

Max-Planck-Institut
für Mathematik
in den Naturwissenschaften
Leipzig

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Preprint no.: 11

2006



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Abstract

We construct a set of PPT (positive partial transpose) states and show that these PPT states are not separable, thus present a class of bound entangled quantum states.

Keywords: Bound Entangled states, PPT

PACS: 03.65.Bz; 89.70.+c

Quantum entangled states are used as key resources in quantum information processing such as quantum teleportation, cryptography, dense coding, error correction and parallel computation [1, 2]. To quantify the degree of entanglement a number of entanglement measures have been proposed for bipartite states. However most proposed measures of entanglement involve extremizations which are difficult to handle analytically. It turns out that to verify the separability of a general mixed states could be extremely difficult.

Among the quantum entangled states, there is a special kind of states that can not be distilled. These states are called bound entangled states. Many powerful separability criteria could not detect the entanglement of these states, e.g. the bound entangled states given in [3, 4, 5]. A few new bound entangled states have been found recently by using the method of positive maps [6].

It has been shown [7] that any state which is entangled and satisfies positive partial transpose (PPT) condition [8] is not distillable. The existence of PPT entangled states was discussed in [9] and explicit examples were provided in [5], based on an elegant separability (range) criterion.

In [10] a special class of quantum states (d -computable states) were constructed. The entanglement of formation of these states can be analytically calculated and it turns out that all the states are entangled. In this paper according to the construction of the d -computable states, we present first a class of PPT states, then by using the range criterion we prove that they are entangled.

Let \mathcal{H} be an N -dimensional complex Hilbert space with orthonormal basis $e_i, i = 1, \dots, N$. A general bipartite pure state on $\mathcal{H} \otimes \mathcal{H}$ is of the form,

$$|\psi\rangle = \sum_{i,j=1}^N a_{ij} e_i \otimes e_j, \quad a_{ij} \in \mathcal{C} \quad (1)$$

with normalization $\sum_{i,j=1}^N a_{ij} a_{ij}^* = 1$. Let A denote the matrix with entries given by a_{ij} in (1). Set

$$A = \begin{bmatrix} 0 & b_1 & a & -c \\ -b_1 & 0 & c & d \\ -a & -c & 0 & -c_1 \\ c & -d & c_1 & 0 \end{bmatrix}, \quad (2)$$

where $a, c, d, b_1, c_1 \in \mathcal{C}$.

It was shown that all pure states $|\psi\rangle$ with A given by (2) have a simple formula of the generalized concurrence C , such that the entanglement of formation is a monotonically increasing function of C [10]. Moreover the entanglement of formation of all mixed states ρ with decompositions on pure states with A given by (2) can be analytically calculated. As all the states with A of (2) are entangled, these mixed states ρ are also entangled.

In fact A is an antisymmetric matrix. Any antisymmetric matrices are equivalent to the following standard form under similarity transformations:

$$A_1 = \begin{bmatrix} 0 & \lambda_1 & 0 & 0 \\ -\lambda_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \lambda_2 \\ 0 & 0 & -\lambda_2 & 0 \end{bmatrix} \quad \text{or} \quad A_2 = \begin{bmatrix} 0 & 0 & \lambda_1 & 0 \\ 0 & 0 & 0 & \lambda_2 \\ -\lambda_1 & 0 & 0 & 0 \\ 0 & -\lambda_2 & 0 & 0 \end{bmatrix}.$$

If we set $\lambda_1 = \pm b, \lambda_2 = -c$ in A_1 , the matrix A_1 gives rise to two pure states

$$|\psi_{\pm b}\rangle = [0 \quad \pm b \quad 0 \quad 0 \quad \mp b \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad -c \quad 0 \quad 0 \quad c \quad 0]^t,$$

and hence two projectors $\rho_{\pm b} = |\psi_{\pm b}\rangle\langle\psi_{\pm b}|$, where t denotes the transposition. We define

$$\rho_b = \frac{1}{2}\rho_{+b} + \frac{1}{2}\rho_{-b}.$$

And if we set $\lambda_1 = \pm a, \lambda_2 = d$ in A_2 we have

$$\rho_a = \frac{1}{2}\rho_{+a} + \frac{1}{2}\rho_{-a},$$

where $\rho_{\pm a} = |\psi_{\pm a}\rangle\langle\psi_{\pm a}|$,

$$|\psi_{\pm a}\rangle = [0 \quad 0 \quad \pm a \quad 0 \quad 0 \quad 0 \quad 0 \quad d \quad \mp a \quad 0 \quad 0 \quad 0 \quad 0 \quad -d \quad 0 \quad 0]^t.$$

We define

$$\rho_0 = \frac{1}{2}\rho_a + \frac{1}{2}\rho_b \quad (3)$$

The state ρ_0 is not separable as its partial transposed matrix has a negative eigenvalue. Below we mix ρ_0 with some separable states in such a way that the resulting state will be

both partial transposition positive and entangled. Let I_4 be a 16×16 matrix with only non-zero entries $(I_4)_{1,1} = (I_4)_{6,6} = (I_4)_{11,11} = (I_4)_{16,16} = 1$. We consider $\rho = (1-\varepsilon)I_4 + \varepsilon\rho_0$, which is of the form:

$$\rho = \begin{bmatrix} x_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & x_3 & 0 & 0 & -x_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & x_2 & 0 & 0 & 0 & 0 & 0 & -x_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -x_3 & 0 & 0 & x_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & x_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & x_5 & 0 & 0 & 0 & 0 & 0 & -x_5 & 0 & 0 \\ 0 & 0 & -x_2 & 0 & 0 & 0 & 0 & 0 & x_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & x_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & x_4 & 0 & 0 & -x_4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -x_5 & 0 & 0 & 0 & 0 & 0 & x_5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -x_4 & 0 & 0 & x_4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & x_1 \end{bmatrix}, \quad (4)$$

where $x_1 = \frac{1-\varepsilon}{4}$, $x_2 = \frac{\varepsilon}{2}|a|^2$, $x_3 = \frac{\varepsilon}{2}|b|^2$, $x_4 = \frac{\varepsilon}{2}|c|^2$, $x_5 = \frac{\varepsilon}{2}|d|^2$.

The eigenvalues of the partial transposed matrix with respect to the second subspace ρ^{T_2} of ρ are

$$0, 0, 0, 0, \frac{\varepsilon}{2}|a|^2, \frac{\varepsilon}{2}|a|^2, \frac{\varepsilon}{2}|b|^2, \frac{\varepsilon}{2}|b|^2, \frac{\varepsilon}{2}|c|^2, \frac{\varepsilon}{2}|c|^2, \frac{\varepsilon}{2}|d|^2, \frac{\varepsilon}{2}|d|^2$$

together with the roots of the following equation:

$$\left(\lambda - \frac{1-\varepsilon}{2}\right)^4 - \frac{\varepsilon^2}{4}(|a|^4 + |b|^4 + |c|^4 + |d|^4)\left(\lambda - \frac{1-\varepsilon}{4}\right)^2 + \frac{\varepsilon^4}{16}(|a|^2|d|^2 - |b|^2|c|^2)^2 = 0, \quad (5)$$

i.e. $\lambda = \frac{1-\varepsilon}{4} \pm \frac{\varepsilon}{4}\sqrt{2[(|a|^4 + |b|^4 + |c|^4 + |d|^4) \pm \sqrt{\Delta_1}]}$, where Δ_1 is defined by the discriminant of (5):

$$\Delta = \frac{\varepsilon^4}{16}[(|a|^2 - |d|^2)^2 + (|b|^2 + |c|^2)^2][(|a|^2 + |d|^2)^2 + (|b|^2 - |c|^2)^2] = \frac{\varepsilon^4}{16}\Delta_1.$$

If

$$\frac{1-\varepsilon}{4} - \frac{\varepsilon}{4}\sqrt{2[(|a|^4 + |b|^4 + |c|^4 + |d|^4) + \sqrt{\Delta_1}]} > 0, \quad (6)$$

then ρ^{T_2} is positive semidefinite. It is possible that the condition (6) is satisfied while keeping the state ρ_4 entangled. For simplicity we take $a = b = c = d = \frac{1}{2}$. In this case

ρ^{T_2} is positive semidefinite when $0 \leq \varepsilon \leq \frac{1}{2}$, and the state becomes

$$\rho = \begin{bmatrix} \frac{1-\varepsilon}{4} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{\varepsilon}{8} & 0 & 0 & -\frac{\varepsilon}{8} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{\varepsilon}{8} & 0 & 0 & 0 & 0 & 0 & -\frac{\varepsilon}{8} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\frac{\varepsilon}{8} & 0 & 0 & \frac{\varepsilon}{8} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1-\varepsilon}{4} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{\varepsilon}{8} & 0 & 0 & 0 & 0 & 0 & -\frac{\varepsilon}{8} & 0 \\ 0 & 0 & -\frac{\varepsilon}{8} & 0 & 0 & 0 & 0 & 0 & \frac{\varepsilon}{8} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1-\varepsilon}{4} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{\varepsilon}{8} & 0 & 0 & -\frac{\varepsilon}{8} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{\varepsilon}{8} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\frac{\varepsilon}{8} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{\varepsilon}{8} & 0 & 0 & \frac{\varepsilon}{8} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1-\varepsilon}{4} \end{bmatrix}. \quad (7)$$

Recall that if a state ρ acting on Hilbert space $\mathcal{H} = \mathcal{H} \otimes \mathcal{H}$ is separable, then there exists a set of product vectors $\{|\psi_i \rangle \otimes |\phi_k \rangle\}$, $\{i, k\} \in I$ (I is a finite set of pairs of indices with number of pairs $M = \#I \leq N^2$) and probabilities p_{ik} such that the ensemble $\{|\psi_i \rangle \otimes |\phi_k \rangle, p_{ik}\}$ ($\{|\psi_i \rangle \otimes |\phi_k^* \rangle, p_{ik}\}$) corresponds to the matrix ρ (ρ^{T_2}), and the vectors $\{|\psi_i \rangle \otimes |\phi_k \rangle\}$ ($\{|\psi_i \rangle \otimes |\phi_k^* \rangle\}$) span the range of ρ (ρ^{T_2}). In particular any vector $\{|\psi_i \rangle \otimes |\phi_k \rangle\}$ ($\{|\psi_i \rangle \otimes |\phi_k^* \rangle\}$) belongs to the range of ρ (ρ^{T_2}), see [5].

Now we calculate all the product (unnormalised) vectors belonging to the range of ρ . With the basis ordered in the following way $e_1 \otimes e_1, e_1 \otimes e_2, e_1 \otimes e_3, e_1 \otimes e_4, e_2 \otimes e_1, e_2 \otimes e_2, \dots, e_4 \otimes e_4$, any vector belonging to the range of ρ can be presented as

$$\mu = [A \ B \ C \ 0 \ -B \ D \ 0 \ E \ -C \ 0 \ F \ G \ 0 \ -E \ -G \ H]^t, \quad (8)$$

where $A, B, C, D, E, F, H \in \mathcal{C}$. On the other hand a separable μ is of the form

$$\mu_{sep} = [b_1 \ b_2 \ b_3 \ b_4]^t \otimes [c_1 \ c_2 \ c_3 \ c_4]^t, \quad (9)$$

$b_1, b_2, b_3, b_4, c_1, c_2, c_3, c_4 \in \mathcal{C}$. Comparing (9) with (8) we have

$$b_1 c_4 = b_2 c_3 = b_3 c_2 = b_4 c_1 = 0, \quad (10)$$

$$b_1 c_2 = -b_2 c_1, \quad (11)$$

$$b_1 c_3 = -b_3 c_1, \quad (12)$$

$$b_2 c_4 = -b_4 c_2, \quad (13)$$

$$b_3 c_4 = -b_4 c_3. \quad (14)$$

To find a set of basic separable vectors that span the range of (8), let us consider the following cases:

I) $b_1 b_2 \neq 0$. Without loss of generality we set $b_1 = 1$ and $c_1 = A, c_2 = B$. From (10), we have $c_4 = c_3 = 0$ and $b_3 = b_4 = 0$. From (11) we have $b_2 = -\frac{B}{A}$, and $b_2 c_2 = D$

by comparing (8) with (9). Therefore $B^2 = -AD$, and $B = \pm\sqrt{-AD}$. Hence we have $b_2 = \mp\frac{\sqrt{-AD}}{A}$, $c_2 = \pm\sqrt{-AD}$. Thus we obtain the states

$$\frac{1}{A}[A \quad -\sqrt{-AD} \quad 0 \quad 0]^t \otimes [A \quad \sqrt{-AD} \quad 0 \quad 0]^t \quad (15)$$

and

$$\frac{1}{A}[A \quad \sqrt{-AD} \quad 0 \quad 0]^t \otimes [A \quad -\sqrt{-AD} \quad 0 \quad 0]^t. \quad (16)$$

II) $b_1 b_2 = 0$

i) $b_1 \neq 0$, $b_2 = 0$. We set $b_1 = 1$.

If $b_3 b_4 \neq 0$ or $b_3 = 0$, $b_4 \neq 0$, then only the null vector satisfies these conditions.

If $b_3 = b_4 = 0$, from (8) and (9), we obtain $c_1 = A$, $c_2 = c_3 = c_4 = 0$ and the vector is of the form

$$A[1 \quad 0 \quad 0 \quad 0]^t \otimes [1 \quad 0 \quad 0 \quad 0]^t. \quad (17)$$

If $b_3 \neq 0$, $b_4 = 0$, then similar to the case I), we have the following vectors :

$$\frac{1}{A}[A \quad 0 \quad -\sqrt{-AF} \quad 0]^t \otimes [A \quad 0 \quad \sqrt{-AF} \quad 0]^t \quad (18)$$

and

$$\frac{1}{A}[A \quad 0 \quad \sqrt{-AF} \quad 0]^t \otimes [A \quad 0 \quad -\sqrt{-AF} \quad 0]^t. \quad (19)$$

ii) $b_1 = 0$, $b_2 \neq 0$. We take $b_2 = 1$. Similar to the previous case, we have the following two cases:

If $b_3 = b_4 = 0$, then $c_1 = c_3 = c_4 = 0$, $c_2 = D$, the vector is

$$D[0 \quad 1 \quad 0 \quad 0]^t \otimes [0 \quad 1 \quad 0 \quad 0]^t. \quad (20)$$

If $b_3 = 0$, $b_4 \neq 0$ then we have

$$\frac{1}{D}[0 \quad D \quad 0 \quad -\sqrt{-DH}]^t \otimes [0 \quad D \quad 0 \quad \sqrt{-DH}]^t \quad (21)$$

and

$$\frac{1}{D}[0 \quad D \quad 0 \quad \sqrt{-DH}]^t \otimes [0 \quad D \quad 0 \quad -\sqrt{-DH}]^t. \quad (22)$$

iii) $b_1 = b_2 = 0$

If $b_3 b_4 \neq 0$, taking $b_3 = 1$, we have

$$\frac{1}{F}[0 \quad 0 \quad F \quad -\sqrt{-FH}]^t \otimes [0 \quad 0 \quad F \quad \sqrt{-FH}]^t \quad (23)$$

and

$$\frac{1}{F} [0 \ 0 \ F \ \sqrt{-FH}]^t \otimes [0 \ 0 \ F \ -\sqrt{-FH}]^t. \quad (24)$$

If $b_3 \neq 0$, $b_4 = 0$, taking $b_3 = 1$, we obtain

$$F [0 \ 0 \ 1 \ 0]^t \otimes [0 \ 0 \ 1 \ 0]^t. \quad (25)$$

If $b_3 = 0$, $b_4 \neq 0$, taking $b_4 = 1$, we get

$$H [0 \ 0 \ 0 \ 1]^t \otimes [0 \ 0 \ 0 \ 1]^t. \quad (26)$$

The vectors (15), (16), (17) and (20) are linear dependent. So we can exclude (16). For the same reason, we can remove (19), (22) and (24). The left vectors (15), (17), (18), (20), (21), (23), (25) and (26) span the separable linear independent vectors of the range of ρ . The partial complex conjugations (PCC) of these vectors, e.g. PCC of (15), $\frac{1}{A} [A \ -\sqrt{-AD} \ 0 \ 0]^t \otimes [A^* \ \sqrt{-AD}^* \ 0 \ 0]^t$, do not span the range of ρ^{T_2} as the vector

$$[1 \ 0 \ 0 \ 0]^t \otimes [0 \ 1 \ 0 \ 0]^t, \quad (27)$$

which does belong to the range of ρ^{T_2} , does not instead belong to their linear span. Hence for any $0 \leq k \leq \frac{1}{2}$ the state ρ violates the separability criterion in [5]. Thus the states (7) are bound entangled ones.

We have provided a class of inseparable states with positive partial transposition by using the range criterion. Although we have taken $a = b = c = d = \frac{1}{2}$ for simplicity, in fact, the state (4) is bound entangled as long as ε is small enough such that all the roots of (5) are positive.

It is verified that the trace norm of the realigned matrix of (7) is one. Hence the realignment separability criterion [11] could not detect the entanglement of this bound entangled state. Moreover the trace norm of ρ^{T_2} is also one. Therefore neither the lower bound of concurrence nor the lower bound for the entanglement of formation [12] could detect the entanglement.

Acknowledgments We thank the referee for pointing out a mistake in the first version. The work is partially supported by NKBRPC (2004CB318000).

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