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Expected volume of intersection of Wiener
sausages and heat kernel norms on compact
Riemannian manifolds with boundary

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EXPECTED VOLUME OF INTERSECTION OF WIENER SAUSAGES AND HEAT KERNEL NORMS ON COMPACT RIEMANNIAN MANIFOLDS WITH BOUNDARY

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ABSTRACT. Estimates are obtained for the expected volume of intersection of independent Wiener sausages in Euclidean space in the small time limit. The asymptotic behaviour of the weighted diagonal heat kernel norm on compact Riemannian manifolds with smooth boundary is obtained in the small time limit.

1. INTRODUCTION

Let B_1, \dots, B_k be k independent Brownian bridges in \mathbb{R}^m associated with the parabolic operator $-\Delta + \frac{\partial}{\partial t}$, and with $B_i(0) = B_i(t) = 0$ for $1 \leq i \leq k$. Let K be a compact, non-polar set in \mathbb{R}^m . For $1 \leq i \leq k$, let

$$W_K^i(t) = \{B_i(s) + y : 0 \leq s \leq t, y \in K\}$$

be the corresponding pinned Wiener sausages. These random sets are Borel measurable with probability one, and we let

$$Z_{k,m}(t) := \mathbb{E}^1 \otimes \dots \otimes \mathbb{E}^k \left[\left| \prod_{i=1}^k \cap W_K^i(t) \right| \right].$$

It is well known that $Z_{k,m}(t)$ is related to the virial coefficients of a quantum system of obstacles K at inverse temperature t . For example, G. E. Uhlenbeck [15] calculated the asymptotic behaviour of $Z_{k,m}(t)$ as $t \rightarrow \infty$ in the special case where $k = 1$, $m = 3$, and K is a ball. I. McGillivray [14] obtained the full asymptotic series as $t \rightarrow \infty$ for $k = 1$, $m \geq 3$, and K an arbitrary, non-polar compact set. The special case where $k = 1$, $m = 2$, and K a ball was studied for $t \rightarrow \infty$ by M. van den Berg and E. Bolthausen [4].

In this paper, we analyse the asymptotic behaviour of $Z_{k,m}(t)$ as $t \rightarrow 0$. Let $p_{\mathbb{R}^m}(\cdot, \cdot; \cdot)$ denote the heat kernel for \mathbb{R}^m given by

$$p_{\mathbb{R}^m}(x, y; t) = (4\pi t)^{-m/2} e^{-|x-y|^2/(4t)},$$

and let $p_{\mathbb{R}^m - K}$ denote the Dirichlet heat kernel for the open set $\mathbb{R}^m - K$. By the Feynman-Kac formula, we have that for $x \in \mathbb{R}^m - K$,

$$(1.a) \quad \mathbb{P}_x[B(s) \cap K = \emptyset, 0 \leq s \leq t] = (4\pi t)^{m/2} p_{\mathbb{R}^m - K}(x, x; t),$$

where $B(s)$, $0 \leq s \leq t$ is a Brownian bridge with $B(0) = B(t) = x$. It is easily seen that

$$(1.b) \quad 0 \leq p_{\mathbb{R}^m - K}(x, x; t) \leq p_{\mathbb{R}^m}(x, x; t),$$

and that

$$(1.c) \quad Z_{k,m}(t) = (4\pi t)^{km/2} \int_{\mathbb{R}^m} \{p_{\mathbb{R}^m}(x, x; t) - p_{\mathbb{R}^m - K}(x, x; t)\}^k dx,$$

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where the Dirichlet heat kernel is extended to all of \mathbb{R}^m by putting

$$p_{\mathbb{R}^m - K}(x, y; t) := 0 \quad \text{for } x \in K \quad \text{and or } y \in K.$$

This extends the equality of formula (1.a) to all of \mathbb{R}^m .

It follows from the results in [5] that if $k = 1$, $m \geq 2$, and K is compact with smooth boundary, then there exists an asymptotic series as $t \rightarrow 0$:

$$Z_{1,m}(t) = \sum_{j=0}^J c_{1,j} t^{j/2} + O(t^{(J+1)/2}),$$

where the coefficients $c_{1,j}$ have been computed for $j = 0, \dots, 4$. For example,

$$c_{1,0} = \int_K 1 dx, \quad c_{1,1} = \frac{\sqrt{\pi}}{2} \int_{\partial K} 1 dy, \quad \text{and} \quad c_{1,2} = \frac{1}{6} \int_{\partial K} L_{aa} dy,$$

where dy is the surface measure on ∂K , L_{aa} is the trace of the second fundamental form on ∂K , and where ∂K is oriented with an outward orientation, i.e. the inward normal is chosen on $\partial(M - K)$.

Let $\xi(t)$ be defined for $t > 0$. We say that $\xi(t) \sim \sum_{n \geq 0} \xi_n t^{n/2}$ as $t \rightarrow 0$ if

$$\xi(t) = \sum_{n=0}^J \xi_n t^{n/2} + O(t^{(J+1)/2}) \quad \text{for any } J \in \mathbb{N}.$$

The main results of this paper are the following two theorems:

Theorem 1.1. *Let K be a compact set in \mathbb{R}^m ($m \geq 2$) with smooth boundary ∂K . Let $k \in \mathbb{N}$ be arbitrary. Then one has that $Z_{k,m}(t) < \infty$ for all $t > 0$, and $Z_{k,m}(t) \sim \sum_{j \geq 0} c_{k,j} t^{j/2}$ as $t \rightarrow 0$ where,*

$$c_{k,0} = \int_K 1 dx, \quad c_{k,1} = \frac{1}{2} \left(\frac{\pi}{k} \right)^{1/2} \int_{\partial K} 1 dy,$$

$$c_{1,2} = -\frac{1}{3} \int_{\partial K} L_{aa} dy,$$

and for $k \geq 2$

$$c_{k,2} = \left\{ -\frac{1}{2k} + \frac{k}{2} \left(\frac{\pi}{4(k-1)^{3/2}} \right. \right. \\ \left. \left. - \frac{1}{2k(k-1)} - \frac{\arctan(k-1)^{-1/2}}{2(k-1)^{3/2}} \right) \right\} \int_{\partial D} L_{aa} dy.$$

Theorem 1.2. *Let M be a compact m -dimensional Riemannian manifold with smooth boundary ∂M . Let $p_M(\cdot, \cdot; \cdot)$ denote the heat kernel for the Laplace-Beltrami operator acting on $L^2(M)$ with Dirichlet boundary conditions on ∂M . Let f be smooth on M and let $k \in \mathbb{N}$.*

- (1) For $t \rightarrow 0$, $\int_M (p_M(x, x; t))^k f(x) dx \sim (4\pi t)^{-km/2} \sum_{j=0}^{\infty} a_{k,j} t^{j/2}$.
- (2) $a_{k,0} = \int_M f dx$.
- (3) $a_{k,1} = \frac{1}{2} \sqrt{\pi} \sum_{\ell=1}^k (-1)^\ell \ell^{-1/2} \binom{k}{\ell} \int_{\partial M} f dy$.
- (4) $a_{k,2} = \frac{k}{6} \int_M f \tau dx$

$$- \frac{1}{2} \sum_{\ell=1}^k \ell^{-1} \int_{\partial M} f^{(1)} dy + \left\{ -\frac{k}{6} + \frac{1}{2} \sum_{\ell=1}^k \binom{k}{\ell} (-1)^{\ell-1} \right.$$

$$\left. \times \left\{ (k-\ell) \left(\frac{\pi}{4\ell^{3/2}} - \frac{1}{2\ell(\ell+1)} - \frac{\arctan \ell^{-1/2}}{2\ell^{3/2}} \right) + \frac{1}{\ell} \right\} \right\} \int_{\partial M} f L_{aa} dy$$

where τ is the scalar curvature, and where $f^{(1)}$ is the normal derivative of f with respect to the inward unit normal vectorfield on ∂M .

It is possible to obtain the formulae for the $c_{k,j}$, $j \geq 1$ in Theorem 1.1 directly from the formulae for the $a_{k,j}$, $j \geq 1$ in Theorem 1.2 by noting that the formulae in Theorem 1.2 also hold for non-compact m -dimensional Riemannian manifolds provided ∂M is compact and smooth and f is smooth and has compact support. Specializing to the case where $M = \mathbb{R}^m - K$, f compactly supported and identically equal to 1 in a neighbourhood of ∂K and noting that $p_{\mathbb{R}^m}(x, x; t) = (4\pi t)^{-m/2}$ yields a closed set of equations relating the $c_{k,j}$ and $a_{k,j}$ by equating powers of t :

$$(1.d) \quad \begin{aligned} c_{k,j} &= \sum_{\ell=1}^k (-1)^\ell \binom{k}{\ell} a_{\ell,j}, \quad j \in \mathbb{N}, \\ a_{k,j} &= \sum_{\ell=1}^k (-1)^\ell \binom{k}{\ell} c_{\ell,j}, \quad j \in \mathbb{N}. \end{aligned}$$

This paper is organized as follows. In Section 2, we consider a half space and a suitable localizing function f . The formulae for $a_{k,0}$ and $a_{k,1}$ in Theorem 1.2 follow from Lemma 2.1. One of the two boundary terms in $a_{k,2}$ also follows from Lemma 2.1.

In Section 3, we study a planar region in Euclidean space. In Lemma 3.3, we determine the coefficient of $\int_{\partial D} L_{aa} dy$ in $c_{k,2}$. This completes the proof of Theorem 1.1. The second boundary term in $a_{k,2}$ is determined in Lemma 3.4 by computing the coefficient of $\int_{\partial M} f L_{aa} dy$. This completes the computation of the boundary term in $a_{k,2}$. It would be a tedious combinatorial exercise to show that these coefficients are consistent with formula (1.d).

In Section 4, we complete the proof of Theorem 1.2 by determining the interior term in $a_{k,2}$. This uses the results of McKean and Singer [13]. The arguments used in this section together with existing formulae [7] would suffice to determine $a_{k,\ell}$ for all k and for $\ell \leq 6$ if the boundary of M is empty. The crucial difficulty in this paper, however, is the determination of the boundary terms. These are not accessible by the methods used in Section 4.

In Section 5 we obtain results for the expected volume of intersection of Wiener sausages which are not pinned, and which complement Theorem 1.1.

2. HALF SPACE CALCULATIONS

Let $p_{\mathbb{R}^+}(x, x; t)$ be the Dirichlet heat kernel for the positive half line \mathbb{R}^+ .

Lemma 2.1. *Let $k \in \mathbb{N}$, and let f be smooth. For $t \rightarrow 0$, we have*

$$\begin{aligned} & \int_{\mathbb{R}^+} \{p_{\mathbb{R}^+}(x, x; t)\}^k f(x) dx \\ \sim & (4\pi t)^{-k/2} \left\{ \int_{\mathbb{R}^+} f(x) dx + \sum_{j=1}^{\infty} t^{j/2} f^{(j-1)}(0) \alpha_{k,j} \right\} \end{aligned}$$

where

$$\alpha_{k,j} = \frac{1}{2} \sum_{\ell=1}^k (-1)^\ell \binom{k}{\ell} \Gamma\left(\frac{j}{2}\right) \Gamma(j)^{-1} \ell^{-j/2}.$$

Proof. We have that

$$\begin{aligned} p(x, x; t) &= (4\pi t)^{-1/2}(1 - e^{-x^2/t}), \\ f(x) &\sim \sum_{j=0}^{\infty} \frac{1}{j!} f^{(j)}(0)x^j \quad \text{as } x \rightarrow 0, \\ (1 - e^{-x^2/t})^k &= 1 + \sum_{\ell=1}^k (-1)^\ell \binom{k}{\ell} e^{-\ell x^2/t}, \\ \int_0^\infty x^j e^{-\ell x^2/t} dx &= \frac{1}{2} \Gamma\left(\frac{j+1}{2}\right) \left(\frac{t}{\ell}\right)^{(j+1)/2}, \quad j > -1. \end{aligned}$$

Lemma 2.1 follows from these identities. \square

Note that by identity (1.45) of [8]

$$\alpha_{k,2} = -\frac{1}{2} \sum_{\ell=1}^k \ell^{-1}.$$

3. COMPUTATIONS FOR PLANAR REGIONS AND PROOF OF THEOREM 1.1

Lemma 3.1. *Let K be a compact set in \mathbb{R}^m , and let $Z_{k,m}(t)$ be given by (1.c). Then $Z_{k,m}(t) < \infty$ for all $t > 0$.*

Proof. Since $p_{\mathbb{R}^m}(x, y; t) \leq (4\pi t)^{-m/2}$ we have by (1.b), (1.c) that

$$\begin{aligned} (3.a) \quad Z_{k,m}(t) &\leq (4\pi t)^{m/2} \int_{\mathbb{R}^m} \{p_{\mathbb{R}^m}(x, x; t) - p_{\mathbb{R}^m - K}(x, x; t)\} dx \\ &= (4\pi t)^{m/2} \int_{\mathbb{R}^m} \int_{\mathbb{R}^m} \{(p_{\mathbb{R}^m}(x, y; t/2))^2 - (p_{\mathbb{R}^m - K}(x, y; t/2))^2\} dx dy \\ &\leq 2^{(2+m)/2} \int_{\mathbb{R}^m} \int_{\mathbb{R}^m} \{p_{\mathbb{R}^m}(x, y; t/2) - p_{\mathbb{R}^m - K}(x, y; t/2)\} dx dy \\ &= 2^{(2+m)/2} \int_{\mathbb{R}^m - K} \left\{ 1 - \int_{\mathbb{R}^m - K} p_{\mathbb{R}^m - K}(x, y; t/2) dy \right\} dx \\ &\quad + 2^{(2+m)/2} |K|, \end{aligned}$$

where we have used the semigroup property of the heat kernels for \mathbb{R}^m and $\mathbb{R}^m - K$ respectively. The first term in the right hand side of (3.a) is the amount of heat in $\mathbb{R}^m - K$ at time $t/2$ if K is kept at fixed temperature $2^{(2+m)/2}$, and if $\mathbb{R}^m - K$ has initial temperature 0. This term is finite by the results of F. Spitzer [16]. \square

A key ingredient in the proofs of Theorems 1.1 and 1.2 is an estimate for the Dirichlet heat kernel obtained by R. Lang [9] and H. R. Lerche and D. Siegmund [11]. For related results see also [12]. Before we state their results (Theorem 3.2) we introduce some further notation. For an open, bounded and connected set D in \mathbb{R}^2 we define the distance function $\delta : D \rightarrow (0, \infty)$ by $\delta(x) = \min\{|x - z| : z \in \partial D\}$. Suppose ∂D is of class C^3 . Then there exists $\varepsilon_D > 0$ such that for all $x \in D$ with $\delta(x) < \varepsilon_D$ there is a unique $s \in \partial D$, depending on x , with $|x - s| = \delta(x)$. Hence for all such $x \in D$ there is a smooth parametrization $x \rightarrow (s, \delta)$ where $s \in \partial D$ is parametrized by arc length.

Theorem 3.2. *Let D be an open, bounded and connected set in \mathbb{R}^2 with C^3 boundary. Let $x \in D$ be such that $\delta(x) \leq \varepsilon_D/2$. Then as $t \rightarrow 0$*

$$(3.b) \quad p_D(x, x; t) = (4\pi t)^{-1} \left\{ 1 - e^{-\delta(x)^2/t} - L_{aa}(s)\delta(x)^2 t^{-1/2} \int_{\delta(x)t^{-1/2}}^{\infty} e^{-\eta^2} d\eta \right\} + O(1),$$

where the remainder $O(1)$ is uniform on $\{x \in D : \delta(x) \leq \varepsilon_D/2\}$, L_{aa} is the curvature of ∂D at s , and where ∂D is oriented by an inward unit vector field.

Lemma 3.3. *Let D be as in Theorem 3.2, and let $k \in \mathbb{N}$. Then as $t \rightarrow 0$*

$$(3.c) \quad (4\pi t)^k \int_D \{p_{\mathbb{R}^2}(x, x; t) - p_D(x, x; t)\}^k dx = \sum_{j=0}^2 c_{k,j} t^{j/2} + o(t),$$

where $c_{k,0} = |D|$, $c_{k,1} = \frac{1}{2} \left(\frac{\pi}{k}\right)^{1/2} \int_{\partial D} 1 ds$,

$$c_{1,2} = -\frac{1}{3} \int_{\partial D} L_{aa}(s) ds,$$

and for $k \geq 2$

$$c_{k,2} = \left\{ -\frac{1}{2k} + \frac{k}{2} \left(\frac{\pi}{4(k-1)^{3/2}} - \frac{1}{2k(k-1)} - \frac{\arctan(k-1)^{-1/2}}{2(k-1)^{3/2}} \right) \right\} \int_{\partial D} L_{aa}(s) ds.$$

Proof. Let $\varepsilon \in (0, \frac{1}{2})$ and denote $D_\varepsilon = \{x \in D : \delta(x) \leq t^\varepsilon\}$. Since

$$(3.d) \quad p_{\mathbb{R}^2}(x, x; t) \geq p_D(x, x; t) \geq p_{\mathbb{R}^2}(x, x; t) - (\pi t)^{-1} e^{-\delta(x)^2/(2t)}$$

(see Theorem 1 and estimate (ii) in [1]) we that for $\varepsilon < \frac{1}{2}$ and $t \rightarrow 0$

$$(3.e) \quad \int_{D \setminus D_\varepsilon} \{p_{\mathbb{R}^2}(x, x; t) - p_D(x, x; t)\}^k dx = O(e^{-t^{2\varepsilon-1}/3}).$$

It remains to calculate the contribution from the set D_ε to the integral in the left hand side of (3.c). Since the $O(1)$ remainder estimate is uniform in x we have that this remainder estimate contributes at most $O(t^{1+\varepsilon})$ to the integral in (3.c). Hence for t sufficiently small we have by (3.e)

$$Z_{k,2}(t) = \int_{\partial D} ds \int_0^{t^\varepsilon} dr \left\{ e^{-r^2/t} + L_{aa}(s) r^2 t^{-1/2} \int_{rt^{-1/2}}^{\infty} e^{-\eta^2} d\eta \right\}^k (1 - r L_{aa}(s)) + O(t^{1+\varepsilon}).$$

Since $\int_0^{t^\varepsilon} dr r^4 t^{-1} = O(t^{5\varepsilon-1})$ we have that for $\varepsilon \in (\frac{2}{5}, \frac{1}{2})$ the contribution from such a term is $o(t)$. Hence for $t \rightarrow 0$

$$(3.f) \quad Z_{k,2}(t) = \int_{\partial D} ds \int_0^{t^\varepsilon} dr \left\{ e^{-kr^2/t} + k e^{-(k-1)r^2/t} L_{aa}(s) r^2 t^{-1/2} \int_{rt^{-1/2}}^{\infty} e^{-\eta^2} d\eta \right\} \times (1 - r L_{aa}(s)) + O(t^{5\varepsilon-1}).$$

Furthermore for $\varepsilon < 1/2$ and $t \rightarrow 0$

$$(3.g) \quad \begin{aligned} & \int_{\partial D} ds \int_0^{t^\varepsilon} dr e^{-kr^2/t} \sim \frac{1}{2} \left(\frac{\pi t}{k}\right)^{1/2} \int_{\partial D} ds, \\ & - \int_{\partial D} ds \int_0^{t^\varepsilon} dr e^{-kr^2/t} r L_{aa}(s) \sim -\frac{t}{2k} \int_{\partial D} ds L_{aa}(s), \\ & \int_{\partial D} ds L_{aa}(s) \int_0^{t^\varepsilon} dr r^2 t^{-1/2} \int_{rt^{-1/2}}^{\infty} e^{-\eta^2} d\eta \sim \frac{t}{6} \int_{\partial D} ds L_{aa}(s), \end{aligned}$$

and for $k \geq 2$

$$\begin{aligned}
& \int_{\partial D} ds L_{aa}(s) \int_0^{t^\varepsilon} dr k e^{-(k-1)r^2/t} r^2 t^{-1/2} \int_{rt^{-1/2}}^\infty e^{-\eta^2} d\eta \\
(3.h) \quad & \sim \frac{kt}{2} \int_{\partial D} ds L_{aa}(s) \int_0^\infty dr k e^{-(k-1)r^2/t} r^2 t^{-1/2} \int_{rt^{-1/2}}^\infty e^{-\eta^2} d\eta \\
& = \frac{kt}{2} \int_{\partial D} ds L_{aa}(s) \int_1^\infty d\eta (\eta^2 + k - 1)^{-2} \\
& = \frac{kt}{2} \left\{ \frac{\pi}{4(k-1)^{3/2}} - \frac{1}{2k(k-1)} - \frac{\arctan(k-1)^{-1/2}}{2(k-1)^{3/2}} \right\} \int_{\partial D} ds L_{aa}(s).
\end{aligned}$$

It is easily seen that the term with $L_{aa}(s)^2$ in (3.f) contributes $O(t^{3/2})$. Collecting the contributions from (3.g – 3.h) we arrive at the conclusion of Lemma 3.3. \square

Proof of Theorem 1.1. Since the contribution to the asymptotic series for $Z_{k,m}(t)$ comes from a $t^\varepsilon, \varepsilon < 1/2$ neighbourhood of ∂K we can identify the coefficients $c_{k,0}, c_{k,1}$ for $m = 2$ by Lemma 3.3. By standard results of invariance theory they also hold for $m > 2$ [7]. Moreover $Z_{k,m}(t)$ has a full asymptotic series. The proof of Theorem 1.1 is complete by Lemma 3.1. \square

Lemma 3.4. *The coefficient of the term $\int_{\partial D} f L_{aa} dy$ in the expression for $a_{k,2}$ in Theorem 1.2 is given by*

$$-\frac{k}{6} + \frac{1}{2} \sum_{\ell=1}^k \binom{k}{\ell} (-1)^{\ell-1} \left\{ (k-\ell) \left(\frac{\pi}{4\ell^{3/2}} - \frac{1}{2\ell(\ell+1)} - \frac{\arctan \ell^{-1/2}}{2\ell^{3/2}} \right) + \frac{1}{\ell} \right\}.$$

Proof. It suffices to consider a planar region with $f \equiv 1$ as in the proof of Lemma 3.3. Let $\varepsilon \in (2/5, 1/2)$. Then by (3.d)

$$\int_{D-D_\varepsilon} dx \{p_D(x, x; t)\}^k \sim (4\pi t)^{-km/2} |D - D_\varepsilon|.$$

The $O(1)$ remainder in (3.b) contributes at most $O(t^{-k+1+\varepsilon})$ to the integral over the set D_ε . Hence for $2/5 < \varepsilon < 1/2$

$$\begin{aligned}
\int_{D_\varepsilon} dx \{p_D(x, x; t)\}^k &= (4\pi t)^{-k} \int_{\partial D} ds \int_0^{t^\varepsilon} dr (1 - r L_{aa}(s)) \\
&\times \left\{ 1 - e^{-r^2/t} - r^2 t^{-1/2} L_{aa}(s) \int_{rt^{-1/2}}^\infty e^{-\eta^2} d\eta \right\}^k + O(t^{-k+1+\varepsilon}) \\
&= (4\pi t)^{-k} \int_{\partial D} ds \int_0^{t^\varepsilon} dr (1 - r L_{aa}(s)) \left\{ (1 - e^{-r^2/t})^k \right. \\
&\left. - k(1 - e^{-r^2/t})^{k-1} r^2 t^{-1/2} L_{aa}(s) \int_{rt^{-1/2}}^\infty e^{-\eta^2} d\eta \right\} + O(t^{5\varepsilon-k-1}),
\end{aligned}$$

by estimates similar to the ones in the proof of Lemma 3.3. For $2/5 < \varepsilon < 1/2$ and $t \rightarrow 0$ we have

$$\begin{aligned}
& (4\pi t)^{-k} \int_{\partial D} ds \int_0^{t^\varepsilon} dr (1 - r L_{aa}(s)) = (4\pi t)^{-k} |D_\varepsilon|, \\
& (4\pi t)^{-k} \int_{\partial D} ds \int_0^{t^\varepsilon} dr \{(1 - e^{-r^2/t})^k - 1\} \\
& \sim 2^{-1} (4\pi t)^{-k} (\pi t)^{1/2} \sum_{\ell=1}^k \binom{k}{\ell} (-1)^\ell \ell^{-1/2},
\end{aligned}$$

and

$$(3.i) \quad \begin{aligned} & -(4\pi t)^{-k} \int_{\partial D} ds L_{aa}(s) \int_0^{t^\varepsilon} dr r \{ (1 - e^{-r^2/t})^k - 1 \} \\ & \sim 2^{-1} (4\pi t)^{-k} t \sum_{\ell=1}^k \binom{k}{\ell} (-1)^{\ell-1} \ell^{-1} \int_{\partial D} ds L_{aa}(s). \end{aligned}$$

Furthermore

$$\begin{aligned} & -k(4\pi t)^{-k} \int_{\partial D} ds L_{aa}(s) \int_0^{t^\varepsilon} dr r^2 t^{-1/2} \int_{rt^{-1/2}}^\infty e^{-\eta^2} d\eta \\ & \sim -\frac{k}{6} (4\pi t)^{-k} t \int_{\partial D} ds L_{aa}(s), \end{aligned}$$

and so it remains to compute (for $k \geq 2$)

$$(3.j) \quad \begin{aligned} & -k(4\pi t)^{-k} \int_{\partial D} ds L_{aa}(s) \int_0^{t^\varepsilon} dr \\ & \quad \times \sum_{\ell=1}^{k-1} \binom{k-1}{\ell} (-1)^\ell e^{-\ell r^2/t} r^2 t^{-1/2} \int_{rt^{-1/2}}^\infty e^{-\eta^2} d\eta \\ & \sim -k(4\pi t)^{-k} \int_{\partial D} ds L_{aa}(s) \int_0^\infty dr \\ & \quad \times \sum_{\ell=1}^{k-1} \binom{k-1}{\ell} (-1)^\ell e^{-\ell r^2/t} r^2 t^{-1/2} \int_{rt^{-1/2}}^\infty e^{-\eta^2} d\eta \\ & = -\frac{k}{2} (4\pi t)^{-k} t \int_{\partial D} ds L_{aa}(s) \sum_{\ell=1}^{k-1} \binom{k-1}{\ell} (-1)^\ell \int_1^\infty (\eta^2 + \ell)^{-2} d\eta \\ & = -\frac{k}{2} (4\pi t)^{-k} t \int_{\partial D} ds L_{aa}(s) \sum_{\ell=1}^{k-1} \binom{k-1}{\ell} \\ & \quad \times (-1)^\ell \left\{ \frac{\pi}{4\ell^{3/2}} - \frac{1}{2\ell(\ell+1)} - \frac{\arctan \ell^{-1/2}}{2\ell^{3/2}} \right\}. \end{aligned}$$

Collecting all of the above we have shown that the formulae for $a_{k,0}$, $a_{k,1}$ and $a_{k,2}$ in Theorem 1.2 hold for planar regions with C^3 boundary. In particular we have shown that, by collecting the terms (3.i-3.j), the coefficient of the term $\int_{\partial D} f L_{aa} dy$ in the expression for $a_{k,2}$ is given by Lemma 3.4. The scalar curvature term in the expression for $a_{k,2}$ in Theorem 1.2 (4) will follow from the results in Section 4 below. \square

4. THE INTERIOR TERMS

In order to obtain the contributions from the interior of M to the coefficients $a_{k,j}$ in Theorem 1.2, it suffices to consider Riemannian manifolds without boundary. The general setting is as follows. Let D be an operator of Laplace type on a smooth vector bundle V over a Riemannian manifold M without boundary. One can express D in terms of geometrical data in an invariant fashion as follows. There is a unique connection ∇ on V and there is a unique endomorphism E of V so that one can express D in the Bochner formalism by writing:

$$D = -\{g^{ij} \nabla_i \nabla_j + E\}.$$

Let $p_M(x, x; t)$ be the heat kernel, and let F be a smooth endomorphism of V . Theorem 1.2 generalizes to this setting to yield the existence of a complete asymptotic

expansion

$$\int_M \text{Tr}_{V_x} \{(p_M(x, x; t))^k F(x)\} dx \sim (4\pi t)^{-km/2} \sum_{j=0}^{\infty} a_{k,2j}(F, D) t^j,$$

where the coefficients $a_{k,2j}$ are locally computable. There are no half integer powers of t since the boundary of M is empty. Consequently $a_{k,2j+1} = 0$.

If $k = 1$, then we have a complete asymptotic expansion

$$(4.a) \quad p_M(x, x; t) \sim (4\pi t)^{-m/2} \sum_{\nu=0}^{\infty} e_{2\nu}(D) t^\nu$$

where the $e_{2\nu}$ are locally computable endomorphisms. Let $\tau := R_{ijji}$ be the scalar curvature. Then one has the following well known result (see, for example, the discussion in [7]):

Lemma 4.1. *Let D be a operator of Laplace type on a closed Riemannian manifold.*

- (1) $a_{1,n}(F, D) = \int_M \text{Tr}(F e_n(x, D)) dx.$
- (2) $e_0(x, D) = \text{id}.$
- (3) $e_2(x, D) = \frac{1}{6} \{\tau \text{id} + 6E\}.$

Remark 4.2. *We note that similar formulae are available for e_4 and e_6 [7].*

Raising the asymptotic expansion in formula (4.a) to the k^{th} power and applying Lemma 4.1 then yields immediately the following:

Corollary 4.3. *Let M be a closed Riemannian manifold.*

- (1) $a_{k,0}(F, D) = a_{1,0}(F, D).$
- (2) $a_{k,2}(F, D) = k a_{1,2}(F, D).$
- (3) $a_{k,4}(F, D) = k a_{1,4}(F, D) + \int_M \text{Tr} \{ F \frac{k(k-1)}{72} (\tau^2 + 12\tau E + 36E^2) \} dx.$

5. EXPECTED VOLUME OF INTERSECTION OF INDEPENDENT WIENER SAUSAGES

In this section we obtain results analogous to Theorem 1.1 for the unpinned Wiener sausage. Let $\beta^1(s), \dots, \beta^k(s)$, $s \geq 0$ be independent Brownian motions in \mathbb{R}^m , and let K be a compact set in \mathbb{R}^m . The Wiener sausages $S_K^1(t), \dots, S_K^k(t)$ are the random sets defined by

$$S_K^i(t) = \{\beta^i(s) + y : 0 \leq s \leq t, y \in K\}, \quad i = 1, \dots, k.$$

These random sets are compact and Borel measurable with probability 1. Define

$$Q_{k,m}(t) = \mathbb{E}_0^1 \otimes \dots \otimes \mathbb{E}_0^k \left[\left| \bigcap_{i=1}^k S_K^i(t) \right| \right].$$

It is well known that

$$Q_{k,m}(t) = \int_{\mathbb{R}^m} dx \left\{ 1 - \int_{\mathbb{R}^m - K} p_{\mathbb{R}^m - K}(x, y; t) dy \right\}^k,$$

and that $Q_{k,m}(t) \leq Q_{1,m}(t) < \infty$ ([16]). We note that $Q_{k,m}(t)$ has the following analytic interpretation. Let $u : \mathbb{R}^m - K \times (0, \infty) \rightarrow \mathbb{R}$ be the unique weak solution of $\Delta u = \frac{\partial u}{\partial t}$ with initial condition $u(x; 0) = 0, x \in \mathbb{R}^m - K$ and boundary condition $u(x; t) = 1, x \in \partial K, t > 0$. We extend u to all of $\mathbb{R}^m \times (0, \infty)$, and note that

$$u(x; t) = 1 - \int_{\mathbb{R}^m - K} p_{\mathbb{R}^m - K}(x, y; t) dy.$$

Then

$$Q_{k,m}(t) = \|u(\cdot; t)\|_k^k.$$

The large t behaviour of $Q_{k,m}(t)$ has been investigated for $m = 2$ in [10] and for $m \geq 3$ in [2].

Theorem 5.1. *Let K be a compact set in \mathbb{R}^m with C^∞ boundary. There exists an asymptotic series as $t \rightarrow 0$*

$$Q_{k,m}(t) \sim \sum_{j=0}^{\infty} b_{k,j} t^{j/2},$$

where

$$(5.a) \quad b_{k,0} = \int_K 1 dx,$$

$$(5.b) \quad b_{k,1} = 2^{2+k} \pi^{(2-k)/2} \Gamma((k+1)/2) \int_1^\infty d\eta_1 \dots \int_1^\infty d\eta_k (\eta_1^2 + \dots + \eta_k^2)^{-(k+1)/2} \int_{\partial K} 1 dy,$$

$$(5.c) \quad b_{k,2} = -2^{2+k} \pi^{(2-k)/2} (k-2) \Gamma((k+2)/2) \\ \times \int_1^\infty d\eta_1 \dots \int_1^\infty d\eta_k (\eta_1^2 + \dots + \eta_k^2)^{-(k+2)/2} \int_{\partial K} L_{aa} dy.$$

Proof. It suffices to prove (5.a–5.c) for a disk in \mathbb{R}^3 . The general case for compact K with C^∞ boundary follows by invariance theory. For the disk D^3 centred at the origin it is well known [6] that

$$u(x;t) = 2\pi^{-1/2} r^{-1} \int_{(r-1)/(2t^{1/2})}^\infty e^{-\eta^2} d\eta, \quad r = |x| \geq 1.$$

Hence for $K = D^3$ we have that

$$Q_{k,3}(t) = \frac{4\pi}{3} + 4\pi \left(\frac{2}{\pi^{1/2}} \right)^k \int_1^\infty dr r^{2-k} \left\{ \int_{(r-1)/(2t^{1/2})}^\infty e^{-\eta^2} d\eta \right\}^k.$$

The main contribution to $Q_{k,3}(t)$, $t \rightarrow 0$ comes from the interval $[1, 1+t^\varepsilon]$, where $0 < \varepsilon < 1/2$. Expanding r^{2-k} near 1 yields

$$Q_{k,3}(t) = \frac{4\pi}{3} + 4\pi \left(\frac{2}{\pi^{1/2}} \right)^k \int_0^\infty dr \{1 + (2-k)r\} \left\{ \int_{r/(2t^{1/2})}^\infty e^{-\eta^2} d\eta \right\}^k + O(t^{3/2}) \\ = \frac{4\pi}{3} + 8\pi \left(\frac{2}{\pi^{1/2}} \right)^k \int_0^\infty dr r^k \int_1^\infty d\eta_1 \dots \int_1^\infty d\eta_k e^{-r^2(\eta_1^2 + \dots + \eta_k^2)} t^{1/2} \\ - 8\pi \left(\frac{2}{\pi^{1/2}} \right)^k (k-2) \int_0^\infty dr r^{k+1} \int_1^\infty d\eta_1 \dots \int_1^\infty d\eta_k e^{-r^2(\eta_1^2 + \dots + \eta_k^2)} t \\ + O(t^{3/2}),$$

and the claim follows by Fubini's theorem. \square

We note that the coefficient of t vanishes for $k = 2$. This jibes with the fact [3] that for any compact K in \mathbb{R}^m , $m \geq 1$, and all $t > 0$

$$Q_{2,m}(t) = 2Q_{1,m}(t) - Q_{1,m}(2t).$$

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