# Max-Planck-Institut
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**Differentiable $L^p$-functional calculus for certain sums of non-commuting operators**

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by


*Michael Gnewuch*



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# Differentiable $L^p$-functional calculus for certain sums of non-commuting operators

Michael Gnewuch *

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Max Planck Institute for Mathematics in the Sciences,
Inselstr. 22, D-04103 Leipzig, Germany
e-mail: gnewuch@mis.mpg.de

**Abstract**

We consider a special class of sums of non-commuting positive operators on $L^2$-spaces and derive a formula for their holomorphic semigroups. The formula enables us to give sufficient conditions for these operators to admit differentiable $L^p$-functional calculus for $1 \leq p \leq \infty$. Our results are in particular applicable to certain sub-Laplacians, Schrödinger operators and sums of even powers of vector fields on solvable Lie groups with exponential volume growth.

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# 1 Introduction

Our investigation is stimulated by a multiplier theorem of S. Mustapha stating the following (for a more detailed discussion see Section 3):

Consider a semidirect product $G$ of the real line $\mathbb{R}$ and a stratified nilpotent Lie group $N$, where $\mathbb{R}$ is acting on $N$ via natural dilations. Then $G$ is a solvable Lie group with exponential volume growth. Let $\Delta$ be a (distinguished type of) left invariant sub-Laplacian on $G$, and let $L^p(G)$ be the $L^p$-space on $G$ with respect to the right invariant Haar measure. Let $\kappa > 2$, and let $H^\kappa(\mathbb{R})$ be the $L^2$-Sobolev space of order $\kappa$ on $\mathbb{R}$. Mustapha showed in [27] that for continuous, compactly supported $f \in H^\kappa(\mathbb{R})$ the restriction of $f(\Delta)$ to $L^2 \cap L^p(G)$ extends to a bounded operator on $L^p(G)$ for all $1 \leq p \leq \infty$. Roughly speaking: it suffices to control a finite number of derivatives of $f$ to get $L^p$-boundedness for $f(\Delta)$. (In this situation we say that $\Delta$ admits *differentiable $L^p$-functional calculus*, and $f$ is called an *$L^p$-multiplier* for $\Delta$.)

This multiplier theorem has many predecessors, starting with the classical results of Mikhlin and Hörmander in the Euclidean setting (cf. [18]). So it is well known that sub-Laplacians on connected Lie groups with polynomial volume growth (including all connected nilpotent Lie groups) have always differentiable $L^p$-functional calculus [1]. But for a sub-Laplacian on a solvable Lie group with exponential volume growth the validity of such a theorem is a priori not clear. In fact there are sub-Laplacians on exponential groups known that do not admit differentiable $L^p$-functional calculus as defined above – they are of so-called *holomorphic $L^p$-type* [5, 17, 23].

A particularly interesting facet of the result of S. Mustapha is the order $\kappa$ of the Sobolev space that is independent of the "size" of the group, which, e.g., can be expressed by the Euclidean dimension of $G$ or the homogeneous dimension $Q$ of $N$. Actually, W. Hebisch proved in [16] a multiplier theorem that is identical with the one of Mustapha, except for the order of the Sobolev space, which is $Q/2 + 5/2 + \varepsilon$ instead of $2 + \varepsilon$. And indeed, in the conditions of the multiplier theorems for solvable groups that have been proven so far there regularly appears some quantity describing the size of the underlying group (see, e.g., [1, 2, 4, 6, 7, 11, 14, 18, 24, 25, 26]). Referring to this, the result of Mustapha seems remarkable.

One aim of this paper is to present a different proof strategy of Mustapha's multiplier theorem than the one used in [27]: Our approach is purely analytic and relies on bounded functional calculus and Bessel functions instead of on stochastic methods like Brownian motion.

Another aim is to extend the multiplier theorem to a larger class of operators. We define this class in a more abstract way, but it includes several interesting differential operators on those Lie groups mentioned above. These particular operators are not necessarily hypoelliptic, left-invariant nor of second order like the sub-Laplacians considered by Mustapha.

The article is organized as follows: In *Section 2* we consider operators

$$T = -\partial_r^2 \otimes I + e^{2\nu r} \otimes L\,, \quad \nu \in \mathbb{R} \setminus \{0\}\,,$$

on $L^2(\mathbb{R} \times X)$, where $X$ is an arbitrary $\sigma$-finite measure space, $\partial_r$ the derivative with respect to the parameter $r \in \mathbb{R}$, $I$ the identity operator on $L^2(X)$, $e^{2\nu r}$ the multiplication operator $\varphi(r) \mapsto e^{2\nu r}\varphi(r)$ on $L^2(\mathbb{R})$ and $L$ a positive selfadjoint operator on $L^2(X)$. $T$ can be realized as a selfadjoint operator, and we deduce a representation of the holomorphic semigroup $(e^{-zT})_{\Re(z)>0}$ of $T$ in terms of $(e^{-tL})_{t\geq 0}$. From this it is almost straightforward to derive our central multiplier result Theorem 2.2.

In *Section 3* we present differential operators on Lie groups, which fulfill the conditions of our multiplier theorem. There we discuss also the result of S. Mustapha in detail.

The appendix (*Section 4*) provides a selfadjointness theorem for generalized sub-Laplacians and Schrödinger operators on Lie groups. "Generalized" means here that the operators are not required to satisfy the so-called Hörmander condition, i.e., they are not necessarily hypoelliptic. The result is probably known in the literature, but we are not aware of any explicit reference. Since we use the theorem a few times in the main text, we decided to give a proof (which is actually of simple conception).

# 2 The general multiplier result

Let $(X,\, dx)$ be a $\sigma$-finite measure space and $L$ a positive selfadjoint operator on $L^2(X)$ with domain $\mathrm{D}(L)$. Let $dr$ be the Lebesgue measure on $\mathbb{R}$ and $dr \otimes dx$ the product measure on $\mathbb{R} \times X$. Let $\nu \in \mathbb{R} \setminus \{0\}$ be fixed. We denote the function $r \mapsto e^{2\nu r}$ on $\mathbb{R}$ briefly by $e^{2\nu r}$. For $\kappa \geq 0$ let

$$H^\kappa(\mathbb{R}) := \{f \in L^2(\mathbb{R}) : (1 + |\cdot|)^\kappa \hat{f} \in L^2(\mathbb{R})\}$$

be the *Sobolev space* of order $\kappa$ on $\mathbb{R}$, and put

$$\mathscr{A} := H^2(\mathbb{R}) \cap \{f \in L^2(\mathbb{R}) : e^{2\nu r} f \in L^2(\mathbb{R})\}\,.$$

The positive operator $-\partial_r^2 + e^{2\nu r} L$ is defined on $\mathscr{A} \otimes \mathrm{D}(L) \subseteq L^2(\mathbb{R} \times X)$ and its closure is denoted by $T$. It is clear that $T$ has a selfadjoint extension, namely its Friedrichs' extension. In fact $T$ itself is already a selfadjoint operator, as we shall prove in Proposition 2.10.

The precise definition of an $L^p$-multiplier that we want to use throughout this article is the following: Let $p \in [1, \infty]$. We call a bounded function $f : \mathbb{R} \to \mathbb{C}$ an *$L^p$-multiplier for $T$* if there exists a positive constant $C$ such that

$$\|f(T)\varphi\|_{L^p(\mathbb{R}\times X)} \leq C \|\varphi\|_{L^p(\mathbb{R}\times X)} \quad \text{for all } \varphi \in L^2 \cap L^p(\mathbb{R} \times X).$$

Notice that in the case $p = \infty$ this definition does not necessarily imply that $f(T)$ can be extended to a bounded operator on the whole space $L^\infty(\mathbb{R} \times X)$, since, for a non-trivial measure $dx$, $L^2 \cap L^\infty(\mathbb{R} \times X)$ is not dense in $L^\infty(\mathbb{R} \times X)$. On the other hand, if $f(T)$ is given explicitly, e.g., by integration against a suitable kernel function, then it is often easy to extend $f(T)$ to a bounded operator on $L^\infty(\mathbb{R} \times X)$ in a canonical way.

Before we state the main results of this section, some remarks concerning notations: We use the abbreviations sh, ch and th for the hyperbolic functions sinh, cosh and tanh respectively. $\Re(z)$ stands for the real part, $\Im(z)$ for the imaginary part and $z^*$ for the complex conjugate of the complex number $z$. For $f \in L^2(\mathbb{R} \times X)$ and $s \in \mathbb{R}$ let $f_s$ denote the function $x \mapsto f(s, x)$ on $X$.

## 2.1 Results of Section 2

First we give a representation of the holomorphic semigroup of $T$ in terms of the semigroup of $L$ with positive time parameter.

**Theorem 2.1.** *Let $z \in \mathbb{C}$ with $\Re(z) > 0$ and $f \in L^2(\mathbb{R} \times X)$. Then*

$$e^{-zT} f(r,x) = \Big( \int_{\mathbb{R}} \int_0^\infty |\nu| \Psi_{\nu^2 z}(\xi) \exp\Big( -\frac{\mathrm{ch}(\nu(r-s))}{\xi} \Big) \exp\Big( -\frac{\xi e^{\nu(r+s)}}{2\nu^2} L \Big) f_s \, d\xi \, ds \Big)(x) \tag{1}$$

*for almost all $(r, x) \in \mathbb{R} \times X$, where the function $\Psi_z : ]0, \infty[ \to \mathbb{C}$ is given by*

$$\Psi_z(\xi) = \frac{\xi^{-2}}{\sqrt{4\pi^3 z}} \exp\Big(\frac{\pi^2}{4z}\Big) \int_0^\infty \mathrm{sh}(\vartheta) \sin\Big(\frac{\pi\vartheta}{2z}\Big) \exp\Big( -\frac{\vartheta^2}{4z} - \frac{\mathrm{ch}(\vartheta)}{\xi} \Big) \, d\vartheta \,. \tag{2}$$

*There exists a $C_z > 0$, only depending on $z$, with $|\Psi_z(\xi)| \leq C_z \xi^{-2}$ for all $\xi > 0$.*

With the help of Theorem 2.1 it is easy to prove estimate (3), which is the key inequality for our main multiplier result:

**Theorem 2.2.** *Let $p \in [1, \infty]$. If there exists a constant $C > 0$ fulfilling $\|e^{-tL}\psi\|_{L^p(X)} \leq C\|\psi\|_{L^p(X)}$ for each $t > 0$ and every $\psi \in L^2 \cap L^p(X)$, then*

$$\|e^{-(\rho+i\sigma)T}\varphi\|_{L^p(\mathbb{R}\times X)} \leq C(2+|\nu|\sqrt{\rho}) \exp\Big(\frac{\pi^2}{4\nu^2\rho}\Big)\Big(1+\frac{|\sigma|}{\rho}\Big)^{3/2}\|\varphi\|_{L^p(\mathbb{R}\times X)} \quad (3)$$

*for all $\rho > 0$, $\sigma \in \mathbb{R}$ and $\varphi \in L^2 \cap L^p(\mathbb{R} \times X)$. Moreover, for every $\kappa > 2$ each continuous $f \in H^\kappa(\mathbb{R})$ with compact support is an $L^q$-multiplier for $T$ if $q$ satisfies $1/q = s + (t - s)/p$ for some $s$, $t \in [0, 1]$ with $s + t = 1$.*

The following theorem states our multiplier result in terms of heat or evolution kernels rather than in terms of semigroups:

**Theorem 2.3.** *Assume that $L$ has a measurable evolution kernel $(p_t)_{t>0}$, i.e., $e^{-tL}\psi(x) = \int_X p_t(x, y)\psi(y)\,dy$ for almost all $x \in X$. If the operators $(\Lambda_t)_{t>0}$, defined by $\psi \mapsto \int_X |p_t(\cdot, y)|\psi(y)\,dy$, are bounded on $L^2(X)$ with $\|\Lambda_t\|_{L^2(X)\to L^2(X)} \leq \kappa$, $\kappa$ independent of $t > 0$, then $T$ has an evolution kernel $(P_z)_{\Re(z)>0}$, given by*

$$P_z((r, x),(s, y)) = \int_0^\infty |\nu|\Psi_{\nu^2 z}(\xi) \exp\Big(-\frac{\mathrm{ch}(\nu(r-s))}{\xi}\Big) p_{\frac{\xi e^{\nu(r+s)}}{2\nu^2}}(x, y)\,d\xi \quad (4)$$

*for almost all $((r, x),(s, y)) \in (\mathbb{R} \times X)^2$.*

*If there is in addition a constant $C > 0$ with $\|p_t(\cdot, x)\|_{L^1(X)} \leq C$ for all $t > 0$ and $x \in X$, then for each $z \in \mathbb{C}$ with $\Re(z) > 0$ and every $g \in \mathbb{R} \times X$*

$$\|P_z(\cdot, g)\|_{L^1(\mathbb{R}\times X)} \leq C(2 + |\nu|\sqrt{\Re(z)}) \exp\Big(\frac{\pi^2}{4\nu^2\Re(z)}\Big)\Big(1 + \frac{|\Im(z)|}{\Re(z)}\Big)^{3/2}. \quad (5)$$

*In particular we have for every $\varepsilon > 0$ and every $p \in [1, \infty]$ that each compactly supported, continuous $f \in H^{2+\varepsilon}(\mathbb{R})$ is an $L^p$-multiplier for $T$.*

**Remark 2.4.** (i) The exponent $3/2$ of $1 + |\Im(z)|/\Re(z)$ in (3) and (5) is optimal in the sense, that there exists an operator $T = -\partial_r^2 + e^{2r}L$ with

$$\|e^{-(1+i\sigma)T}\|_{L^1(\mathbb{R}\times X)\to L^1(\mathbb{R}\times X)} \sim (1 + |\sigma|)^{3/2}.$$

This was shown in [27] for a sub-Laplacian $T$ on some solvable Lie group $G$, given by $\mathbb{R}$ acting on $\mathbb{R}^2$ via natural dilations (cf. Section 3). S. Mustapha used a concrete formula from [3] for the heat kernel $\Phi_z$ of $T$ to demonstrate that $\|\Phi_{1+i\sigma}\|_{L^1(G)} \sim (1+|\sigma|)^{3/2}$.
(ii) One observes that the term $C\exp(\pi^2/4\nu^2\Re(z))(2+|\nu|\sqrt{\Re(z)})$ in (3) and (5) tends to infinity if $|\nu|$ tends to zero. That reflects the fact that for operators of the form $-\partial_r^2 + L$ in general one can not achieve estimates like (3) and (5). If, e.g., $L$ is the Laplacian on $\mathbb{R}^n$, then $-\partial_r^2 + L$ is the Laplacian on $\mathbb{R}^{n+1}$ and for its heat kernel $\Phi_z$ holds

$$\|\Phi_{1+i\sigma}\|_{L^1(\mathbb{R}^{n+1})} \sim (1+|\sigma|)^{\frac{n+1}{2}} .$$

Thus for $n > 2$ an estimate like (5) does not hold for $P_z(\cdot, 0) = \Phi_z$.

## 2.2 Proof of the results

For $a \geq 0$ we consider the operator $A = A(a) := -\partial_r^2 + ae^{2\nu r}$ defined on the space of test functions $C_c^\infty(\mathbb{R})$. We denote its closure again by $A(a)$. Then $A(a)$ is selfadjoint with domain

$$\mathrm{D}(A(a)) = \begin{cases} H^2(\mathbb{R}) & \text{if } a = 0, \\ \mathscr{A} & \text{if } a > 0 \end{cases}$$

(see, e.g., Theorem 4.1).

*Guideline for the proof of Theorem 2.1 and 2.2*: The multiplier statement in Theorem 2.2 follows from estimate (3) by utilizing the spectral theorem and the Fourier inversion formula: There holds

$$f(T) = \frac{1}{2\pi} \int_{\mathbb{R}} \hat{F}(\xi) e^{-(1-i\xi)T} \, d\xi \, ,$$

where $F := f \cdot \exp$. Hence we get for $\varphi \in L^2 \cap L^p(\mathbb{R} \times X)$, with $v(\xi) := 1 + |\xi|$ and $K := C(2+|\nu|)\exp(\pi^2/4\nu^2)$,

$$\begin{aligned} \|f(T)\varphi\|_{L^p(\mathbb{R}\times X)} &\leq \frac{1}{2\pi} \int_{\mathbb{R}} |\hat{F}(\xi)| \|e^{-(1-i\xi)T}\varphi\|_{L^p(\mathbb{R}\times X)} \, d\xi \\ &\leq \frac{K}{2\pi} \int_{\mathbb{R}} |\hat{F}(\xi)| v(\xi)^{\frac{3}{2}} \, d\xi \, \|\varphi\|_{L^p(\mathbb{R}\times X)} \\ &\leq \frac{K}{2\pi} \|v^{-\frac{1}{2}-\varepsilon}\|_{L^2(\mathbb{R})} \|v^{2+\varepsilon}\hat{F}\|_{L^2(\mathbb{R})} \|\varphi\|_{L^p(\mathbb{R}\times X)} \, . \end{aligned}$$

To apply a duality argument, we use the following statement: For all $q, q' \in [1,\infty]$ with $1/q + 1/q' = 1$, and for all $\xi \in L^q(\mathbb{R}\times X)$ we have

$$\|\xi\|_{L^q(\mathbb{R}\times X)} = \sup\{|\langle \xi, \tau\rangle| : \tau \in L^2 \cap L^{q'}(\mathbb{R}\times X), \|\tau\|_{L^{q'}(\mathbb{R}\times X)} = 1\}\,. \qquad (6)$$

Hölder's inequality shows that the right hand side of (6) is bounded from above by the left hand side. For $q \in ]1,\infty[$ equality follows from the theorem of Hahn-Banach, the fact that $L^{q'}$ is isometrically isomporphic to the dual space of $L^q$ and the fact that $L^2 \cap L^{q'}$ is dense in $L^{q'}$. In the remaining cases $q = 1, \infty$ the $\sigma$-finiteness of the measure space $\mathbb{R}\times X$ allows us to construct sequences $(\eta_n)$ in $L^2 \cup L^{q'}(\mathbb{R}\times X)$ satisfying $|\langle \xi, \eta_n\rangle| \to \|\xi\|_{L^q(\mathbb{R}\times X)}$ for $n \to \infty$. (These constructions are straightforward. Nevertheless, the details can be found in [10, §1.1].) From (6) we get, with a simplified notation,

$$\begin{aligned}
&\sup\{\|f(T)\psi\|_{L^{p'}} : \psi \in L^2 \cap L^{p'}, \|\psi\|_{L^{p'}} = 1\} \\
=&\sup\{|\langle f(T)\psi, \tau\rangle| : \psi \in L^2 \cap L^{p'}, \tau \in L^2 \cap L^p, \|\psi\|_{L^{p'}} = 1 = \|\tau\|_{L^p}\} \\
=&\sup\{|\langle \psi, f^*(T)\tau\rangle| : \psi \in L^2 \cap L^{p'}, \tau \in L^2 \cap L^p, \|\psi\|_{L^{p'}} = 1 = \|\tau\|_{L^p}\} \\
=&\sup\{\|f^*(T)\tau\|_{L^p} : \tau \in L^2 \cap L^p, \|\tau\|_{L^p} = 1\} \le \frac{K}{2\pi}\int_{\mathbb{R}} |\hat{F}(\xi)| v(\xi)^{\frac{3}{2}}\, d\xi\,,
\end{aligned}$$

since $(f^* \cdot \exp)^\wedge(\xi) = \hat{F}(-\xi)^*$.

The full multiplier statement follows now from the interpolation theorem of Riesz-Thorin.

Inequality (3) is more or less a direct consequence of identity (1), as we shall show later. Therefore the crucial part of the proof is to verify (1). Actually it is sufficient to establish (1) for $\nu = 1$, since the general case can then be derived using the coordinate transform $s := \nu r$. We consider therefore only the case $\nu = 1$ to make formulas a little bit shorter.

The formal idea to establish (1) is the following: Instead of looking directly at $e^{-zT} = e^{-z(-\partial_r^2 + e^{2r}L)}$, we first consider the family of operators $e^{-z(-\partial_r^2 + ae^{2r})} = e^{-zA(a)}$, $a \in [0,\infty[$. These operators have an integration kernel $q_z(a,\cdot,\cdot)$, i.e.,

$$e^{-zA(a)}\varphi = \int_{\mathbb{R}} q_z(a,\cdot,s)\varphi(s)\, ds \quad \text{for any } \varphi \in L^2(\mathbb{R}),$$

which can be calculated easily. After deriving a suitable representation of $q_z$, we are able to replace the variable $a$ again by the operator $L$ in the sense of bounded functional calculus, and we get

$$e^{-zT}\varphi \otimes \psi = \int_{\mathbb{R}} (q_z(L,\cdot,s)\psi)\varphi(s)\,ds \quad \text{for any } \varphi \in L^2(\mathbb{R}),\ \psi \in L^2(X).$$

On the next pages we realize this proof idea in a mathematically rigorous manner. The first step is to calculate the kernel $q_z(a,\cdot,\cdot)$ of $e^{-zA(a)}$, $a \geq 0$.

**Lemma 2.5.** *Let $a \in [0,\infty[$ and $z \in \mathbb{C}$ with $\Re(z) > 0$. For $n \in \mathbb{N}$ let the curve $\gamma_n : \mathbb{R} \to \mathbb{C}$ be defined by $\gamma_n(\theta) = (\theta + i2^{-n})^2$.*
*(i) There holds*

$$e^{-zA(a)} = -\frac{1}{2\pi i}\int_{\gamma_n} e^{-z\lambda}(\lambda - A(a))^{-1}\,d\lambda\,; \tag{7}$$

*here the parameterized integrand $s \mapsto \gamma_n'(s)e^{-z\gamma_n(s)}(\gamma_n(s) - A(a))^{-1}$ is an $L^1$-mapping from $\mathbb{R}$ into the space of bounded linear operators on $L^2(\mathbb{R})$.*
*(ii) If $\varphi \in C_c^\infty(\mathbb{R})$, then there holds for almost all $r \in \mathbb{R}$*

$$e^{-zA(a)}\varphi(r) = -\frac{1}{2\pi i}\int_{\gamma_n} e^{-z\lambda}((\lambda - A(a))^{-1}\varphi)(r)\,d\lambda\,. \tag{8}$$

*Proof.* Formulas like (7) are well known in the literature, see, e.g., [28, §1.7]. From (7) follows for almost all $r \in \mathbb{R}$

$$e^{-zA(a)}\varphi(r) = -\frac{1}{2\pi i}\Big(\int_{\gamma_n} e^{-z\lambda}(\lambda - A(a))^{-1}\varphi\,d\lambda\Big)(r)\,.$$

Since $\varphi$ is smooth, one can change the order of integration and point evaluation in $r$. Thus (8) holds. □

Equation (8) indicates that we can calculate $q_z$ if we have a concrete formula for the resolvent of $A(a)$. We found such a formula in [20] (it is also stated in [29, Ex. 4.15]): For $a > 0$ and $\theta \in \mathbb{C}$ we define the function $k_\theta^a : \mathbb{R}^2 \to \mathbb{C}$ by

$$k_\theta^a(r,s) = \begin{cases} -I_{-i\theta}(\sqrt{a}e^r)K_{i\theta}(\sqrt{a}e^s) & \text{for } s \geq r\,, \\ -I_{-i\theta}(\sqrt{a}e^s)K_{i\theta}(\sqrt{a}e^r) & \text{for } s < r\,, \end{cases}$$

where $I$ and $K$ are the so called *Bessel functions of imaginary argument.* (Our reference for Bessel functions is [22].) Furthermore, we define $\vartheta(\theta)$ to be the complex number with $\vartheta(\theta)^2 = \theta$ and $\arg(\vartheta(\theta)) \in [0, \pi[$. Then for $\lambda \in \mathbb{C} \setminus [0, \infty[$ and $\varphi \in C_c^\infty(\mathbb{R})$ we have

$$(\lambda - A(a))^{-1}\varphi(r) = \int_{\mathbb{R}} k^a_{\vartheta(\lambda)}(r, s)\varphi(s)\, ds \quad \text{for almost all } r \in \mathbb{R}. \tag{9}$$

Before we calculate $q_z$, we want to prove another auxiliary lemma:

**Lemma 2.6.** *There exists a $C > 0$ such that for all $a > 0$, $r$, $s \in \mathbb{R}$ and all $z \in \mathbb{C}$ with $\Im(z) \geq 0$*

$$|k^a_z(r, s)| \leq C a^{-1/4} e^{-(r+s)/4}\,.$$

*Proof.* Because of symmetry we can confine ourselves to the case $r \geq s$. If $J_0$ denotes the Bessel function of first type and order 0, we have

$$K_{iz}(\sqrt{a}e^r) I_{-iz}(\sqrt{a}e^s) = \frac{1}{2} \int_{r-s}^{+\infty} J_0(\sqrt{2ae^{r+s}}\sqrt{\operatorname{ch}(\sigma) - \operatorname{ch}(r - s)})e^{iz\sigma}\, d\sigma\,.$$

$J_0$ is bounded on $[0, \infty[$ and there exists a $C > 0$ with $|J_0(x)| \leq C/\sqrt{x}$ for all $x \in [0, \infty[$. Therefore

$$|K_{iz}(\sqrt{a}e^r) I_{-iz}(\sqrt{a}e^s)| \leq \frac{C}{2}(2a)^{-1/4} e^{-(r+s)/4} \int_{r-s}^{\infty} (\operatorname{ch}(\sigma) - \operatorname{ch}(r - s))^{-1/4}\, d\sigma\,.$$

With the substitution $\tau := \operatorname{ch}(\sigma)$, $d\sigma = (\tau^2 - 1)^{-1/2}\, d\tau$ we get

$$\int_{r-s}^{\infty} (\operatorname{ch}(\sigma) - \operatorname{ch}(r - s))^{-1/4}\, d\sigma = \int_{\operatorname{ch}(r-s)}^{\infty} (\tau - \operatorname{ch}(r - s))^{-1/4} \frac{d\tau}{\sqrt{\tau^2 - 1}}$$
$$= \int_0^\infty \tau^{-1/4} \frac{d\tau}{\sqrt{(\tau + \operatorname{ch}(r - s))^2 - 1}} \leq \int_0^\infty \tau^{-3/4} \frac{d\tau}{\sqrt{\tau + 2}} < \infty\,.$$

□

**Proposition 2.7.** *Let $a > 0$, $z \in \mathbb{C}$ with $\Re(z) > 0$ and $\varphi \in C_c^\infty(\mathbb{R})$. Then*

$$e^{-zA(a)}\varphi(r) = \int_{\mathbb{R}} q_z(a, r, s)\varphi(s)\, ds \quad \textit{for almost all } r \in \mathbb{R}, \tag{10}$$

*where*

$$q_z(a, r, s) := -\frac{1}{\pi i} \int_{\mathbb{R}} \theta e^{-z\theta^2} k^a_\theta(r, s)\, d\theta\,.$$

*Proof.* Lemma 2.5(ii) and (9) imply for $\varphi \in C_c^\infty(\mathbb{R})$

$$e^{-zA(a)}\varphi(r) = -\frac{1}{2\pi i}\int_{\gamma_n} e^{-z\lambda}\int_{\mathbb{R}} k^a_{\vartheta(\lambda)}(r,s)\varphi(s)\,ds\,d\lambda$$
$$= -\frac{1}{\pi i}\int_{\mathbb{R}}\int_{\mathbb{R}}(\theta + i2^{-n})\exp(-z(\theta+i2^{-n})^2)k^a_{\theta+i2^{-n}}(r,s)\varphi(s)\,ds\,d\theta$$

for almost all $r$ and every $n \in \mathbb{N}$. For fixed $r \in \mathbb{R}$ the integrand of the last term converges for $n \to \infty$ pointwise to the function

$$(\theta, s) \mapsto \theta e^{-z\theta^2} k^a_\theta(r,s)\varphi(s)\,.$$

From

$$|(\theta + i2^{-n})\exp(-z(\theta+i2^{-n})^2)k^a_{\theta+i2^{-n}}(r,s)\varphi(s)|$$
$$\le \frac{C}{a^{1/4}}(1+|\theta|)\exp(-\Re(z)(\theta^2-1) + 2|\Im(z)\theta|)\,e^{-(r+s)/4}|\varphi(s)|$$

(see Lemma 2.6) and the dominated convergence theorem we get

$$e^{-zA}\varphi(r) = -\frac{1}{\pi i}\int_{\mathbb{R}}\int_{\mathbb{R}}\theta e^{-z\theta^2}k^a_\theta(r,s)\varphi(s)\,ds\,d\theta\,.$$

Hence Fubini's theorem verifies (10) for test functions. □

**Proposition 2.8.** *Let $z \in \mathbb{C}$ with $\Re(z) > 0$ and $\Psi_z$ as in (2). For all $a > 0$ and all $r$, $s \in \mathbb{R}$ we have*

$$q_z(a,r,s) = \int_0^\infty \Psi_z(\xi)\exp\Big(-\frac{\mathrm{ch}(r-s)}{\xi}\Big)\exp\Big(-\frac{\xi e^{r+s}}{2}a\Big)\,d\xi\,. \qquad (11)$$

*Proof.* Let $a > 0$. From

$$K_\nu(\zeta) = \frac{\pi}{2}\frac{I_{-\nu}(\zeta) - I_\nu(\zeta)}{\sin(\pi\nu)}\,,\quad \nu \notin \mathbb{Z}\,,\quad \text{and}\quad K_{-\nu} = K_\nu$$

we get for $r \ge s$

$$q_z(a,r,s) = \frac{2}{\pi^2}\int_0^{+\infty}\theta\,\mathrm{sh}(\theta\pi)e^{-z\theta^2}K_{i\theta}(\sqrt{a}e^r)K_{i\theta}(\sqrt{a}e^s)\,d\theta\,.$$

Obviously this result holds still if $r < s$. From the integral representation [22, (5.10.25)] of $K_{i\theta}$ follows for $s, r \in \mathbb{R}$

$$\begin{aligned} q_z(a,r,s) =& \frac{1}{2\pi^2} \int_0^\infty \theta \operatorname{sh}(\theta\pi) e^{-z\theta^2 + i\theta(r-s)} \Big( \int_0^\infty \exp\Big( - v - \frac{ae^{2r}}{4v}\Big) v^{-i\theta-1}\, dv \Big) \\ & \cdot \Big( \int_0^\infty \exp\Big( - w - \frac{ae^{2s}}{4w}\Big) w^{i\theta-1}\, dw \Big)\, d\theta \\ =& \frac{1}{2\pi^2} \int_0^\infty \int_0^\infty \int_0^\infty \theta \operatorname{sh}(\theta\pi) \exp(-z\theta^2 + i\theta(r-s) + i\theta \ln(w/v)) \\ & \cdot \exp\Big( - \frac{a}{4}\Big(\frac{e^{2r}}{v} + \frac{e^{2s}}{w}\Big) - w - v\Big) \frac{dv}{v} \frac{dw}{w}\, d\theta\,. \end{aligned}$$

The identity $q_z(a,r,s) = \frac{1}{2}(q_z(a,r,s) + q_z(a,s,r))$ shows that we can exchange the factor $\exp(i\theta(r-s+\ln(w/v)))$ of the integrand by $\cos(\theta(r-s+\ln(w/v)))$. Then, with substitutions $\nu := 2ve^{-r+s}$, $\mu := 2we^{r-s}$,

$$\begin{aligned} q_z(a,r,s) = & \frac{1}{2\pi^2} \int_0^\infty \int_0^\infty \int_0^\infty \theta \operatorname{sh}(\theta\pi) \cos(\theta(s-r+\ln(\mu/\nu))) e^{-z\theta^2} \\ & \cdot \exp\Big( - \frac{a}{2} e^{r+s}\Big(\frac{1}{\nu} + \frac{1}{\mu}\Big) - \frac{1}{2}\big(\nu e^{r-s} + \mu e^{s-r}\big)\Big) \frac{d\nu}{\nu} \frac{d\mu}{\mu}\, d\theta\,. \end{aligned}$$

Using the coordinate transformation

$$\Phi : \{(\xi,\eta) : \xi > 0, |\eta| < \xi\} \to ]0,\infty[^2, \ \ (\xi,\eta) \mapsto \Big(\frac{2}{\xi+\eta}, \frac{2}{\xi-\eta}\Big) =: (\mu,\nu)\,,$$

we get

$$\begin{aligned} q_z(a,r,s) = & \frac{1}{\pi^2} \int_0^\infty \int_0^\infty \int_{-\xi}^{\xi} \theta \operatorname{sh}(\theta\pi) \cos\Big(\theta\Big(s - r + \ln\Big(\frac{\xi-\eta}{\xi+\eta}\Big)\Big)\Big) e^{-z\theta^2} \\ & \cdot \exp\Big( - \frac{ae^{r+s}\xi}{2} - \frac{e^{r-s}}{\xi-\eta} - \frac{e^{s-r}}{\xi+\eta}\Big) \frac{d\eta\, d\xi}{\xi^2-\eta^2}\, d\theta \\ = & \frac{1}{\pi^2} \int_0^\infty \int_0^\infty \int_{-1}^{+1} \theta \operatorname{sh}(\theta\pi) \cos\Big(\theta\Big(s - r + \ln\Big(\frac{1-w}{1+w}\Big)\Big)\Big) e^{-z\theta^2} \\ & \cdot \exp\Big( - \frac{ae^{r+s}\xi}{2} - \frac{1}{\xi}\Big(\frac{e^{r-s}}{1-w} + \frac{e^{s-r}}{1+w}\Big)\Big) \frac{dw\, d\xi}{\xi(1-w^2)}\, d\theta\,, \end{aligned}$$

where in the last step we employed the transformation of variables $w = \eta/\xi$. If we define $u := \operatorname{arth}(w)$, there holds $\ln((1-w)/(1+w)) = -2\operatorname{arth}(w) = -2u$

and $1-w^2 = 1-\mathrm{th}(u)^2 = \mathrm{ch}(u)^{-2}$. Furthermore, we get

$$\frac{e^{r-s}}{1-w} + \frac{e^{s-r}}{1+w} = 2\,\mathrm{ch}(u)\,\mathrm{ch}(r-s+u)\,,$$

This leads to

$$q_z(a,r,s) = \frac{1}{\pi^2}\int_0^\infty\int_0^\infty\int_{-\infty}^{+\infty} \theta\,\mathrm{sh}(\theta\pi)\cos(\theta(s-r-2u))e^{-z\theta^2} \cdot \frac{1}{\xi}\exp\Big(-\frac{ae^{r+s}\xi}{2} - \frac{2}{\xi}\mathrm{ch}(u)\,\mathrm{ch}(r-s+u)\Big)\,du\,d\xi\,d\theta\,.$$

With $\vartheta := 2u + r - s$ we have the identity

$$2\,\mathrm{ch}(u)\,\mathrm{ch}(r-s+u) = \mathrm{ch}(r-s) + \mathrm{ch}(\vartheta)\,.$$

That implies

$$q_z(a,r,s) = \frac{1}{\pi^2}\int_0^\infty\int_0^\infty \Big(\int_0^\infty \theta\,\mathrm{sh}(\theta\pi)\cos(\theta\vartheta)e^{-z\theta^2}\,d\theta\Big) \cdot \frac{1}{\xi}\exp\Big(-\frac{ae^{r+s}\xi}{2} - \frac{1}{\xi}(\mathrm{ch}(r-s)+\mathrm{ch}(\vartheta))\Big)\,d\vartheta\,d\xi\,.$$

With partial integration and [13, Eq. 4.133] we get for $z \in ]0,\infty[$

$$\begin{aligned}&\int_0^\infty \theta\,\mathrm{sh}(\theta\pi)\cos(\theta\vartheta)e^{-z\theta^2}\,d\theta \\ &= \frac{\sqrt{\pi}}{2\sqrt{z}}\exp\Big(\frac{\pi^2-\vartheta^2}{4z}\Big)\Big(\frac{\pi}{2z}\cos\Big(\frac{\pi\vartheta}{2z}\Big) - \frac{\vartheta}{2z}\sin\Big(\frac{\pi\vartheta}{2z}\Big)\Big) \\ &= \frac{\sqrt{\pi}}{2\sqrt{z}}\exp\Big(\frac{\pi^2}{4z}\Big)\partial_\vartheta\Big(\exp\Big(-\frac{\vartheta^2}{4z}\Big)\sin\Big(\frac{\pi\vartheta}{2z}\Big)\Big)\,.\end{aligned}$$

As all terms in the preceding calculation are holomorphic in $z$, this identity holds for all $z \in \mathbb{C}$ with $\Re(z) > 0$. After partial integration with respect to the $\vartheta$-variable we get eventually formula (11). □

**Corollary 2.9.** *Let $z \in \mathbb{C}$ with $\Re(z) > 0$, and let us define $q_z(a,r,s)$ for $a = 0$ via (11). Then $q_z$ is continuous on $[0,\infty[\times\mathbb{R}^2$, and there exists a $C_z > 0$, just depending on $z$, with*

$$|q_z(a,r,s)| \le C_z\,\mathrm{ch}(r-s)^{-1} \quad \text{for all } a \ge 0 \text{ and } r,s \in \mathbb{R}. \tag{12}$$

*Furthermore, identity (10) holds for all $a \ge 0$ and all $\varphi \in L^2(\mathbb{R})$.*

*Proof.* It is easy to see that there exists a $C_z > 0$, just depending on $z$, with $|\Psi_z| \leq C_z \xi^{-2}$. This implies

$$|q_z(a, r, s)| \leq C_z \int_0^\infty \xi^{-2} \exp\Big(-\frac{\mathrm{ch}(r-s)}{\xi}\Big)\, d\xi = C_z \,\mathrm{ch}(r-s)^{-1}\,.$$

The continuity of $q_z$ follows now from the dominated convergence theorem.

Let $a > 0$. Since (12) holds, both sides of (10) define bounded linear operators on $L^2(\mathbb{R})$, which then have to be equal on the whole space $L^2(\mathbb{R})$.

Let now $a = 0$: From (12) and the dominated convergence theorem we have for $\varphi$, $\psi \in L^2(\mathbb{R})$ and $a' > 0$

$$\begin{aligned}\langle e^{-zA(a')}\varphi, \psi\rangle &= \int_{\mathbb{R}}\int_{\mathbb{R}} q_z(a', r, s)\varphi(s)\psi(r)^*\, ds\, dr\\ &\longrightarrow \int_{\mathbb{R}}\int_{\mathbb{R}} q_z(0, r, s)\varphi(s)\psi(r)^*\, ds\, dr \qquad \text{for } a' \searrow 0\,.\end{aligned}$$

As $e^{-zA(a')}\varphi \to e^{-zA(0)}\varphi$ for $a' \searrow 0$ (see, e.g., [31, Thm. 9.17]), (10) holds also for $a = 0$ and arbitrary $\varphi \in L^2(\mathbb{R})$. □

Now let $z \in \mathbb{C}$ with $\Re(z) > 0$ and $r$, $s \in \mathbb{R}$. According to (12) the function $a \mapsto q_z(a, r, s)$ is bounded on $[0, \infty[$, and in the sense of bounded functional calculus we obtain

$$q_z(L, r, s) = \int_0^\infty \Psi_z(\xi) \exp\Big(-\frac{\mathrm{ch}(r-s)}{\xi}\Big) \exp\Big(-\frac{\xi e^{r+s}}{2}L\Big)\, d\xi\,. \tag{13}$$

Hereby the integrand on the right hand side of (13) is an $L^1$-mapping from $]0, \infty[$ with values in the space of bounded linear operators on $L^2(X)$.

**Proposition 2.10.** *The positive operator $T$ is selfadjoint and its holomorphic semigroup $(e^{-zT})_{\Re(z)>0}$ is given by*

$$e^{-zT} f(r, x) = \Big(\int_{\mathbb{R}} q_z(L, r, s) f_s\, ds\Big)(x) \tag{14}$$

*for all $f \in L^2(\mathbb{R} \times X)$ and almost all $(r, x) \in \mathbb{R} \times X$.*

*Proof.* According to the spectral theorem, $L$ is unitarily equivalent to an operator of multiplication on some $L^2(Y)$, where $Y$ is another $\sigma$-finite measure space. It is therefore sufficient to consider the case where $L\psi = m\psi$ for some measurable $m : X \to [0,\infty[$. For $z \in \mathbb{C}$ with $\Re(z) > 0$ let us define the operator $S(z)$ by

$$S(z)f(r,x) = \int_{\mathbb{R}} q_z\big(m(x),r,s\big)f(s,x)\,ds$$

for $f \in L^2(\mathbb{R}\times X)$. It is easy to see from the properties of the holomorphic semigroups $(e^{-zA(a)})_{\Re(z)>0}$, $a \geq 0$, and their kernels $q_z(a,r,s)$ that $(S(z))_{\Re(z)>0}$ is a semigroup of operators on $L^2(\mathbb{R}\times X)$ and that the mapping $z \mapsto \langle S(z)f, g\rangle_{L^2(\mathbb{R}\times X)}$ is holomorphic for all $f$, $g \in L^2(\mathbb{R}\times X)$. Since the semigroup $(S(t))_{t\geq 0}$ is selfadjoint, the same holds for its infinitesimal generator $-G$. It is straightforward to show that $\mathscr{A}\otimes \mathrm{D}(L) \subseteq \mathrm{D}(G)$ and that $T = G$ on $\mathscr{A}\otimes \mathrm{D}(L)$. Thus $T = G|_{\mathrm{D}(T)}$.

To conclude the proof, we only have to verify that $T$ is a selfadjoint operator, since then $T = G$ and $e^{-zT} = S(z)$. One can surely realize that in several ways, here we want to sketch the following approach: We prove that $\mathrm{D}(T)$ is a core of $G$ (which implies $T = G$). For this it is sufficient to show that $\mathrm{D}(T)$ is invariant under $(S(t))_{t\geq 0}$ [8, Thm. 1.9]. The technical problem is here that our description of $\mathrm{D}(T)$ is somewhat abstract, namely that $\mathrm{D}(T)$ is the completion of $\mathscr{A}\otimes \mathrm{D}(L)$ with respect to the graph norm of $T$. To handle this, we define some "discrete approximation" of $S(t)$: For $k = 1,\ldots,2^{2n}$ put $a_{n,k} = n^{-1} + k2^{-n}$ and $V(n,k) = m^{-1}([a_{n,k}, a_{n,k+1}[)$. Furthermore, let $a_{n,0} = 0$ and $V(n,0) = m^{-1}(\{0\})$. We define the operator $\pi_n$ on $L^2(\mathbb{R})\otimes L^2(X)$ by

$$\pi_n \eta\otimes\xi = \sum_{k=0}^{2^{2n}} e^{-tA(a_{n,k})}\eta\otimes 1_{V(n,k)}\xi\,,$$

where $1_{V(n,k)}$ is the characteristic function of $V(n,k)$. It is not hard to see that the range of $\pi_n$ is contained in $\mathrm{D}(T)$. Moreover, the properties of $q_t$ and the theorem of dominated convergence ensure that $\lim_{n\to\infty}\pi_n\eta\otimes\xi = S(t)\eta\otimes\xi$ for all $\eta \in L^2(\mathbb{R})$, $\xi \in L^2(X)$.

We prove now $\lim_{n\to\infty} T(\pi_n\varphi\otimes\psi) = S(t)(T\varphi\otimes\psi)$ for all $\varphi \in \mathscr{A}$, $\psi \in \mathrm{D}(L)$, which then establishes $S(t)(\mathscr{A}\otimes\mathrm{D}(L)) \subseteq \mathrm{D}(T)$. Since a semigroup and its generator commute on the domain of the generator, we get for $k = 0$

$$\begin{aligned} &T(e^{-tA(0)}\varphi\otimes 1_{V(n,0)}\psi) = -(e^{-tA(0)}\varphi)''\otimes 1_{V(n,0)}\psi \\ =&e^{-tA(0)}(-\varphi'')\otimes 1_{V(n,0)}\psi = (e^{-tA(0)}\otimes 1_{V(n,0)})(T\varphi\otimes\psi) \end{aligned}$$

and for $k > 0$

$$T(e^{-tA(a_{n,k})}\varphi \otimes 1_{V(n,k)}\psi) = (e^{-tA(a_{n,k})} \otimes 1_{V(n,k)})(T\varphi \otimes \psi) + \\ + \big(e^{-tA(a_{n,k})}\varphi'' - (e^{-tA(a_{n,k})}\varphi)''\big) \otimes \Big(1 - \frac{m}{a_{n,k}}\Big) 1_{V(n,k)}\psi \,.$$

From this follows almost directly

$$\lim_{n\to\infty} T(\pi_n\varphi \otimes \psi) = \lim_{n\to\infty} \pi_n(T\varphi \otimes \psi) = S(t)(T\varphi \otimes \psi)\,.$$

Let us now consider an arbitrary $f \in \mathrm{D}(T)$. Then there exists a sequence $(f_n)$ in $\mathscr{A} \otimes \mathrm{D}(L)$ with $f_n \to f$ and $Tf_n \to Tf$. Since $S(t)$ is bounded, we have $S(t)f_n \to S(t)f$. From what we have shown so far follows $TS(t)f_n = S(t)Tf_n \to S(t)Tf$. Thus the sequence $(S(t)f_n)$ in $\mathrm{D}(T)$ converges to $S(t)f$ in the graph norm of $T$. This eventually proves $S(t)(\mathrm{D}(T)) \subseteq \mathrm{D}(T)$ for all $t \geq 0$. □

Theorem 2.1 follows now from Proposition 2.10 and identity (13).

*Proof of Theorem 2.2.* Let $\varphi \in L^2 \cap L^p(\mathbb{R} \times X)$. From Proposition 2.10 and (13) follows

$$\begin{aligned} &\|e^{-zT}\varphi\|_{L^p(\mathbb{R}\times X)} \\ &\leq \Big\| \int_{\mathbb{R}} \int_0^\infty \Big|\Psi_z(\xi) \exp\Big(-\frac{\mathrm{ch}(r-s)}{\xi}\Big)\Big| \Big\| e^{-\frac{\xi e^{r+s}}{2}L}\varphi_s \Big\|_{L^p(X)} \, d\xi \, ds \Big\|_{L^p(\mathbb{R},\, dr)} \\ &\leq C \Big\| \int_{\mathbb{R}} \int_0^\infty \Big|\Psi_z(\xi) \exp\Big(-\frac{\mathrm{ch}(s)}{\xi}\Big)\Big| \|\varphi_{s+r}\|_{L^p(X)} \, d\xi \, ds \Big\|_{L^p(\mathbb{R},\, dr)} \\ &\leq C \Big( \int_{\mathbb{R}} \int_0^\infty \Big|\Psi_z(\xi) \exp\Big(-\frac{\mathrm{ch}(s)}{\xi}\Big)\Big| \, d\xi \, ds \Big) \|\varphi\|_{L^p(\mathbb{R}\times X)} \,. \end{aligned}$$

It was shown in [27, Sec. 3] that

$$\int_{\mathbb{R}} \int_0^\infty \Big|\Psi_z(\xi) \exp\Big(-\frac{\mathrm{ch}(s)}{\xi}\Big)\Big| \, d\xi \, ds \leq (2 + \sqrt{\Re(z)}) e^{\frac{\pi^2}{4\Re(z)}} \Big(1 + \frac{|\Im(z)|}{\Re(z)}\Big)^{3/2} .$$

That proves (3). As we demonstrated earlier, (3) implies the multiplier statement of Theorem 2.2. Hence the proof of Theorem 2.2 is complete. □

**Remark 2.11.** In a similar manner as in the proof of Theorem 2.2 one can try to use representation (1) to obtain (under certain conditions on $L$) also bounds for $\|e^{-zT}\|_{L^p(\mathbb{R}\times X)\to L^q(\mathbb{R}\times X)}$, $p \neq q$. In [12] we derived, e.g., some sort of ultracontractivity result for $e^{-zT}$. Depending on $L$, such a result may be of interest for some operators $T = T(L)$.

*Proof of Theorem 2.3.* Let the first assumptions on $(p_t)_{t>0}$ and $(\Lambda_t)_{t>0}$ hold. Let $\eta$, $\tau \in L^2(\mathbb{R}\times X)$. We want to be able to write the integrals appearing in $\langle e^{-zT}\eta, \tau\rangle_{L^2(\mathbb{R}\times X)}$ in any order, so first we show that the theorem of Fubini-Tonelli is applicable: Using the Cauchy-Schwarz inequality we get, with a constant $M_z$ just depending on $z$,

$$\int_{\mathbb{R}}\int_{\mathbb{R}}\int_0^\infty \int_X\int_X \Big|\Psi_z(\xi)\exp\Big(-\frac{\mathrm{ch}(r-s)}{\xi}\Big)p_{\frac{\xi e^{r+s}}{2}}(x,y)\eta(s,y)\tau(r,x)\Big|\,dy \cdot dx\,d\xi\,ds\,dr$$
$$\leq \int_{\mathbb{R}}\int_{\mathbb{R}}\int_0^\infty \Big|\Psi_z(\xi)\exp\Big(-\frac{\mathrm{ch}(r-s)}{\xi}\Big)\Big|\Big|\big\langle\Lambda_{\frac{\xi e^{r+s}}{2}}|\eta_s|,|\tau_r|\big\rangle_{L^2(X)}\Big|\,d\xi\,ds\,dr$$
$$\leq \kappa\int_{\mathbb{R}}\Big(\int_0^\infty\Big|\Psi_z(\xi)\exp\Big(-\frac{\mathrm{ch}(s)}{\xi}\Big)\Big|\,d\xi\Big)\Big(\int_{\mathbb{R}}\|\eta_{s+r}\|_{L^2(X)}\|\tau_r\|_{L^2(X)}\,dr\Big)\,ds$$
$$\leq \kappa\, M_z\|\eta\|_{L^2(\mathbb{R}\times X)}\|\tau\|_{L^2(\mathbb{R}\times X)}\,.$$

The theorem of Fubini-Tonelli ensures now the existence of the integral in (4) for almost all $((r,x),(s,y))\in(\mathbb{R}\times X)^2$. From (13) and (14) we get

$$\langle e^{-zT}\eta,\tau\rangle_{L^2(\mathbb{R}\times X)} = \int_X\int_{\mathbb{R}}\Big(\int_X\int_{\mathbb{R}} P_z((r,x),(s,y))\eta(s,y)\,ds\,dy\Big)\tau(r,x)^*\,dr\,dx\,.$$

Thus $P_z$ is the integration kernel of $e^{-zT}$.

Let now in addition $\|p_t(\cdot,x)\|_{L^1(X)}$ be bounded independently of $x$ and $t$. Then (5) can be proven in a similar way as (3). As (5) implies (3) for $p=1$, the multiplier statement in Theorem 2.3 follows from Theorem 2.2. □

# 3 Differential operators on solvable Lie groups with exponential volume growth

Let $\mathfrak{n}$ be a real stratified nilpotent Lie algebra, i.e., there exist subspaces $V_1,...,V_q$ of $\mathfrak{n}$ with $\mathfrak{n}=V_1\oplus...\oplus V_q$ and $[V_i,V_j]\subseteq V_{i+j}$ (convention: $V_k=0$ if $k>q$), and $V_1$ generates the whole Lie algebra $\mathfrak{n}$. (For stratified nilpotent Lie groups and algebras we refer to [9].) Let the derivation $D$ on $\mathfrak{n}$ be defined by $Dv_j = jv_j$ for all $v_j\in V_j$ and the group homomorphism $\theta:\mathbb{R}\to\mathrm{Aut}(\mathfrak{n})$ by $\theta(r)=e^{rD}$. Furthermore, let $N$ be the set $\mathfrak{n}$, endowed with the Campbell Hausdorff multiplication; thus $N$ is, up to isomorphism, the uniquely determined connected and simply connected nilpotent Lie group with Lie algebra $\mathfrak{n}$. The exponential mapping $\exp_N$ is the identity $I_{\mathfrak{n}}$ on $\mathfrak{n}$. The Lebesgue measure $dn$ on the Euclidean space $\mathfrak{n}$ is a biinvariant Haar measure on $N$. If $Q$ denotes the trace of $D$ (the so-called *homogeneous dimension* of $N$), then

$$\int_N f(e^{rD}n)\,dn = e^{-rQ}\int_N f(n)\,dn \quad \text{for all } f \in L^1(N).$$

We define the solvable Lie group $G$ by $G := N \rtimes_\theta \mathbb{R}$ ("$\mathbb{R}$ is acting on $N$ via natural dilations"). If $dr$ denotes the Lebesgue measure on $\mathbb{R}$, then $dg := dn \otimes dr$ is a right invariant Haar measure on $G$. The modular function $m$ on $G$ is given by $m(n,r) = e^{rQ}$, hence $G$ has exponential volume growth (see [30, §IX.1]).

In the section *"Improvements and open problems"* of [16] W. Hebisch asked whether the evolution kernel $P_z$ of a sum of even powers of vector fields or of a Schrödinger operator $T$ on $G$ satisfies an estimate like $\|P_{1+i\xi}(\cdot, 1_G)\|_{L^1(G)} \le C(1+|\xi|^{\frac{Q+4}{2}})$. (In [16] Hebisch derived such an inequality for sub-Laplacians on $G$. This result is obviously not as good as the estimate (5), which is independent of the homogeneous dimension of $X := N$ and was shown by Mustapha in [27] in the case of sub-Laplacians (cf. Subsection 3.1). But the methods of Hebisch have the advantage to extend to a reasonably larger class of Lie groups [11].)

For some special cases we can give a positive answer (independent of the homogeneous dimension of $N$) by employing Theorem 2.2. We use the following notation: For $\mathcal{X} \in \mathfrak{n}$ define vector fields $\mathcal{X}^N$ on $N$ and $\mathcal{X}^G$ on $G$ by

$$\mathcal{X}^N\psi(x) = \frac{d}{dt}\psi(x \cdot \exp_N(t\mathcal{X}))|_{t=0} \quad \text{for } \psi \in C^1(N),$$
$$\mathcal{X}^G\varphi(x,r) = \frac{d}{dt}\varphi((x,r) \cdot \exp_G(t\mathcal{X}))|_{t=0} \quad \text{for } \varphi \in C^1(G).$$

Furthermore, put $\mathcal{X}_0^G := \partial_r$. Then $\mathcal{X}_0^G$ is a left invariant vector field on $G$.

## 3.1 Sub-Laplacians

Let $\mathcal{X}_1, ..., \mathcal{X}_m \in V_1$ fulfill *Hörmander's condition*, i.e., generate $\mathfrak{n}$ as a Lie algebra. The operator $-\sum_{j=1}^m (\mathcal{X}_j^N)^2$ is defined on $C_c^\infty(N)$; let $L$ denote its closure. $L$ is positive and selfadjoint on $L^2(N)$ (see, e.g., Theorem 4.1), and hypoelliptic [19]. The semigroup of $L$ is given by convolution from the right with a smooth heat kernel $\phi_t$, $t > 0$, which satisfies $\|\phi_t\|_{L^1(N)} = 1$:

$$e^{-tL}\varphi = \varphi * \phi_t \,.$$

Thus $L$ induces a semigroup of contractions on $L^1(N)$, i.e.,

$$\|e^{-tL}\varphi\|_{L^1(N)} \leq \|\varphi\|_{L^1(N)}\|\phi_t\|_{L^1(N)} = \|\varphi\|_{L^1(N)} \,.$$

Therefore the sub-Laplacian $T := -\sum_{j=0}^{m}(\mathcal{X}_j^G)^2$ on $G$, which is of the form $T = -\partial_r^2 + e^{2r}L$, fulfills the conditions of Theorem 2.2. Consequently, each compactly supported continuous $f \in H^\kappa(\mathbb{R})$, $\kappa > 2$, is an $L^p$-multiplier with respect to $T$ for any $p \in [1, \infty]$.

This multiplier result was verified by S. Mustapha in [27]. Mustapha derived the representation (4) for the heat kernel of $T$ by using stochastic methods and a formula from [32].

## 3.2 Non-hypoelliptic sums of squares of vector fields

Here $\mathcal{X}_1, ..., \mathcal{X}_m \in V_1$ are not required to fulfill Hörmander's condition. Thus the closure $L$ of $-\sum_{j=1}^{m}(\mathcal{X}_j^N)^2$ is still positive and selfadjoint (see again Theorem 4.1), but in general not hypoelliptic. Although we cannot expect $L$ to have a smooth heat kernel, it still induces a semigroup of contractions on $L^1(N)$, because

$$e^{-tL}\varphi = \int_N \varphi \circ \rho_x \, dp_t(x) \,,$$

where $\rho_x$ is the right translation on $N$ by $x$ and $(p_t)_{t>0}$ is a convolution semigroup of probability measures on $N$ [21]. Therefore $T := -\sum_{j=1}^{m}(\mathcal{X}_j^G)^2$ fulfills still the conditions of Theorem 2.2. Again each compactly supported continuous $f \in H^\kappa(\mathbb{R})$, $\kappa > 2$, is an $L^p$-multiplier with respect to $T$ for every $p \in [1, \infty]$.

Theorem 2.2 and this subsection show that the hypoellipticity of the sub-Laplacians plays no crucial role in Mustapha's multiplier result.

## 3.3 Schrödinger operators

Let $\mathcal{X}_1, ..., \mathcal{X}_m \in V_1$ and $\tilde{f}_1, ..., \tilde{f}_m \in C^1(N, \mathbb{R})$. We consider

$$\tilde{L} := -\sum_{j=1}^{m}(\mathcal{X}_j^N + i\tilde{f}_j)^2 \,.$$

Moreover, define $f_0 := 0$ and $f_j := \tilde{f}_j \otimes e^r$ for $j = 1, ..., m$. The operator

$$T := -\sum_{j=0}^{m} (\mathcal{X}_j^G + i f_j)^2$$

is of the form $T = -\partial_r^2 + e^{2r}\tilde{L}$. $\tilde{L}$ and $T$ are essentially selfadjoint (see Theorem 4.1), and we denote their closures again by $\tilde{L}$ and $T$. (If $\{\mathcal{X}_1, ..., \mathcal{X}_m\}$ spans $V_1$, then $\tilde{L}$ and $T$ are called *Schrödinger operators*.)

Now let $L$ be the main part of $\tilde{L}$, i.e., $L = -\sum_{j=1}^{m} (\mathcal{X}_j^N)^2$. Then

$$\|e^{-t\tilde{L}}\varphi\|_{L^1(N)} \leq \|e^{-tL}|\varphi|\|_{L^1(N)} \leq \|\varphi\|_{L^1(N)}$$

for any $\varphi \in L^1 \cap L^2(N)$ since $|e^{-t\tilde{L}}\varphi| \leq e^{-tL}|\varphi|$ pointwise almost everywhere (see, e.g., [15, Lemma 1.3] and its proof or, for a detailed proof of the whole statement, [10, Lemma 3.21]). Hence Theorem 2.2 implies once again that each $f \in C_c \cap H^\kappa(\mathbb{R})$, $\kappa > 2$, is an $L^p$-multiplier with respect to $T$ for every $p \in [1, \infty]$.

Notice that, in contrast to sub-Laplacians, the operators $T$ defined in this subsection are not left invariant.

## 3.4 Rockland operators on $N$

A left invariant differential operator $L$ on $N$ is called *homogeneous of degree* $d \in \mathbb{N}$ if $L(\varphi \circ \theta(r)) = e^{dr}(L\varphi) \circ \theta(r)$ holds for all $\varphi \in C_c^\infty(N)$ and all $r \in \mathbb{R}$. If in addition for every non-trivial irreducible unitary representation $\pi$ of $N$ the operator $d\pi(L)$ is injective on the space of $C^\infty$-vectors of $\pi$ (i.e., the set of elements $\varphi$ of the representation space $\mathscr{H}_\pi$, where $N \ni x \mapsto \pi(x)\varphi$ is a $C^\infty$-function), $L$ is called *Rockland operator*. (For Rockland operators see [9].)

We consider here a Rockland operator $L$ which is positive and formally selfadjoint on $C_c^\infty(N)$. Then $L|_{C_c^\infty(N)}$ is essentially selfadjoint and its closure shall again be denoted by $L$. $L$ induces a semigroup on $L^2(N)$ by $e^{-tL}f = f * \phi_t$, $t > 0$, where $\phi_t$ is in $C^\infty \cap L^1(N)$ and satisfies $\|\phi_t\|_{L^1(N)} = \|\phi_1\|_{L^1(N)} =: C$ for all $t > 0$.

In the notation of Theorem 2.3 we have $p_t(x, y) = \phi_t(y^{-1}x)$, which implies

$$\|p_t(\cdot, m)\|_{L^1(N)} = \int_N |\phi_t(m^{-1}n)|\, dn = \int_N |\phi_t(n)|\, dn = C\,.$$

By

$$(\tilde{L}f)(n,r) := e^{rd}(Lf(\,\cdot\,,r))(n)$$

we obtain a left invariant differential operator $\tilde{L}$ on $G$. If we define the operator $T$ on $G$ by $T = -\partial_r^2 + \tilde{L}$, then $T = -\partial_r^2 + e^{rd}L$. From Theorem 2.3 we get for the convolution kernel $(\Phi_z)_{\Re(z)>0}$ of $e^{-zT}$

$$\Phi_z(n,r) = P_z((n,r),1_G) = \int_0^\infty \nu\Psi_{\nu^2 z}(\xi)\exp\Big(-\frac{\mathrm{ch}(\nu r)}{\xi}\Big)\phi_{\frac{\xi e^{\nu r}}{2\nu^2}}(n)\,d\xi\,,$$

where $\nu := d/2$ and $1_G$ is the neutral element of $G$. Furthermore,

$$\|\Phi_{1+i\xi}\|_{L^1(G)} \leq C_\nu(1+|\xi|)^{3/2}\,,$$

$C_\nu$ independent of $\xi$, and all compactly supported, continuous $f \in H^{2+\varepsilon}(\mathbb{R})$, $\varepsilon > 0$, are $L^p$-multipliers of $T$ for any $p \in [1,\infty]$.

The last example is a special case of the preceding class of differential operators:

## 3.5 Sums of even powers of vector fields

Let $\mathcal{X}_1,...,\mathcal{X}_m \in \mathfrak{n}$ generate the Lie algebra $\mathfrak{n}$, and let $k_1,...,k_m \in \mathbb{N}$. Moreover, let the differential operator

$$L := \sum_{j=1}^m (-1)^{k_j}(\mathcal{X}_j^N)^{2k_j}$$

on $C_c^\infty(N)$ be homogeneous of degree $2\nu$, $\nu = \max\{k_j : 1 \leq j \leq m\}$. (Hence $\mathcal{X}_j \in V_l$ implies $lk_j = \nu$.) $L$ is then a positive Rockland operator. If we define the sum of even powers of vector fields $T$ on $G$ by

$$T = -\partial_r^2 + \sum_{j=1}^m (-1)^{k_j}(\mathcal{X}_j^G)^{2k_j}\,,$$

then $T = -\partial_r^2 + e^{2\nu r}L$. We have the results of Subsection 3.4 for $d = 2\nu$.

Subsection 3.5 stresses that a differential operator $T$ has not necessarily to be of second order to satisfy the conditions of our multiplier theorem.

# 4 Appendix: A selfadjointness theorem

Let $G$ be a real Lie group with a countable number of connection components, $\mathfrak{g}$ its Lie algebra, $dg$ a right invariant Haar measure on $G$ and $L^2 = L^2(G, dg)$. Let $\langle\cdot,\cdot\rangle$ be the scalar product and $\|\cdot\|$ the norm on $L^2$. We shall identify each $\mathcal{X} \in \mathfrak{g}$ with a left invariant vector field by

$$\mathcal{X} f(g) = \frac{d}{dt} f(g \cdot \exp(t\mathcal{X}))|_{t=0} \,,$$

where exp denotes the exponential function with respect to $G$ and $\mathfrak{g}$.

Let $J$ be a finite index set, $\mathcal{X}_j$, $j \in J$, left invariant vector fields and $V_j$, $j \in J$, real valued, continuously differentiable functions on $G$. The operator

$$T := -\sum_{j\in J} (\mathcal{X}_j + iV_j)^2$$

is well defined on $C_c^\infty = C_c^\infty(G)$, the space of test functions on $G$.

For $f \in L^2$ and for operators $\Phi$, $\Phi^*$ defined on $C_c^\infty$ with $\langle \Phi\varphi, \psi\rangle = \langle \varphi, \Phi^*\psi\rangle$ for all $\varphi$, $\psi \in C_c^\infty$ we shall say that $\Phi f$ *exists in a weak sense* if there is a function $\tilde{f} \in L^2$ with $\langle f, \Phi^*\varphi\rangle = \langle \tilde{f}, \varphi\rangle$ for all $\varphi \in C_c^\infty$. In this case we define $\Phi_w f := \tilde{f}$, so the domain of the operator $\Phi_w$ is given by

$$\mathrm{D}(\Phi_w) = \{f \in L^2 : \Phi f \quad \text{exists in a weak sense}\} \,.$$

A helpful tool for the proof of our selfadjointness theorem is the *convolution* of two (suitable) functions $\varphi$, $\psi$ on $G$, defined by

$$\varphi * \psi(x) = \int_G \varphi(xg^{-1})\psi(g)\, dg \,.$$

We call a sequence $(\varphi_n)$ in $C_c^\infty$ a *Dirac sequence* if $\varphi_n \geq 0$, $\int \varphi_n\, dg = 1$ for all $n \in \mathbb{N}$ and if for each neighborhood $U$ of the unit element of $G$ there exists an $n_0 \in \mathbb{N}$ with $\mathrm{supp}(\varphi_n) \subseteq U$ for every $n \geq n_0$.

We also shall utilize a sequence $(\psi_n)$ in $C_c^\infty$ that approximates the characteristic function of $G$ in the following way:

(i) For any $n \in \mathbb{N}$ the condition $0 \leq \psi_n \leq 1$ holds.

(ii) $(\psi_n^{-1}(\{1\}))$ is an increasing sequence of sets with $\cup\psi_n^{-1}(\{1\}) = G$.

(iii) For each left invariant differential operator $L$ there exists a $C > 0$, independent of $n$, with $|L\psi_n| \leq C$ for all $n$.

It is easy to see that such a sequence exists. (However, a construction can be found in [10, Lemma 2.28].)

Furthermore, we will make use of the theory of quadratic forms of selfadjoint operators and the well known theorem about Friedrichs' extension. This can, e.g., be found in [8, Ch. 4] and [31, §5.5].

**Theorem 4.1.** *$T$ is essentially selfadjoint on $C_c^\infty \subseteq L^2$. Its selfadjoint closure $\overline{T}$ is given by $T_w$. The domain of the quadratic form of $T_w$ is*

$$\mathrm{Q}(T_w) = \cap_{j\in J} \mathrm{D}((\mathcal{X}_j + iV_j)_w)$$

*and the inclusion $\mathrm{D}(T_w) \subseteq \mathrm{Q}(T_w)$ holds.*

*Proof.* From the definition of the adjoint operator $T^*$ of $T$ on $L^2$ it is clear that $T^* = T_w$. As $T$ is positive and symmetric on $C_c^\infty$, there exists Friedrichs' extension $S$ of $T$ with $\mathrm{D}(S) = \mathrm{D}(T_w) \cap \mathrm{Q}(S)$. Here the domain $\mathrm{Q}(S)$ of the quadratic form of $S$ is the set of all $f \in L^2(G)$ for which there exists a sequence $(f_n)$ in $C_c^\infty$ with $L^2\text{-}\lim f_n = f$ such that $((\mathcal{X}_j + iV_j)f_n)$ is an $L^2$-Cauchy sequence for each $j \in J$.

Obviously we have $\overline{T} \subseteq S = S^* \subseteq T^*$. Our aim is to show that $\overline{T} = S$, because that would imply $\overline{T} = T^* = T_w$. Before we do so, we verify that $\mathrm{Q}(S)$ is equal to $\Lambda := \cap_{j\in J} \mathrm{D}((\mathcal{X}_j + iV_j)_w)$. Since it is easy to see that $\mathrm{Q}(S) \subseteq \Lambda$ holds, we just have to prove

*Statement* $(a)$: $\Lambda \subseteq \mathrm{Q}(S)$.

To verify Statement (a), we first consider a compactly supported function $f \in \Lambda$ and a Dirac sequence $(\varphi_n)$. Then $(\varphi_n * f)$ is a sequence in $C_c^\infty$ with $\|f - \varphi_n * f\| \to 0$ for $n \to \infty$. Let $j \in J$. From $f \in \Lambda$ and $V_j f \in L^2$ follows $f \in \mathrm{D}((\mathcal{X}_j)_w)$. Therefore the left invariance of $\mathcal{X}_j$ leads us to $\mathcal{X}_j(\varphi_n * f) = \varphi_n * (\mathcal{X}_j)_w f$, and

$$\begin{aligned}&\|(\mathcal{X}_j + iV_j)_w f - (\mathcal{X}_j + iV_j)(\varphi_n * f)\| \\ \leq &\|(\mathcal{X}_j + iV_j)_w f - \varphi_n * (\mathcal{X}_j + iV_j)_w f\| + \|\varphi_n * V_j f - V_j(\varphi_n * f)\| .\end{aligned}$$

The first term on the right vanishes if $n \to \infty$. In general for continuous $\vartheta$ on $G$ and compactly supported $\tilde{f} \in L^2$ there holds

$$\|\varphi_n * \vartheta \tilde{f} - \vartheta(\varphi_n * \tilde{f})\| \leq \|\varphi_n * |\tilde{f}|\| \sup |\vartheta(x^{-1}g) - \vartheta(g)| , \qquad (15)$$

where the supremum is taken over all $x \in \mathrm{supp}(\varphi_n)$ and $g \in \mathrm{supp}(\varphi_n)\,\mathrm{supp}(\tilde{f})$. The expression on the right hand side in (15) tends to zero for $n \to \infty$. It follows that $f$ is an element of $\mathrm{Q}(S)$.

Let us now consider a general $f \in \Lambda$, and let $(\psi_n)$ be an approximating sequence of the characteristic function of $G$ as described above. It is trivial that $\psi_n f \in \Lambda$ with

$$(\mathcal{X}_j + iV_j)_w(\psi_n f) = \psi_n(\mathcal{X}_j + iV_j)_w f + (\mathcal{X}_j\psi_n)f\,.$$

Since $\operatorname{supp}(\psi_n f)$ is compact, we have $\psi_n f \in \mathrm{Q}(S)$. Obviously $\psi_n f \to f$ and

$$\|(\mathcal{X}_j + iV_j)_w(f - \psi_n f)\| \le \|(1-\psi_n)(\mathcal{X}_j + iV_j)_w f\| + \|(\mathcal{X}_j\psi_n)f\|\,.$$

The choice of $(\psi_n)$ implies $\|(\mathcal{X}_j+iV_j)_w(f-\psi_n f)\| \to 0$ for $n \to \infty$. It follows $f \in \mathrm{Q}(S)$, i.e., Statement (a) holds.

*Statement* (*b*): $\mathrm{D}(S) \subseteq \mathrm{D}(\overline{T})$.

Our strategy is similar as in the proof of Statement (a): First let $f \in \mathrm{D}(S)$ be compactly supported. Then $V_j f \in L^2$ and $f \in \mathrm{D}((\mathcal{X}_j)_w)$ for each $j \in J$, because $f$ is contained in $\mathrm{Q}(S) = \Lambda$ (recall that $\mathrm{D}(S) \subseteq \mathrm{Q}(S)$). Moreover, $f$ is contained in $\mathrm{D}(T_w)$, $\mathrm{D}((V_j\mathcal{X}_j)_w)$ and $V_j^2 f$, $(\mathcal{X}_j V_j)f \in L^2$ for each $j \in J$, which implies $f \in \mathrm{D}((\sum_{j\in J} X_j^2))_w)$.

Let us again consider a Dirac sequence $(\varphi_n)$. The sequence $(f_n)$, where $f_n := \varphi_n * f$, is a sequence in $C_c^\infty$ with $f_n \to f$ in $L^2$. We obtain

$$\begin{aligned}\|T_w(f-f_n)\| \le&\|T_w f - \varphi_n * T_w f\| + \sum_{j\in J}\big(2\|\varphi_n * (V_j\mathcal{X}_j)_w f - V_j(\varphi_n *\\ &*(\mathcal{X}_j)_w f)\| + \|\varphi_n * (V_j^2 - i(\mathcal{X}_j V_j))f - (V_j^2 - i(\mathcal{X}_j V_j))\varphi_n * f\|\big)\,.\end{aligned}$$

With (15) we observe that $Tf_n \to T_w f$ in $L^2$, which means $f \in \mathrm{D}(\overline{T})$ and $\overline{T}f = T_w f$.

Now let $f$ be an arbitrary element in $\mathrm{D}(S)$, and let $(\psi_n)$ be as in the proof of Statement (a). Obviously $\psi_n f \in \mathrm{D}(S)$ for each $n \in \mathbb{N}$. As $\psi_n f$ has compact support, we have $\psi_n f \in \mathrm{D}(\overline{T})$. Moreover, $\psi_n f \to f$ in $L^2$ and

$$\begin{aligned}&\|T_w f - \overline{T}(\psi_n f)\|\\ \le&\|(1-\psi_n)T_w f\| + \sum_{j\in J}\big(\|(\mathcal{X}_j^2\psi_n)f\| + 2\|(\mathcal{X}_j\psi_n)(\mathcal{X}_j + iV_j)_w f\|\big)\,.\end{aligned}$$

From our definition of $(\psi_n)$ follows $\|T_w f - \overline{T}(\psi_n f)\| \to 0$ for $n \to \infty$. As $\mathrm{D}(\overline{T})$ is closed with respect to the graph norm of $T$, $f$ is an element of $\mathrm{D}(\overline{T})$. This proves Statement (b), which implies $\overline{T} = S$. □

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