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**Metriplectic Structure, Leibniz Dynamics and Dissipative Systems**

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by


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# Metriplectic Structure, Leibniz Dynamics and Dissipative Systems

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**Abstract**

A metriplectic (or Leibniz) structure on a smooth manifold is a pair of skew-symmetric Poisson tensor $P$ and symmetric metric tensor $G$. The dynamical system defined by the metriplectic structure can be expressed in terms of Leibniz bracket. This structure is used to model the geometry of the dissipative systems. The dynamics of purely dissipative systems are defined by the geometry induced on a phase space via a metric tensor. The notion of Leibniz brackets is extendable to infinite-dimensional spaces. We study metriplectic structure compatible with the Euler-Poincaré framework of the Burgers and Whitham-Burgers equations. This means metricplectic structure can be constructed via Euler-Poincaré formalism. We also study the Euler-Poincaré frame work of the Holm-Staley equation, and this exhibits different type of metriplectic structure. Finally we study the 2D Navier-Stokes using metriplectic techniques.

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# 1 Introduction

Let $M$ be the phase space of a Hamiltonian system. Let $H$ be a Hamiltonian function on $M$. The Hamiltonian dynamics are defined on any function $f$ on $M$ by

$$f_t = \{f, H\} = \sum_{j,k=1}^{n} P^{jk}(z) \frac{\partial f}{\partial z_j} \frac{\partial H}{\partial z_k} \equiv \nabla f . P \, \nabla H, \tag{1}$$

where the Poisson matrix $P(z) = (P^{jk}(z))_{j,k=1}^{n}$ is a skew-symmetric square matrix, satisfying a technical condition

$$\sum_{l=1}^{n} P^{li} \frac{\partial P^{jk}}{\partial z^l} + P^{lj} \frac{\partial P^{ki}}{\partial z^l} + P^{lk} \frac{\partial P^{ij}}{\partial z^l} = 0$$

enforced by the Jacobi identity.

Using the coordinate representation, the Hamiltonian equation is given by

$$\dot{z_j} = \{z_j, H\} = \sum_{k=1}^{n} P^{jk}(z) \frac{\partial H}{\partial z_k}, \tag{2}$$

and this reduces to the standard definition of a Hamiltonian system if canonical coordinates are chosen. It is clear that the conservative classical mechanics can be formulated in terms of Poisson structure [1,19,21]. The properties of the Poisson bracket have important consequences on the dynamical features of the Hamiltonian vector field $X_H$ in ordinary mechanics, for example, the skewsymmetric condition implies that the Hamiltonian function $H$ is a constant of the motion for $X_H$.

In otherwords, we say that $F$ and $H$ are in involution if their Poisson bracket is trivial, $\{F, H\} = 0$. This is equivalent to saying that $F$ is constant along the flows of the system corresponding to the Hamiltonian $H$.

The notion of Poisson brackets is extendable to infinite-dimensional phase spaces. In this case the Poisson matrix $P^{ij}$ is replaced by a skew-adjoint differential operator $\mathcal{O}_{skew}$.

The Poisson bracket between any two functionals $f = \int F(u, u_x, \cdots)dx$ and $g = \int G(u, u_x, \cdots)dx$ is given by

$$\{f, g\} = \int_M \frac{\delta f}{\delta u} \mathcal{O}_{skew} \frac{\delta g}{\delta u} \, dx, \tag{3}$$

where $u(x,t)$ acts like a coordinate in an infinite-dimensional phase space, and $\frac{\delta f}{\delta u}$ stands for the Fréchet derivative of f. The Frechet derivative $\frac{\delta f}{\delta u}$ is defined by

$$\frac{\delta f}{\delta u}(v) = \frac{d}{d\epsilon}|_{\epsilon=0} f(u + \epsilon v).$$

The element $\frac{\delta f}{\delta u}(v)$ is called the variational derivative of $f$ at $v$.

These families of Poisson brackets are all known to be compatible, that is, any linear combination of these brackets is also a Poisson bracket. This Poisson bracket satisfies all the axioms of Poisson structure and the Jacobi identity is automatically satisfied due to skew-symmetric nature of the operator $\mathcal{O}_{skew}$.

For example, the operator

$$\mathcal{O}^2_{skew} = \partial \qquad \partial \equiv \frac{\partial}{\partial x}$$

and

$$\mathcal{O}^2_{skew} = \partial^3 + 2u\partial + 2\partial u$$

are used to define first and second Poisson brackets of the celebrated KdV equation

$$u_t = 6uu_x + u_{xxx}.$$

Let us turn back our attention to dissipative system. If a dynamical system exhibits dissipative characteristics, then the system will not be represented by a Hamiltonian model. We have to adopt different geometric approach. This geometrical description of disspative systems goes back to the work of Brockett, Morrison, Grmela, Kaufman [5,10,16,17,23,24]. They have developed a technique to attach a dissipation terms to Hamiltonian systems (conservative part) geometrically. The dynamics of dissipative part are defined by the geometry induced on a phase space via a metric tensor. This is known as a metric or gradient system.

There are two classes of metriplectic systems [8,25]. The first class of metriplectic was described by Bloch, Krishnaprasad, Marsden and Ratiu [4] where as the second class of systems were proposed by Morrison, Kaufman and Grmela [10,16,23,24].

An **example** that fits into Leibniz mechanics framework [4] is the Landau-Lifschitz equation for the magnetization vector $\mathbf{m}$ in an external vector field $\mathbf{B}$,

$$\dot{\mathbf{m}} = \gamma \mathbf{m} \times \mathbf{B} + \frac{\beta}{||\mathbf{m}||^2}(\mathbf{m} \times (\mathbf{m} \times \mathbf{B})) \tag{4}$$

where $\gamma$ and $\beta$ are physical parameters. The Leibniz bracket on $\mathbf{R}^3$ associated to the Landau-Lifschitz equation is given by the sum of the two brackets

$$\{f, g\}_{skew}(\mathbf{m}) := \mathbf{m} \cdot (\nabla f(\mathbf{m}) \times \nabla g(\mathbf{m}))$$

$$\{\{f, g\}\}_{sym}(\mathbf{m}) := \frac{\beta(\mathbf{m} \times \nabla f(\mathbf{m}))(\mathbf{m} \times \nabla g(\mathbf{m}))}{\gamma ||\mathbf{m}||^2}.$$

**Motivation and result of the paper** The notion of generalized brackets is extendable to infinite-dimensional phase spaces. In this case the symmetric matrix or metric tensor $G^{ij}$ is replaced by various powers of Laplacian $\Delta$ and the Poisson tensor is replaced by skew-adjoint operators, and thus the ordinary Leibniz bracket goes over to field theoretic Leibniz bracket.

As a prototypical example, we see that the Burgers equation

$$u_t = \nu u_{xx} + u u_x \tag{5}$$

and the Navier-Stokes equation

$$\Omega_t + curl[\Omega \times u] = \nu \nabla^2 \Omega, \tag{6}$$

expressed in terms of vorticity $\Omega = \nabla \times u$, fit into this model. In the case of one dimensional Burgers equation the dissipation term is

$$\nu u_{xx} = \nu \partial^2 u,$$

and the dissipation term for Navier-Stokes equation is

$$\nu \nabla^2 \Omega = \nu \nabla^2 curl u = \nu \nabla^2 curl curl \frac{\delta H}{\delta u} = -\nu \nabla^4 \frac{\delta H}{\delta u}.$$

Thus, everytime the metric tensor $G^{ij}$ for finite-dimensional system is replaced by various powers of Laplacian, that is,

$$G^{ij} \Longrightarrow (\Delta)^n \qquad \forall n \geq 1.$$

In this paper we study a field theoretic analog of two types of metriplectic systems. Actually, we give a geometrical method to construct metriplectic systems via the Euler-Poincaré framework. We show that the Burgers hierarchy associated to the EP flows on space of first order differential operators exhibits first category of infinite-dimensional metriplectic systems. The Lie-Poisson or Leibniz-Poisson brackets consist of two parts: symmetric and skew symmetric parts. We also show how the EP framework of the Holm-Staley leads us to construct second category of infinite-dimensional metriplectic systems. Finally we study 2D Navier-Stokes equation. We show that although it exhibits metriplectic structure but this can not be described via Euler-Poincaré framework.

This paper is **organized** as follows: We give a brief introduction to metriplectic systems in Section 2. In the Section 3 we briefly describe the basic features of Leibniz brackets. We show its connection to metriplectic structure. Section 4 is devoted to infinite-dimensional analog of metriplectic geometry. We show that various types of the Burgers equation fits into the first category of (infinite-dimensional) metriplectic systems. We describe the Holm-Staley type system in Section 5. This equation fits into the second category of infinite-dimensional metriplectic systems. We consider 2D Navier-Stokes equation in Section 5. Several other applications are described in this Section.

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# 2 A brief description of metriplectic dynamics

The representation of a dynamical system as a metriplectic system [8,25] requires two geometrical structures - a Poisson structure $P$ and a covariant metric tensor $G$. In other words, the dynamics is controlled by a skew-symmetric Poisson tensor $P$ and a symmetric tensor $G$:

$$T(df, dg) = P(df, dg) + \lambda G(df, dg),$$

and the generalized Hamiltonian dynamics on any function $f$ is given by

$$f_t = \sum_{j,k=1}^{n} P^{jk}(z) \frac{\partial f}{\partial z_j} \frac{\partial H}{\partial z_k} + \lambda G^{jk} \frac{\partial f}{\partial z_j} \frac{\partial H}{\partial z_k}, \tag{7}$$

where $\lambda$ is any number. The bracket associated to contravariant symmetric tensor field of the type $(2,0)$

$$G(z) = \sum_{i,j}^{n} G^{ij}(z) \frac{\partial}{\partial z_i} \otimes \frac{\partial}{\partial z_j} \qquad \text{where } f, g \in C^\infty(M)$$

is called symmetric bracket.

A symmetric bracket becomes a semimetric bracket iff it is non-negative, i.e. the matrix $[G^{jk}]$ is non-negative definite. A semimetric bracket is called metric bracket if $G$ is positive definite, that is, with constant maximal rank.

The generalized equation of motion in terms of local coordinate systems is given by

$$\dot{z}_i = P^{ij} \frac{\partial H}{\partial z_j} + G^{ij} \frac{\partial H}{\partial z_j}. \tag{8}$$

It is clear that in order the system to describe dissipation we require that

$$\frac{dH}{dt} = G^{ij} \partial_i H \partial_j H \leq 0.$$

Let us fix $\lambda = -1$. Then the above condition compelled us to consider a semimetric type Leibniz structure. The semimetric Leibniz manifold is a pair $(M, P - G)$, where $P$ is a Poisson bivector and $G$ is a semimetric tensor.

**Second class of metriplectic systems**: A second class of metriplectic system is defined by the equations

$$\dot{z} = PdH + GdS, \tag{9}$$

where $H$ and $S$ are functions on $M$. If the functions $H$ and $S$ are such that

$$PdS = GdH = 0$$

then this system can be expressed in terms of free energy $F = H - S$ in a simple form

$$\dot{z} = P^{ij}\frac{\partial F}{\partial x^j} + G^{ij}\frac{\partial F}{\partial x^j}. \tag{10}$$

In this situation $H$ remains a conserved quantity and the dissipation is described by motion transverse to the level surface $S$.

If $G$ is positive-definite, then the function $S$ can be regarded as a entropy type function, and the dissipative behavior of the system can be interpreted as an increase of entropy along trajectories

$$\dot{S} = \frac{\partial S}{\partial z^i}\dot{z} = \frac{\partial S}{\partial z^i}G^{ij}\frac{\partial S}{\partial z^j} = (S, S) \geq 0.$$

# 3 Leibniz bracket and Dynamics

Let us give a quick introduction to Leibniz bracket and Leibniz vector fields [26] and its connection to our metriplectic structure.

It has been noticed recently that a different type of Poisson bracket is sometimes necessary to incorporate dissipative type systems. A well known example is almost Poisson brackets, the brackets do not satisfy Jacobi identity, are employed to study non-holonomic constrained system [6]. Morrison [23,24] and Brockett [see for example, 5] have proposed the modeling of certain dissipative phenomena by adding a symmetric bracket to a known antisymmetric one. This new bracket is called Leibniz bracket, given as

$$[.,.]_{Leibniz} = \{.,.\}_{skew} + \{\{.,.\}\}_{sym},$$

where the bracket $\{.,.\}_{skew}$ is skewsymmetric, $\{\{.,.\}\}_{sym}$ is symmetric, and the sum is a Leibniz bracket. In the infinite-dimensional case, this bracket captures the modeling of a surprising number of physical examples [11-13]. The Leibniz dynamics has a profound applications in nonholonomic mechanics [3, references therein] and in conformal Hamiltonian systems [22,27].

A Leibniz algebra is a vector space $\mathfrak{g}$ equipped with a binary operation $[,] : \mathfrak{g}\otimes\mathfrak{g} \to \mathfrak{g}$ which satisfies the Leibniz identity

$$[x, [y, z]] = [[x, y], z] - [[x, z], y], \qquad x, y, z \in \mathfrak{g}.$$

Lie algebras are trivial examples of Leibniz algebras, those with antisymmetric bracket.

**Definition 3.1** *Let $M$ be a smooth manifold and let $C^\infty(M)$ be the ring of smooth functions on it. A Leibniz bracket on $M$ is a bilinear map $[.,.]_{Leib} : C^\infty(M) \times C^\infty(M) \longrightarrow C^\infty(M)$ that is a derivation in each entry:*

$$[fg, h]_{Leib} = [f, h]_{Leib} g + f[g, h]_{Leib}, \qquad [f, gh]_{Leib} = g[f, h]_{Leib} + h[f, g]_{Leib}.$$

This bracket was first introduced by Grabowski and Urbanski [9], and one should not be confused with the Loday's famous construction [20] of Leibniz structure.

Let $H$ be a smooth function on $M$. There exist two vector fields $\mathcal{X}_H^R$ and $\mathcal{X}_H^L$ on $M$, defined as

$$\mathcal{X}_H^R[f] = [f, H]_{Leib} \qquad \text{and} \qquad \mathcal{X}_H^L[f] = -[H, f]_{Leib} \qquad f \in C^\infty(M).$$

These two vector fields appear as a direct consequence of the Leibniz structure. Given two smooth functions $g, h \in C^\infty(M)$ there exists a unique vector field $\mathcal{X}_{g,h}$ on $M$ such that

$$\mathcal{X}_{g,h}^R[f](m) = [[f, g], h] + [[g, h], f] + [[h, f], g],$$

where all the brackets are Leibniz brackets.

The flow $\phi : M \times \mathbf{R} \longrightarrow M$ of the vector field $\mathcal{X}_H^R$ satisfies

$$\mathcal{X}_H^R(f) = \frac{d}{dt}|_{t=0} f(\phi_t(x) = [f, H](\phi_t(x)) \qquad f \in C^\infty(M).$$

**Definition 3.2** *An almost Poisson manifold is a pair $(M, [.,.]_{Leib})$ and the bracket $[.,.]_{Leib}$ does not satisfy Jacobi identity. An almost Poisson structure on $M$ will be Poisson manifold if its Jacobiator $\mathcal{J} : C^\infty(M) \times C^\infty(M) \times C^\infty(M) \longrightarrow C^\infty(M)$ given by*

$$\mathcal{J}(f, g, h) = [[f, g], h] + [[g, h], f] + [[h, f], g]$$

*vanishes.*

In fact, Definition 3.2 suggest us to use the name almost Poisson manifold for a manifold endowed with a Leibniz bracket (not Jacobi identity but Leibniz property - derivation on each argument).

Let us define a contravariant rank 2 tensor $F$. We can define tensor map $F : T^*M \times T^*M \longrightarrow \mathbf{R}$ by

$$F(df, dg) = [f, g]_{Leib} \qquad f, g \in C^\infty(M). \tag{11}$$

We can associate two vector bundle maps

$$F_R^\sharp : T^*M \longrightarrow TM, \qquad F_L^\sharp : T^*M \longrightarrow TM$$

corresponding to two vector fields $\mathcal{X}_f^R$, $\mathcal{X}_f^L$ respectively, these are defined as

$$F(\gamma,\delta) = <\gamma, F_R^\sharp(\delta)> \qquad \text{and} \qquad F(\gamma,\delta) = - <\delta, F_L^\sharp(\gamma)>,$$

for any $\gamma, \delta \in T^*M$. It is known that if the Leibniz bracket $[.,.]_{Leib}$ is either symmetric or antisymmetric then both distributions coincide. Interested readers are advised to consult Ortega et. al. [26].

## 3.1 Connection to metriplectic geometry

It has been found that the Leibniz bracket contains both the skew-symmetric and symmetric part. A dissipative vector field generated by $H$ is a vector field defined by a symmetric bracket

$$\mathcal{X}_H^D(f) = \{\{f, H\}\} \qquad \text{for all} \;\; f \in C^\infty(M), \tag{12}$$

and the flow generated by $\mathcal{X}_H^D$ is called dissipative flow.

Unlike Hamiltonian vector fields, dissipative vector fields do not act like derivation [cf. 24]. It satisfies

$$2\mathcal{X}_H^D\{\{f,g\}\}_{sym} = \{\{\mathcal{X}_H^D f, g\}\}_{sym} + \{\{f, \mathcal{X}_H^D g\}\}_{sym}$$

$$\Longrightarrow \qquad 2\{\{\{\{f,g\}\}_{sym}, h\}\}_{sym} = \{\{\{\{f,h\}\}_{sym}, g\}\}_{sym} + \{\{\{\{g,h\}\}_{sym}, f\}\}_{sym},$$

this is an analog of the Jacobi identity for symmetric bracket.

In this paper we only consider systems where $(2,0)$ tensor $F$ is sum of skew-symmetric (Poisson) or symmetric tensor $F = P + G$. The inverse of the $(2,0)$ tensors $P$ and $G$ give rise to symplectic form and Riemannian metric tensor respectively, and this justifies the nomenclature " metriplectic structure". Of course, the definition of metriplectic structure still holds good for any generic Poisson bracket, not necessarily it has to be non-degenerate structure.

# 4 Euler-Poincaré framework of the Burgers equation

In this section we consider the Burgers equation as an infinite-dimensional analogue of the ( first category) metriplectic geometry. We will study the Euler-Poincaré framework of the Burgers equation and study its Hamiltonian structure. The Lie-Poisson structure yields the Leibniz bracket. We tacitly apply the orbit method [18].

## 4.1 Computation of Hamiltonian structure

Let us consider a manifold $M$. Consider the determinant bundle $\wedge^n TM \longrightarrow M$. The group $\mathbf{R}^*$ acts on the fibres by multiplication.

**Definition 4.1** *A homogeneous function of degree $\mu$ on the complement $\wedge^n TM/M$ of the zero section of the determinant bundle*

$$F(\kappa z) = \kappa^\mu F(z) \tag{13}$$

*is called tensor density of degree $\mu$ on $M$.*

Let us denote $\mathcal{F}_\mu(M)$ the space of tensor-densities of degree $\mu$, hence,

$$\mathcal{F}_\lambda = \{a(x)dx^\lambda \mid a(x) \in C^\infty(S^1).$$

Let us consider a first order differential operator on the circle $S^1$

$$\Delta_1 = \frac{d}{dx} + u(x). \tag{14}$$

This $\Delta_1$ satisfies

$$\Delta_1 = \nu\frac{d}{dx} + u(x) \ : \ \mathcal{F}_{-\frac{1}{2}} \longrightarrow \mathcal{F}_{\frac{1}{2}}. \tag{15}$$

**Definition 4.2** *The $Vect(S^1)$- action on $\Delta_1$ is defined by the commutator with the Lie derivative*

$$[\mathcal{L}_{f(x)\frac{d}{dx}}, \Delta_1] \ := \ \mathcal{L}^{\frac{1}{2}}_{f(x)\frac{d}{dx}} \circ \Delta_1 - \Delta_1 \circ \mathcal{L}^{-\frac{1}{2}}_{f(x)\frac{d}{dx}}, \tag{16}$$

*where LHS denotes the Lie derivative action of vector fields on $\Delta_1$.*

The result of this action is a scalar operator, i.e. the operator of multiplication by a function.

**Lemma 4.3**

$$[\mathcal{L}_{f(x)\frac{d}{dx}}, \Delta_1] \ = \ \nu\frac{1}{2}f''(x) + uf_x(x) + u_x f(x). \tag{17}$$

**Proof**: By direct computation.
□

Hence, the (second) Hamiltonian operator of the Burgers equation (after rescaling) is given by

$$\mathcal{O}_{Burgers} = \nu\frac{d^2}{dx^2} + 2u\frac{d}{dx} + 2u_x. \tag{18}$$

**Lemma 4.4** *The Lie-Poisson bracket associated to the operator (18) is given us Leibniz bracket*

$$[f,g]_{Leib} := \int_{S^1} \frac{\delta f}{\delta u} \mathcal{O}_{Burgers} \frac{\delta g}{\delta u}\, dx, \tag{19}$$

*where*

$$\mathcal{O}_{Burgers} = \partial u + \nu \partial^2.$$

**Proof**: It is easy to see that

$$\mathcal{O}^{skew}_{Burgers} = \partial u, \qquad \mathcal{O}^{sym}_{Burgers} = \nu \partial^2.$$

So the diffusion part is coming from the symmetric part of $\mathcal{O}_{Burgers}$.
□

Let us study the dynamics involved with the above bracket. Thus using Eqn. (19) we obtain the Burgers equation

$$u_t = [u, H]_{Leib} = \nu u_{xx} + 2uu_x,$$

for $H = \frac{1}{2}\int_{S^1} u^2\, dx$.

## 4.2 Recursion operator and Burgers hierarchy

By inspection it is easy to see that the first Hamiltonian structure of the Burgers equation is

$$\mathcal{O}^1_{Burgers} \equiv \mathcal{O}_1 = \partial. \tag{20}$$

**Proposition 4.5** *The Burgers system is bi-Hamiltonian*

$$u_t = \mathcal{O}_1 \frac{\delta H_0}{\delta u} = \mathcal{O}_{Burgers} \frac{\delta H_1}{\delta u}$$

*where* $\frac{\delta H_0}{\delta u} = \nu u_x + u^2$ *and* $\frac{\delta H_1}{\delta u} = u$.

A recursion operator for a system is a linear operator $\mathcal{R}$ in the space of differential function with the property that if $Q$ is a generalized symmetry then $\tilde{Q} = \mathcal{R}Q$ is also a generalized symmetry.

Geometrically, $\mathcal{R}$ be a tangent-valued one-form ( a type $(1,1)$ tensor field) on a manifold $M$. It yields a bundle endomorphisms

$$\mathcal{R} : TM \longrightarrow TM, \qquad \mathcal{R}^* : T^*M \longrightarrow T^*M$$

over $M$. Let $\mathcal{O}_1$ and $\mathcal{O}_2$ be two Hamiltonian structures on $M$. A recursion operator is defined as

$$\mathcal{R} = \mathcal{O}_2\mathcal{O}_1^{-1}. \tag{21}$$

Therefore, the recursion operator ( or Nijenhuis tensor) associated to Burgers hierarchy is given by

$$\mathcal{N}[u] = \nu\partial + \partial u\partial^{-1}. \tag{22}$$

The Burgers hierarchy is then obtained by repeated applications on the translation group generator $Q_0 = u_x$, of the tensor operator expressed by

$$Q_k = \mathcal{R}^k Q_0,$$

where

$$Q_1 = 2uu_x + \nu u_{xx}$$
$$Q_2 = (3u^3 + 3uu_x + u_{xx})_x.$$

It is shown in Vilasi [28], that the dissipation hierarchy is generated by the odd powers and $\mathcal{R}^k$, where as, the Hamiltonian hierarchy is given by even powers of $\mathcal{R}^k$.

**Remark** : It is well known that the Burgers equation is connected to heat equation $v_t = v_{xx}$ via the Cole-Hopf transformation

$$u = \nu\frac{d}{dx}(ln\ v).$$

Thus, a Nijenhuis tensor or recursion operator for the heat hierarchy $\mathcal{R}_{heat}$ is readily obtained as

$$\mathcal{R}_{heat} = (\frac{\delta v}{\delta u})\mathcal{R}_{Burgers}(\frac{\delta v}{\delta u})^{-1}.$$

## 4.3 Geometrical interpretation of $Vect(S^1)$ action

The dual space of the Virasoro algebra or the centrally extended $Vect(S^1)$ algebra can be identified with the space of Hill's operator

$$\Delta = \frac{d^2}{dx^2} + u(x), \tag{23}$$

where $u$ is a periodic potential: $u(x+2\pi) = u(x) \in C^\infty(\mathbb{R})$. The Hill's operator maps

$$\Delta : \mathcal{F}_{-\frac{1}{2}} \longrightarrow \mathcal{F}_{\frac{3}{2}}. \tag{24}$$

The action of $Vect(S^1)$ on the space of Hill's operator is equivalent to the relation

$$\frac{d^2}{dx^2} + u(x) = (\frac{d}{dx} - \frac{1}{2}v(x))(\frac{d}{dx} + \frac{1}{2}v(x)), \tag{25}$$

where

$$u = \frac{1}{2}(v_x - \frac{1}{2}v^2).$$

This yields the formal factorization of the Hill's operator.

Geometrically this can be realized as

$$\mathcal{F}_{\frac{1}{2}} \xrightarrow{\Delta_1} \mathcal{F}_{-\frac{1}{2}} \xrightarrow{\Delta^1} \mathcal{F}_{-\frac{3}{2}},$$

where $\Delta = \Delta^1\Delta_1 = (\partial - \frac{1}{2}v)(\partial + \frac{1}{2}v)$. This is compatible with $\Delta : \mathcal{F}_{\frac{1}{2}} \longrightarrow \mathcal{F}_{-\frac{3}{2}}$.

Thus the Burgers operator can be related to the Hamiltonian operator obtained from the action of $Vect(S^1)$ on the square root of dual.

## 4.4 The Whitham-Burgers equation

Let us study the $H^1$ analogue of Lemma (4.3), and this would lead to different type of dispersive systems, the Whitham-Burgers equation

$$m_t + m_{xx} + (mu)_x = 0. \tag{26}$$

It is clear that the formula (18) is valid for $L^2$ norm. We are interested to get the same type of formula for the $H^1$ case.

**Lemma 4.6**

$$[\mathcal{L}_{f(x)\frac{d}{dx}}, \frac{d}{dx} + \frac{1}{2}u]_{H^1} \;=\; f''(x) + mf'(x) + m'f(x), \qquad \textit{with } m = u - u_{xx} \tag{27}$$

Again, we can interpret this equation as an action of the vector field $\mathcal{L}_{f(x)\frac{d}{dx}}$ on the space of modified first order scalar differential operator $\frac{d}{dx} + \frac{1}{2}m$.

The L.H.S. denotes coadjoint action with respect to $H^1$ norm. Once again we convert this to $L^2$ action, given as

$$[\mathcal{L}_{f(x)\frac{d}{dx}}, \frac{d}{dx} + \frac{1}{2}u]_{L^2} = (1-\partial^2)^{-1}[\mathcal{L}_{f(x)\frac{d}{dx}}, \frac{d}{dx} + \frac{1}{2}u]_{H^1}.$$

Therefore, the Hamiltonian operator of the Whitham-Burgers equation becomes

$$\mathcal{O}_{WB}^{H^1} = (1-\partial^2)^{-1}(\partial^2 + \partial m). \tag{28}$$

**Lemma 4.7** *The Euler–Poincaré flow on the space with respect to $H^1$ norm on the first order differential operators yields the Whitham Burgers equation*

$$m_t + m_{xx} + (mu)_x = 0.$$

**Proof**:

$$u_t = \mathcal{O}_{WB}^{H^1}\frac{\delta H}{\delta u}]$$
$$= -(1-\partial^2)^{-1}(\partial^2+\partial m)\frac{\delta H}{\delta u}$$
$$\Longrightarrow \qquad (1-\partial^2)u_t = -(\partial^2+\partial m)\frac{\delta H}{\delta u}$$

for Hamiltonian $H = \frac{1}{2}\int_{S^1} u^2\, dx$.
□

In [7] Degasperis et. al. presented a large family of peakon equations, with a Legendre transformation leading to the Lie-Poisson structure and Hamiltonian equation as

$$m_t = \{m, \tilde{H}\}_{LP} = \hat{\mathcal{O}}\frac{\delta \tilde{H}}{\delta m} \tag{29}$$

with the quadratic Hamiltonian

$$\tilde{H} = \frac{1}{2}\int mg * m\, dx, \qquad \frac{\delta \tilde{H}}{\delta m} = u,$$

where $g*$ denotes convolution with a symmetric integral kernel $g(x)$ defined on the real line $\mathbf{R}$

$$u(x) = g * m(x) = \int_{-\infty}^{+\infty} g(x-y)m(y)\, dy.$$

The Lie-Poisson bracket associated to the operator (14) can be written in form of Leibniz bracket

$$[f,g]_{Leib} := \int_{S^1} \frac{\delta f}{\delta m}\mathcal{O}_{WB}\frac{\delta g}{\delta m}\, dx, \tag{30}$$

where $\mathcal{O}_{WB} = \partial m + \nu\partial^2$, and the skew and symmetric part of $\mathcal{O}_{WB}$ are given by

$$\mathcal{O}_{WB}^{skew} = \partial m, \qquad \mathcal{O}_{WB}^{sym} = \nu\partial^2.$$

# 5 Infinite-dimensional analogue of Brockett-Grmela-Kaufman-Morrison type of metriplectic systems

In this Section we will study an infinite-dimensional analogue of dynamical system of the form

$$\dot{x} = TdF, \qquad \text{where } T = P(x) + G(x) \qquad \text{and } F = H - S$$

such that $PdS = GdH = 0$.

This type of metriplectic systems have been propsed by Morrison and studied by Kaufman, Grmela and others.

**Proposition 5.1** *Let us consider* $T = \partial u + \nu\partial^2$. *Suppose we consider Hamiltonian* $H$ *and entropy function* $S$ *such that*

$$\frac{\delta F}{\delta u} = u - u^{-1}.$$

*The dynamical system associated to this metriplectic system is governed by the Euler-Poincaré flow related to generalized free energy* $F$ *is given by*

$$u_t = 2uu_x + \nu(u^{-1})_{xx}. \tag{31}$$

**Proof**: It is readily clear that $S$ is a Casimir for the Poisson tensor $P$ and $dH$ is a null vector for one-dimensional Laplacian $\Delta = \partial^2$.

Thus, the Euler-Poincaré flow is given by

$$u_t = (\partial u + \nu\partial^2)\frac{\delta F}{\delta u}$$

$$= (\partial u + \nu\partial^2)(u - u^{-1})$$

$$= 2uu_x + \nu(u^{-1})_{xx}.$$

□

**Remark** By simple transformation $u = \rho^{-1}$, Eqn. (25) can be transformed to

$$\rho_t = 2\rho^{-1}\rho_x - \nu\rho_{xx}. \tag{32}$$

## 5.1 The Holm-Staley Equation

In this section we show that Euler-Poincaré framework of the Holm-Staley equation

$$m_t + \kappa m_{xx} + m_x u + 3m_x u = 0.$$

There is a dispersive term added to the Degasperis-Procesi equation [7].

The solutions of the Holm-Staley family of equations [14,15] exhibit stable Burgers ramp/cliff solution structure. We will see that Hamiltonian framework of the Holm-Staley equation exhibits second category of metriplectic structure proposed by Morrison satisfying slightly relaxed condition

$$PdS \neq 0, \qquad GdH \neq 0.$$

This means, the Holm-Staley cannot be explained via generalized free energy $F = H - S$. Thus it *cannot be written* as

$$u_t = (P + G)\frac{\delta F}{\delta u}.$$

The Hamiltonian structure of the Holm-Staley equation is the combination of the Hamiltonian operators of the Camassa-Holm equation and the Whitham-Burgers equation.

Let us recall the Hamiltonian operator associated to the Camassa-Holm equation

$$\mathcal{O}_{CH} = (1 - \partial^2)^{-1}(\partial m + m\partial).$$

**Definition 5.2** *The Hamiltonian structure of the Holm-Staley equation is given by*

$$\mathcal{O}_{HS} = (1 - \partial^2)^{-1}(\lambda \mathcal{O}_{CH} + \mu \mathcal{O}_{WB}). \tag{33}$$

Therefore, we obtain

$$\mathcal{O}_{HS} = (1 - \partial^2)^{-1}[(\lambda \partial m + m\partial) + \mu(\nu\partial^2 + \partial m)].$$

Let us set $\lambda = \mu = 1$ and $\mu = 1$.

The Lie-Poisson bracket associated to the Holm-Staley Hamiltonian operator can be expressed in terms of Leibniz bracket

$$[f, g]_{Leib} := \int_{S^1} \frac{\delta f}{\delta u} \mathcal{O}_{HS} \frac{\delta g}{\delta u}\, dx, \tag{34}$$

where the skew and symmetric part of $\mathcal{O}_{HS}$ are given by

$$\mathcal{O}_{HS}^{skew} = (1 - \partial^2)^{-1}(m\partial + 2\partial m), \qquad \mathcal{O}_{HS}^{sym} = \nu(1 - \partial^2)^{-1}\partial^2.$$

**Proposition 5.3** *The following Euler-Poincaré flow*

$$u_t = -\mathcal{O}_{HS}^{skew} \frac{\delta H}{\delta u} - \mathcal{O}_{HS}^{sym} \frac{\delta S}{\delta u}, \tag{35}$$

*where* $H = \frac{1}{2} \int_{S^1} u^2\, dx$ *and*

$$S = \int_{S^1} (u^2 + u_x^2)\, dx. \tag{36}$$

The second Hamiltonian or entropy function $S$ takes values on $H^1$ norm of $u$. Thus, we find that the Holm-Staley equation fits into the category of Morrison class of metriplectic system which does not satisfy $PdS \neq 0$ and $GdH \neq 0$.

# 6 Applications to Hydrodynamics, Discussion and Outlook

In this paper we explore the infinite-dimensional analogue of two classes of metriplectic systems proposed by Morrison and Bloch et. al. respectively. We found that in the infinite-dimensional case the covariant metric tensor $G$ is replaced by symmetric operator, Viz. Laplacian. Therefore, the Lie-Poisson structure associated to such systems are

All the equations considered in this paper related to a infinite-dimensional manifold $M$ on which both a Poisson or anti-symmetric and a symmetric tensor. We have seen that the Burgers fits into the category of metriplectic system proposed by Bloch, Krishnaprasad, Marsden and Ratiu and the Holm-Staley equation can be explained via the method of Morrison and Kaufman.

## 6.1 Applications to hydrodynamic systems

Our next aim is to generalize this metriplectic construction to the Navier-Stokes equation. The 2D Navier-Stokes equation is given as

$$\partial_t \Omega + \{\Psi, \Omega\} = \nu \nabla^2 \Omega,$$

where $\Psi$ and $\Omega$ are known as stream function and vorticity of fluid, and these are related by $\Omega = \nabla^2 \Psi$. It is difficult to see that this equation can be recasted into Poisson part (2D Euler equation) which yields skew-symmetric part and vorticity part is connected to symmetric part. Again, the metric tensor for finite dimensional case is replaced by Laplacian.

Let us briefly discuss the geometry of the group $SDiff(\mathcal{A})$ of area preserving diffeomorphisms of the annulus

$$\mathcal{A} := \{0 \leq x \leq 1\} \times \{exp(2\pi i \theta) | 0 \leq \theta \leq 1\}$$

from the paper of Bloch, Flaschka and Ratiu [2].

The Lie algebra $\mathfrak{g} = sdiff(2)$ of $SDiff(\mathcal{A})$ is the algebra of divergence-free vector fields tangent to the boundary of $\mathcal{A}$. These vector fields are Hamiltonian with respect to the area form $dx \wedge d\theta$ and their Hamiltonian functions $H(x, \theta)$ satisfy

$$\frac{\partial H(x_0, \theta)}{\partial \theta} = 0 \qquad x_0 = 0, 1. \tag{37}$$

The action of the vector field $X_H$ on $\mathcal{A}$ is given by

$$\mathcal{L}_{X_H} F =< dF, X_H >= \{F, H\} := \frac{\partial F}{\partial x}\frac{\partial H}{\partial \theta} - \frac{\partial F}{\partial \theta}\frac{\partial H}{\partial x}. \tag{38}$$

Let us fix the ad-invariant quantity

$$TrL = \int_{\mathcal{A}} Ldxd\theta. \tag{39}$$

Thus, we can define a weakly nondegenerate invariant inner product [8] on $sdiff(2)$ by

$$< L, M >= Tr(LM) = \int_{\mathcal{A}} LM\ dxd\theta \qquad L, M \in sdiff(2). \tag{40}$$

Thus, if $g \in SDiff(\mathcal{A})$ we have

$$\begin{aligned} < L \circ g, M \circ g >&= \int_{\mathcal{A}} (L \circ g)(M \circ g)dxd\theta \\ &= \int_{\mathcal{A}} ((LM) \circ g)dxd\theta =< L, M > . \end{aligned}$$

The infinitesimal version of invariance is given by

$$< ad^*_N L, M >=< L, ad_N M > \tag{41}$$

or

$$< \{L, N\}, M >=< L, \{N, M\} >$$

where $ad_F(G) = \{F, G\}$.

Hence, the coadjoint action is

**Lemma 6.1**

$$ad^*_F(G) = -\{F, G\} \equiv -(\frac{\partial F}{\partial x}\frac{\partial G}{\partial \theta} - \frac{\partial F}{\partial \theta}\frac{\partial G}{\partial x}). \tag{42}$$

Therefore, the Lie-Poisson bracket on $sdiff(2)$ is given by

$$\{f, g\}_{LP}(\alpha) =< \alpha, \{\frac{\delta f}{\delta \alpha}, \frac{\delta g}{\delta \alpha}\} >, \tag{43}$$

where $\frac{\delta f}{\delta \alpha}$ denotes the Frechét derivative.

Our next task is to derive 2D Navier-Stokes equation. Here we apply the Morrison-Brockett prescription by adding a symmetric bracket, induced by a symmetric operator, to our known antisymmetric Poisson structure.

**Proposition 6.2** *The 2D Navier-Stokes equation is a Leibniz flow associated to the area preserving group of diffeomorphism for Hamiltonian*

$$H_1 =< \Omega, \Psi >=< |\nabla \Psi|^2 >,$$

*where* $\Psi$ *is the stream function given by* $\Omega = \Delta \Psi$, *and " entropy function"* $S = \frac{1}{2} < \Omega, \Omega >$ *The Leibniz dynamics is defined as*

$$\frac{d}{dt}\Omega(t) = -ad^*_{\frac{\delta H_1}{\delta \Omega}} \Omega(t) + \nu \nabla^2 \frac{\delta S}{\delta \Omega}. \tag{44}$$

## 6.2 Final outlook

There are several dissipative systems can be described via metriplectic construction. But the question arises as to whether all these constructions meet the requirement of the Euler-Poincaré framework. Certainly, 2D Navier-Stokes fails to satisfy this requirement.

It would be more challenging to work with the 3D Navier-Stokes equation

$$\partial_t \Omega + (u \cdot \nabla)\Omega - (\Omega . \nabla) u = \nu \nabla^2 \Omega.$$

The metricplectic structure here would give rise to many surprising results. It is essential to understand the interaction between conservative and dissipative forces of such systems. Certainly, this would enable us to study the chaotic and complex behaviour of the real world system.

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