

Max-Planck-Institut
für Mathematik
in den Naturwissenschaften
Leipzig

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group branchings

by

Bertfried Fauser, Peter D. Jarvis, and Ronald C. King

Preprint no.: 77

2005



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Bertfried Fauser

*Max Planck Institut für Mathematik in den Naturwissenschaften, Inselstr. 22-26,
D-04103 Leipzig, Germany; fauser@mis.mpg.de*

Peter D. Jarvis

*University of Tasmania, School of Mathematics and Physics, GPO Box 252-21,
7001 Hobart, TAS, Australia; Peter.Jarvis@utas.edu.au*

Ronald C. King

*School of Mathematics, University of Southampton, Southampton SO17 1BJ, Eng-
land; R.C.King@maths.soton.ac.uk*

Abstract. We describe a Hopf algebraic approach to the Grothendieck ring of representations of subgroups H_π of the general linear group $\mathrm{GL}(n)$ which stabilize a tensor of Young symmetry $\{\pi\}$. It turns out that the representation ring of the subgroup can be described as a Hopf algebra twist, with a 2-cocycle derived from the Cauchy kernel 2-cocycle using plethysms. Due to Schur-Weyl duality we also need to employ the coproduct of the inner multiplication. A detailed analysis including combinatorial proofs for our results can be found in [4]. In this paper we focus on the Hopf algebraic treatment, and a more formal approach to representation rings and symmetric functions.

1. Group representation rings

We are interested in the representation rings of $\mathrm{GL}(n)$ and its subgroups described as stabilizers of certain elements T^π of Young symmetry π . A matrix representation $\rho : \mathrm{GL}(n) \rightarrow \mathrm{GL}(m)$, $m \geq n$ is polynomial if the entries of $\rho(g) \in \mathrm{GL}(m)$ are polynomials in the entries of $g \in \mathrm{GL}(n)$. The character of the representation ρ is the central function $\chi_\rho : \mathrm{GL}(n) \rightarrow \mathbb{C}$, $\chi_\rho(g) = \mathrm{tr}(\rho(g))$. Representations form an Abelian semigroup under the direct sum $V^\lambda \oplus V^\mu$, which is completed to form the Grothendieck group $\mathcal{R}^n = \mathcal{R}_{\mathrm{GL}(n)}(\{V^\lambda\}, \oplus)$ using virtual representations $-V^\lambda$ [9]. The tensor product $V^\lambda \otimes V^\mu = \oplus^\nu C_{\lambda\mu}^\nu V^\nu$ turns this structure into a ring $\mathcal{R}^n = \mathcal{R}_{\mathrm{GL}(n)}(\{V^\lambda\}, \oplus, \otimes)$. We proceed to the

inductive limit $\mathcal{R}_{\mathrm{GL}} = \lim_{\leftarrow} R^n$ since finitely generated representation rings develop syzygies while the limit ring is free. Finite examples thus require establishing these syzygies by so-called modification rules.

We follow [15]. $\mathrm{GL}(n)$ acts by conjugation on the Lie algebra $\mathfrak{gl}(n)$ of all $n \times n$ -matrices, hence acts on the invariant ring $\mathrm{Pol}(\mathfrak{gl}(n))^{\mathrm{GL}(n)}$ with integer coefficients (in Lie theory real coefficients). Under some topological restrictions one can identify the characters $\chi \in \mathcal{R}$ bijectively with elements in $\mathrm{Pol}(\mathfrak{gl}(n))^{\mathrm{GL}(n)}$ via the isomorphism $\phi : \mathcal{R}_{\mathrm{GL}(n)} \rightarrow \mathrm{Pol}(\mathfrak{gl}(n))^{\mathrm{GL}(n)}$ of class functions.

A particular basis of the representation ring is given by equivalence classes of irreducible representations V^λ , labelled by integer partitions λ (see below). For each partition label λ there exists a Schur map (Schur endofunctor on $\mathbf{FinVect}_{\mathbb{C}}$, [11]) mapping the vector space $V = \mathbb{C}^n$ the corresponding Schur module V^λ , a highest weight $\mathrm{GL}(n)$ -module in Lie theory. The character of V^λ is the Schur polynomial s_λ that is polynomial in the eigenvalues of $g \in \mathrm{GL}(n)$.

Let $\dim V = n$. On any tensor product $W = \otimes^p V$ we have the left $\mathrm{GL}(n)$ action and right action of the symmetric group \mathcal{S}_p . As bimodule we have $W = \otimes^p V = \bigoplus^\lambda V^\lambda \otimes S^\lambda$ by Schur-Weyl duality [17, 6]. One considers Young symmetrizers $Y_\lambda = c_\lambda r_\lambda$ where c_λ is the column anti-symmetrizer of the tableau of shape λ and r_λ is the row symmetrizer. The Y_λ are idempotents and reduce W into irreducible parts with respect to $\mathrm{GL}(n) \times \mathcal{S}_p$. The Schur module V^λ is the image of the identity morphisms $W \rightarrow W$ defined as right multiplication by Y_λ . This twofold nature of Schur polynomials will cause the remarkable self duality of the Hopf algebra studied in the next paragraph.

2. Symmetric functions

2.1. THE HOPF ALGEBRA OF SYMMETRIC FUNCTIONS

An introduction to symmetric functions can be found e.g. in [12, 7, 11], the well known Hopf algebra structure is discussed in [5, 19, 16, 8]. Here we focus on the Hopf algebraic aspects of symmetric functions related to representation rings of the GL groups and their Weyl groups $\lim_{\leftarrow} \oplus \mathcal{S}_p$ using the isomorphism between $\mathrm{GL}(n)$ and \mathcal{S}_p representation rings and $\Lambda = \mathbb{Z}[x_1, x_2, \dots]^{\mathcal{S}} = \lim_{\leftarrow} \oplus^n \mathbb{Z}[x_1, \dots, x_n]^{\mathcal{S}^n}$ of symmetric functions in infinitely many variables.

Schur functions s_λ , or in Littlewood's bracket notation $\{\lambda\}$, are indexed by integer partitions $\lambda = (\lambda_1, \dots, \lambda_k) = [1^{r_1}, \dots, p^{r_p}]$, where the

λ_i , ordered by magnitude $\lambda_i \geq \lambda_{i+1}$, are called parts, the r_i are multiplicities, and are conveniently displayed by Ferrers diagrams (also called Young diagrams). Schur functions are given by $s_\lambda(x) = \sum_{T \in ST^\lambda} x^{\text{wgt}(T)}$, where the sum is over all tableaux (fillings) T belonging to the set ST^λ of semi-standard tableaux (column strict, row semistrict) of shape λ . Each summand is a monomial in the variables x_1, x_2, \dots, x_n of degree $n = |\lambda| = \sum \lambda_i$. The module underlying Λ is spanned by \mathbb{Z} -linear combinations of Schur functions (irreducible representations). To establish the ring structure we introduce the *outer multiplication*

$$V^\lambda \otimes V^\mu = \bigoplus^\nu C_{\lambda\mu}^\nu V^\nu \quad \stackrel{\phi}{\cong} \quad s_\lambda(x) \cdot s_\mu(x) = \sum_\nu C_{\lambda\mu}^\nu s_\nu(x). \quad (1)$$

Where the nonnegative integer constants $C_{\lambda\mu}^\nu$ are the famous Littlewood-Richardson coefficients determined e.g. combinatorially.

Schur functions are important because they encode characters of irreducible representations of the $\text{GL}(n)$ groups which by Schur's lemma decompose into isoclasses. The Schur-Hall scalar product encodes this fact letting Schur functions be orthogonal by definition $\langle \cdot | \cdot \rangle : \Lambda \otimes_{\mathbb{Z}} \Lambda \rightarrow \mathbb{Z}$, $\langle s_\lambda | s_\mu \rangle = \delta_{\lambda,\mu}$. This implies an elementwise identification of the module underlying Λ with the dual module $\Lambda^* = \text{Hom}(\Lambda, \mathbb{Z})$. Λ^* is *a priori* not an algebra! However, inspection of classical results shows [3] that we can introduce the *same* outer product on the dual Λ^* reflecting the Frobenius reciprocity. Using the Milnor-Moore theorem this induces a coalgebraic structure on Λ fulfilling the axioms of a Hopf algebra [3]. Schur functions s_λ have a life as characters of $\text{GL}(n)$ -modules V^λ and through the Schur-Weyl duality mentioned previously are also associated with irreducible representations of \mathcal{S}_p , a remarkable incidence. With $f, h, g \in \Lambda$ we define the outer coproduct $\Delta : \Lambda \rightarrow \Lambda \otimes \Lambda$ as

$$\langle \Delta(f) | g \otimes h \rangle := \langle f | g \cdot h \rangle = \langle f_{(1)} | g \rangle \langle f_{(2)} | h \rangle \quad \Delta(s_\lambda) = \sum_{\eta, \xi} C_{\eta\xi}^\lambda s_\eta \otimes s_\xi \quad (2)$$

where we have introduced Sweedler indices $\Delta(f) = \sum_{(f)} f_{(1)} \otimes f_{(2)}$ [14] neatly to encode the double sum. The unit of the outer product is 1, the constant Schur function $s_{(0)}$, given by injection of the underlying ring $\mathbb{Z} \xrightarrow{\eta} \Lambda$. The coproduct has a counit $\Lambda \xrightarrow{\epsilon} \mathbb{Z}$ given as $\epsilon(s_\lambda) = \delta_{\lambda,(0)}$. Product and coproduct fulfill the homomorphism property

$$\Delta(f \cdot g) = (fg)_{(1)} \otimes (fg)_{(2)} = f_{(1)} \cdot g_{(1)} \otimes f_{(2)} \cdot g_{(2)} = \Delta(f)\Delta(g) \quad (3)$$

showing Λ 's bialgebra structure. The Hopf algebra Λ admits an antipode $S(s_\lambda) = (-)^{|\lambda|} s_{\lambda'}$ where $|\lambda| = \sum \lambda_i$, and λ' is the partition conjugate to λ (that is, the partition associated with the Young diagram specified by λ reflected through its diagonal).

We define the adjoint with respect to the Schur-Hall scalar product, called skew operation, of the outer multiplication

$$\begin{aligned} \langle s_\mu^\perp \cdot s_\lambda \mid s_\pi \rangle &= \langle s_{\lambda/\mu} \mid s_\pi \rangle = \langle s_\lambda \mid s_\mu \cdot s_\pi \rangle \\ s_\mu^\perp(x) \cdot s_\nu &\equiv s_{\nu/\mu}(x) = \sum_\lambda C_{\lambda\mu}^\nu s_\lambda(x). \end{aligned} \quad (4)$$

Through their association with \mathcal{S}_p -modules, Schur functions inherit a second, *inner*, product determined by the product $\chi^\lambda \chi^\mu = \sum_\nu \gamma_{\lambda\mu}^\nu \chi^\nu$ of characters of \mathcal{S}_p -modules, where λ , μ and ν are all partitions of p . We denote this product as $\star : \mathcal{S}_p \times \mathcal{S}_p \rightarrow \mathcal{S}_p$ and it can be dualized using the Schur-Hall scalar product as $\langle \delta(f) \mid g \otimes h \rangle := \langle f \mid g \star h \rangle$, $\delta(f) = \sum_{[f]} f_{[1]} \otimes f_{[2]}$. We should not be astonished to note that the convolution (\star, δ) is neither Hopf nor even a bialgebra [3]. However, we will need the coproduct δ in a prominent place!

2.2. PLETHYSM OR COMPOSITION

Schur maps (functors) enjoy composition $s_\lambda \otimes s_\mu \equiv s_\mu[s_\lambda]$ (Note the index order!) A linear map $f : V \rightarrow U$ determines a linear map $s_\lambda(f) : s_\lambda(V) \rightarrow s_\lambda(U)$ being functorial, i.e. $s_\lambda(f_1 \circ f_2) = s_\lambda(f_1) \circ s_\lambda(f_2)$, and $s_\lambda(\text{Id}_V) = \text{Id}_{s_\lambda(V)}$. We use the symbol $\underline{\otimes}$ to distinguish plethysms from ordinary tensor products. In terms of Schur functions the plethysm, or composition, of the Schur function s_μ with the Schur function s_λ , is given by $s_\mu[s_\lambda](x) = s_\mu(y) = \sum_{T \in ST^\mu} y^{\text{wgt}(T)}$ where the entries in each tableau are now taken from the set $\{y_i \mid i = 1, 2, \dots, m\}$ of monomials of the Schur function $s_\lambda(x)$.

2.1 Example:

Consider $s_{(2)}[s_{(1^2)}](x_1, \dots, x_4)$. Expand $s_{(1^2)}$ as $s_{(1^2)}(x_1, \dots, x_4) = x_1x_2 + x_1x_3 + x_1x_4 + x_2x_3 + x_2x_4 + x_3x_4 = y_1 + y_2 + y_3 + y_4 + y_5 + y_6$ which leads to the expansion of the composition $s_{(2)}[s_{(1^2)}]$ as

$$\begin{aligned} s_{(2)}[s_{(1^2)}](x_1, \dots, x_4) &= s_{(2)}(y_1, \dots, y_6) = s_{(2)}(x_1x_2, \dots, x_3x_4) \\ &= y_1^2 + \dots + y_6^2 + y_1y_2 + \dots + y_5y_6 \\ &= x_1^2x_2^2 + \dots + x_3^3x_4^2 + x_1^2x_2x_3 + \dots + x_4^2x_2x_3 + 3x_1x_2x_3x_4 \\ &= s_{(2^2)}(x_1, \dots, x_4) + s_{(1^4)}(x_1, \dots, x_4) \end{aligned} \quad (5)$$

The problem of the evaluation of the plethysm is to expand $s_{(2)}(y)$ in the Schur function basis $s_\nu(x)$ with ν a partition of $4 = |\{2\}| + |\{1^2\}|$. ■

Plethysm is tied to representation theory: Consider two groups $\mathrm{GL}(m)$, $\mathrm{GL}(n)$, with $m > n$. Let a Schur function $\{\lambda\}$ represent the character of a m -dimensional irrep of $\mathrm{GL}(n)$ surjectively embedded in the fundamental representation of the group $\mathrm{GL}(m)$ whose character is $\{1\}$. The branching process $\mathrm{GL}(m) \rightarrow \mathrm{GL}(n)$ is then described by the injective map $\{1\} \rightarrow \{\lambda\}$ which leads to the general formula explaining physical origin of the symbol $\underline{\otimes}$

$$\mathrm{GL}(m) \rightarrow \mathrm{GL}(n) : \quad \{\mu\} \rightarrow \{\lambda\} \underline{\otimes} \{\mu\} \quad (6)$$

The connection with the previous definition of a plethysm as a composition comes about because the $\mathrm{GL}(n)$ character $\{\lambda\}$ is nothing other than the Schur function $s_\lambda(x)$, with a suitable identification of x . Its dimension m , obtained by setting all $x_i = 1$, is just the number of monomials y_i in $s_\lambda(x)$, and $s_\mu(y)$ is the corresponding $\mathrm{GL}(m)$ character $\{\mu\}$. In terms of modules, let V^λ be the $\mathrm{GL}(n)$ -module with character $\{\lambda\}$ and dimension m . This module may be identified with the defining $\mathrm{GL}(m)$ -module V on which $\mathrm{GL}(m)$ acts naturally. Then the plethysm $\{\lambda\} \underline{\otimes} \{\mu\}$ arises as the character of the $\mathrm{GL}(m)$ -module $V^\mu = (V^\lambda)^\mu$ viewed as a $\mathrm{GL}(n)$ -module. As a result of this interpretation it is sometimes convenient to adopt a suggestive exponential notation for plethysms $\{\lambda\} \underline{\otimes} \{\mu\} = \{\lambda\}^{\underline{\otimes} \{\mu\}}$.

2.3. SCHUR FUNCTION SERIES

Littlewood [10] introduced formal power series $\Lambda[[t]]$ (S -function series) to encode the process of group branching [18], i.e. passing over to subgroup characters or taking ‘traces’ out. Grothendieck introduced λ -rings, a special formal power series ring assigned to any commutative ring formalizing the idea of taking antisymmetric powers. λ -rings contain the full information about representations [9]. We will use only basic series of mutually inverse M and L series $M_t \cdot L_t = 1$ defined as

$$\begin{aligned} M_t(x) &= \prod_{i \geq 1} (1 - x_i t)^{-1} = \sum \{m\} t^m \\ L_t(x) &= \prod_{i \geq 1} (1 - x_i t) = \sum (-)^m \{1^m\} t^m \end{aligned} \quad (7)$$

S -function series enjoy point wise sum and product. We consider only series derived from the basic L and M -series applying plethysms, e.g:

$$A_t(x) = L_t(x_i x_j)_{(i < j)} = \prod_{i < j} (1 - x_i x_j t) = \sum_{\alpha} (-1)^{\frac{|\alpha|}{2}} \{\alpha\} t^{\omega_{\alpha}} = \{1^2\} \underline{\otimes} L_t$$

$$C_t(x) = L_t(x_i x_j)_{(i \leq j)} = \prod_{i \leq j} (1 - x_i x_j t) = \sum_{\gamma} (-1)^{\frac{|\gamma|}{2}} \{\gamma\} t^{\omega_{\gamma}} = \{2\} \underline{\otimes} L_t$$

together with the inverse series $B = A^{-1}$ and $D = C^{-1}$. In general we will consider M_{π} -series, $M_{\pi} = \{\pi\} \underline{\otimes} M$. These series differ dramatically in their Schur function content when expanded as S -function series. E.g. the A -series (C -series) will encode the process of extracting all traces w.r.t. an antisymmetric (symmetric) second rank tensor.

To every Schur function series Φ_t we assign a linear form ϕ_t via $\phi_t(s_{\lambda}) = \langle \Phi_t | s_{\lambda} \rangle$. We are interested in the sum of Schur functions contained in Φ_t dropping t assuming it is evaluated at $t = 1$.

3. Morphisms between group representation rings

We model subgroups H_{π} of $GL(n)$ groups fixing a certain tensor T^{π} of Young symmetry type $\{\pi\}$. The simplest examples are given by stabilizing a vector¹ v^i , a symmetric rank 2 tensor² $g^{(i,j)}$ or an antisymmetric rank 2 tensor $f^{[i,j]}$. These tensors give rise to the classical subgroup branchings

$$GL(n-1) \subset GL(n) \quad \{\lambda\} \mapsto \{\lambda / (\{1\} \underline{\otimes} L)\} = \{\lambda / M^{-1}\} \quad (8)$$

$$O(n) \subset GL(n) \quad [\lambda] \mapsto \{\lambda / (\{2\} \underline{\otimes} L)\} = \{\lambda / C\} \quad (9)$$

$$Sp(n) \subset GL(n) \quad \langle \lambda \rangle \mapsto \{\lambda / (\{1^2\} \underline{\otimes} L)\} = \{\lambda / A\} \quad (10)$$

where $[\cdot]$ and $\langle \cdot \rangle$ brackets denote $O(n)$ and $Sp(n)$ characters. Considering the first case, we need to extract from every $GL(n)$ tensor all components in the subspace spanned by v^i . One finds easily

$$u^i = (u^i - \langle u^i | v_j \rangle v^j) + \langle u^i | v_j \rangle v^j \quad \Leftrightarrow \quad \{1\} \downarrow \{1\} + \{0\}$$

$$T^{(i,j)} = (T^{(i,j)} - (T^{(l,j)} v^i + T^{(i,l)} v^j) v_l - T^{(l,k)} v_l v_k v^i v^j)$$

$$+ (T^{(l,j)} v^i + T^{(i,l)} v^j) v_l + T^{(l,k)} v_l v_k v^i v^j \quad \Leftrightarrow \quad \{2\} \downarrow \{2\} + \{1\} + \{0\}$$

¹In some basis e_i .

² (\cdot, \cdot) and $[\cdot, \cdot]$ brackets denote symmetry and antisymmetry of tensors.

This is systematically done by skewing with the M^{-1} -series as shown in eqn. (8). In a similar way the extraction of traces w.r.t. the symmetric and antisymmetric rank 2 tensors can be formalized along the same way as the classical results (9) and (10) show. It is noteworthy that this process encodes the Wick theorem of QFT [2]. The branchings are bijections since we have $M \cdot M^{-1} = 1$ and $(s_\lambda/\mu)/\nu = s_\lambda/(\mu \cdot \nu)$ Which leads to $(\{\lambda\}/\Phi)/\Phi^{-1} = \{(\lambda/\Phi)/\Phi^{-1}\} = \{\lambda/(\Phi \cdot \Phi^{-1})\} = \{\lambda\}$ obtaining the branchings $\mathrm{GL}(n-1) \uparrow \mathrm{GL}(n)$, $\mathrm{O}(n) \uparrow \mathrm{GL}(n)$ and $\mathrm{Sp}(n) \uparrow \mathrm{GL}(n)$; all these are classical results.

The skew by a Series can be modeled by using the Hopf algebraic framework $s_\lambda/\Phi = (\phi \otimes \mathrm{Id})\Delta(s_\lambda)$ and the $/\Phi$ operators are called *branching operators* for that reason. All of the $M_\pi = \{\pi\} \otimes M$ series have an inverse $M_\pi^{-1} = \{\pi\} \otimes M^{-1}$. The new characters of the subgroups $H_\pi(n) \subset \mathrm{GL}(n)$ stabilizing a tensor T^π of Young symmetry $\{\pi\}$ are denoted by double parentheses $((\lambda))_\pi$ where the subscript is usually dropped or even be replaced by the dimension of the irreducible representation space. We find the generalization of the classical branchings

$$H_\pi(n) \subset \mathrm{GL}(n) \quad ((\lambda)) \mapsto \{\lambda/(\{\pi\} \otimes L)\} = \{\lambda/M_\pi^{-1}\} \quad (11)$$

$$\mathrm{GL}(n) \supset H_\pi(n) \quad \{\lambda\} \mapsto \{\lambda/(\{\pi\} \otimes M)\} = ((\lambda/M_\pi)) \quad (12)$$

Hence we find isomorphisms of the modules underlying the representation spaces

$$\mathcal{R}_{\mathrm{GL}} = \mathcal{R}(\{\lambda\}, \oplus) \begin{array}{c} \xrightarrow{\{\lambda\}/M_\pi^{-1}} \\ \xleftarrow{((\lambda))/M_\pi} \end{array} \mathcal{R}_{H_\pi} = \mathcal{R}(((\lambda)), \oplus) \quad (13)$$

For labelling problems of many particle states in quantum mechanics this process allows to compute states and energy levels. However, one would like to reobtain the ring structure of the subgroup representation rings $\mathcal{R}_{H_\pi} = \mathcal{R}(((\lambda))_\pi, \oplus, \otimes_\pi)$, which is considered to be a hard problem. Due to the restrictions imposed by extracting traces the tensor product of two reduced, say $\mathrm{O}(n)$, representations is in general no longer in an obvious way a direct sum of $\mathrm{O}(n)$ representations. Also the Weyl group changes [6] and the Schur-Weyl duality needs a different Weyl group, e.g. the hyperoctahedral group in case of $\mathrm{O}(n)$.

4. Hopf algebra twists of group representation rings

4.1. TWISTS OF REPRESENTATION RINGS

The main aim of the present paper is to explain how Hopf algebra twists can be used to establish the desired product formulae for subgroup

characters.

To do this, we introduce some Hopf algebra cohomology along classical lines [13, 1]. We define n -cochains $c_n : \Lambda^n \rightarrow \mathbb{Z}$, multilinear forms of n -arguments. The linear forms ϕ associated to the branching operator $/\Phi$ is hence a 1-cochain. Having a Hopf algebra Λ , we can define on all endomorphisms $f, g : \Lambda \rightarrow \Lambda$ the convolution product $* : \text{End}\Lambda \times \text{End}\Lambda \rightarrow \text{End}\Lambda$ $(f * g)(x) = f(x_{(1)})g(x_{(2)})$ this product can be generalized in a straight forward manner to n -cochains with multiplication taken in \mathbb{Z} .

We define a multiplicatively written coboundary operator ∂_n mapping n -cochains c_n to $(n + 1)$ -cochains c_{n+1} as

$$c_{n+1} = \partial^i c_n(x_0, \dots, x_n) = \begin{cases} \epsilon(x_0)c_n(x_1, \dots, x_n) & i = 0 \\ c_n(x_1, \dots, x_i x_{i+1}, \dots, x_n) & i \in \{1, \dots, n-1\} \\ c_n(x_1, \dots, x_{n-1})\epsilon(x_n) & i = n \end{cases} \quad (14)$$

which can be used to give $\partial_n = \partial^0 c_n * \partial^2 c_n^{-1} * \dots * \partial_n^n c_n^{\pm 1}$ having alternating signs. If $\partial_n c_n = \epsilon^{n+1}$, c_n is closed, if $c_{n+1} = \partial_n c_n$ c_{n+1} is exact. Cohomology neatly classifies the linear 1-cochains associated to Schur function series. Group like series having closed associated 1-chains, e.g. M, L, V , etc. give rise to homomorphisms with respect to the branching, having an unaltered product structure.

4.2 Theorem [3]: Let G be a group like Schur function series, i.e. $\Delta(G) = G \otimes G$. The associated 1-cochain is closed $\partial_1 g = \epsilon^2$. The associated representation ring $\mathcal{R}_{\text{GL} \downarrow \text{H}_G} \left(((\lambda))_G, \oplus, \otimes \right)$ has the same tensor product and is homomorphic to the representation ring of $\text{GL}(n)$. ■

An example is the $\text{GL}(n) \downarrow \text{GL}(n-1)$ branching and its inverse. More subtle situations are obtained with 2-cocycles which are exact hence derivable from 1-cochains. The 2-cocycle condition assures that the new twisted product remains associative. Let $(\partial_1 \phi)(x, y) = \phi^{-1}(x_{(1)}) \phi^{-1}(y_{(1)}) \phi(x_{(2)} \cdot y_{(2)})$ One introduces the twisted product w.r.t. this 2-cocycle. We give two equivalent formulae

$$((\lambda))_\phi \cdot_\phi ((\mu))_\phi = \left(((\lambda))_\phi / \Phi \cdot ((\mu))_\phi / \Phi \right) / \Phi^{-1} \quad (15)$$

$$((\lambda))_\phi \cdot_\phi ((\mu))_\phi = (\partial \phi) \left(\{\lambda\}_{(1)}, \{\mu\}_{(2)} \right) \left(\left(\{\lambda\}_{(2)} \cdot \{\mu\}_{(2)} \right) \right)_\phi \quad (16)$$

These twists induce not graded but only filtered multiplications due to the traces which are extracted from the original characters. We state our

main result about formal characters restricting ourselves to S -function series M_π .

4.3 [π -Newell-Littlewood] Theorem [4]: The representation ring $\mathcal{R}_{\mathbf{H}_\pi}$ of a subgroup \mathbf{H}_π of $\mathbf{GL}(n)$ stabilizing a tensor T^π of Schur symmetry $\{\pi\}$ is obtained by a Hopf algebra twist of the character ring $\mathcal{R}_{\mathbf{GL}}(\{\lambda\}, \oplus, \otimes)$ of the $\mathbf{GL}(n)$ with respect to the subgroup characters $((\lambda))_\pi$ and the product deformation induced by the Schur function series $M_\pi = \{\pi\} \underline{\otimes} M$ resp. its associated 1-cochain m_π ($\otimes_{m_\pi} \equiv \cdot m_\pi$)

$$\mathcal{R}_{\mathbf{GL}}(\{\lambda\}, \oplus, \otimes) \downarrow \mathcal{R}_{\mathbf{H}_\pi}(((\lambda))_\pi, \oplus, \otimes_{m_\pi}) \quad (17)$$

These branchings are ring isomorphisms, hence the \uparrow direction is obtained from M_π^{-1} . \blacksquare

4.2. TECHNICAL DETAILS

Passing to examples needs concrete combinatorial formulae. The key result is the outer coproduct of M_π series. While [4] provides formulae for coproducts of the plethysm of a Schur polynomial by a Schur function series, we stick to a

4.4 Corollary [4]: For any partition π , the coproduct of the series $M_\pi = \{\pi\} \underline{\otimes} M(x) \equiv M^{\{\pi\}}$ reads

$$\begin{aligned} \Delta M_\pi(x) &= (M_\pi)_{(1)} \otimes (M_\pi)_{(2)} = M_\pi(x, y) \\ &= (M \otimes M)^{\Delta\{\pi\}}(x, y) = M^{\{\pi\}_{(1)}} \otimes M^{\{\pi\}_{(2)}}(x, y) \\ &= M_\pi(x) M_\pi(y) \sum_{\sigma_1, \dots, \sigma_k} \prod_{\xi} \prod_{\eta < \pi}^{C_{\xi\eta}^\pi} s_{\{\xi\} \underline{\otimes} \{\sigma_k\}}(x) s_{\{\eta\} \underline{\otimes} \{\sigma_k\}}(y) \quad (18) \end{aligned}$$

where the $C_{\xi\eta}^\pi$ are the Littlewood-Richardson coefficients of outer multiplication. The summations over the σ_j are formally over all Schur functions. The proper cut part of the coproduct is defined as $K(x, y) = (M_\pi)_{(1')} \otimes (M_\pi)_{(2')} = \sum \prod \prod s_{\xi \underline{\otimes} \sigma}(x) s_{\eta \underline{\otimes} \sigma}(y)$. \blacksquare

This remarkable and formidable formula hides somehow a peculiarity. Namely that it contains inner coproducts stemming from the Weyl group \mathcal{S}_p . This can be seen from a Lemma which was needed to prove the corollary

4.5 Lemma [4]: Let $\delta(x) = x_{[1]} \otimes x_{[2]}$, $\Delta(x) = x_{(1)} \otimes x_{(2)}$, then $\Delta(\{\mu\} \underline{\otimes} \{\lambda\}) = \{\mu\} \underline{\otimes} (\Delta(\{\lambda\})) = (\{\mu\}_{[1]} \underline{\otimes} \{\lambda\}_{(1)}) \otimes (\{\mu\}_{[2]} \underline{\otimes} \{\lambda\}_{(2)})$ \blacksquare

This generalizes to Corollary (4.4) by right distributivity of the plethysm. The interplay between Weyl group and structure of the twist on the representation ring employs the famous Cauchy kernel $C(x, y) = \sum_{\xi} s_{\xi}(x)s_{\xi}(y)$, which enters the twists in a complicated fashion. It is the *inner coproduct* of the M series $\delta M = M_{[1]} \otimes M_{[2]} = C(x, y)$ and the coefficients $\{m\}$ of the M series are the units of inner products of irreps of \mathcal{S}_m . This result allows an easy

Proof of Theorem 4.3: We make use of duality, and of the fact that the proper cut part of the coproduct of M_{π} has an inverse to calculate the product of $H_{\pi}(\mathfrak{n})$ characters directly in terms of Schur functions:

$$\begin{aligned}
 \langle ((\mu))_{\pi} \cdot ((\nu))_{\pi} \mid s_{\rho} \rangle &= \langle \mu \otimes \nu \mid L_{\pi} \otimes L_{\pi} \cdot \Delta s_{\rho} \rangle \\
 &= \langle \mu \otimes \nu \mid (M_{\pi'_{(1)}} L_{\pi'_{(1)}}) \otimes (M_{\pi'_{(2)}} L_{\pi'_{(2)}}) L_{\pi} \otimes L_{\pi} \cdot \Delta s_{\rho} \rangle \\
 &= \langle \mu \otimes \nu \mid M_{\pi'_{(1)}} \otimes M_{\pi'_{(2)}} \cdot L_{\pi'_{(1)}} \otimes L_{\pi'_{(2)}} \cdot L_{\pi} \otimes L_{\pi} \cdot \Delta s_{\rho} \rangle \\
 &= \langle \mu/M_{\pi'_{(1)}} \otimes \nu/M_{\pi'_{(2)}} \mid \Delta L_{\pi} \cdot \Delta s_{\rho} \rangle \\
 &= \langle \mu/M_{\pi'_{(1)}} \otimes \nu/LM_{\pi'_{(2)}} \mid \Delta(L_{\pi} \cdot s_{\rho}) \rangle \\
 &= \langle ((\mu/M_{\pi'_{(1)}} \cdot \nu/M_{\pi'_{(2)}}))_{\pi} \mid s_{\rho} \rangle
 \end{aligned} \tag{19}$$

The conclusion follows from nondegeneracy of the Schur scalar product, and completeness of the Schur basis. \square

5. Examples

This section shall give a look-and-feel idea of what kind of groups may be expected and how their character theory looks like.

5.1. REPRESENTATION RINGS OF CLASSICAL GROUPS

Special cases of the general formula are the classical results. For the branchings w.r.t. $L = M_{\{1\}}^{-1}$, $A = M_{\{1^2\}}^{-1}$ and $C = M_{\{2\}}^{-1}$. The M series is group like inducing no twist providing representation rings intertwined by a change of basis (irreducibles) and identity maps on direct sum and tensor product.

Remarkably the symplectic and orthogonal cases lead to the same deformation! All M_{π} coproducts are of the form $\Delta(M_{\pi})(x) = M_{\pi}(x)M_{\pi}(y)K(x, y)$ where $K(x, y)$ is a complicated expression obtained from proper cuts of the coproduct, i.e. such parts which do not contain an identity component and Cauchy kernels. The proper cuts of $\{2\}$ and $\{1^2\}$ are identical! $\Delta'(\{2\}) = \Delta'(\{1^2\}) = \{1\} \otimes \{1\}$ The theorem 4.3 simplifies to

5.6 Newell-Littlewood Theorem:

$$\mathrm{O}(n) \quad [\lambda] \cdot_c [\mu] = \sum_{\xi} [\lambda/\xi \cdot \mu/\xi] \quad (20)$$

$$\mathrm{Sp}(n) \quad \langle \lambda \rangle \cdot_a \langle \mu \rangle = \sum_{\xi} \langle \lambda/\xi \cdot \mu/\xi \rangle \quad (21)$$

■

Having used formal characters, finite dimensional examples have to cope with syzygies reinduced by so-called modification rules. Classically one has only two cases dealt with by case-by-case studies [8]. The general theory, having infinitely many cases, needs a not yet available formalism.

5.2. NONCLASSICAL GROUPS

If we fix in $\mathrm{GL}(n)$ an epsilon tensor of Young symmetry $\{1^n\}$ unique up to a multiplicative constant, then we find $\mathrm{H}_{1^n} = \mathrm{SL}(n)$ providing a systematic treatment of $\mathrm{SL}(n)$ groups. However the representation ring $\mathcal{R}_{\mathrm{H}_{1^n}}$ is *not* that of the $\mathrm{SL}(n)$ groups in the inductive limit.

Consider $\mathrm{H}_{1^3}(4) \subset \mathrm{GL}(4)$ choosing a basis for $T^{1^3} = \eta$ given as

$$\eta_{pqr} = \begin{cases} \epsilon_{abc} & a \wedge b \wedge c \in 1, 2, 3 \\ 0 & \text{else} \end{cases} \quad (22)$$

The subgroup H_{1^3} is characterized by $A_p^x A_q^y A_r^z \eta_{pqr} = \eta_{xyz}$ which splits into parts containing an index 4 or not. This gives a $3+1$ decomposition of the matrices into blocks leading to

$$\mathrm{H}_{1^3} \ni [\mathbf{M}] = \begin{pmatrix} B_{3 \times 3} & D_{3 \times 1} \\ 0_{1 \times 3} & C_{1 \times 1} \end{pmatrix} \quad (23)$$

where 0 is the 1×3 zero-matrix, $C \neq 0$ and $\det(B) = 1$. This is an affine algebra, D playing the role of translations. In general, the groups H_{π} are semisimple and may not be reductive or can be discrete.

The interpretation of formal characters of $\mathrm{H}_{1^3}(4)$ needs modification rules, required for all $((\mu))$ with μ of length $\ell(\mu) = 4$. In our example this can only arise in those cases for which λ also has length 4. Since $\{1^4\} = \varepsilon$ is the character of the 1-dimensional determinant representation of $\mathrm{GL}(4)$ it follows that $\{\lambda_1, \lambda_2, \lambda_3, \lambda_4\} = \varepsilon^{\lambda_4} \{\lambda_1 - \lambda_4, \lambda_2 - \lambda_4, \lambda_3 - \lambda_4, 0\}$.

Applying this to the H_1^3 characters $((\lambda))_{\dim}$ gives:

$$\begin{aligned}
 ((1111))_{-3} &= \{1^4\}_1 - ((1))_4 = \varepsilon((0))_1 - ((1))_4 \\
 ((2111))_{-12} &= \{2111\}_4 - ((2))_{10} - ((11))_6 = \varepsilon((1))_4 - ((2))_{10} - ((11))_6 \\
 ((2211))_{-17} &= \{2211\}_6 - ((21))_{20} - ((111))_3 \\
 &= \varepsilon((11))_6 - ((21))_{20} - ((111))_3 \\
 ((2221))_{-8} &= \{2221\}_4 - ((211))_{11} - ((1111))_{-3} - ((1))_4 \\
 &= \varepsilon((111))_3 - ((211))_{11} \\
 ((2222))_3 &= \{2222\}_1 - ((2111))_{-12} - ((2))_{10} \\
 &= \varepsilon^2((0))_1 - \varepsilon((1))_4 + ((11))_6 \\
 ((3111))_{-30} &= \{3111\}_{10} - ((3))_{20} - ((21))_{20} = \varepsilon((2))_{10} - ((3))_{20} - ((21))_{20}
 \end{aligned} \tag{24}$$

This provides a collection of modification rules to be applied in the case $((\mu))$ with $\ell(\mu) = 4$. A complete set of modification rules, including those appropriate to $((\mu))$ with $\ell(\mu) > 4$ should be established.

Character tables and further details may be found in [4]. Examples for products of subgroup characters for $H_1^3(4)$ are:

·	$((2))_{10}$
$((2))_{10}$	$((4))_{35} + ((31))_{45} + ((22))_{30}$
$((11))_6$	$((31))_{45} + ((211))_{11} + ((1))_4$
$((3))_{20}$	$((5))_{56} + ((41))_{84} + ((32))_{60}$
$((21))_{20}$	$((41))_{84} + ((32))_{60} + ((311))_{26} + ((221))_{14} + ((2))_{10} + ((11))_6$
$((111))_3$	$((311))_{26} + ((2111))_{-12} + ((2))_{10} + ((11))_6$

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