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by

Fathi Dkhil, and Angela Stevens

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F. Dkhil* and A. Stevens†

Dedicated to K.P. Hadeler

Abstract

In this paper we study the effects of periodically varying heterogeneous media on the speed of traveling waves in reaction-diffusion equations. Under suitable conditions the traveling wave speed of the non-homogenized problem can be calculated in terms of the speed of the homogenized problem. We discuss a variety of examples and focus especially on the influence of the symmetric and antisymmetric part of the diffusion matrix on the wave speed.

Keywords Reaction-diffusion equations, homogenization, traveling wave speed

AMS classification 35B20, 35B27, 35B50, 35C20, 35K55, 35K57, 41A60

1 Introduction

Front propagation is a phenomenon occuring in various scientific contexts. Examples are: chemical kinetics, spread of epidemics, traveling population fronts in ecology and in population genetics, transport in porous media, transport of chemical signals through tissues, propagation of action potentials in coupled nerve cells, shear flows in cylinders, and reactive flows in composite materials. Several of these processes are modeled by reaction-diffusion-advection equations.

Many results are available on front propagation in homogeneous media and enviroments. The study of front propagation in heterogeneous media is more recent. Since heterogeneities occur in every natural environment, an important problem is to understand how these heterogeneities influence the traveling fronts, e.g. their location, profile, and their speed. In this paper we therefore consider a general ansatz first, and calculate the speed of traveling fronts of reaction-diffusion equations with rapidly oscillating diffusion and drift coefficients. For an excellent reference on mathematical results of propagation phenomena in heterogeneous media, especially on existence and stability of traveling waves, and further references not cited here, compare the survey paper [13].

*Département de Mathématiques, Institut Supérieur d'Informatique, Université Tunis El-Manar, 2 rue Abou Raihan Bayrouni, 2080 Ariana, Tunisie. (fathi.dkhil@isi.rnu.tn)

†Max-Planck-Institute for Mathematics in the Sciences, Inselstr. 22-26, D-04103 Leipzig, Germany. (stevens@mis.mpg.de)

The basic typ of equation we are dealing with is the following

$$\begin{array}{l} u_t = \nabla_z \left(A(z)\nabla_z u\right) + f(u) \qquad z = (z_1, ..., z_n) \in D = \mathbb{R} \times \Omega \subset \mathbb{R}^n, \;\; t > 0. \\ \text{with inital conditions } u(0,z) = u_*(z). \end{array} \tag{1}$$

Here $A(z)$ is a positive definite matrix, $f \in C^2([0,1])$ fulfills $f(0) = f(1) = 0$ and is either of bistable or combustion type. The cross section of the cylinder $\Omega \subset \mathbb{R}^{n-1}$ is bounded with $C^{1,\alpha}$ boundary. For simplicity we will consider waves in direction $e_1 = (1, 0, ..., 0)$.

Concerning existence and stability of traveling waves, when A is independent of z_1, the shear flow problem has been studied most. Existence of unique monotone waves for bistable nonlinearity and a convex cross section Ω, as well as for the combustion nonlinearity were shown in [2], [1]. Global stability in dimensions higher than one where shown in [9], extending the classical one-dimensional result by Fife and McLeod, [3].

For A depending on all spatial variables, $z_1, ..., z_n$ the existence of a wave for combustion nonlinearities was shown in [12], and in [11] for bistable nonlinearities, when the matrix $A(z)$ is close to a constant. In the later case also local stability in one space dimension could be proved. In [14] it was shown that waves do not exist in the bistable case if the oscillations of $A(z)$ are too large. The existence of traveling waves for systems in periodically perforated domains close to the homogenization limit was proved by Heinze [7]. This technique also covers the case of rapidly oscillating coefficients.

First results on the qualitative behavior of propagating waves, namely variational principles for the speed of front propagation for KPP-type nonlinearities, respectively Fisher's population genetic model, were given by Hadeler and Rothe in [5] and Gärtner and Freidlin in [4].

In the following we choose $z = \frac{x}{\varepsilon}$ and consider the problem

$$u_t = \nabla(A(\tfrac{x}{\varepsilon})\nabla u) + f(u), \qquad x \in \mathbb{R}^n, \qquad t > 0 \tag{2}$$

with initial data

$$u(0,x) = u_*(x).$$

We assume $A(z)$ to be 1-periodic in all variables, thus the cross section Ω is a unit torus. We suppose that this equation admits a unique monotone and stable traveling wave in direction e_1. As mentioned before waves to exist in the bistable case if ε is sufficiently small, but their stability has not yet been proven.
The variational characterization given in [8] allows to calculate the deviation of the wave speed of (2), if existing, from the wave speed of the related homogenized problem. A similar variational characterization was also given in [6] for A depending on $(z_2, .., z_n)$ only.

In [8] it was shown that in many cases the acceleration or slowing down of the wave speed for problem (2) can not be decided upon the first order expansion of the wave speed. Here we have a more detailed look at the problem and consider the second order coefficients of the expansion where necessary.

2 The variational characterization of the wave speed

DEFINITION 2.1 *(compare [8]). We define a wave u^* of (1) with velocity $c \neq 0$ in direction e_1 connecting the two zeros 0 and 1 of f by:*

- u^* *is a solution of (1)*
- $u^*\left(t+\frac{1}{c}, x-e_1\right) = u^*(t,x)$ *for all* $t \in \mathbb{R},\ x \in \mathbb{R}^n$
- $u^*(-\infty, x) = 0, \quad u^*(\infty, x) = 1.$

The wave u^* is stable with respect to some subset I_s of initial data if for the solution u of (1) with $u_* \in I_s$

$$\lim_{t\to\infty} \sup_{x\in\mathbb{R}^n} |u(t,x) - u^*(t, x+\theta e_1)| = 0$$

holds for some shift $\theta \in \mathbb{R}$.

Suppose that there exists a unique monotone and stable traveling wave (u^*, c) in direction of e_1. Let

$$\mathcal{K} = \left\{ v \in C^1(\mathbb{R}, C^2(D)) \middle| \partial_t v(t,x) > 0, 0 < v(t,x) < 1, v(0,.) \in I_s, v(t+1, x-e_1) = v(t,x) \right\}$$

be the sets of admissible comparison functions. For $v \in \mathcal{K}$ define

$$\psi(v(x)) = \frac{\nabla_x (A(x) \nabla_x v(t,x)) + f(v(x))}{\partial_t v(t,x)}.$$

In [8] a variational characterization of the wave speed was given for more general situations. Here we will state a specific version of the result as a lemma.

LEMMA 2.2 *[8] Suppose that there exists a unique stable traveling wave for problem (1) then the traveling wave speed c is given by*

$$\sup_{v\in\mathcal{K}} \inf_{(t,x)\in(\mathbb{R}\times D)} \psi(v(t,x)) = c = \inf_{v\in\mathcal{K}} \sup_{(t,x)\in(\mathbb{R}\times D)} \psi(v(t,x)).$$

The proof needs a maximum principle and relates technically to results given by Vol'pert et al in [10] for monotone systems of ODE's.

A traveling wave u_ε for problem (2) with wave speed c_ε has of course to satisfy

$$u_\varepsilon^*(t + \frac{\varepsilon}{c}, x - \varepsilon e_1) = u^*(t,x)$$

in Definition 2.1. Thus the condition for admissible test functions in $\mathcal{K} = \mathcal{K}_\varepsilon$ changes to

$$v(t+\varepsilon, x - \varepsilon e_1) = v(t,x).$$

By means of Lemma 2.2 we obtain an asymptotic expansion for c_ε. Here we consider terms up to second order. The first order expansion of the wave speed was already proved in [8]. We have to calculate cell problems of higher order and thus use different test functions.

3 Asymptotic expansion of c_ε

THEOREM 3.1 *Assume that the homogenized problem of (2) has a unique traveling front (up to translation) with nonzero speed and that there exists a unique monotone and stable traveling wave $(u_\varepsilon, c_\varepsilon)$ in direction e_1 which is a solution of (2). Then the wave speed c_ε has the following expansion*

$$c_\varepsilon = c_0 + \varepsilon c_1 + \varepsilon^2 c_2 + O(\varepsilon^3)$$

where c_0 is the speed of the homogenized problem.

Proof. We consider the test function:

$$v(t,x) = u_0(\xi) + \varepsilon u_1(\xi,z) + \varepsilon^2 u_2(\xi,z) + \varepsilon^3 u_3(\xi,z) + \varepsilon^4 u_4(\xi,z)$$

where $\xi = x_1 + t$, $z = \frac{x}{\varepsilon}$ and u_0 is a solution of the following homogenized problem

$$\begin{cases} A_0 u_0'' - c_0 u_0' + f(u_0) &= 0 \\ u_0(-\infty) = 0, \quad u_0(\infty) &= 1 \end{cases} \tag{3}$$

with

$$\begin{aligned}
u_1(\xi,z) &= \chi_1(z) u_0'(\xi) + \phi_1(\xi) \\
u_2(\xi,z) &= \chi_2(z) u_0''(\xi) + \chi_1(z)\phi_1'(\xi) + \phi_2(\xi) \\
u_3(\xi,z) &= \chi_3(z) u_0'''(\xi) + \chi_2(\xi)\phi_1''(\xi) + \chi_1(z)\phi_2'(\xi) \\
u_4(\xi,z) &= \chi_4(z) u_0^{(4)}(\xi) + \chi_3(z)\phi_1'''(\xi) + \chi_2(\xi)\phi_2''(\xi) + h u_0'^2(\xi) f''(u_0(\xi)).
\end{aligned}$$

Here χ_i, h solve the following cell problem (unique, 1-periodic, with zero average solutions) and ϕ_1, ϕ_2 will be chosen later

$$\begin{aligned}
\nabla_z A \nabla_z \chi_1 &= -\nabla_z A e_1 \\
\nabla_z A \nabla_z \chi_2 &= A_0 - e_1 A e_1 - e_1 A \nabla_z \chi_1 - \nabla_z A e_1 \chi_1 \\
\nabla_z A \nabla_z \chi_3 &= A_0 \chi_1 + A_1 - e_1 A e_1 \chi_1 - e_1 A \nabla_z \chi_2 - \nabla_z A e_1 \chi_2 \\
\nabla_z A \nabla_z \chi_4 &= A_0 \chi_2 + A_1 \chi_1 + A_2 - e_1 A e_1 \chi_2 - e_1 A \nabla_z \chi_3 - \nabla_z A e_1 \chi_3 \\
\nabla_z A \nabla_z h &= \chi_2 - \frac{\chi_1^2}{2} - \oint (\chi_2 - \frac{\chi_1^2}{2})
\end{aligned} \tag{4}$$

with

$$\begin{aligned}
A_0 &= \oint e_1 A(\nabla_z \chi_1 + e_1) = \oint (\nabla_z \chi_1 + e_1) A (\nabla_z \chi_1 + e_1) \\
A_1 &= \oint e_1 A(\nabla_z \chi_2 + e_1 \chi_1) \\
A_2 &= \oint e_1 A(\nabla_z \chi_3 + e_1 \chi_2).
\end{aligned} \tag{5}$$

Since A is positive definite we have $A_0 > 0$. Further on we know $v(t+\varepsilon, x - \varepsilon e_1) = v(t,x)$. Near $\pm\infty$ all derivatives of u_0 have the same exponential decay rate as u_0'. Also ϕ_1 and

ϕ_2, which will be specified below, have this decay rate. Hence $\partial_t v > 0$ for small ε and thus $v \in \mathcal{K}_\varepsilon$. Since $\nabla = e_1 \partial_\xi + \frac{1}{\varepsilon}\nabla_z$, an easy computation gives that

$$\begin{aligned}
\nabla(A\nabla v) + f(v) &= c_0 u_0' + \varepsilon\left[A_1 u_0''' + c_0 u_1' - c_0\phi_1' + A_0\phi_1'' + \phi_1 f'(u_0)\right] \\
&+\varepsilon^2\left[A_2 u_0^{(4)} + \chi_1\left(A_1 u_0^{(4)} + a_0\phi_1''' + \phi_1' f'(u_0) + u_0'\phi_1 f''(u_0)\right)\right. \\
&\quad +\chi_2\left(A_0 u_0^{(4)} + u_0'' f'(u_0) + {u_0'}^2 f''(u_0)\right) + A_1\phi_1''' - \beta {u_0'}^2 f''(u_0) \\
&\quad \left. + A_0\phi_2'' + \phi_2 f'(u_0) + \frac{\phi_1^2}{2} f''(u_0)\right] + O(\varepsilon^3)
\end{aligned}$$

where $\beta = \oint(\chi_2 - \frac{\chi_1^2}{2}) = -\frac{1}{2}\oint\chi_1^2$. We have

$$\frac{1}{\partial_t v} = \frac{1}{u_0'}\left(1 - \frac{\varepsilon u_1'}{u_0'} - \frac{\varepsilon^2 u_2'}{u_0'} + \frac{\varepsilon^2 {u_1'}^2}{{u_0'}^2}\right) + O(\varepsilon^3).$$

Thus

$$\begin{aligned}
\psi(v) &= c_0 + \frac{\varepsilon}{u_0'}\left[A_1 u_0''' + A_0\phi_1'' - c_0\phi_1' + \phi_1 f'(u_0)\right] \\
&+\tfrac{\varepsilon^2}{u_0'}\left[\tfrac{u_1'}{u_0'}\left(-A_1 u_0''' - A_0\phi_1'' + c_0\phi_1' - \phi_1 f'(u_0)\right)\right. \\
&\quad +\chi_1\left(A_1 u_0^{(4)} + A_0\phi_1''' - c_0\phi_1'' + \phi_1' f'(u_0) + u_0'\phi_1 f''(u_0)\right) \\
&\quad +A_2 u_0^{(4)} + A_1\phi_1''' - \beta {u_0'}^2 f''(u_0) \\
&\quad \left. + \frac{\phi_1^2}{2} f''(u_0) + A_0\phi_2'' - c_0\phi_2' + \phi_2 f'(u_0)\right] + O(\varepsilon^3).
\end{aligned}$$

We choose ϕ_1 and ϕ_2 such that the coefficients of ε and ε^2 for $\psi(v)$ are constants. This requires that ϕ_1 and ϕ_2 are solutions of the following two equations

$$\begin{aligned}
A_0\phi_1'' - c_0\phi_1' + \phi_1 f'(u_0) &= -A_1 u_0''' + c_1 u_0' \\
A_0\phi_2'' - c_0\phi_2' + \phi_2 f'(u_0) &= c_2 u_0' - A_2 u_0^{(4)} + \beta {u_0'}^2 f''(u_0) - \frac{\phi_1^2}{2} f''(u_0) - A_1\phi_1''' + c_1\phi_1'.
\end{aligned}$$

The solvability conditions imply that

$$c_1 \int_{\mathbb{R}} {u_0'}^2(x) e^{-\frac{c_0}{A_0}x} dx = A_1 \int_{\mathbb{R}} u_0'(x) u_0'''(x) e^{-\frac{c_0}{A_0}x} dx, \tag{6}$$

$$\begin{aligned}
c_2 \int_{\mathbb{R}} {u_0'}^2(x) e^{-\frac{c_0}{A_0}x} dx &= A_2 \int_{\mathbb{R}} u_0'(x) u_0^{(4)}(x) e^{-\frac{c_0}{A_0}x} dx - c_1 \int_{\mathbb{R}} u_0'(x)\phi_1'(x) e^{-\frac{c_0}{A_0}x} dx \\
&\quad + \int_{\mathbb{R}} u_0'(x)\left(A_1\phi_1''' - \beta {u_0'}^2 f''(u_0) + \frac{\phi_1^2}{2} f''(u_0)\right) e^{-\frac{c_0}{A_0}x} dx.
\end{aligned} \tag{7}$$

If the solvability condition (6) is satisfied then $\phi_1 = \phi + \lambda u_0'$ with $\lambda \in \mathbb{R}$; ϕ is uniquely determined, bounded and satisfies the following equations:

$$A_0\phi'' - c_0\phi' + \phi f'(u_0) = -A_1 u_0''' + c_1 u_0' \tag{8}$$

and

$$\int_{\mathbb{R}} u_0'\phi e^{-\frac{c_0}{A_0}x}dx = 0. \tag{9}$$

Hence the solvability condition (7) can be written (with some effort) as

$$\begin{aligned}
c_2\int_{\mathbb{R}} {u_0'}^2(x)e^{-\frac{c_0}{A_0}x}dx &= A_2\int_{\mathbb{R}} u_0'(x)u_0^{(4)}(x)e^{-\frac{c_0}{A_0}x}dx - c_1\int_{\mathbb{R}} u_0'(x)\phi'(x)e^{-\frac{c_0}{A_0}x}dx \\
&\quad + \int_{\mathbb{R}} u_0'(x)\left(A_1\phi''' + \frac{\phi^2}{2}f''(u_0)\right)e^{-\frac{c_0}{A_0}x}dx. \\
&= A_2\int_{\mathbb{R}} u_0'(x)u_0^{(4)}(x)e^{-\frac{c_0}{A_0}x}dx - c_1\int_{\mathbb{R}} u_0'(x)\phi'(x)e^{-\frac{c_0}{A_0}x}dx \\
&\quad + A_1\int_{\mathbb{R}} \phi(x)\left(-2u_0^{(4)}(x) - \frac{7c_0}{2A_0}u_0'''(x) + \frac{2c_0^2}{A_0^2}u_0''(x)\right)e^{-\frac{c_0}{A_0}x}dx. \qquad (10)
\end{aligned}$$

From the solvability conditions (6) and (10) we obtain c_1 and c_2 respectively. Therefore we have

$$\psi(v) = c_0 + \varepsilon c_1 + \varepsilon^2 c_2 + O(\varepsilon^3).$$

To conclude the result of the theorem, note that the asymptotic expansion of $\psi(v)$ up to the second order in ε is a constant. This ends the proof of Theorem 3.1.

PROPOSITION 3.2 *Let $\alpha > 0$ and $c_\alpha = c_{0,\alpha} + \varepsilon c_{1,\alpha} + \varepsilon^2 c_{2,\alpha} + O(\varepsilon^3)$ be the asymptotic expansion of the wave speed defined in Theorem 3.1 for the matrix αA. Then we have*

$$c_{0,\alpha} = \sqrt{\alpha}c_0, \quad c_{1,\alpha} = c_1 \quad \text{and} \quad c_{2,\alpha} = \frac{c_2}{\sqrt{\alpha}}$$

where c_0, c_1 and c_2 are given in Theorem 3.1.

Thus multiplication of the matrix A by a positive constant has no effect on the signs of c_0, c_1 and c_2. This will be used later when we give explicit examples.

Proof. Let $(v, c_{0,\alpha})$ be the unique solution of

$$\begin{cases} \alpha A_0 v'' - c_{0,\alpha}v' + f(v) = 0 \\ v(-\infty;\infty;0) = u_0(-\infty;\infty;0). \end{cases}$$

We know $A_i = A_i(A)$ for $i \in \{0,1,2\}$, and thus for all $\alpha > 0$ we have

$$A_i(\alpha A) = \alpha A_i(A).$$

By uniqueness, it follows that $c_{0,\alpha} = \sqrt{\alpha} c_0$ and for all $x \in \mathbb{R}$ we have

$$v(x) = u_0(\frac{x}{\sqrt{\alpha}}).$$

Equality (6) implies that

$$c_{1,\alpha} \int_{\mathbb{R}} v'^2(x) e^{-\frac{c_{0,\alpha}}{\alpha A_0} x} dx = \alpha A_1 \int_{\mathbb{R}} v'(x) v'''(x) e^{-\frac{c_{0,\alpha}}{\alpha A_0} x} dx.$$

By change of variable we get

$$\frac{c_{1,\alpha}}{\sqrt{\alpha}} \int_{\mathbb{R}} {u_0'}^2(x) e^{-\frac{c_0}{A_0} x} dx = \frac{\alpha A_1}{(\sqrt{\alpha})^3} \int_{\mathbb{R}} u_0'(x) u_0'''(x) e^{-\frac{c_0}{A_0} x} dx$$

which implies $c_{1,\alpha} = c_1$.

Let ϕ_α be the unique solution of

$$\alpha A_0 \phi_\alpha'' - c_{0,\alpha} \phi_\alpha' + \phi_\alpha f'(v) = -\alpha A_1 v''' + c_{1,\alpha} v'$$

with

$$\int_{\mathbb{R}} v' \phi_\alpha e^{-\frac{c_{0,\alpha}}{\alpha A_0} x} dx = 0.$$

Then $\phi_\alpha(x) = \frac{1}{\sqrt{\alpha}} \phi(\frac{x}{\sqrt{\alpha}})$ where ϕ is defined by (8) and (9).
After a change of variable the solvability condition (10) for $c_{2,\alpha}$ results in $c_{2,\alpha} = \frac{c_2}{\sqrt{\alpha}}$.

REMARK 3.3 *If $A_1 = 0$ then $c_1 = 0$ and c_2 is given by*

$$c_2 \int_{\mathbb{R}} {u_0'}^2(x) e^{-\frac{c_0}{A_0} x} dx = A_2 \int_{\mathbb{R}} u_0'(x) u_0^{(4)}(x) e^{-\frac{c_0}{A_0} x} dx. \tag{11}$$

This formulation will be used later to calculate the sign of c_2.

In the following we first try to get some information about A_2.

PROPOSITION 3.4 *Let $B = A^T$ and B_i be the corresponding constants defined by (5) for the matrix A^T. Then we have*

1. *$(i) B_0 = A_0, \quad (ii) B_1 = -A_1, \quad (iii) B_2 = A_2$*
2. *If A is a symmetric matrix then $A_1 = 0$, so $c_1 = 0$ and A_2 is given by*

$$A_2 = \oint \nabla_z (\chi_2 - \frac{\chi_1^2}{2}) A \nabla_z (\chi_2 - \frac{\chi_1^2}{2}).$$

In this case we have $A_2 \geq 0$ and c_2 can be calculated from (11).

The following assertions are equivalent:

(i) $A_2 = 0$

(ii) $\chi_2 - \frac{\chi_1^2}{2} = const.$

(iii) $h = 0$

(iv) $(e_1 + \nabla_z \chi_1) A (e_1 + \nabla_z \chi_1) = const.$*, which is equal to* A_0.

Proof. Let ψ_i be the solutions of the cell problem (4) with the matrix B in place of A.
Case 1.(i)

$$\begin{aligned} B_0 &= \oint e_1 B(\nabla_z \psi_1 + e_1) = \oint e_1 A e_1 - \oint \psi_1 \nabla_z A e_1 \\ &= \oint e_1 A e_1 + \oint \psi_1 \nabla_z A \nabla_z \chi_1 = \oint e_1 A e_1 + \oint \chi_1 \nabla_z B \nabla_z \psi_1 \\ &= \oint e_1 A e_1 - \oint \chi_1 \nabla_z B e_1 = \oint e_1 A(\nabla_z \chi_1 + e_1) = A_0. \end{aligned}$$

Case 1.(ii)

$$\begin{aligned} \oint e_1 A \nabla_z \chi_2 &= \oint \psi_1 \nabla_z A \nabla_z \chi_2 \\ &= \oint \psi_1 (A_0 - \nabla_z A e_1 \chi_1 - e_1 A \nabla_z \chi_1 - e_1 A e_1) \\ &= \oint \chi_1 (e_1 B \nabla_z \psi_1 + \nabla_z B e_1 \psi_1) - \oint e_1 B e_1 \psi_1 \\ &= \oint \chi_1 (-\nabla_z B \nabla_z \psi_2 + B_0 - e_1 B e_1) - \oint e_1 B e_1 \psi_1 \\ &= -\oint \psi_2 \nabla_z A \nabla_z \chi_1 - \oint e_1 A e_1 \chi_1 - \oint e_1 B e_1 \psi_1 \\ &= -\oint e_1 B \nabla_z \psi_2 - \oint e_1 A e_1 \chi_1 - \oint e_1 B e_1 \psi_1 = -B_1 - \oint e_1 A e_1 \end{aligned}$$

which implies that $A_1 = -B_1$.

Case 1.(iii)

$$\begin{aligned} \oint e_1 A \nabla_z \chi_3 &= \oint \psi_1 \nabla_z A \nabla_z \chi_3 \\ &= \oint \psi_1 (A_0 \chi_1 + A_1 - \nabla_z A e_1 \chi_2 - e_1 A \nabla_z \chi_2 - e_1 A e_1 \chi_1) \\ &= \oint \psi_1 \chi_1 (A_0 - e_1 A e_1) - \oint \chi_2 e_1 A e_1 - \oint \psi_2 \nabla_z A \nabla_z \chi_2 \\ &= \oint \psi_1 \chi_1 (A_0 - e_1 A e_1) - \oint \chi_2 e_1 A e_1 + \oint \psi_2 e_1 B e_1 \\ &\quad - \oint \chi_1 (e_1 B \nabla_z \psi_2 + \nabla_z B e_1 \psi_2). \end{aligned}$$

This implies that

$$
\begin{aligned}
A_2 &:= \oint e_1 A(\nabla_z \chi_3 + e_1 \chi_2) \\
&= \oint \psi_2 e_1 B e_1 + \oint \chi_1 (A_0 \psi_1 - e_1 A e_1 \psi_1 - e_1 B \nabla_z \psi_2 - \nabla_z B e_1 \psi_2) \\
&= \oint \psi_2 e_1 B e_1 + \oint \chi_1 \nabla_z B \nabla_z \psi_3 \\
&= \oint \psi_2 e_1 B e_1 + e_1 B \nabla_z \psi_3 = B_2.
\end{aligned}
$$

2. If A is a symmetric matrix then $A = B$ and $\psi_i = \chi_i$ and we have

$$
\begin{aligned}
A_2 &= \oint e_1 A \nabla_z \chi_3 + \oint \chi_2 e_1 A e_1 \\
&= \oint \psi_1 \chi_1 (A_0 - e_1 A e_1) - \oint \psi_2 \nabla_z A \nabla_z \chi_2 \\
&= \oint \chi_1^2 (A_0 - e_1 A e_1 + \nabla_z \chi_1 A \nabla_z \chi_1) - \oint \chi_1^2 \nabla_z \chi_1 A \nabla_z \chi_1 + \oint \nabla_z \chi_2 A \nabla_z \chi_2 \\
&= \oint \chi_1^2 (A_0 - e_1 A e_1 + \nabla_z \chi_1 A \nabla_z \chi_1) + \oint \nabla_z (\chi_2 - \frac{\chi_1^2}{2}) A \nabla_z (\chi_2 + \frac{\chi_1^2}{2}) \\
&= \oint \chi_1^2 \left(A_0 - e_1 A e_1 + \nabla_z \chi_1 A \nabla_z \chi_1 - \nabla_z A \nabla_z \chi_2 + \frac{1}{2} \nabla_z A \nabla_z \chi_1^2 \right) \\
&\quad + \oint \nabla_z (\chi_2 - \frac{\chi_1^2}{2}) A \nabla_z (\chi_2 - \frac{\chi_1^2}{2}) \\
&= \oint \chi_1^2 \left(e_1 A \nabla_z \chi_1 + \nabla_z A e_1 \chi_1 + \nabla_z \chi_1 A \nabla_z \chi_1 + \frac{1}{2} \nabla_z A \nabla_z \chi_1^2 \right) \\
&\quad + \oint \nabla_z (\chi_2 - \frac{\chi_1^2}{2}) A \nabla_z (\chi_2 - \frac{\chi_1^2}{2}) \\
&= \oint \chi_1^2 (-e_1 A \nabla_z \chi_1 - \nabla_z \chi_1 A \nabla_z \chi_1) + \oint \nabla_z (\chi_2 - \frac{\chi_1^2}{2}) A \nabla_z (\chi_2 - \frac{\chi_1^2}{2}).
\end{aligned}
$$

We multiply the first equation of the cell problem (4) by χ_1^3 and by integration by parts we get

$$\oint \chi_1^2 (e_1 A \nabla_z \chi_1 + \nabla_z \chi_1 A \nabla_z \chi_1) = 0$$

which implies that

$$A_2 = \oint \nabla_z (\chi_2 - \frac{\chi_1^2}{2}) A \nabla_z (\chi_2 - \frac{\chi_1^2}{2}).$$

Since the matrix A is positive definite we have $A_2 \geq 0$ and $A_2 = 0$ if and only if

$$\nabla_z (\chi_2 - \frac{\chi_1^2}{2}) = 0 \tag{12}$$

which implies that

$$\chi_2 - \frac{\chi_1^2}{2} = const. = \oint (\chi_2 - \frac{\chi_1^2}{2}).$$

The last equation of the cell problem (4) implies that

$$\nabla_z (\chi_2 - \frac{\chi_1^2}{2}) = 0 \Longleftrightarrow h = 0.$$

Since A is positive definite, equation (12) is equivalent to

$$\begin{aligned} 0 &= \nabla_z A \nabla_z (\chi_2 - \frac{\chi_1^2}{2}) \\ &= A_0 - e_1 A e_1 - e_1 A \nabla_z \chi_1 - \nabla_z A e_1 \chi_1 - \nabla_z (A \chi_1 \nabla_z \chi_1) \\ &= A_0 - e_1 A (e_1 + \nabla_z \chi_1) - (e_1 + \nabla_z \chi_1) A \nabla_z \chi_1 \\ &= A_0 - (e_1 + \nabla_z \chi_1) A (e_1 + \nabla_z \chi_1). \end{aligned}$$

This is equivalent to

$$A_0 = (e_1 + \nabla_z \chi_1) A (e_1 + \nabla_z \chi_1), \text{ which is constant.}$$

This ends the proof of Proposition 3.4.

4 Specific cases

To understand the effects of specific types of periodically varying heterogeneous media on the wave speed we study some cases in more detail.

Case 1: $A = A(x_1)$.
First we consider the 1-dimensional situation. Let $A = a(x_1) > 0$ be a 1-periodic function defined in $\mathbb{R}$, then $A_1 = 0$ and thus $c_1 = 0$. An easy computation gives that

$$A_0 = \oint a(\chi_1' + 1) = \frac{1}{\int_0^1 \frac{1}{a}}$$

and

$$A_2 = A_0 \int_0^1 \chi_1^2 \geq 0$$

$$A_2 = 0 \Longleftrightarrow \chi_1 = 0 \Longleftrightarrow a = const.$$

The last equivalence results from the first equation of the cell problem (4).
The same results are true for $A = A(x_1)$ being an $n \times n$ matrix with entries just depending on x_1, namely A_0 and A_2 just depend on the first component $a_{1,1}$ of the matrix A.

Later we will use the formulation given above for A_2 to determine the sign of c_2 when f is of exact bistable type.

Case 2: 2-D case.
For simplicity let

$$A = A(x_2) = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{13}$$

be a positive definite matrix. Since a, b, c, d all depend only on x_2, we have $\chi_i = \chi_i(x_2)$ for all i. Define f_{c-b} and g by

$$f_{c-b} = \frac{c-b}{d} - \frac{1}{d} \frac{\int_0^1 \frac{c-b}{d}}{\int_0^1 \frac{1}{d}} \quad \text{and} \quad g = a - \frac{bc}{d} + \frac{1}{d} \frac{\int_0^1 \frac{c}{d} \int_0^1 \frac{b}{d}}{(\int_0^1 \frac{1}{d})^2}. \tag{14}$$

After some computations (details are given in the Appendix) we obtain

$$A_0 = \int_0^1 g,$$

$$A_1 = \int_0^1 (A_0 - g) \int_0^x f_{c-b} ds dx$$

and

$$\begin{aligned} A_2 &= \int_0^1 \frac{1}{d} \left(\int_0^x (A_0 - g) \right)^2 - \frac{1}{\int_0^1 \frac{1}{d}} \left(\int_0^1 \frac{1}{d} \int_0^x (A_0 - g) \right)^2 \\ &+ \int_0^1 \left(\int_0^x (A_0 - g) \right) \left(\frac{c-b}{d} \int_0^x f_{c-b} - f_{c-b} \int_0^1 \int_0^x f_{c-b} - \frac{1}{d} \frac{\int_0^1 \frac{c-b}{d}}{\left(\int_0^1 \frac{1}{d} \right)^2} \int_0^1 \frac{1}{d} \int_0^x f_{c-b} \right) \\ &- \frac{\int_0^1 \frac{c}{d} \int_0^1 \frac{b}{d}}{\left(\int_0^1 \frac{1}{d} \right)^2} \left(\int_0^1 \frac{1}{d} \left(\int_0^x f_{c-b} \right)^2 - \frac{1}{\int_0^1 \frac{1}{d}} \left(\int_0^1 \frac{1}{d} \int_0^x f_{c-b} \right)^2 \right). \end{aligned}$$

The first term of the right hand side of the equation is positive. To simplify the second term and to determine the sign of the third term we define $c^* = c - k$ and $b^* = b + k$ with

$$k = \frac{1}{2 \int_0^1 \frac{1}{d}} \int_0^1 \frac{c-b}{d}.$$

We have

$$f_{c-b} = f^* = \frac{c^* - b^*}{d} \quad \text{and} \quad g - kf^* = g^* = a - \frac{c^* b^*}{d} + \frac{1}{d} \frac{(\int_0^1 \frac{c^*}{d})^2}{(\int_0^1 \frac{1}{d})^2}.$$

Thus $f^* = 0$ is equivalent to $A - A^T = const$. An easy computation gives that

$$A_0 = \int_0^1 g^*, \quad A_1 = \int_0^1 A_0 - g^* \int_0^x f^* \quad \text{and}$$

$$\begin{aligned} A_2 &= \int_0^1 \frac{1}{d} \left(\int_0^x (A_0 - g^*) \right)^2 - \frac{1}{\int_0^1 \frac{1}{d}} \left(\int_0^1 \frac{1}{d} \int_0^x (A_0 - g^*) \right)^2 \\ &+ \int_0^1 \left(\int_0^x (A_0 - g^*) \right) f^* \left(\int_0^x f^* - \int_0^1 \int_0^x f^* \right) \\ &- \frac{\left(\int_0^1 \frac{c^*}{d} \right)^2}{\left(\int_0^1 \frac{1}{d} \right)^2} \left(\int_0^1 \frac{1}{d} \left(\int_0^x f^* \right)^2 - \frac{1}{\int_0^1 \frac{1}{d}} \left(\int_0^1 \frac{1}{d} \int_0^x f^* \right)^2 \right) \end{aligned} \tag{15}$$

With this we can see that the first term on the right hand side of (15) is positive and the third term is negative. The second term is simplified and equal to zero if $A_0 - g^* = Cf^*$. To analyze the sign of A_2, we first state a useful corollary.

COROLLARY 4.1 *Suppose that there exists a constant C such that $A_0 - g^* = Cf^*$ (if d is constant this is equivalent to $ad - bc = C'(c-b) + const.$ for a $C' \in \mathbb{R}$) while $c - b$ is not constant, then $A_1 = 0$ and the sign of A_2 is well determined as a function of the constant C and of $\int_0^1 \frac{c+b}{d}$. More precisely*

1. *If* $C^2 = \left(\frac{\int_0^1 \frac{c+b}{d}}{2\int_0^1 \frac{1}{d}}\right)^2$ *then* $A_2 = 0$.
   For $d = const.$ the condition reads $C' = -\int_0^1 b$ or $C' = \int_0^1 c$.

2. *If* $C^2 > \left(\frac{\int_0^1 \frac{c+b}{d}}{2\int_0^1 \frac{1}{d}}\right)^2$ *then* $A_2 > 0$.
   For $d = const.$: $C' < \min\{-\int_0^1 b, \int_0^1 c\}$ *or* $C' > \max\{-\int_0^1 b, \int_0^1 c\}$.

3. *If* $C^2 < \left(\frac{\int_0^1 \frac{c+b}{d}}{2\int_0^1 \frac{1}{d}}\right)^2$ *then* $A_2 < 0$.
   For $d = const.$: $\min\{-\int_0^1 b, \int_0^1 c\} < C' < \max\{-\int_0^1 b, \int_0^1 c\}$.

We omit the proof of this Corollary.

Equality (15) implies that
1) If $f^* = 0$, respectively $A - A^T = const.$, then we have $A_1 = 0$ and A_2 is given by

$$A_2 = \int_0^1 \frac{1}{d}\left(\int_0^x (A_0 - g^*)\right)^2 - \frac{1}{\int_0^1 \frac{1}{d}}\left(\int_0^1 \frac{1}{d}\int_0^x (A_0 - g^*)\right)^2 \geq 0;$$

$A_2 = 0 \Longleftrightarrow g^* = const. \Longleftrightarrow g = const.$. This means that $c_1 = c_2 = 0$.

2) If $g^* = const.$ then we have $A_1 = 0$ and A_2 is given by

$$A_2 = -\frac{\left(\int_0^1 \frac{c^*}{d}\right)^2}{\left(\int_0^1 \frac{1}{d}\right)^2}\left(\int_0^1 \frac{1}{d}\left(\int_0^x f^*\right)^2 - \frac{1}{\int_0^1 \frac{1}{d}}\left(\int_0^1 \frac{1}{d}\int_0^x f^*\right)^2\right) \leq 0;$$

$A_2 = 0$ if and only if ($f^* = 0$ or $\int_0^1 \frac{c^*}{d} = 0$) which means ($A - A^T = const.$ or $\int_0^1 \frac{c+b}{d} = 0$).

To understand the effects of the symmetric and antisymmetric part of A on c_1 and c_2 better, we discuss some examples in more detail.

5 Examples

We know from [11] that for bistable nonlinearities f and for small oscillations of A there exists a traveling wave for (2) but for large oscillations of A there is no traveling wave. Therefore we have to chose our examples accordingly. By multiplication with positive constans we can transform large oscillations into a small ones and we know from Proposition 3.2 that this multiplication does not change the sign of c_1 and c_2 but only affects the absolute value of c_2. Hence for the sign considerations in our examples we do not take care of these factors.

In the following let k_i, b_i and K be real constants and $n \neq m$ nonzero integers. The constant K is chosen large enough in comparison to the other constants in order to fulfill the positive definiteness condition for the matrix A.

We know from Proposition 3.4 that b and c play the same role in the computation of A_0 and A_2, so we can exchange b and c in those calculations.

Example 1: Let the matrix $A = A(x_2)$ in (13) be defined as follows

$$b = const., \quad d = const. > 0, \quad c = k_1 \sin(2\pi n x_2) + b$$

and

$$a = k_2 \sin(2\pi n x_2) + k_3 \cos(2\pi n x_2) + k_4 \sin(2\pi m x_2) + k_5 \cos(2\pi m x_2) + K$$

with $n \neq m$. Using the fact that for all $n \neq m$ we have

$$\int_0^1 \sin(2\pi n x)\cos(2\pi m x)dx = \int_0^1 \sin(2\pi n x)\sin(2\pi m x)dx = \int_0^1 \cos(2\pi n x)\cos(2\pi m x)dx = 0$$

an easy computation gives that

$$A_0 = K, \quad A_1 = \frac{k_1 k_3}{4\pi n d}$$

and

$$A_2 = \frac{1}{8d\pi^2}\left[\frac{1}{n^2}(k_2^2 + k_3^2 - \frac{2bk_1k_2}{d}) + \frac{1}{m^2}(k_4^2 + k_5^2) + \frac{k_1^2 k_5}{2dmn}\delta_{m=2n}\right].$$

If k_1, k_3 have the same sign, then $A_1 > 0$. Identical oscillations of a and c have no effect on A_1, and also differences in periodicity of the oscillations are not affecting it, whereas the right type of “counter-oscillations” with the same periodicity do. We remark that

$$A_1 = 0 \Longleftrightarrow k_1 = 0 \text{ or } k_3 = 0.$$

If $k_1 = 0$ then A is a symmetric matrix and we have $A_1 = 0$ and

$$A_2 = \frac{1}{8d\pi^2}\left[\frac{1}{n^2}(k_2^2 + k_3^2) + \frac{1}{m^2}(k_4^2 + k_5^2)\right] \geq 0.$$

So the sign of c_2 does not change for different constants $k_2, ..., k_5$. The last inequality is already known from Proposition 3.4;
In this case $A_2 = 0$ if and only if $k_2 = k_3 = k_4 = k_5 = 0$ (which means the matrix A has constant entries).

If $k_3 = 0$ in the original situation of this example, then $A_1 = 0$, thus $c_1 = 0$ and

$$A_2 = \frac{1}{8d\pi^2}\left[\frac{1}{n^2}(k_2^2 - \frac{2bk_1k_2}{d}) + \frac{1}{m^2}(k_4^2 + k_5^2) + \frac{k_1^2k_5}{2dmn}\delta_{m=2n}\right].$$

which can be negative in case the identical oscillations of a and c are strong and/or b is large. Thus also c_2 can change sign in this case.

Example 2: Let the matrix A be defined like in (13) with

$$d = const. > 0, \quad b = k_1 \sin(2\pi n x_2) + k_2 \cos(2\pi n x_2) + b_1$$

$$c = k_3 \sin(2\pi n x_2) + k_4 \cos(2\pi n x_2) + b_2$$

and

$$a = k_5 \sin(2\pi n x_2) + k_6 \cos(2\pi n x_2) + K.$$

An easy computation gives that

$$A_0 = K - \frac{k_1k_3 + k_2k_4}{2d},$$

$$A_1 = \frac{1}{4\pi nd}\left(k_6(k_3 - k_1) - k_5(k_4 - k_2) + \frac{b_1 + b_2}{d}(k_1k_4 - k_2k_3)\right)$$

and

$$\begin{aligned} A_2 &= \frac{1}{8\pi^2n^2d}\Big[k_5^2 + k_6^2 - k_5\frac{b_1+b_2}{d}(k_1 + k_3) - k_6\frac{b_1+b_2}{d}(k_2 + k_4) \\ &\quad + \frac{k_1k_3 + k_2k_4}{8d^2}\left(k_1^2 + k_2^2 + k_3^2 + k_4^2 + 8(b_1 + b_2)^2\right) - \frac{3(k_1^2 + k_2^2)(k_3^2 + k_4^2)}{16d^2}\Big] \end{aligned}$$

where again K is supposed to be large enough and especially $A_0 > 0$. Since the explicit expression of A_2 is quite complex, we discuss its sign only when $A_1 = 0$ and in some simplified cases.

Case 1: $k_1 = k_2 = 0$

$$A_0 = K, \quad A_1 = \frac{k_3k_6 - k_4k_5}{4\pi nd}, \quad \text{and} \quad A_2 = \frac{1}{8d\pi^2n^2}\left[k_5^2 + k_6^2 - \frac{b_1 + b_2}{d}(k_3k_5 + k_4k_6)\right].$$

We remark that

$$A_1 = 0 \Longleftrightarrow k_3k_6 = k_4k_5.$$

Here the counter-oscillations of a and c are balanced.
If k_3, k_4, k_5, k_6 are all positive, then A_1 changes sign if the counter-oscillations of a and c are imbalanced in a suitable way.

Suppose that $A_1 = 0$ and there exists $i \in \{3,4,5,6\}$ such that $k_i \neq 0$ (for example k_3) then

$$A_2 = \frac{k_5(k_3^2 + k_4^2)}{8d\pi^2 n^2 k_3^2} \left(k_5 - \frac{b_1 + b_2}{d} k_3 \right).$$

This case fulfill the conditions of Corollary 4.1 namely $ad - bc = C'(c - b) + const.$ with $C' = b_1 - \frac{k_5 d}{k_3}$ therefore

1. $A_2 = 0$ if and only if $k_5 = 0$ or $k_5 = \frac{b_1+b_2}{d} k_3$.
   This is an example for $c_1 = c_2 = 0$ and the matrix A is not constant.
2. $A_2 > 0$ if and only if $k_5 < \min\{0, \frac{b_1+b_2}{d} k_3\}$ or $k_5 > \max\{0, \frac{b_1+b_2}{d} k_3\}$
3. $A_2 < 0$ if and only if $\min\{0, \frac{b_1+b_2}{d} k_3\} < k_5 < \max\{0, \frac{b_1+b_2}{d} k_3\}$.

Case 2: $k_5 = \frac{b_1+b_2}{2d}(k_1 + k_3)$ and $k_6 = \frac{b_1+b_2}{2d}(k_2 + k_4)$. Then we have $A_1 = 0$ and

$$\begin{aligned} A_2 &= \frac{-1}{128\pi^2 n^2 d^3} \Big[4(b_1+b_2)^2 \Big((k_3 - k_1)^2 + (k_4 - k_2)^2 \Big) + 3(k_1^2 + k_2^2)(k_3^2 + k_4^2) . \\ &\qquad -2(k_1 k_3 + k_2 k_4) \left(k_1^2 + k_2^2 + k_3^2 + k_4^2 \right) \Big]. \end{aligned}$$

Depending on the parameters A_2 changes sign.

If $k_1 k_3 + k_2 k_4 \leq 0$ then $A_2 \leq 0$;
$A_2 = 0$ if one of the following assertions holds

1. $k_1 = k_2 = k_3 = k_4 = 0$, in this case the matrix A is constant.
2. $k_1 = k_2 = 0$ and $b_1 + b_2 = 0$, thus $c = const.$
   or
3. $k_3 = k_4 = 0$ and $b_1 + b_2 = 0$, thus $b = const.$
   Both are examples for $c_1 = c_2 = 0$ and the matrix A is not constant.

For Case 2, 2.,3. we have $a = const.$ and b or c is constant which means that $ad - bc = C'(c - b) + const.$ with $C' = c$ or $C' = -b$ (i.e. the constant one), so we know from Corollary 4.1 that there is no influence on the wave speed up to the second order.

In particular, if the matrix A has exact antisymmetric oscillations, i.e. $k_3 = -k_1$ and $k_4 = -k_2$, then $A_2 < 0$.

For large $b_1 + b_2$ and ($k_1 \neq k_3$ or $k_2 \neq k_4$) we also have $A_2 < 0$.

Note here that the symmetric part of the matrix A has an effect towards A_2 being positive and the antisymmetric part has an effect towards A_2 being negative.

6 The exact bistable case

Here we are back to the general n-dimensional case. In this section, for exact bistable nonlinearity f, we determine the value of c_1 as a function of A_0 and A_1, and if $A_1 = 0$ (e.g. when A is a symmetric matrix) we determine the value of c_2 as a function of A_0 and A_2. The nonlinearity f of equations (2) and (3) is assumed to be of the form

$$f(u) = u(1-u)(u-\mu) \quad \mu \in (0,1) \setminus \{\frac{1}{2}\}.$$

For a unique solution of equation (3), we fix a point, for example $u_0(0) = \frac{1}{2}$. An easy computation gives that

$$c_0 = \sqrt{\frac{A_0}{2}}(1-2\mu) \quad \text{and} \quad u_0(x) = \frac{1}{1+e^{-\frac{x}{\sqrt{2A_0}}}}.$$

Consider the three terms

$$I_0 = \int_{\mathbb{R}} {u_0'}^2 e^{-\frac{c_0}{A_0}x} dx, \quad I_1 = \int_{\mathbb{R}} u_0' u_0''' e^{-\frac{c_0}{A_0}x} dx \quad \text{and} \quad I_2 = \int_{\mathbb{R}} u_0' u_0^{(4)} e^{-\frac{c_0}{A_0}x} dx.$$

We have

$$\begin{aligned} I_0 &= \int_{\mathbb{R}} {u_0'}^2 e^{-\frac{c_0}{A_0}x} dx = \frac{1}{2A_0} \int_{\mathbb{R}} \frac{e^{-\frac{2x}{\sqrt{2A_0}}}}{(1+e^{-\frac{x}{\sqrt{2A_0}}})^4} e^{-\frac{c_0}{A_0}x} dx \\ &= \frac{1}{\sqrt{2A_0}} \int_0^\infty \frac{t^{2-2\mu}}{(1+t)^4} dt \\ &= \frac{1}{\sqrt{2A_0}} \int_0^\infty \frac{t^{1-2\mu}}{(1+t)^3} dt - \frac{1}{\sqrt{2A_0}} \int_0^\infty \frac{t^{1-2\mu}}{(1+t)^4} dt. \end{aligned}$$

Using the fact that for all integers $n \geq 2$ we have

$$\int_0^\infty \frac{t^{1-2\mu}}{(1+t)^{n+1}} dt = (1 - 2\frac{1-\mu}{n}) \int_0^\infty \frac{t^{1-2\mu}}{(1+t)^n} dt$$

we obtain

$$I_0 = \frac{2\mu(1-\mu)}{3\sqrt{2A_0}} \int_0^\infty \frac{t^{1-2\mu}}{(1+t)^2} dt.$$

A similarly easy computation gives that

$$I_1 = \int_{\mathbb{R}} u_0' u_0''' e^{-\frac{c_0}{A_0}x} dx = \frac{\sqrt{2A_0}}{60A_0^2} \mu(1-\mu)(12\mu^2 - 12\mu + 1) \int_0^\infty \frac{t^{1-2\mu}}{(1+t)^2} dt.$$

Hence

$$c_1 = \frac{A_1}{20A_0}(12\mu^2 - 12\mu + 1) = \frac{A_1}{10A_0^2}(3c_0^2 - A_0).$$

Like in the computation of I_0 and I_1 we obtain that

$$I_2 = \int_{\mathbb{R}} u_0' u_0^{(4)} e^{-\frac{c_0}{A_0}x} dx = \frac{\mu(1-\mu)}{90A_0^2}(1-2\mu)(8\mu^2 - 8\mu - 1)\int_0^\infty \frac{t^{1-2\mu}}{(1+t)^2} dt.$$

which implies that

$$I_2 \cdot c_0 < 0.$$

Conclusion. For the exact bistable nonlinearity we have

$$c_1 = \frac{3A_1}{20A_0}\left((1-2\mu)^2 - \frac{2}{3}\right).$$

For $\mu = \frac{1}{2} \pm \sqrt{\frac{1}{6}}$ we have $c_1 = 0$. If $A_1 \neq 0$ then c_1 can change sign depending on the exact value of the parameter μ, so the nonlinearity has an effect on the wave speed which means slowing it down or speeding it up. For the effect of A_1 on the sign of c_1 compare the discussion and details given previously.
If $A_1 = 0$ then $c_1 = 0$ and c_2 is given by

$$c_2 = \frac{A_2}{30A_0\sqrt{2A_0}}(1-2\mu)(8\mu^2 - 8\mu - 1)$$

which implies that

$$c_0 c_2 = \frac{A_2(1-2\mu)^2}{30A_0}\left((1-2\mu)^2 - \frac{3}{2}\right).$$

In this case we see that the sign of c_0c_2 depends only on the sign of A_2 since $(1-2\mu)^2 - \frac{3}{2} < 0$. The nonlinearity has an effect on the wave speed but the wave speed slows down or speeds up depending only on the diffusivity (and not on the nonlinearity).
In particular if A is a symmetric matrix then $A_1 = 0$, $c_1 = 0$ and $A_2 \geq 0$. Since $0 < \mu < 1$ in this case we have $c_0c_2 \leq 0$. For any $A = A(x_1)$ with $a_{1,1}$ non-constant we have $c_0c_2 < 0$ in any space dimension.

7 Discussion

In this paper we studied the effects of periodically varying heterogeneous media on the speed of traveling waves of reaction diffusion equation. The speed of the wave can be expanded in terms of the space periodicity. Since the first order expansion often cannot clarify the influence of the medium on the speed of the front, in particular if the diffusion matrix A is symmetric, we have analyzed also the second order expansion where necessary

$$c_\varepsilon = c_0 + c_1\varepsilon + c_2\varepsilon^2 + O(\varepsilon^3).$$

If A is symmetric, then $A_1 = 0$, thus $c_1 = 0$ and $A_2 \geq 0$.
Generally, if $A_1 = 0$ then $c_1 = 0$ and c_2 is given by (11).

If $A = A(x_1)$ then $A_1 = c_1 = 0$ and A_0 and A_2 are depending only on the first component $a_{1,1,}$ of the matrix A. For an exact bistable f we have $c_0 c_2 \leq 0$.

In two dimensions and for $A = A(x_2)$ we obtained the following results. If $A - A^T = const.$ then $c_1 = 0$ and $A_2 \geq 0$. So no sign change of c_2 is possible. For constant $a_{2,2}$ and identical oscillations of $a_{1,1}$ and $a_{2,1}$ there is no effect on A_1 and thus not on c_1. The right type of counter-oscillations with the same periodicity do have an effect on the wave speed.

In the n-dimensional setting and for exact bistable f, the nonlinearity does influence the wave speed if $A_1 \neq 0$. In case $A_1 = 0$, thus $c_1 = 0$, then f does not influence the sign of c_2. Explicit formulas for c_1, c_2 in the expansion of the wave speed could be given.

8 Appendix

Here we compute A_0, A_1 and A_2 for the two-dimensional situation as given in section 4, case 2. Let A, f_{c-b} and g be given like in (13) and (14). We have for all $i \in \{1, 2, 3, 4\}$, $\chi_i = \chi_i(x_2)$. Define f_c and f_b by

$$f_c = \frac{c}{d} - \frac{1}{d}\frac{\int_0^1 \frac{c}{d}}{\int_0^1 \frac{1}{d}} \quad \text{and} \quad f_b = \frac{b}{d} - \frac{1}{d}\frac{\int_0^1 \frac{b}{d}}{\int_0^1 \frac{1}{d}}.$$

From the first equation of the cell problem (4) we have

$$\nabla A(\nabla \chi_1 + e_1) = 0 \Longrightarrow \chi_1' = -\frac{c}{d} + \frac{1}{d}\frac{\int_0^1 \frac{c}{d}}{\int_0^1 \frac{1}{d}} = -f_c$$

$$A_0 = \int_0^1 a + b\chi_1' = \int_0^1 (a - \frac{bc}{d}) + \frac{\int_0^1 \frac{c}{d} \int_0^1 \frac{b}{d}}{\int_0^1 \frac{1}{d}} = \int_0^1 g \ .$$

We have $\int_0^1 \chi_1 = 0$ and $\chi_1(1) = \chi_1(0)$ which implies that

$$\chi_1 = -\int_0^x f_c + \int_0^1 \int_0^x f_c.$$

The equation satisfied by χ_2 is

$$\nabla A(\nabla \chi_2 + e_1 \chi_1) = A_0 - e_1 A(\nabla \chi_1 + e_1),$$

after integration we get

$$\begin{aligned} \chi_2' \quad &= \quad \frac{1}{d}\int_0^x (A_0 - g) - \frac{1}{d}\frac{\int_0^1 \frac{c}{d}}{\int_0^1 \frac{1}{d}} \int_0^x f_b - \frac{c}{d}\chi_1 + \frac{1}{d}\frac{\int_0^1 \frac{c}{d}}{\int_0^1 \frac{1}{d}} \int_0^1 \int_0^x f_c \\ &\quad -\frac{1}{d}\frac{\int_0^1 \frac{c}{d}}{(\int_0^1 \frac{1}{d})^2} \int_0^1 \frac{1}{d}\int_0^x f_{c-b} - \frac{1}{d}\frac{\int_0^1 \frac{1}{d}\int_0^x (A_0 - g)}{\int_0^1 \frac{1}{d}}. \end{aligned}$$

Using this expression we get from (5)

$$\begin{aligned}
A_1 &= \int_0^1 a\chi_1 + b\chi_2' = \int_0^1 \chi_1 \left(g + \frac{bc}{d} - \frac{1}{d}\frac{\int_0^1 \frac{c}{d}\int_0^1 \frac{b}{d}}{(\int_0^1 \frac{1}{d})^2} \right) - \int_0^1 \chi_1 \frac{bc}{d} + \int_0^1 \frac{b}{d}\int_0^x (A_0 - g) \\
&\quad - \frac{\int_0^1 \frac{b}{d}}{\int_0^1 \frac{1}{d}} \int_0^1 \frac{1}{d}\int_0^x (A_0 - g) - \frac{\int_0^1 \frac{c}{d}}{\int_0^1 \frac{1}{d}} \int_0^1 \frac{b}{d}\int_0^x f_b + \frac{\int_0^1 \frac{b}{d}\int_0^1 \frac{c}{d}}{\int_0^1 \frac{1}{d}} \int_0^1 \int_0^x f_c \\
&\quad - \frac{\int_0^1 \frac{b}{d}\int_0^1 \frac{c}{d}}{(\int_0^1 \frac{1}{d})^2} \int_0^1 \frac{1}{d}\int_0^x f_{c-b} \\
&= \int_0^1 f_b \int_0^x (A_0 - g) + \int_0^1 g \left(-\int_0^x f_c + \int_0^1 \int_0^x f_c \right) \\
&\quad - \frac{\int_0^1 \frac{b}{d}\int_0^1 \frac{c}{d}}{(\int_0^1 \frac{1}{d})^2} \int_0^1 \frac{1}{d} \left(-\int_0^x f_c + \int_0^1 \int_0^x f_c \right) \\
&\quad + \frac{\int_0^1 \frac{b}{d}\int_0^1 \frac{c}{d}}{\int_0^1 \frac{1}{d}} \int_0^1 \int_0^x f_c - \frac{\int_0^1 \frac{b}{d}\int_0^1 \frac{c}{d}}{(\int_0^1 \frac{1}{d})^2} \int_0^1 \frac{1}{d}\int_0^x f_c \\
&= \int_0^1 f_b \int_0^x (A_0 - g) + \int_0^1 (A_0 - g) \int_0^x f_c = \int_0^1 (A_0 - g) \int_0^x f_{c-b}.
\end{aligned}$$

Here we have used $\int_0^1 f_b \int_0^x f_b = 0$.
Now we integrate the third equation of the cell problem (4) we get for χ_3

$$\begin{aligned}
\chi_3' &= -\frac{c}{d}\chi_2 + \frac{1}{d}\int_0^x \left(A_0\chi_1 + A_1 - a\chi_1 - b\chi_2' \right) \\
&\quad + \frac{1}{d\int_0^1 \frac{1}{d}} \int_0^1 \left(\frac{c}{d}\chi_2 - \frac{1}{d}\int_0^x (A_0\chi_1 + A_1 - a\chi_1 - b\chi_2') \right).
\end{aligned}$$

Therefore we get

$$\begin{aligned}
A_2 &= \int_0^1 a\chi_2 + b\chi_3' = \int_0^1 \left[a\chi_2 - \frac{bc}{d}\chi_2 + \frac{b}{d}\int_0^x (A_0\chi_1 + A_1 - a\chi_1 - b\chi_2')\right] \\
&\quad + \frac{\int_0^1 \frac{b}{d}}{\int_0^1 \frac{1}{d}} \int_0^1 \left[\frac{c}{d}\chi_2 - \frac{1}{d}\int_0^x A_0\chi_1 + A_1 - a\chi_1 - b\chi_2'\right] \\
&= \int_0^1 \chi_2 \left(-A_0 + g + \frac{\int_0^1 \frac{b}{d}}{\int_0^1 \frac{1}{d}} f_c\right) + \int_0^1 f_b \int_0^x (A_0\chi_1 + A_1 - a\chi_1 - b\chi_2') \\
&= \int_0^1 \chi_2' \left(\int_0^x (A_0 - g) - \frac{\int_0^1 \frac{b}{d}}{\int_0^1 \frac{1}{d}} \int_0^x f_c + b\int_0^x f_b\right) - \int_0^1 (A_0\chi_1 + A_1 - a\chi_1) \int_0^x f_b \\
&= \int_0^1 \left[-\frac{c}{d}\chi_1 + \frac{1}{d}\int_0^x (A_0 - g) - \frac{1}{d\int_0^1 \frac{1}{d}} \int_0^1 \frac{1}{d}\int_0^x (A_0 - g)\right. \\
&\qquad - \frac{1}{d}\frac{\int_0^1 \frac{c}{d}}{\int_0^1 \frac{1}{d}} \int_0^x f_b + \frac{1}{d}\frac{\int_0^1 \frac{c}{d}}{\int_0^1 \frac{1}{d}} \int_0^1 \int_0^x f_c \\
&\qquad \left. - \frac{1}{d}\frac{\int_0^1 \frac{c}{d}}{(\int_0^1 \frac{1}{d})^2} \int_0^1 \frac{1}{d}\int_0^x f_{c-b}\right] \left(\int_0^x (A_0 - g) - \frac{\int_0^1 \frac{b}{d}}{\int_0^1 \frac{1}{d}} \int_0^x f_c + b\int_0^x f_b\right) \\
&\quad - \int_0^1 \left(A_0\chi_1 + A_1 - \left(g + \frac{bc}{d} - \frac{1}{d}\frac{\int_0^1 \frac{c}{d}\int_0^1 \frac{b}{d}}{(\int_0^1 \frac{1}{d})^2}\right)\chi_1\right) \int_0^x f_b
\end{aligned}$$

$$
\begin{aligned}
= \quad & \int_0^1 \left(\int_0^x (A_0-g)\right)\Bigg[-\frac{c}{d}\int_0^1\int_0^x f_c + \frac{c}{d}\int_0^x f_c + \frac{1}{d}\int_0^x (A_0-g) - \frac{1}{d\int_0^1 \frac{1}{d}}\int_0^1 \frac{1}{d}\int_0^x (A_0-g) \\
& \quad -\frac{1}{d}\frac{\int_0^1 \frac{c}{d}}{\int_0^1 \frac{1}{d}}\int_0^x f_b + \frac{1}{d}\frac{\int_0^1 \frac{c}{d}}{\int_0^1 \frac{1}{d}}\int_0^1\int_0^x f_c - \frac{1}{d}\frac{\int_0^1 \frac{c}{d}}{(\int_0^1 \frac{1}{d})^2}\int_0^1 \frac{1}{d}\int_0^x f_{c-b} - \frac{1}{d}\frac{\int_0^1 \frac{b}{d}}{\int_0^1 \frac{1}{d}}\int_0^x f_c \\
& \quad +\frac{1}{d}\frac{\int_0^1 \frac{b}{d}}{(\int_0^1 \frac{1}{d})^2}\int_0^1 \frac{1}{d}\int_0^x f_c + \frac{b}{d}\int_0^x f_b - \frac{1}{d\int_0^1 \frac{1}{d}}\int_0^1 \frac{b}{d}\int_0^x f_b + f_b\int_0^1\int_0^x f_c + f_{c-b}\int_0^1\int_0^x f_b\Bigg] \\
& + \int_0^1 (A_0-g)\int_0^x f_c \int_0^x f_b \\
& + \int_0^1 \left[-\frac{1}{d}\frac{\int_0^1 \frac{c}{d}}{\int_0^1 \frac{1}{d}}\int_0^x f_b + \frac{1}{d}\frac{\int_0^1 \frac{c}{d}}{\int_0^1 \frac{1}{d}}\int_0^1\int_0^x f_c - \frac{1}{d}\frac{\int_0^1 \frac{c}{d}}{(\int_0^1 \frac{1}{d})^2}\int_0^1 \frac{1}{d}\int_0^x f_{c-b}\right]\left[b\int_0^x f_b - \frac{\int_0^1 \frac{b}{d}}{\int_0^1 \frac{1}{d}}\int_0^x f_c\right] \\
& + \int_0^1 \chi_1 \left(-\frac{bc}{d}\int_0^x f_b + \frac{c}{d}\frac{\int_0^1 \frac{b}{d}}{\int_0^1 \frac{1}{d}}\int_0^x f_c + \frac{bc}{d}\int_0^x f_b - \frac{1}{d}\frac{\int_0^1 \frac{c}{d}\int_0^1 \frac{b}{d}}{(\int_0^1 \frac{1}{d})^2}\int_0^x f_b\right) \\
= \quad & \int_0^1 \frac{1}{d}\left(\int_0^x (A_0-g)\right)^2 - \frac{1}{\int_0^1 \frac{1}{d}}\left(\int_0^1 \frac{1}{d}\int_0^x (A_0-g)\right)^2 - \frac{1}{2}\int_0^1 (A_0-g)\left(\int_0^x f_{c-b}\right)^2 \\
& + \int_0^1 \left(\int_0^x (A_0-g)\right)\left[-\frac{1}{d}\frac{\int_0^1 \frac{c-b}{d}}{\left(\int_0^1 \frac{1}{d}\right)^2}\int_0^1 \frac{1}{d}\int_0^x f_{c-b} + \frac{1}{d}\frac{\int_0^1 \frac{c-b}{d}}{\int_0^1 \frac{1}{d}}\int_0^x f_{c-b} - f_{c-b}\int_0^1\int_0^x f_{c-b}\right] \\
& + \frac{\int_0^1 \frac{c}{d}\int_0^1 \frac{b}{d}}{\left(\int_0^1 \frac{1}{d}\right)^2}\int_0^1 \Bigg[\frac{1}{d}\int_0^x f_b \int_0^x f_{c-b} - \frac{1}{d}\left(\int_0^1\int_0^x f_c\right)\int_0^x f_{c-b} + \frac{1}{d\int_0^1 \frac{1}{d}}\int_0^x f_{c-b}\int_0^1 \frac{1}{d}\int_0^x f_{c-b} \\
& \qquad -\frac{1}{d}\left(\int_0^x f_c\right)^2 + \frac{1}{d}\int_0^x f_c \int_0^1\int_0^x f_c + \frac{1}{d}\int_0^x f_c\int_0^x f_b - \frac{1}{d}\int_0^x f_b \int_0^1\int_0^x f_c\Bigg] \\
= \quad & \int_0^1 \frac{1}{d}\left(\int_0^x (A_0-g)\right)^2 - \frac{1}{\int_0^1 \frac{1}{d}}\left(\int_0^1 \frac{1}{d}\int_0^x (A_0-g)\right)^2 \\
& + \int_0^1 \left(\int_0^x (A_0-g)\right)\left(\frac{c-b}{d}\int_0^x f_{c-b} - f_{c-b}\int_0^1\int_0^x f_{c-b} - \frac{1}{d}\frac{\int_0^1 \frac{c-b}{d}}{\left(\int_0^1 \frac{1}{d}\right)^2}\int_0^1 \frac{1}{d}\int_0^x f_{c-b}\right) \\
& + \frac{\int_0^1 \frac{c}{d}\int_0^1 \frac{b}{d}}{\left(\int_0^1 \frac{1}{d}\right)^2}\left(-\int_0^1 \frac{1}{d}\left(\int_0^x f_{c-b}\right)^2 + \frac{1}{\int_0^1 \frac{1}{d}}\left(\int_0^1 \frac{1}{d}\int_0^x f_{c-b}\right)^2\right).
\end{aligned}
$$

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References

[1] H, BERESTYCKI, B. LARROUTUROU, P.L. LIONS, "*Multi-dimensional travelling-wave solutions of a flame propagation model*," Arch. Rat. Mech. Anal. 111 (1990), no.1, 33-49.

[2] H. BERESTYCKI, L. NIRENBERG, "*Travelling fronts in cylinders*," Ann. Inst. Henri Poincare, Anal. Non Lineaire 9 (1992), no.5, 497-572.

[3] P.C. FIFE, J.B. MCLEOD, "*The approach of solutions of nonlinear diffusion equations to travelling front solutions*," Arch. Ration. Mech. Anal. 65 (1977), 335-361.

[4] J. GÄRTNER, M.I. FREIDLIN, "*On the propagation of concentration waves in periodic and random media*," Soviet. Math. Dokl., 20 (1979), 1282-1286.

[5] K.P. HADELER, F. ROTHE, "*Traveling fronts in nonlinear diffusion equations*," J. Math. Biol. 2 (1975), 251-263.

[6] F. HAMEL, "*Folmules min-max pour les vitesse d'ondes progressive multidimensionelles*," Ann. Fac. Sci. Toulouse Math. (6), 8 (1999), 259-280.

[7] S. HEINZE, "*Wave solutions to reaction-diffusion systems in perfored domains*," Z. Anal. Anwendungen, 20 (2001), no.3, 661–676.

[8] S. HEINZE, G. PAPANICOLAOU, A. STEVENS, "*Variational principle for propagation speeds in inhomogeneous media*," SIAM J. Appl. Math. vol. 62, no.1, (2001), 129-148.

[9] J.M. ROQUEJOFFRE, "*Eventual monotonicity and convergence to travelling fronts for the solutions of parabolic equations in cylinders*, " Ann. Inst. Henri Poincare, Anal. Non Lineaire 14, No.4, 499-552 (1997).

[10] A.I. VOL'PERT, V. VOL'PERT, V.A. VOL'PERT, "*Traveling wave solutions of parabolic systems*," Translations of Mathematical Monographs. 140. Providence, RI: American Mathematical Society, (1994).

[11] J. XIN, "*Existence and stability of traveling waves in periodic media governed by a bistable nonlinearity*," J. Dynam. Differential Equations, 3 (1991), 541-573.

[12] J. XIN, "*Existence of planar flame fronts in convective-diffusive periodic media*," Arch. Ration. Mech. Anal., 121, (1992), 205-233.

[13] J. XIN, "*Front propagation in heterogeneous media.*" SIAM Rev., 42 (2000), 161-230.

[14] J. XIN, J. ZHU, "*Quenching and propagation of bistable reaction-diffusion front in multidimensional periodic media*," Phys. D., 81 (1995), 94-110.