is obtained by writing $\displaystyle\int w_m(r)\mu(r) = \int e_m p_m^{-1}$ which, by (9.6) and (9.7), is bounded. Moreover, recalling that $u = \psi w_m$,

$$\frac{\tilde{u}(r)}{w_m(r)} = \tilde{\psi}(r), \quad \tilde{\psi} := \psi - e_m \langle \psi, e_m \rangle_m \tag{9.23}$$

By (9.22) and (9.8),

$$|\int A_m^n(r_0, r')\tilde{\psi}(r')| \le e_m(r_0)\{\|\tilde{\psi}\|_\infty c[\lambda_m e^{-(\omega/L^2)}]^n \big(w_m(r_0) + C'\big)\} \tag{9.24}$$

which using (9.6) proves:

Theorem 9.4. *There are positive constants L^*, c and ω so that for any of the above chains, any $L > L^*$, $n \ge 1$ and any ψ such that $\langle \psi, e_m \rangle_m = 0$,*

$$\|A_m^n \psi\|_\infty \le c'[\lambda_m e^{-(\omega/L^2)}]^n \|\psi\|_\infty \tag{9.25}$$

where $c' = c\left[1 + C'\|e_m\|_\infty\right]$ and for any $t > 0$, (recalling that $\Omega_m = A_m - 1$)

$$\|e^{\Omega_m t}\psi\|_\infty \le e^{-t} c' \|\psi\|_\infty \sum_{n=0}^{\infty} \frac{(\lambda_m e^{-(\omega/L^2)} t)^n}{n!} \le c' e^{-(\omega/2L^2)t} \|\psi\|_\infty \tag{9.26}$$

The last bound follows for L large enough by bounding $-1 + \lambda_m e^{-x} < |\lambda_m - 1| + e^{-x} - 1$, $e^{-x} - 1 \le -3x/4$ ($x > 0$ small enough), $|\lambda_m - 1| \le \omega/(4L^2)$, by (9.4) for L large enough.

9.6 A preliminary lemma

In the proof of Theorem 9.3 and in Section 10 as well we will use Lemma 9.5 below. With reference to (9.6), define for $m \in \mathcal{M}_L$

$$w_{m;a}(r) = w_m(r) e^{a|r\cdot e_1 - x_m|}, \quad a \ge 0 \tag{9.27}$$

$$k_m(n, r) := \min\{n, (|r \cdot e_1 - x_m| - (s+1))\mathbf{1}_{|r\cdot e_1 - x_m| - (s+1) > 0}\}, \tag{9.28}$$

where s is as in (9.5).

Lemma 9.5. *Let δ be as in (9.5). Then there exist positive constants L^*, a_0, c and $\delta_1 \in (\delta, 1)$ such that for any $0 < a < a_0$ and $L^* > L$ the following holds. If $m \in \mathcal{M}_L$ then for any $n \ge 1$*

$$\int K_m^n(r, r') w_{m;a}(r') dr' \le c\delta_1^{k_m(n,r)} w_{m;a}(r) \tag{9.29}$$

All the above coefficients can be taken uniformly in $m \in \mathcal{M}_L$.

Proof. Call $x = r \cdot e_1$ and $P_s(r,r') = K_m(r,r')\mathbf{1}_{|x-x_m|\ge s}$ and $=0$ otherwise; let E_r be the expectation of the Markov process with transition probability K_m starting from r so that

$$\int K_m^n(r,r')w_{m;a}(r')dr' = E_r\Big(w_{m;a}(r(n))\Big)$$

We decompose the expectation on the r.h.s. by using the sets $A_0 = \{r(\cdot) : |x(0) - x_m| \le s\}$,

$$\begin{aligned}
A_k &:= \Big\{r(\cdot) : |x(t) - x_m| > s, \;\; t = 0,..,k-1; \;\;\; |x(k) - x_m| \le s\Big\}, \;\; k \ge 1 \\
B_h &:= \Big\{r(\cdot) : |x(t) - x_m| > s, \;\; t = h,..,n; \;\;\; |x(h-1) - x_m| \le s\Big\}, \;\; h \ge 1 \\
C_n &= \Big\{r(\cdot) : |x(t) - x_m| > s, t = 0,..,n\Big\}, \;\; D_n = \Big\{r(\cdot) : |x(n) - x_m| \le s\Big\}
\end{aligned}$$

Then,

$$\begin{aligned}
\int K_m^n(r,r')w_{m;a}(r')dr' &= \int P_s^n(r,r')w_{m;a}(r')dr' \\
&+ \sum_{n\ge h>k} \int P_s^k(r,r_0)\mathbf{1}_{|x_0-x_m|\le s}E_{r_0}\Big(\mathbf{1}_{|r(h-k-1)\cdot e_1 - x_m|\le s}\phi_{n-h}(r(h-k))\Big)dr_0
\end{aligned} \tag{9.30}$$

where $\phi_l(r) := \displaystyle\int P_s^l(r,r')w_{m;a}(r')dr'$ for $l \in \mathbb{N}$. By (9.8), (9.5) and (9.7) there is c so that

$$\int P_s^l(r,r')w_{m;a}(r')dr' \le c[\lambda_m^{-1}e^a\delta]^l w_{m;a}(r) \tag{9.31}$$

because $|x'-x| \le l$. By (9.4) for L large and a small enough $\lambda_m^{-1}e^a\delta =: \delta_1 < 1$. Note that only for $k \ge k_m(r,n)$ the corresponding terms in (9.30) are nonzero, hence

$$\int K_m^n(r,r')w_{m;a}(r')dr' \le c \sum_{n\ge h>k_m(r,n)} \delta_1^{k+n-h} w_{m;a}(r),$$

and (9.29) then follows. By the last item in Subsection 9.1 all coefficients in the above bounds can be chosen uniformly in $m \in \mathcal{M}_L$ so that the proof of the lemma is complete.

□

9.7 Proof of Theorem 9.3

Given n call n_0 the integer part of $n/2$ and shorthand $\xi_n = (r(n), r'(n))$. Then

$$\mathcal{E}_{\xi_0}\big(W[\xi_n]\big) = \mathcal{E}_{\xi_0}\Big(\mathcal{E}_{\xi_{n_0}}\big(W[\xi_n]\big)\{\mathbf{1}_{x_{n_0}\ne x'_{n_0}} + \mathbf{1}_{x_{n_0}=x'_{n_0}}\}\Big) \tag{9.32}$$

When $x_{n_0} \ne x'_{n_0}$ we bound $W[\xi_n] \le w_m\big(x(n)\big) + w_m\big(x'(n)\big)$, namely we drop the characteristic function that $r(n) \ne r'(n)$ so that the expectations relative to $r(\cdot)$ and $r'(\cdot)$ uncouple. Then by Lemma 9.5 with $a=0$, $\mathcal{E}_{\xi_{n_0}}\big(W[\xi_n]\big)\mathbf{1}_{x_{n_0}\ne x'_{n_0}} \le cW^{(1)}[x_{n_0}, x'_{n_0}]$, hence by Theorem 9.1

$$\mathcal{E}_{\xi_0}\Big(\mathcal{E}_{\xi_{n_0}}\big(W[\xi_n]\big)\mathbf{1}_{x_{n_0}\ne x'_{n_0}}\Big) \le c'W[\xi_0]e^{-\omega^{(1)}n_0} \tag{9.33}$$

To bound $\mathcal{E}_{\xi_{n_0}}\big(W[\xi_n]\big)$ with $x_{n_0} = x'_{n_0}$ we recall from Theorem 9.1 and the definition of $\mathcal{P}_{\xi_0}$ that $x(i) = x'(i)$ for all $i \geq n_0$, so that

$$W[\xi_n] = 2w_m(x(n))\mathbf{1}_{y_{n_0}\neq y'_{n_0}} \tag{9.34}$$

We distinguish two cases:

First case, $|x_0 - x_m| > n$.

We bound $W[\xi_n] \leq 2w_m(x(n))$ and get

$$\mathcal{E}_{\xi_0}\Big(\mathcal{E}_{\xi_{n_0}}\big(W[\xi_n]\big)\mathbf{1}_{x_{n_0}\neq x'_{n_0}}\Big) \leq 2E_{r_0}\Big(w_m(r(n))\Big) \leq c\delta_1^n w_m(r_0) \tag{9.35}$$

having used Lemma 9.5 with $a = 0$ and with c above a suitable constant.

Second case, $|x_0 - x_m| \leq n$.

To decouple x from (y, y') we use Hölder. Let $p^{-1} + q^{-1} = 1$ then, supposing (for instance) $w_m(x_0) \leq w_m(x'_0)$

$$\mathcal{E}_{\xi_0}\Big(\mathcal{E}_{\xi_{n_0}}\big(W[\xi_n]\big)\mathbf{1}_{x_{n_0}=x'_{n_0}}\Big) \leq 2E_{r_0}\Big(w_m(r(n))^p\Big)^{1/p}\mathcal{P}_{\xi_0}\Big(\{x(n_0) = x'(n_0); y(n) \neq y'(n)\}\Big)^{1/q} \tag{9.36}$$

We use the second inequality in (9.6) to write

$$w_m(r)^p \leq [(c\sqrt{L})e^{\alpha''_m|x-x_m|}]^{p-1}\, e_m^{-1}(r) = (c\sqrt{L})^{p-1}w_{m;a}(r), \quad a = \alpha''_m(p-1) \tag{9.37}$$

Taking $p - 1 > 0$ small enough we can apply Lemma 9.5 and recalling that $|x_0 - x_m| \leq n$ we get

$$E_{r_0}\Big(w_m(x(n))^p\Big) \leq c'(\sqrt{L})^{p-1}w_{m;a}(r_0)\delta_1^{|x_0-x_m|} \leq c''(\sqrt{L})^{p-1}w_m(r_0) \tag{9.38}$$

The last inequality is valid for $p - 1 > 0$ small enough. Then

$$\Big\{E_{r_0}\Big(w_m(x(n))^p\Big)\Big\}^{1/p} \leq C(\sqrt{L})^{1-1/p}w_m(r_0)e_m(r_0)^{1-1/p} \leq C'w_m(r_0) \tag{9.39}$$

having used the first inequality in (9.6).

Conclusions.

In the first case, $|x_0 - x_m| > n$, the bound (9.35) concludes the proof, while in the second case we need to prove that $\mathcal{P}_{\xi_0}\Big(\{x(n_0) = x'(n_0); y(n) \neq y'(n)\}\Big)$ is exponentially small, which is done in the next subsection.

9.8 Coupling the y coordinates

In this subsection we suppose $r_0 = (x_0, y_0)$ and $r'_0 = (x_0, y'_0)$, namely that the initial x coordinates are the same. This is indeed what happens at time n_0 in the case we have to study and, to simplify notation, we have just reset time n_0 equal to 0. We will prove that at a time cL^2 the y coordinates are the same with probability not smaller than a number $\pi > 0$, uniformly in ξ_0 and L. By iteration this will prove that (shorthanding $\xi_0 = (r_0, r_0')$)

$$\mathcal{P}_{\xi_0}\Big(\{y(n) \neq y'(n)\}\Big) \leq (1-\pi)e^{-n/(cL^2)} \tag{9.40}$$

which inserted in (9.36) will conclude the proof of (9.18). Let

$$\tau = \inf\{n \in \mathbb{N} : |y(n) - y'(n)| \leq 1\} \tag{9.41}$$

We will prove that

Proposition 9.6. *There are $k_1 > 0$ and $\pi_1 > 0$ so that*

$$\inf_{x_0, y_0, y_0'} \mathcal{P}_{\xi_0}\Big(\{\tau \leq k_1 L^2\}\Big) \geq \pi_1 \tag{9.42}$$

Proposition 9.6 and Lemma 9.2 prove (9.40) with $\pi = \pi_0\pi_1$ and $cL^2 > k_0 + k_1 L^2$. In the sequel we will prove Proposition 9.6. Since $y(n)$ and $y'(n)$ are independent of each other till τ, we may as well and will in the sequel consider $\mathcal{P}_{\xi_0}$ defined so that $y(n)$ and $y'(n)$ are independent of each other at all times. Shorthand

$$Z_n = [y'(n) - y'(0)] - [y(n) - y(0)]$$

and call

$$\sigma := \inf_x \int P_(x, x') q_{x,x'}(z) z^2 \, dx' dz > 0$$

Positivity follows because there is c so that $\dfrac{e_m(r')}{e_m(r)} \leq c$ for any $|(r'-r)\cdot e_1| \leq 2$, see [12].

Lemma 9.7. *There is c so that for any $n \geq 1$ for any ξ_0 with $x_0 = x_0'$,*

$$\mathcal{E}_{\xi_0}\big(Z_n\big) = 0, \quad \mathcal{E}_{\xi_0}\big(Z_n^2\big) \geq 2\sigma\, n, \quad \mathcal{E}_{\xi_0}\big(Z_n^4\big) \leq c\, n^2 \tag{9.43}$$

Proof. We write $z_n = Z_n - Z_{n-1}$, $n \geq 1$, so that $Z_n = z_1 + \cdots + z_n$. For any k, n with $k < n$ and any measurable function f on $\mathbb{R}$, using that $J(0, r)$ depends on $|r|$ and $q_{x,x'}(z) = q_{x,x'}(-z)$,

$$\begin{aligned}\mathcal{E}_{\xi_0}\big(f(z_k) z_n\big) &= \mathcal{E}_{\xi_0}\Big(f(z_k) \int (u'-u) q_{x_{n-1},x}(u) q_{x_{n-1},x}(u') P(x_{n-1}, x) du du' dx\Big) \\ &= 0\end{aligned}$$

hence the first equality in (9.43) after setting $f = 1$. Analogously, recalling also the definition of σ,

$$\begin{aligned}\mathcal{E}_{\xi_0}\big(z_n^2\big) &= \mathcal{E}_{r_0, r_0'}\Big(\int (u'-u)^2 q_{x_{n-1},x}(u) q_{x_{n-1},x}(u') P(x_{n-1}, x) du du' dx\Big) \\ &= \mathcal{E}_{\xi_0}\Big(\int (u'^2 + u^2) q_{x_{n-1},x}(u) q_{x_{n-1},x}(u') P(x_{n-1}, x) du du' dx\Big) \geq 2\sigma\end{aligned}$$

hence the lower bound in (9.43). The upper bound in (9.43) is derived by noticing that by symmetry

$$\mathcal{E}_{\xi_0}\big(Z_n^4\big) = \mathcal{E}_{\xi_0}\Big(\sum_{j\le n} z_j^4 + 12\sum_{i<j\le n} z_i^2 z_j^2\Big) \le cn^2$$

$\square$

Proof of Proposition 9.6.
We have $\{\tau \le n\} \supseteq \{|Z_n| > L, \mathrm{sign}(Z_n) \ne \mathrm{sign}(y'(0) - y(0))\}$ because $y'(k) - y(k)$ jumps at most by 2. By symmetry,

$$\mathcal{P}_{\xi_0}\big(|Z_n| > L, \mathrm{sign}(Z_n) \ne \mathrm{sign}(y'_0 - y_0)\big) = \frac{1}{2}\mathcal{P}_{\xi_0}\big(|Z_n| > L\big)$$

so that

$$\mathcal{P}_{\xi_0}\big(\tau \le n\big) \ge \frac{1}{2}\mathcal{P}_{\xi_0}\big(|Z_n| > L\big)$$

We have

$$\begin{aligned}\mathcal{E}_{\xi_0}(Z_n^2) &= \mathcal{E}_{\xi_0}(Z_n^2 1_{|Z_n|\le L}) + \mathcal{E}_{\xi_0}(Z_n^2 \mathbf{1}_{|Z_n|>L})\\ &\le L^2 + \mathcal{E}_{\xi_0}(Z_n^4)^{1/2}\mathcal{P}_{\xi_0}(|Z_n| > L)^{1/2}\end{aligned}$$

Moreover, using (9.43) and the choice of σn, we obtain that for $n > L^2\sigma^{-1}$

$$\mathcal{P}_{\xi_0}(|Z_n| > L)^{1/2} \ge \frac{2\sigma n - L^2}{(cn^2)^{1/2}} \ge \frac{\sigma}{\sqrt{c}},$$

hence (9.42).

$\square$

10 Spectral gap

We regard here $\Omega_m = A_m - 1$, $m \in \mathcal{M}_L$, as an operator on the weighted L^2-spaces $L^2(Q_L, p_m^{-1}dr)$ or on $L^2(Q_{\infty,L}, p^{-1}dr)$ if $m = \bar{m}^e$ and denote by $\langle\cdot,\cdot\rangle_m$ the scalar product. On such spaces Ω_m is self-adjoint, it has eigenvalue $\lambda_m - 1$ with eigenvector the planar function e_m. We will prove here that:

Theorem 10.1. *There is $a > 0$ so that for all L large enough,*

$$\sup_{f:\langle f, e_m\rangle_m = 0} \frac{\langle f, \Omega_m f\rangle_m}{\langle f, f\rangle_m} \le -\frac{a}{L^2} \tag{10.1}$$

A crucial point in the proof of Theorem 10.1, which is given in the remaining of this section, is that the operator Ω_m is self-adjoint. The mere existence of a spectral gap then follows from Weyl's theorem by the same argument used in [15] for the $d = 1$ case. The argument is however abstract and does not allow to determine the dependence on L of the spectral gap. Notice on the other hand that for the Allen-Cahn equation $m_t = \Delta m - V'(m)$ the question trivializes because the linearized operator is a sum of two commuting operators, $\{\frac{d^2}{dx^2} - V''(\bar{m}(x))\} + \frac{d^2}{dy^2}$, so that it is the non local nature of the interaction which is behind all difficulties we find here.

Notation. To simplify notation we fix $m \in \mathcal{M}_L$ and shorthand $\langle \cdot, \cdot \rangle$ for $\langle \cdot, \cdot \rangle_m$. We call M the self adjoint operator equal to A_m on $\{f : \langle f, e_m \rangle = 0\}$, while $M e_m = 0$. We denote by $\|M\|$ its norm:

$$\|M\| = \sup_{f \neq 0} \frac{|\langle f, Mf \rangle|}{\langle f, f \rangle} = \sup_{f : \langle f, e_m \rangle = 0} \frac{|\langle f, A_m f \rangle|}{\langle f, f \rangle} \tag{10.2}$$

Lemma 10.2. *If there is $a > 0$ so that for all L large enough,*

$$\log \|M\| \leq -\frac{2a}{L^2} \tag{10.3}$$

then (10.1) *holds.*

Proof. If (10.3) holds, then

$$\sup_{f : \langle f, e_m \rangle = 0} \frac{\langle f, (A_m - 1) f \rangle}{\langle f, f \rangle} \leq -1 + \|M\| \leq -1 + e^{-2a/L^2} \leq -\frac{a}{L^2}$$

for L large enough.

□

To bound $\log \|M\|$ we use the spectral theorem:

Proposition 10.3.

$$\log \|M\| = \sup_{f \neq 0, \|f\|_\infty < \infty} \liminf_{n \to \infty} \frac{1}{2n} \log \left\{ \frac{\langle f, M^{2n} f \rangle}{\langle f, f \rangle} \right\} \tag{10.4}$$

Equality holds with limsup as well.

Proof. (10.4) is a direct consequence of the spectral theorem for self-adjoint operators, as we are going to see. Let $\langle f, f \rangle = 1$ and n even. Since $\langle f, M^n f \rangle \leq \|M\|^n$,

$$\limsup_{n \to \infty} \frac{1}{2n} \log \langle f, M^{2n} f \rangle \leq \log \|M\| \tag{10.5}$$

For the reverse inequality we use the spectral theorem to say that for any $0 \le \lambda < \|M\|$ there is a non zero orthogonal projection P_λ which commutes with M and such that for any $n \ge 1$, $M^{2n} P_\lambda \ge \lambda^{2n} P_\lambda$. Since L^∞ is dense in L^2, given any $0 \le \lambda < \|M\|$ there are f and R such that $\|f\|_\infty < R$ and $P_\lambda f \ne 0$. Then writing f in $\langle f, M^{2n} f\rangle$ as $f = P_\lambda f + (1-P_\lambda) f$ and expanding,

$$\langle f, M^{2n} f\rangle \ge \langle P_\lambda f, M^{2n} P_\lambda f\rangle \ge \lambda^{2n} \langle P_\lambda f, P_\lambda f\rangle, \quad \langle P_\lambda f, P_\lambda f\rangle > 0,$$

the first inequality using that M and P_λ commute, so that the mixed terms vanish. Hence

$$\sup_{f:\langle f,f\rangle=1,\|f\|_\infty<\infty} \liminf_{n\to\infty} \frac{1}{2n} \log\langle f, M^{2n} f\rangle \ge \log\lambda$$

thus

$$\sup_{f:\langle f,f\rangle=1,\|f\|_\infty<\infty} \liminf_{n\to\infty} \frac{1}{2n} \log\langle f, M^{2n} f\rangle_m \ge \log\|M\|$$

which, together with (10.5), yields (10.4).

$\square$

Proof of (10.3).

We consider f such that $\langle f, e_m\rangle = 0$, $\langle f, f\rangle = 1$ and $\|f\|_\infty \le R$ and we look for upper bounds on $\langle f, M^n f\rangle$, n even. We define $g = f/e_m$. Recalling (9.19), $\int g\mu = \langle f, e_m\rangle = 0$. By (9.8) and shorthanding K for K_m,

$$\begin{aligned}\lambda_m^{-n}\langle f, M^n f\rangle &= \int g[K^n g]\mu\, dr = \int g(r) g(r')[K^n(r,r') - \mu(r')]\, dr' \mu(r) dr \\ &= \int g(r) g(r')[K^n(r,r') - K^n(r'',r')]\, \mu(r'')\, dr'' dr' \mu(r) dr\end{aligned}$$

where we have used the invariance of μ with respect to K, see (9.19). Calling $Q^n_{r_0,r_0'}(r,r')$ the kernel $Q^n(r_0, r_0'; r, r')$, see subsec:n11.2,

$$\int g(r')[K^n(r,r') - K^n(r'',r')] dr' = \int [g(r_1) - g(r_2)]\, Q^n_{r,r''}(dr_1 dr_2). \tag{10.6}$$

With such notation,

$$\lambda_m^{-n}\langle f, M^n f\rangle = \int g(r)[g(r_1) - g(r_2)]\, Q^n_{r,r''}(dr_1 dr_2)\mu(r'')\, dr'' \mu(r) dr$$

With reference to (9.5)-(9.6), we split the domain of integration into the two sets $\{|x - x_m| \le n, |x'' - x_m| \le n\}$ and its complement, denoting by x and x'' the x coordinates of r and r''. We call

$$I := \int_{\{|x-x_m|\le n, |x''-x_m|\le n\}} g(r)[g(r_1) - g(r_2)]\, Q^n_{r,r''}(dr_1 dr_2)\mu(r'')\, dr'' \mu(r) dr \tag{10.7}$$

Recalling that $g = f/e_m$, $\|f\|_\infty \le R$ and with W defined in (9.16), proceeding as in (9.21),

$$\begin{aligned} I &\le R^2 \int_{\{|x-x_m|\le n, |x''-x_m|\le n\}} \frac{W[r_1,r_2]}{e_m(r)} Q^n_{r,r''}(dr_1 dr_2)\mu(r'')\, dr''\mu(r)dr \\ &\le cR^2 e^{-(\omega/L^2)n} \int_{\{|x-x_m|\le n, |x''-x_m|\le n\}} \frac{W[r,r'']}{e_m(r)} \mu(r'')\, dr''\mu(r)dr \end{aligned}$$

where we have used (9.18). By the definition of $W[r,r'']$ we have for $r \neq r''$

$$\frac{W[r,r'']}{e_m(r)} = \frac{1}{e_m^2(r)} + \frac{1}{e_m(r'')e_m(r)}$$

hence

$$\int_{\{|x-x_m|\le n, |x''-x_m|\le n\}} \frac{W[r,r'']}{e_m(r)} \mu(r'')\, dr''\mu(r)dr \le \int_{\{|x-x_m|\le n\}} p_m^{-1} dr + \left[\int e_m p_m^{-1} dr\right]^2$$

By (9.6) and (9.7), $\int e_m \le c\sqrt{L}$, so that for a suitable constant c

$$I \le cR^2 e^{-(\omega/L^2)n} nL \tag{10.8}$$

Note that in the case Q_L the better bound

$$\langle f, M^n f\rangle \le cR^2 e^{-(\omega/L^2)n} L^2$$

follows directly form the spectral gap in L^∞, see (9.25). For the channel Q_L, however, the entire analysis presented in this section is necessary.
We will see below that all other contributions to $\lambda_m^{-n}\langle f, M^n f\rangle$ are smaller (for L large enough). In the complement of $\{|x - x_m| \le n, |x'' - x_m| \le n\}$ we use (10.6) backwards to rewrite the integrals in terms of $g(r')[K^n(r, r') - K^n(r'', r')]$. We will not exploit the minus sign and bound separately the two terms in the difference. We start with the term

$$\begin{aligned} Z &:= \int_{\{|x-x_m|\ge n; |x''-x_m|\le n\}} g(r)g(r')K^n(r,r')\,\mu(r'')\, dr''dr'\mu(r)dr \\ &\le \int_{\{|x-x_m|\ge n\}} |g(r)g(r')|K^n(r,r')\, dr'\mu(r)dr =: \hat{Z} \end{aligned} \tag{10.9}$$

Call $K_s(r,r') = K(r,r')$ if $|x - x_m| \ge s$ and 0 otherwise. Then $\hat{Z} = \sum_{h=0}^{n} Z_h$, where

$$\begin{aligned} Z_0 &= \int_{\{|x-x_m|\ge n\}} |g(r)g(r')|K^n_s(r,r')\, dr'\mu(r)dr \\ Z_h &= \int_{\{|x-x_m|\ge n, |x''-x_m|\le s\}} |g(r)g(r')|K^{n-h}_s(r,r'')K^h(r'',r')\, dr''dr'\mu(r)dr \end{aligned} \tag{10.10}$$

To bound Z_0 we use (9.5) to write

$$K_s(r,r') \le \delta J^{\rm neum}(r,r')\frac{e_m(r')}{\lambda_m e_m(r)}$$

and get, with $\|f\|_2^2 = \displaystyle\int f^2$,

$$\begin{aligned} Z_0 &\le \lambda_m^{-n}\delta^{n-1}\int |f(r)f(r')|(J^{\rm neum})^n(r,r')\,dr'dr \\ &\le \|f\|_2^2\lambda_m^{-n}\delta^{n-1} \end{aligned} \tag{10.11}$$

$$\begin{aligned} Z_h &\le \gamma_h R\int_{\{|x-x_m|\ge n,|x''-x_m|\le s\}} |g(r)|K_s^{n-h}(r,r'')\,dr''\mu(r)dr \le \gamma_h R^2\int_{\{|x-x_m|\ge n\}} e_m(r) \\ \gamma_h &:= \sup_{|x''-x_m|\le s}\int e_m(x')^{-1}K^h(r'',r')dr' \le c\sqrt{L} \end{aligned} \tag{10.12}$$

The last inequality follows from Lemma 9.5 and (9.6), with $c = c(s)$ a constant independent of h. By (9.6),

$$\sup_{|x-x_m|\ge n}\int e_m(r) \le c\sqrt{L}e^{-(\alpha_m/2)n} \tag{10.13}$$

so that $Z_h \le cR^2Le^{-(\alpha_m/2)n}$. In conclusion, there is c so that

$$Z \le c\Big(\|f\|_2^2\lambda_m^{-n}\delta^n + nR^2Le^{-(\alpha_m/2)n}\Big) \tag{10.14}$$

The next term we examine is

$$\begin{aligned} B &:= \int_{\{|x-x_m|\ge n;|x''-x_m|\le n\}} |g(r)g(r')|K^n(r'',r')\,\mu(r'')\,dr''dr'\mu(r)dr \\ &\le cR^2\sqrt{L}e^{-(\alpha_m/2)n}\int_{\{|x''-x_m|\le n\}}\frac{K^n(r'',r')}{e_m(x')}\,\mu(r'')\,dr''dr' \\ &\le cR^2\sqrt{L}e^{-(\alpha_m/2)n}\int\frac{\mu(r')}{e_m(x')}dr' \le c'R^2Le^{-(\alpha_m/2)n} \end{aligned} \tag{10.15}$$

where we have used (10.13).
The next term is

$$C := \int_{\{|x-x_m|\le n;|x''-x_m|\ge n\}} |g(r)g(r')|K^n(r'',r')\,\mu(r'')\,dr''dr'\mu(r)dr$$

which is equal to Z, see (10.9). The next one is

$$D := \int_{\{|x-x_m|\le n;|x''-x_m|\ge n\}} |g(r)g(r')|K^n(r,r')\,\mu(r'')\,dr''dr'\mu(r)dr$$

which is equal to B, see (10.15). The last two terms are G and H :

$$G := \int_{\{|x-x_m|\geq n; |x''-x_m|\geq n\}} |g(r)g(r')| K^n(r,r')\, \mu(r'')\, dr'' dr' \mu(r) dr$$
$$\leq c e^{-2\alpha_m n} \int_{\{|x-x_m|\geq n\}} |g(r)g(r')| K^n(r,r')\, dr' \mu(r) dr = c e^{-2\alpha_m n} \hat{Z}$$

where $\hat{Z}$ is defined in (10.9). By (10.13) and (9.19),

$$H := \int_{\{|x-x_m|\geq n; |x''-x_m|\geq n\}} |g(r)g(r')| K^n(r'',r')\, \mu(r'')\, dr'' dr' \mu(r) dr$$
$$\leq c R^2 \sqrt{L} e^{-(\alpha_m/2)n} \int \frac{K^n(r'',r')}{e_m(r')}\, \mu(r'')\, dr'' dr' \leq c' R^2 L e^{-(\alpha_m/2)n}$$

In conclusion we have proved that there is a constant c so that

$$\langle f, M^n f\rangle \leq c \lambda_m^n \left(R^2 e^{-(\omega/L^2)n} n L + \|f\|^2 \lambda_m^{-n} \delta^n + (n+1) R^2 L e^{-(\alpha_m/2)n} \right).$$

Hence for L large enough,

$$\limsup_{n\to\infty} \frac{1}{2n} \log\langle f, M^{2n} f\rangle \leq \log \lambda_m - \frac{\omega}{L^2}$$

which by (9.4) yields and Proposition 10.3 yields (10.3).

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