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für Mathematik
in den Naturwissenschaften
Leipzig

Fast and Exact Projected Convolution of Piecewise Linear Functions on Non-equidistant Grids - Extended Version

(revised version: December 2006)

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Preprint no.: 110 2006

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Abstract

Usually, the fast evaluation of a convolution integral $\int_{\mathbb{R}} f(y)g(x-y)\mathrm{d}y$ requires that the functions f, g are discretised on an equidistant grid in order to apply the fast Fourier transform. Here we discuss the efficient performance of the convolution in locally refined grids. More precisely, f and g are assumed to be piecewise linear and the convolution result is projected into the space of linear functions in a given locally refined grid. Under certain conditions, the overall costs are still $\mathcal{O}(N \log N)$, where N is the sum of the dimensions of the subspaces containing f, g and the resulting function.

AMS Subject Classifications: 44A35, 42A55
Key words: convolution integral, non-uniform grids, discrete convolution

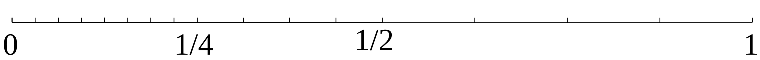
1 Introduction

We consider the convolution integral

$$\omega_{\mathrm{exact}}(x) := (f * g)(x) := \int_{\mathbb{R}} f(y)g(x-y)\mathrm{d}y \tag{1.1}$$

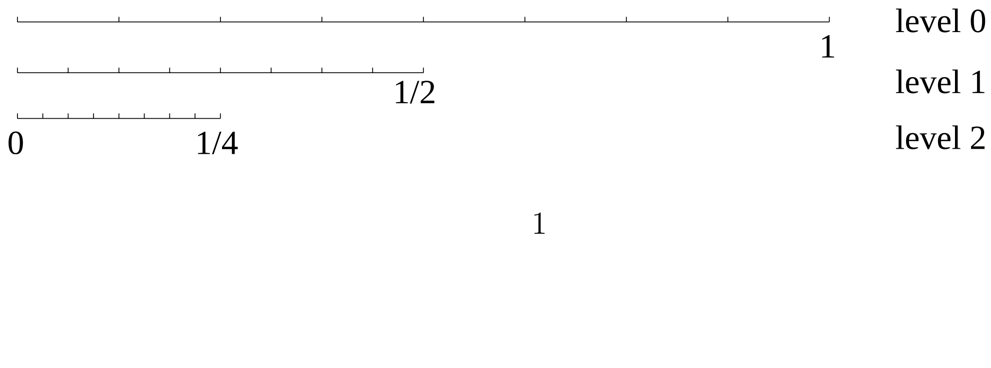
for functions f, g of bounded support.

The computations are restricted to functions f, g which are piecewise linearly defined on *locally refined meshes* with possible discontinuities at the grid points. A simple example of such a locally refined mesh is depicted below:



(1.2a)

The mesh size 1/8 in $[1/2, 1]$, 1/16 in $[1/4, 1/2]$ and 1/16 in $[0, 1/4]$ is a typical refinement towards $x = 0$. The depicted mesh can be decomposed into different levels as indicated below:



(1.2b)

The latter representation uses several levels, but each level ℓ is associated with an equidistant grid of size

$$h_\ell := 2^{-\ell}h \qquad (0 \leq \ell \leq L) \tag{1.3}$$

(here, $h = 1/8$). The largest level number appearing in the grid hierarchy will be denoted by L.

Such a refinement approach is well-known from the adaptive wavelet technique. Grids like the depicted ones are obtainable from a first coarse grid with mesh size h at level $\ell = 0$ which is locally refined by halving certain subintervals recursively. The exact description of the locally refined mesh will be given in Section 2.

The standard tool for convolutions is the Fast Fourier Transform (FFT), which, however, applies only for data in a uniform grid.

In principle, one can approximate $f * g$ via Fourier transform: Also for non-equidistant grids there are ways to approximate the Fourier transform $\hat{f}$ and $\hat{g}$ (see [5]) and the back transform of $\hat{f} \cdot \hat{g}$ would yield an approximation of $f * g$. A fast algorithm for a generalised convolution is described in [6]. However, in these approaches the approximation error depends on the interpolation error and we do not guarantee any smoothness of f or g. In contrary, the use of locally refined meshes indicates a nonsmooth situation. In our approach we avoid interpolation errors, since the quantities of interest are computed exactly.

To cover the general case, we will allow that the functions f and g involved in the convolution belong to possibly different locally refined meshes. Also the resulting convolution will be described by a third locally refined grid, which may be different from the grids for f and g.

Since the locally refined meshes have the purpose to approximate some functions f_{exact} and g_{exact} in an adaptive way by f and g, it is only natural to approximate ω_{exact} also by a piecewise linear function ω in a third given locally refined mesh. We use the L^2-orthogonal projection onto this space to obtain the final result ω from ω_{exact}.

Therefore, the goal of the algorithm is to compute ω as the L^2-orthogonal projection of $f * g$. Note that we compute the *exact* L^2-orthogonal projection, i.e., there is no approximation error except the unavoidable projection error.

The computation of $f * g$ for functions from locally refined grids is also discussed in the previous papers [2], [3], [4]. In [2, §§3-6], f, g are assumed to be piecewise constant. The paper describes how to evaluate $\omega_{\text{exact}} := f * g$ at the grid points of the given locally refined grid. [3] explains in general how the convolution of piecewise polynomial functions f, g defined on locally refined grids can be projected L^2-orthogonally into another piecewise polynomial space. However, the concrete algorithm given in [3] is restricted to piecewise constant functions, while for polynomial degree $p > 0$ only a general guideline is given (cf. [3, §8]). The present paper can be considered as the concretisation of the technique for polynomial degree $p = 1$. The paper [4] is also dealing with the piecewise constant case, but describes the necessary modifications in order to obtain local mass conservation. As stated in [4], mass conservation (without any modification of the algorithm) holds for piecewise polynomials of at least degree 1, hence in particular for the piecewise linear ansatz discussed here.

The organisation of this paper is as follows.

Section 2 gives a precise definition of locally refined meshes and of the corresponding ansatz spaces $\mathcal{S}^f, \mathcal{S}^g$ which f and g belong to and of the target space $\mathcal{S}^\omega$ for the projection

ω of $\omega_{\text{exact}} = f * g$. In particular, the basis functions are introduced in §2.3.

Section 3 introduces some notations and formulates the basic problem.

Section 4 introduces the γ, G, and Γ-coefficients which are essential for the representation of the projected values. The Γ-coefficients will appear in the algorithm.

The main chapter of this paper is Section 5 which describes the algorithm. There are three disjoint cases A, B, C which must be treated differently (see §§5.1-5.3).

In all three cases mentioned above, we have to perform a discrete convolution of sequences by means of the FFT technique. The precise implementation of the discrete convolution is described in [3, §6] and need not repeated here. Also the estimation of the computational cost is contained in [3, §7]. There, under certain conditions, the bound

$$\mathcal{O}\left(N \log N\right)$$

is derived, where N describes the data size of the factors f, g and the projected convolution $\omega = P(f * g)$.

So far we have treated the case of piecewise linear but discontinuous functions. The last Section 6 discusses the case of globally continuous piecewise linear functions and proposes an efficient algorithm for the projected convolution.

2 Spaces

2.1 The Locally Refined Meshes

The grids depicted in (1.2b) are embedded into infinite grids $\mathcal{M}_\ell$ which are defined below. With h_ℓ from (1.3) we denote the subintervals of level ℓ by

$$I_\nu^\ell := [\nu h_\ell, (\nu + 1)\, h_\ell) \qquad \text{for } \nu \in \mathbb{Z},\ \ell \in \mathbb{N}_0. \tag{2.1}$$

This defines the meshes

$$\mathcal{M}_\ell := \left\{ I_\nu^\ell : \nu \in \mathbb{Z} \right\} \qquad \text{for } \ell \in \mathbb{N}_0. \tag{2.2}$$

The (equidistant) grid points of $\mathcal{M}_\ell$ are $\{\nu h_\ell : \nu \in \mathbb{Z}\}$.

A finite and *locally refined mesh* $\mathcal{M}$ is a set of finitely many disjoint intervals from various levels, i.e.,[1]

$$\mathcal{M} \subset \bigcup\nolimits_{\ell \in \mathbb{N}_0} \mathcal{M}_\ell, \qquad \text{all } I, I' \in \mathcal{M} \text{ with } I \neq I' \text{ are disjoint}, \qquad \#\mathcal{M} < \infty\,. \tag{2.3}$$

Definition (2.3) corresponds to the representation (1.2a), whereas (1.2b) gives rise to the following dynamic definition. Let $\mathcal{M}_0' \subset \mathcal{M}_0$ be a finite part of the infinite grid $\mathcal{M}_0$. Certain intervals $I_\nu^0 \in \mathcal{M}_0'$ are refined, i.e., I_ν^0 is removed from $\mathcal{M}_0'$ and the resulting new subintervals $I_{2\nu}^1$ and $I_{2\nu+1}^1$ of halved size are added to $\mathcal{M}_1' \subset \mathcal{M}_1$ (initialised by $\mathcal{M}_1' := \emptyset$). Recursively, certain $I_\nu^\ell \in \mathcal{M}_\ell'$ are replaced by $I_{2\nu}^{\ell+1}, I_{2\nu+1}^{\ell+1} \in \mathcal{M}_{\ell+1}'$. Let $\mathcal{M}_\ell''$ be the final value of $\mathcal{M}_\ell'$ after terminating the refinement process. Then $\mathcal{M} := \bigcup_{\ell \in \mathbb{N}_0} \mathcal{M}_\ell''$ yields the set from (2.3).

[1]The sign # denotes the cardinality of a set.

2.2 The Ansatz Spaces

The piecewise linear space $\mathcal{S}$ corresponding to the mesh $\mathcal{M}$ is defined by

$$\mathcal{S} = \mathcal{S}(\mathcal{M}) = \{\phi \in L^\infty(\mathbb{R}) : \phi|_I \text{ linear if } I \in \mathcal{M},\ \phi(x) = 0 \text{ if } x \notin I \text{ for all } I \in \mathcal{M}\}.$$

The two factors f, g of the convolution ω_{exact} from (1.1) as well as the (projected) image ω may be organised by three different locally refined meshes

$$\mathcal{M}^f, \quad \mathcal{M}^g, \quad \mathcal{M}^\omega, \tag{2.4}$$

which are all of the form (2.3). These meshes give rise to three spaces

$$\mathcal{S}^f := \mathcal{S}\left(\mathcal{M}^f\right), \quad \mathcal{S}^g := \mathcal{S}\left(\mathcal{M}^g\right), \quad \mathcal{S}^\omega := \mathcal{S}\left(\mathcal{M}^\omega\right).$$

We recall that $f \in \mathcal{S}^f$ and $g \in \mathcal{S}^g$ are the input data, while the result is the exact L^2-orthogonal projection of $f * g$ onto $\mathcal{S}^\omega$.

Applications involving convolutions can be found in [2], where the underlying problem takes the principal form $\mathrm{d}f/\mathrm{d}t = \ldots + f * f$ (here $\mathcal{S}^f = \mathcal{S}^g = \mathcal{S}^\omega$ is the appropriate choice of spaces) and in [1], where a fixed point equation $f = \ldots + f * g$ is to be solved (therefore $\mathcal{S}^f = \mathcal{S}^\omega$, while $\mathcal{S}^g$ may be chosen differently).

2.3 Basis Functions

Functions from $\mathcal{S}(\mathcal{M})$ may be discontinuous at the grid points of the mesh. This fact has the advantage that the basis functions spanning $\mathcal{S}(\mathcal{M})$ have minimal support (the support is just one interval of $\mathcal{M}$).

Here, we consider the case of piecewise linear functions. The following basis functions of level ℓ are derived from the Legendre polynomials:

$$\Phi^\ell_{i,0}(x) := \begin{cases} 1/\sqrt{h_\ell} & \text{if } x \in I^\ell_i, \\ 0 & \text{otherwise,} \end{cases} \tag{2.5a}$$

$$\Phi^\ell_{i,1}(x) := \begin{cases} \sqrt{12}\left(x - x^\ell_{i+1/2}\right)/h^{3/2}_\ell & \text{if } x \in I^\ell_i, \\ 0 & \text{otherwise,} \end{cases} \tag{2.5b}$$

where

$$x^\ell_{i+1/2} := (i + 1/2)\, h_\ell$$

is the midpoint of the interval I^ℓ_i, which is the support of $\Phi^\ell_{i,\alpha}$. Note that $\Phi^\ell_{i,\alpha}$ $(\alpha = 0, 1)$ are orthonormal. Since the intervals $I \in \mathcal{M}$ are non-overlapping, the functions in the right-hand side of

$$\mathcal{S}(\mathcal{M}) = \operatorname{span}\left\{\Phi^\ell_{i,\alpha} : \alpha \in \{0,1\},\ I^\ell_i \in \mathcal{M}\right\}$$

form an orthonormal system of $\mathcal{S}(\mathcal{M})$.

Let $\mathcal{S}_\ell$ be the space of piecewise linear functions of level ℓ (on the infinite mesh $\mathcal{M}_\ell$ from (2.2)):

$$\mathcal{S}_\ell := \operatorname{span}\left\{\Phi^\ell_{i,\alpha} : \alpha \in \{0,1\},\ i \in \mathbb{Z}\right\} \qquad (\ell \in \mathbb{N}_0). \tag{2.6}$$

For fixed ℓ, the basis

$$\left\{\Phi_{i,\alpha}^{\ell} : \alpha \in \{0,1\},\ i \in \mathbb{Z}\right\}$$

is orthonormal due to the chosen scaling.

The spaces $\mathcal{S}_\ell$ are nested, i.e.,

$$\mathcal{S}_\ell \subset \mathcal{S}_{\ell+1}.$$

In particular, $\Phi_{i,\alpha}^{\ell}$ can be represented by means of $\Phi_{i,\alpha}^{\ell+1}$:

$$\Phi_{i,0}^{\ell} = \frac{1}{\sqrt{2}}\left(\Phi_{2i,0}^{\ell+1} + \Phi_{2i+1,0}^{\ell+1}\right), \tag{2.7a}$$

$$\Phi_{i,1}^{\ell} = \frac{1}{2\sqrt{2}}\left(\Phi_{2i,1}^{\ell+1} + \Phi_{2i+1,1}^{\ell+1} + \sqrt{3}\left(\Phi_{2i+1,0}^{\ell+1} - \Phi_{2i,0}^{\ell+1}\right)\right). \tag{2.7b}$$

3 Notations and Definition of the Problem

3.1 Representations of $f \in \mathcal{S}^f$ and $g \in \mathcal{S}^g$

Following the definition of $\mathcal{S}^f$, we have $\mathcal{S}^f = \operatorname{span}\left\{\Phi_i^\ell : I_i^\ell \in \mathcal{M}^f\right\}$. We can decompose the set $\mathcal{M}^f$ into different levels:

$$\mathcal{M}^f = \bigcup\nolimits_{\ell=0}^{L^f} \mathcal{M}_\ell^f, \qquad \text{where } \mathcal{M}_\ell^f := \mathcal{M}^f \cap \mathcal{M}_\ell.$$

This gives rise to the related index set

$$\mathcal{I}_\ell^f := \left\{i \in \mathbb{Z} : I_i^\ell \in \mathcal{M}_\ell^f\right\} \tag{3.1}$$

and to the corresponding decomposition

$$\mathcal{S}^f = \bigcup\nolimits_{\ell=0}^{L^f} \mathcal{S}_\ell^f \qquad \text{with } \mathcal{S}_\ell^f = \operatorname{span}\left\{\Phi_{i,\alpha}^\ell : \alpha = 0, 1,\ i \in \mathcal{I}_\ell^f\right\}.$$

Here, L^f is the largest level ℓ with $\mathcal{M}_\ell^f \neq \emptyset$. Similarly, $\mathcal{I}_\ell^g$, $\mathcal{S}_\ell^g$ and L^g correspond to $\mathcal{M}^g$ and $\mathcal{S}^g$.

We start from the representation

$$f = \sum_{\ell=0}^{L^f} f_\ell, \qquad f_\ell = \sum_{i \in \mathcal{I}_\ell^f} \sum_{\alpha=0}^{1} f_{i,\alpha}^\ell \Phi_{i,\alpha}^\ell \in \mathcal{S}_\ell^f, \tag{3.2a}$$

$$g = \sum_{\ell=0}^{L^g} g_\ell, \qquad g_\ell = \sum_{i \in \mathcal{I}_\ell^g} \sum_{\alpha=0}^{1} g_{i,\alpha}^\ell \Phi_{i,\alpha}^\ell \in \mathcal{S}_\ell^g, \tag{3.2b}$$

of the factors f, g of the convolution. Similarly, the final L^2-orthogonal projection ω of $f * g$ will have the form

$$\omega = \sum_{\ell=0}^{L^\omega} \omega_\ell, \qquad \omega_\ell = \sum_{i \in \mathcal{I}_\ell^\omega} \sum_{\alpha=0}^{1} \omega_{i,\alpha}^\ell \Phi_{i,\alpha}^\ell \in \mathcal{S}_\ell^\omega. \tag{3.2c}$$

3.2 Projections P and P_ℓ

The L^2-orthogonal projection P onto $\mathcal{S}_\ell^\omega$ is defined by

$$P\varphi := \sum_{i \in \mathcal{I}_\ell^\omega} \sum_{\alpha=0}^{1} \left\langle \varphi, \Phi_{i,\alpha}^\ell \right\rangle \Phi_{i,\alpha}^\ell \in \mathcal{S}_\ell^\omega$$

with the L^2-scalar product $\langle \varphi, \psi \rangle = \int_{\mathbb{R}} \varphi\psi \mathrm{d}x$. We will also use the L^2-orthogonal projection P_ℓ onto the space $\mathcal{S}_\ell$ from (2.6) defined by

$$P_\ell \, \varphi := \sum_{i \in \mathbb{Z}} \sum_{\alpha=0}^{1} \left\langle \varphi, \Phi_{i,\alpha}^\ell \right\rangle \Phi_{i,\alpha}^\ell . \tag{3.3}$$

3.3 Definition of the Basic Problem

We use the decomposition into scales expressed by

$$f = \sum_{\ell'=0}^{L^f} f_{\ell'} \quad \text{and} \quad g = \sum_{\ell=0}^{L^g} g_\ell \qquad \text{(see (3.2a,b))}.$$

The convolution $f * g$ can be written as

$$f * g = \sum_{\ell'=0}^{L^f} \sum_{\ell=0}^{L^g} f_{\ell'} * g_\ell \, .$$

Since the convolution is symmetric, we can rewrite the sum as

$$f * g = \sum_{\ell' \le \ell} f_{\ell'} * g_\ell + \sum_{\ell < \ell'} g_\ell * f_{\ell'} \, , \tag{3.4}$$

where ℓ', ℓ are restricted to the level intervals $0 \le \ell' \le L^f$, $0 \le \ell \le L^g$. Hence, the basic task is as follows.

Problem 3.1 *Let $\ell' \le \ell$, $f_{\ell'} \in \mathcal{S}_{\ell'}$, $g_\ell \in \mathcal{S}_\ell$, and $\ell'' \in \mathbb{N}_0$ a further level. Then, the projection $P_{\ell''}(f_{\ell'} * g_\ell)$ is to be computed. More precisely, only the restriction of $P_{\ell''}(f_{\ell'} * g_\ell)$ to $\bigcup_{i \in \mathcal{I}_{\ell''}^\omega} I_i^{\ell''}$ is needed, since only this part appears in $\mathcal{S}_{\ell''}^\omega$.*

Because of the splitting (3.4), we may assume $\ell' \le \ell$ without loss of generality. In the case of the second sum one has to interchange the rôles of the symbols f and g.

Before we present the solution algorithm in Section 5, we introduce some further notations in the next Section 4.

4 Auxiliary Coefficients

4.1 γ-Coefficients

For level numbers $\ell'', \ell', \ell \in \mathbb{N}_0$ and integers $i, j, k \in \mathbb{Z}$ we define

$$\gamma^{\ell'',\ell',\ell}_{(i,\alpha),(j,\beta),(k,\varkappa)} := \iint \Phi^{\ell''}_{i,\alpha}(x)\, \Phi^{\ell'}_{j,\beta}(y)\, \Phi^{\ell}_{k,\varkappa}(x-y) \mathrm{d}x \mathrm{d}y \tag{4.1}$$

(all integrations over $\mathbb{R}$). We remark that $\gamma^{\ell'',\ell',\ell}_{(i,\alpha),(j,\beta),(k,\varkappa)} = \left\langle \Phi^{\ell''}_{i,\alpha}, \Phi^{\ell'}_{j,\beta} * \Phi^{\ell}_{k,\varkappa} \right\rangle$ is the L^2-scalar product of the basis function $\Phi^{\ell''}_{i,\alpha}$ and the convolution $\Phi^{\ell'}_{j,\beta} * \Phi^{\ell}_{k,\varkappa}$.

The connection to the computation of the projection

$$\omega_{\ell''} = P_{\ell''} \left(f_{\ell'} * g_\ell \right) \tag{4.2}$$

of the convolution $f_{\ell'} * g_\ell$ from Problem 3.1 is as follows. $\omega_{\ell''}$ is represented by

$$\omega_{\ell''} = \sum_{i \in \mathbb{Z}}^{\ell} \sum_{\alpha=0}^{1} \omega^{\ell''}_{i,\alpha} \Phi^{\ell''}_{i,\alpha},$$

where the coefficients $\omega^{\ell''}_{i,\alpha}$ result from

$$\begin{aligned} \omega^{\ell''}_{i,\alpha} &= \int (f_{\ell'} * g_\ell)(x)\, \Phi^{\ell''}_{i,\alpha}(x)\, \mathrm{d}x \qquad (4.3) \\ &= \int \Phi^{\ell''}_{i,\alpha}(x) \left(\sum_{j\in\mathbb{Z}} \sum_{\beta=0}^{1} f^{\ell'}_{j,\beta} \Phi^{\ell'}_{j,\beta} * \sum_{k\in\mathbb{Z}} \sum_{\varkappa=0}^{1} g^{\ell}_{k,\varkappa} \Phi^{\ell}_{k,\varkappa} \right)(x)\, \mathrm{d}x \\ &= \sum_{j,k\in\mathbb{Z}} \sum_{\beta,\varkappa=0}^{1} f^{\ell'}_{j,\beta}\, g^{\ell}_{k,\varkappa} \iint \Phi^{\ell''}_{i,\alpha}(x) \Phi^{\ell'}_{j,\beta}(y) \Phi^{\ell}_{k,\varkappa}(x-y)\, \mathrm{d}x \mathrm{d}y \\ &= \sum_{j,k\in\mathbb{Z}} \sum_{\beta,\varkappa=0}^{1} f^{\ell'}_{j,\beta}\, g^{\ell}_{k,\varkappa}\, \gamma^{\ell'',\ell',\ell}_{(i,\alpha),(j,\beta),(k,\varkappa)} . \end{aligned}$$

The recursion formulae (2.7a,b) can be applied to all three basis functions in the integrand $\Phi^{\ell''}_{(i,\alpha)}(x)\, \Phi^{\ell'}_{(j,\beta)}(y)\, \Phi^{\ell}_{(k,\varkappa)}(x-y)$ of $\gamma^{\ell'',\ell',\ell}_{(i,\alpha),(j,\beta),(k,\varkappa)}$. Some of the resulting recursions for $\gamma^{\ell'',\ell',\ell}_{(i,\alpha),(j,\beta),(k,\varkappa)}$ are given in the next Remark.

Remark 4.1 *For all $\ell'', \ell', \ell \in \mathbb{N}_0$ and all $i, j, k \in \mathbb{Z}$ we have*

$$\gamma^{\ell'',\ell',\ell}_{(i,0),(j,\beta),(k,\varkappa)} = \frac{1}{\sqrt{2}} \left(\gamma^{\ell''+1,\ell',\ell}_{(2i,0),(j,\beta),(k,\varkappa)} + \gamma^{\ell''+1,\ell',\ell}_{(2i+1,0),(j,\beta),(k,\varkappa)} \right), \tag{4.4a}$$

$$\gamma^{\ell'',\ell',\ell}_{(i,1),(j,\beta),(k,\varkappa)} = \frac{1}{2\sqrt{2}} \begin{pmatrix} -\sqrt{3}\,\gamma^{\ell''+1,\ell',\ell}_{(2i,0),(j,\beta),(k,\varkappa)} + \sqrt{3}\,\gamma^{\ell''+1,\ell',\ell}_{(2i+1,0),(j,\beta),(k,\varkappa)} \\ +\gamma^{\ell''+1,\ell',\ell}_{(2i,1),(j,\beta),(k,\varkappa)} + \gamma^{\ell''+1,\ell',\ell}_{(2i+1,1),(j,\beta),(k,\varkappa)} \end{pmatrix}, \tag{4.4b}$$

$$\gamma^{\ell'',\ell',\ell}_{(i,\alpha),(j,0),(k,\varkappa)} = \frac{1}{\sqrt{2}} \left(\gamma^{\ell'',\ell'+1,\ell}_{(i,\alpha),(2j,0),(k,\varkappa)} + \gamma^{\ell'',\ell'+1,\ell}_{(i,\alpha),(2j+1,0),(k,\varkappa)} \right), \tag{4.4c}$$

$$\gamma^{\ell'',\ell',\ell}_{(i,\alpha),(j,1),(k,\varkappa)} = \frac{1}{2\sqrt{2}} \begin{pmatrix} -\sqrt{3}\,\gamma^{\ell'',\ell'+1,\ell}_{(i,\alpha),(2j,0),(k,\varkappa)} + \sqrt{3}\,\gamma^{\ell'',\ell'+1,\ell}_{(i,\alpha),(2j+1,0),(k,\varkappa)} \\ +\gamma^{\ell'',\ell'+1,\ell}_{(i,\alpha),(2j,1),(k,\varkappa)} + \gamma^{\ell'',\ell'+1,\ell}_{(i,\alpha),(2j+1,1),(k,\varkappa)} \end{pmatrix}. \tag{4.4d}$$

4.2 Simplified γ-Coefficients

For levels ℓ, ℓ', ℓ'' with $\ell \geq \max\{\ell', \ell''\}$ we set

$$\gamma^{\ell'',\ell',\ell}_{\nu,(\alpha,\beta,\varkappa)} := \iint \Phi^{\ell''}_{0,\alpha}(x)\, \Phi^{\ell'}_{0,\beta}(y)\, \Phi^{\ell}_{\nu,\varkappa}(x-y)\mathrm{d}x\mathrm{d}y \qquad (\nu \in \mathbb{Z}), \tag{4.5}$$

i.e., $\gamma^{\ell'',\ell',\ell}_{\nu,(\alpha,\beta,\varkappa)} = \gamma^{\ell'',\ell',\ell}_{(0,\alpha),(0,\beta),(k,\varkappa)} = \left\langle \Phi^{\ell''}_{0,\alpha}, \Phi^{\ell'}_{0,\beta} * \Phi^{\ell}_{\nu,\varkappa} \right\rangle$. We call these coefficients simplified γ-coefficients, since only one subindex ν is involved instead of the triple (i, j, k).

Under the condition $\ell \geq \max\{\ell', \ell''\}$, it suffices to use the quantities $\gamma^{\ell'',\ell',\ell}_{\nu,(\alpha,\beta,\varkappa)}$ from (4.5) as shown in the next Lemma whose proof is based on the shift property of the basis functions.

Lemma 4.2 *Let $\ell \geq \max\{\ell', \ell''\}$. Then*

$$\gamma^{\ell'',\ell',\ell}_{(i,\alpha),(j,\beta),(k,\varkappa)} = \gamma^{\ell'',\ell',\ell}_{k-i2^{\ell-\ell''}+j2^{\ell-\ell'},(\alpha,\beta,\varkappa)} \qquad \textit{for any } i,j,k \in \mathbb{Z},\ \alpha,\beta,\varkappa \in \{0,1\}\,. \tag{4.6}$$

Remark 4.3 *The values of $\gamma^{\ell,\ell,\ell}_{\nu,(\alpha,\beta,\varkappa)}$ are*

$$\gamma^{\ell,\ell,\ell}_{0,(0,0,0)} = \gamma^{\ell,\ell,\ell}_{-1,(0,0,0)} = \sqrt{h_\ell}/2,$$
$$\gamma^{\ell,\ell,\ell}_{0,(1,0,0)} = \sqrt{h_\ell/12}, \quad \gamma^{\ell,\ell,\ell}_{-1,(1,0,0)} = -\sqrt{h_\ell/12},$$

$$\gamma^{\ell,\ell,\ell}_{0,(0,1,0)} = \gamma^{\ell,\ell,\ell}_{0,(0,0,1)} = -\sqrt{h_\ell/12}, \quad \gamma^{\ell,\ell,\ell}_{-1,(0,1,0)} = \gamma^{\ell,\ell,\ell}_{-1,(0,0,1)} = \sqrt{h_\ell/12},$$
$$\gamma^{\ell,\ell,\ell}_{0,(1,1,0)} = \gamma^{\ell,\ell,\ell}_{-1,(1,1,0)} = \gamma^{\ell,\ell,\ell}_{0,(1,0,1)} = \gamma^{\ell,\ell,\ell}_{-1,(1,0,1)} = 0,$$

$$\gamma^{\ell,\ell,\ell}_{0,(0,1,1)} = \gamma^{\ell,\ell,\ell}_{-1,(0,1,1)} = 0,$$
$$\gamma^{\ell,\ell,\ell}_{0,(1,1,1)} = -\sqrt{3h_\ell}/5, \quad \gamma^{\ell,\ell,\ell}_{-1,(1,1,1)} = \sqrt{3h_\ell}/5,$$

and $\gamma^{\ell,\ell,\ell}_{\nu,(\alpha,\beta,\varkappa)} = 0$ for $\nu \notin \{0,1\}$.

Proof. First, one has to determine the convolution $\int \Phi^{\ell}_{0,\beta}(y)\, \Phi^{\ell}_{0,\varkappa}(x-y)\mathrm{d}y$. This function has its support in $[0, 2h_\ell]$:

β	$\varkappa$	values in $[0, h_\ell]$	values in $[h_\ell, 2h_\ell]$
0	0	x/h_ℓ	$(1 - x/h_\ell)$
1	0	$\sqrt{3}\,(x-h_\ell)\,x/h_\ell^2$	$\sqrt{3}\,(h_\ell - x)\,(x-2h_\ell)\,/h_\ell^2$
0	1	$\sqrt{3}\,(x-h_\ell)\,x/h_\ell^2$	$\sqrt{3}\,(h_\ell - x)\,(x-2h_\ell)\,/h_\ell^2$
1	1	$(3h_\ell^2 - 6h_\ell x + 2x^2)\,x/h_\ell^3$	$(2h_\ell x + h_\ell^2 - 2x^2)\,(x-2h_\ell)\,/h_\ell^3.$

(4.7)

Integration with $\Phi^{\ell}_{0,\alpha}$ yields $\gamma^{\ell,\ell,\ell}_{0,(\alpha,\beta,\varkappa)}$. To obtain $\gamma^{\ell,\ell,\ell}_{-1,(\alpha,\beta,\varkappa)}$ one has to integrate with $\Phi^{\ell}_{1,\alpha}$. ∎

4.3 G- and Γ-Coefficients

As stated in (4.3), we have to compute $\sum_{j,k\in\mathbb{Z}}\sum_{\beta,\varkappa=0}^{1} f_{j,\beta}^{\ell'}\, g_{k,\varkappa}^{\ell}\, \gamma_{(i,\alpha),(j,\beta),(k,\varkappa)}^{\ell'',\ell',\ell}$. Performing only the summation over k and $\varkappa$, leads us to

$$G_{(i,\alpha),(j,\beta)}^{\ell'',\ell',\ell} = \sum_{k\in\mathbb{Z}}\sum_{\varkappa=0}^{1} g_{k,\varkappa}^{\ell}\, \gamma_{(i,\alpha),(j,\beta),(k,\varkappa)}^{\ell'',\ell',\ell}. \tag{4.8}$$

From $G_{(i,\alpha),(j,\beta)}^{\ell',\ell',\ell} = \sum_{k,\varkappa} g_{k,\varkappa}^{\ell}\, \gamma_{k-(i-j)2^{\ell-\ell'},(\alpha,\beta,\varkappa)}^{\ell',\ell',\ell}$ one concludes in the case $\ell''=\ell'\le\ell$ that

$$G_{(i,\alpha),(j,\beta)}^{\ell',\ell',\ell} = G_{(i-j,\alpha),(0,\beta)}^{\ell',\ell',\ell}. \tag{4.9}$$

Using the recursions (4.4a-d) from Remark 4.1, one proves the following result.

Remark 4.4 *For all $\ell'',\ell',\ell\in\mathbb{N}_0$ and all $i,j\in\mathbb{Z}$ we have*

$$\begin{aligned}
\alpha=0: &\quad G_{(i,0),(j,\beta)}^{\ell'',\ell',\ell} = \tfrac{1}{\sqrt{2}}\left(G_{(2i,0),(j,\beta)}^{\ell''+1,\ell',\ell} + G_{(2i+1,0),(j,\beta)}^{\ell''+1,\ell',\ell}\right),\\
\beta=0: &\quad G_{(i,\alpha),(j,0)}^{\ell'',\ell',\ell} = \tfrac{1}{\sqrt{2}}\left(G_{(i,\alpha),(2j,0)}^{\ell'',\ell'+1,\ell} + G_{(i,\alpha),(2j+1,0)}^{\ell'',\ell'+1,\ell}\right),\\
\alpha=1: &\quad G_{(i,1),(j,\beta)}^{\ell'',\ell',\ell} = \frac{1}{2\sqrt{2}}\left(\begin{array}{c} -\sqrt{3}\,G_{(2i,0),(j,\beta)}^{\ell''+1,\ell',\ell} + \sqrt{3}\,G_{(2i+1,0),(j,\beta)}^{\ell''+1,\ell',\ell} \\ +\,G_{(2i,1),(j,\beta)}^{\ell''+1,\ell',\ell} + G_{(2i+1,1),(j,\beta)}^{\ell''+1,\ell',\ell}\end{array}\right),\\
\beta=1: &\quad G_{(i,\alpha),(j,1)}^{\ell'',\ell',\ell} = \frac{1}{2\sqrt{2}}\left(\begin{array}{c} -\sqrt{3}\,G_{(i,\alpha),(2j,0)}^{\ell'',\ell'+1,\ell} + \sqrt{3}\,G_{(i,\alpha),(2j+1,0)}^{\ell'',\ell'+1,\ell} \\ +\,G_{(i,\alpha),(2j,1)}^{\ell'',\ell'+1,\ell} + G_{(i,\alpha),(2j+1,1)}^{\ell'',\ell'+1,\ell}\end{array}\right).
\end{aligned} \tag{4.10}$$

If the first two levels are equal: $\ell''=\ell'\le\ell$, the coefficients are denoted by

$$\Gamma_{i,(\alpha,\beta)}^{\ell',\ell} := G_{(0,\alpha),(-i,\beta)}^{\ell',\ell',\ell} = \sum_{k\in\mathbb{Z}}\sum_{\varkappa=0}^{1} g_{k,\varkappa}^{\ell}\, \gamma_{k-i2^{\ell-\ell'},(\alpha,\beta,\varkappa)}^{\ell',\ell',\ell}. \tag{4.11}$$

Lemma 4.5 *a) For $\ell'=\ell$, the values*

$$\Gamma_{i,(\alpha,\beta)}^{\ell,\ell} = \sum_{\varkappa=0}^{1}\left(g_{i,\varkappa}^{\ell}\,\gamma_{0,(\alpha,\beta,\varkappa)}^{\ell,\ell,\ell} + g_{i-1,\varkappa}^{\ell}\,\gamma_{-1,(\alpha,\beta,\varkappa)}^{\ell,\ell,\ell}\right)$$

can be computed from the γ-values given in Remark 4.3.

b) For $\ell'<\ell$, one can make use of the following recursions:

$$\begin{aligned}
\Gamma_{i,(0,0)}^{\ell',\ell} &= \frac{1}{2}\Gamma_{2i-1,(0,0)}^{\ell'+1,\ell} + \Gamma_{2i,(0,0)}^{\ell'+1,\ell} + \frac{1}{2}\Gamma_{2i+1,(0,0)}^{\ell'+1,\ell},\\
\Gamma_{i,(1,0)}^{\ell',\ell} &= \frac{\sqrt{3}}{4}\left(\Gamma_{2i+1,(0,0)}^{\ell'+1,\ell} - \Gamma_{2i-1,(0,0)}^{\ell'+1,\ell}\right) + \frac{1}{4}\left(\Gamma_{2i-1,(1,0)}^{\ell'+1,\ell} + \Gamma_{2i+1,(1,0)}^{\ell'+1,\ell}\right) + \frac{1}{2}\Gamma_{2i,(1,0)}^{\ell'+1,\ell},\\
\Gamma_{i,(0,1)}^{\ell',\ell} &= \frac{\sqrt{3}}{4}\left(\Gamma_{2i-1,(0,0)}^{\ell'+1,\ell} - \Gamma_{2i+1,(0,0)}^{\ell'+1,\ell}\right) + \frac{1}{4}\left(\Gamma_{2i-1,(0,1)}^{\ell'+1,\ell} + \Gamma_{2i+1,(0,1)}^{\ell'+1,\ell}\right) + \frac{1}{2}\Gamma_{2i,(0,1)}^{\ell'+1,\ell},\\
\Gamma_{i,(1,1)}^{\ell',\ell} &= -\frac{3}{8}\left(\Gamma_{2i-1,(0,0)}^{\ell'+1,\ell} + \Gamma_{2i+1,(0,0)}^{\ell'+1,\ell}\right) + \frac{3}{4}\Gamma_{2i,(0,0)}^{\ell'+1,\ell} + \frac{1}{8}\left(\Gamma_{2i-1,(1,1)}^{\ell'+1,\ell} + \Gamma_{2i+1,(1,1)}^{\ell'+1,\ell}\right) + \frac{1}{4}\Gamma_{2i,(1,1)}^{\ell'+1,\ell}\\
&\quad + \frac{\sqrt{3}}{8}\left(-\Gamma_{2i-1,(0,1)}^{\ell'+1,\ell} + \Gamma_{2i+1,(0,1)}^{\ell'+1,\ell} + \Gamma_{2i-1,(1,0)}^{\ell'+1,\ell} - \Gamma_{2i+1,(1,0)}^{\ell'+1,\ell}\right).
\end{aligned} \tag{4.12}$$

Proof of part b). Case of $(\alpha,\beta)=(0,0)$: In order to compute $\Gamma^{\ell',\ell}_{i,(0,0)}$ from $\Gamma^{\ell'+1,\ell}_{i,(0,0)}$, one has to combine the first two lines of (4.10):

$$\begin{aligned}
\Gamma^{\ell',\ell}_{i,(0,0)} &= G^{\ell',\ell',\ell}_{(i,0),(0,0)} \underset{(4.10_1)}{=} \frac{1}{\sqrt{2}}\left(G^{\ell'+1,\ell',\ell}_{(2i,0),(0,0)} + G^{\ell'+1,\ell',\ell}_{(2i+1,0),(0,0)}\right)\\
&\underset{(4.10_2)}{=} \frac{1}{\sqrt{2}}\left(\tfrac{1}{\sqrt{2}}\left(G^{\ell'+1,\ell'+1,\ell}_{(2i,0),(0,0)} + G^{\ell'+1,\ell'+1,\ell}_{(2i,0),(1,0)}\right) + \tfrac{1}{\sqrt{2}}\left(G^{\ell'+1,\ell'+1,\ell}_{(2i+1,0),(0,0)} + G^{\ell'+1,\ell'+1,\ell}_{(2i+1,0),(1,0)}\right)\right)\\
&\underset{(4.9)}{=} \frac{1}{\sqrt{2}}\left(\tfrac{1}{\sqrt{2}}\left(G^{\ell'+1,\ell'+1,\ell}_{(2i,0),(0,0)} + G^{\ell'+1,\ell'+1,\ell}_{(2i-1,0),(0,0)}\right) + \tfrac{1}{\sqrt{2}}\left(G^{\ell'+1,\ell'+1,\ell}_{(2i+1,0),(0,0)} + G^{\ell'+1,\ell'+1,\ell}_{(2i,0),(0,0)}\right)\right)\\
&= G^{\ell'+1,\ell'+1,\ell}_{(2i,0),(0,0)} + \frac{1}{2}G^{\ell'+1,\ell'+1,\ell}_{(2i-1,0),(0,0)} + \frac{1}{2}G^{\ell'+1,\ell'+1,\ell}_{(2i+1,0),(0,0)}\\
&= \frac{1}{2}\Gamma^{\ell'+1,\ell}_{2i-1,(0,0)} + \Gamma^{\ell'+1,\ell}_{2i,(0,0)} + \frac{1}{2}\Gamma^{\ell'+1,\ell}_{2i+1,(0,0)}.
\end{aligned}$$

Case of $(\alpha,\beta)=(1,0)$:

$$\Gamma^{\ell',\ell}_{i,(1,0)} = G^{\ell',\ell',\ell}_{(i,1),(0,0)} \underset{(4.10_3)}{=} \frac{1}{2\sqrt{2}}\begin{pmatrix} -\sqrt{3}\,G^{\ell'+1,\ell',\ell}_{(2i,0),(0,0)} + \sqrt{3}\,G^{\ell'+1,\ell',\ell}_{(2i+1,0),(0,0)}\\ +G^{\ell'+1,\ell',\ell}_{(2i,1),(0,0)} + G^{\ell'+1,\ell',\ell}_{(2i+1,1),(0,0)}\end{pmatrix}$$

$$\underset{(4.10_2)}{=} \frac{1}{2\sqrt{2}}\begin{pmatrix} -\sqrt{3}\,\frac{1}{\sqrt{2}}\left(G^{\ell'+1,\ell'+1,\ell}_{(2i,0),(0,0)} + G^{\ell'+1,\ell'+1,\ell}_{(2i,0),(1,0)}\right) + \sqrt{3}\,\frac{1}{\sqrt{2}}\left(G^{\ell'+1,\ell'+1,\ell}_{(2i+1,0),(0,0)} + G^{\ell'+1,\ell'+1,\ell}_{(2i+1,0),(1,0)}\right)\\ +\frac{1}{\sqrt{2}}\left(G^{\ell'+1,\ell'+1,\ell}_{(2i,1),(0,0)} + G^{\ell'+1,\ell'+1,\ell}_{(2i,1),(1,0)}\right) + \frac{1}{\sqrt{2}}\left(G^{\ell'+1,\ell'+1,\ell}_{(2i+1,1),(0,0)} + G^{\ell'+1,\ell'+1,\ell}_{(2i+1,1),(1,0)}\right)\end{pmatrix}$$

$$\underset{(4.9)}{=} \frac{1}{4}\begin{pmatrix} -\sqrt{3}\left(G^{\ell'+1,\ell'+1,\ell}_{(2i,0),(0,0)} + G^{\ell'+1,\ell'+1,\ell}_{(2i-1,0),(0,0)}\right) + \sqrt{3}\left(G^{\ell'+1,\ell'+1,\ell}_{(2i+1,0),(0,0)} + G^{\ell'+1,\ell'+1,\ell}_{(2i,0),(0,0)}\right)\\ +\left(G^{\ell'+1,\ell'+1,\ell}_{(2i,1),(0,0)} + G^{\ell'+1,\ell'+1,\ell}_{(2i-1,1),(0,0)}\right) + \left(G^{\ell'+1,\ell'+1,\ell}_{(2i+1,1),(0,0)} + G^{\ell'+1,\ell'+1,\ell}_{(2i,1),(0,0)}\right)\end{pmatrix}$$

$$= \frac{1}{4}\begin{pmatrix} -\sqrt{3}\left(\Gamma^{\ell'+1,\ell}_{2i,(0,0)} + \Gamma^{\ell'+1,\ell}_{2i-1,(0,0)}\right) + \sqrt{3}\left(\Gamma^{\ell'+1,\ell}_{2i+1,(0,0)} + \Gamma^{\ell'+1,\ell}_{2i,(0,0)}\right)\\ +\left(\Gamma^{\ell'+1,\ell}_{2i,(1,0)} + \Gamma^{\ell'+1,\ell}_{2i-1,(1,0)}\right) + \left(\Gamma^{\ell'+1,\ell}_{2i+1,(1,0)} + \Gamma^{\ell'+1,\ell}_{2i,(1,0)}\right)\end{pmatrix}.$$

Case of $(\alpha,\beta)=(0,1)$:

$$\Gamma^{\ell',\ell}_{i,(0,1)} = G^{\ell',\ell',\ell}_{(i,0),(0,1)} \underset{(4.10_1)}{=} \frac{1}{\sqrt{2}}\left(G^{\ell'+1,\ell',\ell}_{(2i,0),(0,1)} + G^{\ell'+1,\ell',\ell}_{(2i+1,0),(0,1)}\right)$$

$$\underset{(4.10_4)}{=} \frac{1}{\sqrt{2}}\begin{pmatrix} \frac{1}{2\sqrt{2}}\left(-\sqrt{3}\,G^{\ell'+1,\ell'+1,\ell}_{(2i,0),(0,0)} + \sqrt{3}\,G^{\ell'+1,\ell'+1,\ell}_{(2i,0),(1,0)} + G^{\ell'+1,\ell'+1,\ell}_{(2i,0),(0,1)} + G^{\ell'+1,\ell'+1,\ell}_{(2i,0),(1,1)}\right)\\ +\frac{1}{2\sqrt{2}}\left(-\sqrt{3}\,G^{\ell'+1,\ell'+1,\ell}_{(2i+1,0),(0,0)} + \sqrt{3}\,G^{\ell'+1,\ell'+1,\ell}_{(2i+1,0),(1,0)} + G^{\ell'+1,\ell'+1,\ell}_{(2i+1,0),(0,1)} + G^{\ell'+1,\ell'+1,\ell}_{(2i+1,0),(1,1)}\right)\end{pmatrix}$$

$$\underset{(4.9)}{=} \frac{1}{4}\begin{pmatrix} -\sqrt{3}\,G^{\ell'+1,\ell'+1,\ell}_{(2i,0),(0,0)} + \sqrt{3}\,G^{\ell'+1,\ell'+1,\ell}_{(2i-1,0),(0,0)} + G^{\ell'+1,\ell'+1,\ell}_{(2i,0),(0,1)} + G^{\ell'+1,\ell'+1,\ell}_{(2i-1,0),(0,1)}\\ -\sqrt{3}\,G^{\ell'+1,\ell'+1,\ell}_{(2i+1,0),(0,0)} + \sqrt{3}\,G^{\ell'+1,\ell'+1,\ell}_{(2i,0),(0,0)} + G^{\ell'+1,\ell'+1,\ell}_{(2i+1,0),(0,1)} + G^{\ell'+1,\ell'+1,\ell}_{(2i,0),(0,1)}\end{pmatrix}$$

$$= \frac{1}{4}\left(\sqrt{3}\,\Gamma^{\ell'+1,\ell}_{2i-1,(0,0)} - \sqrt{3}\,\Gamma^{\ell'+1,\ell}_{2i+1,(0,0)} + \Gamma^{\ell'+1,\ell}_{2i-1,(0,1)} + 2\Gamma^{\ell'+1,\ell}_{2i,(0,1)} + \Gamma^{\ell'+1,\ell}_{2i+1,(0,1)}\right),$$

Case of $(\alpha, \beta) = (1, 1)$:

$$
\begin{aligned}
\Gamma^{\ell',\ell}_{i,(1,1)} &= G^{\ell',\ell',\ell}_{(i,1),(0,1)} \\
&\underset{(4.10_3)}{=} \frac{1}{2\sqrt{2}} \left(-\sqrt{3}\, G^{\ell'+1,\ell',\ell}_{(2i,0),(0,1)} + \sqrt{3}\, G^{\ell'+1,\ell',\ell}_{(2i+1,0),(0,1)} + G^{\ell'+1,\ell',\ell}_{(2i,1),(0,1)} + G^{\ell'+1,\ell',\ell}_{(2i+1,1),(0,1)} \right) \\
&\underset{(4.10_4)}{=} \frac{1}{2\sqrt{2}} \left(\begin{array}{l}
-\sqrt{3}\, \frac{1}{2\sqrt{2}} \left(\begin{array}{c} -\sqrt{3}\, G^{\ell'+1,\ell'+1,\ell}_{(2i,0),(0,0)} + \sqrt{3}\, G^{\ell'+1,\ell'+1,\ell}_{(2i,0),(1,0)} \\ +G^{\ell'+1,\ell'+1,\ell}_{(2i,0),(0,1)} + G^{\ell'+1,\ell'+1,\ell}_{(2i,0),(1,1)} \end{array} \right) \\
+\sqrt{3}\, \frac{1}{2\sqrt{2}} \left(\begin{array}{c} -\sqrt{3}\, G^{\ell'+1,\ell'+1,\ell}_{(2i+1,0),(0,0)} + \sqrt{3}\, G^{\ell'+1,\ell'+1,\ell}_{(2i+1,0),(1,0)} \\ +G^{\ell'+1,\ell'+1,\ell}_{(2i+1,0),(0,1)} + G^{\ell'+1,\ell'+1,\ell}_{(2i+1,0),(1,1)} \end{array} \right) \\
+\frac{1}{2\sqrt{2}} \left(\begin{array}{c} -\sqrt{3}\, G^{\ell'+1,\ell'+1,\ell}_{(2i,1),(0,0)} + \sqrt{3}\, G^{\ell'+1,\ell'+1,\ell}_{(2i,1),(1,0)} \\ +G^{\ell'+1,\ell'+1,\ell}_{(2i,1),(0,1)} + G^{\ell'+1,\ell'+1,\ell}_{(2i,1),(1,1)} \end{array} \right) \\
+\frac{1}{2\sqrt{2}} \left(\begin{array}{c} -\sqrt{3}\, G^{\ell'+1,\ell'+1,\ell}_{(2i+1,1),(0,0)} + \sqrt{3}\, G^{\ell'+1,\ell'+1,\ell}_{(2i+1,1),(1,0)} \\ +G^{\ell'+1,\ell'+1,\ell}_{(2i+1,1),(0,1)} + G^{\ell'+1,\ell'+1,\ell}_{(2i+1,1),(1,1)} \end{array} \right)
\end{array} \right) \\
&\underset{(4.9)}{=} \frac{1}{8} \left(\begin{array}{l}
-\sqrt{3} \left(-\sqrt{3}\, G^{\ell'+1,\ell'+1,\ell}_{(2i,0),(0,0)} + \sqrt{3}\, G^{\ell'+1,\ell'+1,\ell}_{(2i-1,0),(0,0)} + G^{\ell'+1,\ell'+1,\ell}_{(2i,0),(0,1)} + G^{\ell'+1,\ell'+1,\ell}_{(2i-1,0),(0,1)} \right) \\
+\sqrt{3} \left(-\sqrt{3}\, G^{\ell'+1,\ell'+1,\ell}_{(2i+1,0),(0,0)} + \sqrt{3}\, G^{\ell'+1,\ell'+1,\ell}_{(2i,0),(0,0)} + G^{\ell'+1,\ell'+1,\ell}_{(2i+1,0),(0,1)} + G^{\ell'+1,\ell'+1,\ell}_{(2i,0),(0,1)} \right) \\
-\sqrt{3}\, G^{\ell'+1,\ell'+1,\ell}_{(2i,1),(0,0)} + \sqrt{3}\, G^{\ell'+1,\ell'+1,\ell}_{(2i-1,1),(0,0)} + G^{\ell'+1,\ell'+1,\ell}_{(2i,1),(0,1)} + G^{\ell'+1,\ell'+1,\ell}_{(2i-1,1),(0,1)} \\
-\sqrt{3}\, G^{\ell'+1,\ell'+1,\ell}_{(2i+1,1),(0,0)} + \sqrt{3}\, G^{\ell'+1,\ell'+1,\ell}_{(2i,1),(0,0)} + G^{\ell'+1,\ell'+1,\ell}_{(2i+1,1),(0,1)} + G^{\ell'+1,\ell'+1,\ell}_{(2i,1),(0,1)}
\end{array} \right) \\
&= \frac{1}{8} \left(\begin{array}{l}
-3\,\Gamma^{\ell'+1,\ell}_{2i-1,(0,0)} + 6\,\Gamma^{\ell'+1,\ell}_{2i,(0,0)} - 3\,\Gamma^{\ell'+1,\ell}_{2i+1,(0,0)} - \sqrt{3}\,\Gamma^{\ell'+1,\ell}_{2i-1,(0,1)} + \sqrt{3}\,\Gamma^{\ell'+1,\ell}_{2i+1,(0,1)} \\
+\sqrt{3}\,\Gamma^{\ell'+1,\ell}_{2i-1,(1,0)} - \sqrt{3}\,\Gamma^{\ell'+1,\ell}_{2i+1,(1,0)} + \Gamma^{\ell'+1,\ell}_{2i-1,(1,1)} + 2\,\Gamma^{\ell'+1,\ell}_{2i,(1,1)} + \Gamma^{\ell'+1,\ell}_{2i+1,(1,1)}
\end{array} \right).
\end{aligned}
$$

■

4.4 Notation for ℓ^2 Sequences

Let, e.g., $f^{\ell}_{i,\alpha}$ be the coefficients of $f_\ell = \sum_{i\in\mathcal{I}^f_\ell} \sum_{\alpha=0}^{1} f^{\ell}_{i,\alpha} \Phi^{\ell}_{i,\alpha}$. We extend these coefficients by $f^{\ell}_{i,\alpha} := 0$ for $i \notin \mathcal{I}^f_\ell$ and obtain an ℓ^2 sequence

$$f_{\ell,\alpha} := \left(f^{\ell}_{i,\alpha}\right)_{i\in\mathbb{Z}}.$$

We use the convention that an upper level index ℓ indicates a coefficient, while the sequence has a lower index ℓ. Here $\alpha \in \{0, 1\}$ is a further parameter. The same pattern of notation is used for many more sequences.

For general sequences $a, b \in \ell^2$ (i.e, $a = (a_i)_{i\in\mathbb{Z}}$, $b = (b_i)_{i\in\mathbb{Z}}$), the *discrete convolution* $c := a * b$ is defined by

$$c_i = \sum\nolimits_{j\in\mathbb{Z}} a_j b_{i-j}.$$

For its computation using FFT compare [3, §6].

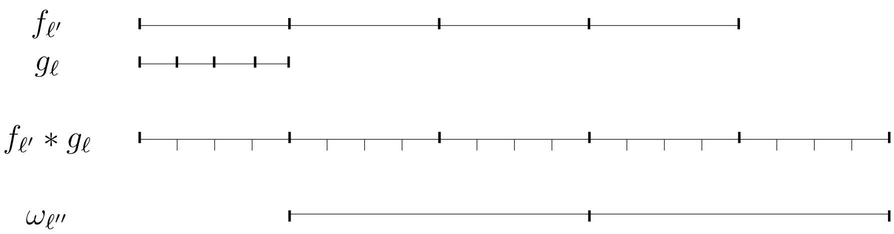
5 Algorithm

In Problem 3.1 three level numbers ℓ'', ℓ', ℓ appear. Without loss of generality $\ell' \leq \ell$ holds. Below we have to distinguish the following three cases:

$$\begin{array}{llllll} (A) & \ell'' & \leq & \ell' & \leq & \ell, \\ (B) & \ell' & < & \ell'' & \leq & \ell, \\ (C) & \ell' & \leq & \ell & < & \ell''. \end{array} \tag{5.1}$$

5.1 Case A: $\ell'' \leq \ell' \leq \ell$

Case A is illustrated by the following figure.



In this figure, the difference $\ell - \ell' = 2$ corresponds to the fact that g_ℓ is given on a grid of step size $h_\ell = h_{\ell'}/4$. The given intervals should show the support of the functions $f_{\ell'}$ and g_ℓ. The convolution $f_{\ell'} * g_\ell$ is a piecewise linear function, where the pieces correspond to the smaller step size h_ℓ. The projection $P_{\ell''}$ of $f_{\ell'} * g_\ell$ is required in two intervals. Because of $\ell'' \leq \ell' \leq \ell$ the step size $h_{\ell''}$ is equal or larger than the other ones. In the figure, $\ell'' = \ell' - 1$ is chosen. Note that in each interval of level ℓ'' the function $\omega_{\ell''}$ is a certain average of 8 pieces of $f_{\ell'} * g_\ell$.

The following algorithm has to compute the projection of $\omega_{\ell''} = P_{\ell''}\omega_{\text{exact}}$ of $\omega_{\text{exact}} := f_{\ell'} * g_\ell$. A straightforward but naive approach would be to compute ω_{exact} first and then its projection. The problem is that in the case $\ell' \ll \ell$, the product $f_{\ell'} * g_\ell$ requires too many data (corresponding to the fine grid in the third line of the figure above). The projection $P_{\ell''}$ would map the many data into few ones. The essence of the following algorithm is to incorporate the projection before a discrete convolution is performed.

5.1.1 Computation of Γ-Coefficients (Step 1)

We start with the sequences $\Gamma_{\ell,\ell,(\alpha,\beta)} = (\Gamma^{\ell,\ell}_{i,(\alpha,\beta)})_{i\in\mathbb{Z}}$. Following Lemma 4.5a, we obtain

$$\Gamma^{\ell,\ell}_{i,(0,0)} = \frac{\sqrt{h_\ell}}{2}\left(g^\ell_{i,0} + g^\ell_{i-1,0}\right) + \frac{\sqrt{h_\ell}}{12}\left(g^\ell_{i-1,1} - g^\ell_{i,1}\right), \tag{5.2a}$$

$$\Gamma^{\ell,\ell}_{i,(1,0)} = \frac{\sqrt{h_\ell}}{12}\left(g^\ell_{i,0} - g^\ell_{i-1,0}\right), \tag{5.2b}$$

$$\Gamma^{\ell,\ell}_{i,(0,1)} = \frac{\sqrt{h_\ell}}{12}\left(g^\ell_{i-1,0} - g^\ell_{i,0}\right), \tag{5.2c}$$

$$\Gamma^{\ell,\ell}_{i,(1,1)} = \frac{\sqrt{3h_\ell}}{5}\left(g^\ell_{i-1,1} - g^\ell_{i,1}\right) \tag{5.2d}$$

for all $i \in \mathbb{Z}$. Then we compute the sequences $\Gamma_{\ell',\ell,(\alpha,\beta)} := (\Gamma^{\ell',\ell}_{i,(\alpha,\beta)})_{i\in\mathbb{Z}}$ for $\ell' = \ell-1, \ell-2, \ldots, 0$ using the recursions from Lemma 4.5b.

5.1.2 Step 2a

Let ℓ' be any level in $[0,\ell]$. For each $\ell'' = \ell', \ell'-1, \ldots, 0$ the projection $P_{\ell''}(f_{\ell'} * g_\ell)$ is to be computed (see Problem 3.1). Following (4.2) and (4.3), the coefficients $\omega_{i,\alpha}^{\ell''} = \sum_{j,k,\beta,\varkappa} f_{j,\beta}^{\ell'} g_{k,\varkappa}^{\ell} \gamma_{(i,\alpha),(j,\beta),(k,\varkappa)}^{\ell'',\ell',\ell}$ are needed. The sequence is denoted by $\omega_{\ell'',\alpha} = (\omega_{i,\alpha}^{\ell''})_{i\in\mathbb{Z}}$.

For the starting value $\ell'' = \ell'$ we have

$$\omega_{i,\alpha}^{\ell'} = \sum_{j,k\in\mathbb{Z}} \sum_{\beta,\varkappa=0}^{1} f_{j,\beta}^{\ell'} g_{k,\varkappa}^{\ell} \gamma_{(i,\alpha),(j,\beta),(k,\varkappa)}^{\ell',\ell',\ell} \underset{(4.6)}{=} \sum_{j,k\in\mathbb{Z}} \sum_{\beta,\varkappa=0}^{1} f_{j,\beta}^{\ell'} g_{k,\varkappa}^{\ell} \gamma_{k-(i-j)2^{\ell-\ell'},(\alpha,\beta,\varkappa)}^{\ell',\ell',\ell}$$

$$\underset{(4.11)}{=} \sum_{\beta=0}^{1} \sum_{j\in\mathbb{Z}} f_{j,\beta}^{\ell'} \Gamma_{i-j,(\alpha,\beta)}^{\ell',\ell} \qquad \text{for all } i \in \mathbb{Z}.$$

The four sums $\sum_{j\in\mathbb{Z}} f_{j,\beta}^{\ell'} \Gamma_{i-j,(\alpha,\beta)}^{\ell',\ell}$ for all combinations of $\alpha,\beta \in \{0,1\}$ describe the discrete convolution of the two sequences $f_{\ell',\beta} := (f_{j,\beta}^{\ell'})_{j\in\mathbb{Z}}$ and $\Gamma_{\ell',\ell,(\alpha,\beta)} := (\Gamma_{k,(\alpha,\beta)}^{\ell',\ell})_{k\in\mathbb{Z}}$. Concerning the performance of the discrete convolutions in

$$\omega_{\ell',\alpha} = \sum_{\beta=0}^{1} f_{\ell',\beta} * \Gamma_{\ell',\ell,(\alpha,\beta)} \qquad (\alpha = 0,1;\ 0 \le \ell' \le \ell) \tag{5.3}$$

we refer to [3, §6].

5.1.3 Step 2b

Given $\omega_{\ell',\alpha}$ from (5.3), we compute $\omega_{\ell'',\alpha}$ for $\ell'' = \ell'-1, \ldots, 0$ by the following recursions.

Lemma 5.1 *The recursions*

$$\omega_{i,0}^{\ell''} = \frac{1}{\sqrt{2}} \left(\omega_{2i,0}^{\ell''+1} + \omega_{2i+1,0}^{\ell''+1} \right), \tag{5.4a}$$

$$\omega_{i,1}^{\ell''} = \frac{\sqrt{3}}{2\sqrt{2}} \left(\omega_{2i+1,0}^{\ell''+1} - \omega_{2i,0}^{\ell''+1} \right) + \frac{1}{2\sqrt{2}} \left(\omega_{2i,1}^{\ell''+1} + \omega_{2i+1,1}^{\ell''+1} \right) \tag{5.4b}$$

holds for all $i \in \mathbb{Z}$ and for all $0 \le \ell'' \le \ell'$.

Proof. Use $\omega_{i,0}^{\ell''} = \sum_{j,k,\beta,\varkappa} f_{j,\beta}^{\ell'} g_{k,\varkappa}^{\ell} \gamma_{(i,0),(j,\beta),(k,\varkappa)}^{\ell'',\ell',\ell}$ and apply (4.4a):

$$\omega_{i,0}^{\ell''} = \frac{1}{\sqrt{2}} \sum_{j,k,\beta,\varkappa} f_{j,\beta}^{\ell'} g_{k,\varkappa}^{\ell} \left(\gamma_{(2i,0),(j,\beta),(k,\varkappa)}^{\ell''+1,\ell',\ell} + \gamma_{(2i+1,0),(j,\beta),(k,\varkappa)}^{\ell''+1,\ell',\ell} \right) = \frac{1}{\sqrt{2}} \left(\omega_{2i,0}^{\ell''+1} + \omega_{2i+1,0}^{\ell''+1} \right).$$

Similarly,

$$\omega_{i,1}^{\ell''} = \sum_{j,k,\beta,\varkappa} f_{j,\beta}^{\ell'} g_{k,\varkappa}^{\ell} \gamma_{(i,1),(j,\beta),(k,\varkappa)}^{\ell'',\ell',\ell}$$

$$\underset{(4.4b)}{=} \frac{1}{2\sqrt{2}} \sum_{j,k,\beta,\varkappa} f_{j,\beta}^{\ell'} g_{k,\varkappa}^{\ell} \begin{pmatrix} -\sqrt{3}\,\gamma_{(2i,0),(j,\beta),(k,\varkappa)}^{\ell''+1,\ell',\ell} + \sqrt{3}\,\gamma_{(2i+1,0),(j,\beta),(k,\varkappa)}^{\ell''+1,\ell',\ell} \\ +\gamma_{(2i,1),(j,\beta),(k,\varkappa)}^{\ell''+1,\ell',\ell} + \gamma_{(2i+1,1),(j,\beta),(k,\varkappa)}^{\ell''+1,\ell',\ell} \end{pmatrix}$$

yields the result for $\omega_{i,1}^{\ell''}$. ■

5.1.4 Intertwining the Computations for all $\ell'' \le \ell' \le \ell$

The superindex ℓ in $\Gamma^{\ell',\ell}_{i,(\alpha,\beta)}$ indicates that this sequence at level ℓ' is originating from the data $g_{\ell,\varkappa}$. Since the further treatment of $\Gamma^{\ell',\ell}_{i,(\alpha,\beta)}$ does not depend on ℓ, we can gather all $\Gamma^{\ell',\ell}_{i,(\alpha,\beta)}$ into

$$\Gamma^{\ell'}_{i,(\alpha,\beta)} := \sum_{\ell=\ell'}^{L^g} \Gamma^{\ell',\ell}_{i,(\alpha,\beta)} \qquad (0 \le \ell' \le L^g)\,. \tag{5.5}$$

Hence, their computation is performed by the loop[2]

	explanations:	
for $\ell' := L^g$ **downto** 0 **do**		
begin if $\ell' = L^g$ **then** $\Gamma^{L^g}_{i,(\alpha,\beta)} := 0$	starting values,	
else compute $\Gamma^{\ell'}_{i,(\alpha,\beta)}$ from $\Gamma^{\ell'+1}_{i,(\alpha,\beta)}$ using (4.12);	see Lemma 4.5b	(5.6)
$\Gamma^{\ell'}_{i,(\alpha,\beta)} := \Gamma^{\ell'}_{i,(\alpha,\beta)} + \Gamma^{\ell',\ell'}_{i,(\alpha,\beta)}$	$\Gamma^{\ell',\ell'}_{i,(\alpha,\beta)}$ defined in (5.2a-d),	
end;	all $\Gamma^{\ell'}_{i,(\alpha,\beta)}$ defined.	

Having available $\Gamma^{\ell'}_{i,(\alpha,\beta)}$ for all $0 \le \ell' \le L^g$, we can compute $\omega_{\ell',\alpha}$ for any ℓ' (cf. (5.3)). For a moment, we use the symbols $\omega_{\ell',\ell',\alpha}$, $\omega_{\ell'-1,\ell',\alpha}, \ldots, \omega_{\ell'',\ell',\alpha}$ for the quantities computed in Step 2a,b. Here, the additional second index ℓ' expresses the fact that the data stem from $f_{\ell'}$ at level ℓ' (see (5.3)).

The coarsening $\omega_{\ell',\ell',\alpha} \mapsto \omega_{\ell'-1,\ell',\alpha} \mapsto \ldots \mapsto \omega_{\ell'',\ell',\alpha}$ can again be done jointly for the different ℓ', i.e., we form[3]

$$\omega_{\ell'',\alpha} := \sum_{\ell'=\ell''}^{L^g} \omega_{\ell'',\ell',\alpha} \qquad \left(0 \le \ell'' \le \min\{L^\omega, L^f, L^g\}\right).$$

The algorithmic form is

	explanations:	
for $\ell'' := \min\{L^f, L^g\}$ **downto** 0 **do**		
begin if $\ell'' = \min\{L^f, L^g\}$ **then** $\omega_{\ell'',\alpha} := 0$	starting values,	
else compute $\omega^{\ell''}_{i,\alpha}$ from $\omega^{\ell''+1}_{i,\alpha}$ via (5.4a,b);	see Lemma 5.1,	(5.7)
$\omega_{\ell'',\alpha} := \omega_{\ell'',\alpha} + \omega_{\ell'',\ell'',\alpha}$	$\omega_{\ell'',\ell'',\alpha}$ defined in (5.3),	
end;	all $\omega_0, \ldots, \omega_{\min\{L^f,L^g\}}$ defined.	

Note that this algorithm yields

$$\omega_{\ell''} = P_{\ell''}\left(\sum_{\ell=\ell''}^{L^g} \sum_{\ell'=\ell''}^{\ell} f_{\ell'} * g_\ell\right) \qquad \left(0 \le \ell'' \le \min\{L^\omega, L^f, L^g\}\right)$$

involving all combinations of indices with $\ell'' \le \ell' \le \ell$.

[2] In this notation, $\Gamma^{\ell'}_{i,(\alpha,\beta)}$ stands for the whole sequence $\Gamma_{\ell',(\alpha,\beta)} = (\Gamma^{\ell'}_{i,(\alpha,\beta)})_{i\in\mathbb{Z}}$ and all combinations of the indices $\alpha, \beta \in \{0,1\}$, i.e., the loop over i, α, β is not explicitly written. More precisely, the i-loop is to be performed for all i belonging to the support of $\Gamma_{\ell',(\alpha,\beta)}$.

[3] The restriction $\ell'' \le \min\{L^\omega, L^f, L^g\}$ follows from $\ell'' \le \ell' \le \ell$, since $\ell' \le L^f$, $\ell \le L^g$, and $\ell'' \le L^\omega$.

5.2 Case B: $\ell' < \ell'' \leq \ell$

In Case B the step size $h_{\ell''}$ used by the projection $P_{\ell''}$ is smaller than the step size $h_{\ell'}$ but larger than h_ℓ.

5.2.1 Explanations for $\ell'' = \ell' + 1$

We will use a loop of ℓ'' from $\ell' + 1$ to ℓ. Here we discuss the first value $\ell'' = \ell' + 1$ and assume $\ell' + 1 \leq \ell$.

We recall that $\mathcal{S}_{\ell'} \subset \mathcal{S}_{\ell'+1}$ (see end of §2.3). The function $f_{\ell'} = \sum_{j,\beta} f_{j,\beta}^{\ell'} \Phi_{j,\beta}^{\ell'} \in \mathcal{S}_{\ell'}$ can be rewritten as a function of level $\ell' + 1$ by using (2.7a,b):

$$f_{\ell'} = \sum_{\beta=0}^{1} \sum_{j} \hat{f}_{j,\beta}^{\ell'+1} \Phi_{j,\beta}^{\ell'+1} \qquad \text{with} \tag{5.8}$$

$$\hat{f}_{2j,0}^{\ell'+1} := \tfrac{1}{\sqrt{2}} f_{j,0}^{\ell'} - \tfrac{\sqrt{3}}{2\sqrt{2}} f_{j,1}^{\ell'}, \quad \hat{f}_{2j+1,0}^{\ell'+1} := \tfrac{1}{\sqrt{2}} f_{j,0}^{\ell'} + \tfrac{\sqrt{3}}{2\sqrt{2}} f_{j,1}^{\ell'},$$
$$\hat{f}_{2j,1}^{\ell'+1} := \tfrac{1}{2\sqrt{2}} f_{j,1}^{\ell'}, \qquad \hat{f}_{2j+1,1}^{\ell'+1} := \tfrac{1}{2\sqrt{2}} f_{j,1}^{\ell'}.$$

Let $\hat{f}_{\ell'+1,\beta} := (\hat{f}_{j,\beta}^{\ell'+1})_{j\in\mathbb{Z}}$ be the sequences of the newly defined coefficients. Since $\ell'' = \ell' + 1 \leq \ell$, the three level numbers $\ell'', \ell' + 1, \ell$ satisfy the inequalities of Case A. As in Step 2a of Case A (see §5.1.2) the desired coefficients of the projection at level $\ell'' = \ell' + 1$ are $\omega_{i,\alpha}^{\ell'+1} = \sum_{j,\beta} \hat{f}_{j,\beta}^{\ell'+1} \Gamma_{i-j,(\alpha,\beta)}^{\ell'+1,\ell}$, i.e., discrete convolutions $\hat{f}_{\ell'+1,\beta} * \Gamma_{\ell'+1,\ell,\beta}$ are to be performed.

In the given formulation, the reinterpretation of $f_{\ell'}$ as a function of level $\ell' + 1$ seems dangerous, since by (5.8) the number of coefficients is doubled. If we repeat this procedure up to level L, the number of coefficients would be multiplied by $2^{L-\ell'}$. A remedy is a restriction of (5.8) to those coefficients $\hat{f}_j^{\ell'+1}$ which are really needed. The coefficients $\omega_{i,\alpha}^{\ell'+1}$ are required only for $i \in \mathcal{I}_\ell^\omega$ (cf. (3.1)), say for $i \in \{i_1^\omega, \ldots, i_2^\omega\}$. Let the nonzero coefficients $\Gamma_{j,(\alpha,\beta)}^{\ell'+1,\ell}$ lie in $i_1^\Gamma \leq j \leq i_2^\Gamma$. The sum in $\omega_{i,\alpha}^{\ell'+1} = \sum_{j,\beta} \hat{f}_{j,\beta}^{\ell'+1} \Gamma_{i-j,(\alpha,\beta)}^{\ell'+1,\ell}$ for $i \in \{i_1^\omega, \ldots, i_2^\omega\}$ involves only $\hat{f}_{j,\beta}^{\ell'+1}$-coefficients with $i_1^\omega - i_2^\Gamma \leq j \leq i_2^\omega - i_1^\Gamma$. Hence, the number of $\hat{f}_{j,\beta}^{\ell'+1}$-coefficients is bounded by $i_2^\omega - i_1^\omega + i_2^\Gamma - i_1^\Gamma + 1$. A similar number appears for later levels.

Since in the further recursions also $\omega_{i,\alpha}^{\ell''}$ for $\ell'' = \ell' + 2, \ldots, \min\{L^\omega, L^g\}$ are to be determined, the interval $[i_1^\omega, i_2^\omega]$ from above is to be increased a bit (using the notation of [3, §6.1], we have to replace $S_c(\omega_{\ell'+1})$ by $S_{cc}(\omega_{\ell'+1})$).

5.2.2 Complete Recursion

Step 1 in Case A has already produced the coefficients $\Gamma_{j,(\alpha,\beta)}^{\ell'}$ gathering all $\Gamma_{j,(\alpha,\beta)}^{\ell',\ell}$ ($\ell \geq \ell'$, cf. (5.5)). For $\ell'' = \ell' + 1, \ell' + 2, \ldots, \ell$ we represent the function $f_{\ell'}$ at these levels ℓ'' by computing the coefficients $\hat{f}_{j,\beta}^{\ell''}$ as in (5.8):

$$\hat{f}_{j,\beta}^{\ell'} := f_{j,\beta}^{\ell'} \qquad \text{(starting value)}, \tag{5.9a}$$

$$\text{compute } \hat{f}_{j,\beta}^{\ell''} \text{ from } \hat{f}_{j,\beta}^{\ell''-1} \text{ via (5.8)} \qquad (\ell' + 1 \leq \ell'' \leq \ell)\,. \tag{5.9b}$$

Note, however, that only those coefficients are to be determined which are really needed in the next step, which are four discrete convolutions

$$\omega_{\ell'',\alpha} = \sum_{\beta=0}^{1} \hat{f}_{\ell'',\beta} * \Gamma_{\ell'',(\alpha,\beta)} \qquad (\ell'+1 \le \ell'' \le \ell) \tag{5.9c}$$

of the sequences $\hat{f}_{\ell'',\beta} := (\hat{f}^{\ell''}_{j,\beta})_{j\in\mathbb{Z}}$ with $\Gamma_{\ell'',(\alpha,\beta)}$.

5.2.3 Combined Computations for all $\ell' < \ell'' \le \ell$

The resulting algorithm is

Algorithm	*explanations:*
$\hat{f}^0_j := 0;$	
for $\ell'' := 1$ **to** $\min\{L^\omega, L^g\}$ **do**	
begin $\hat{f}^{\ell''-1}_{j,\beta} := \hat{f}^{\ell''-1}_{j,\beta} + f^{\ell''-1}_{j,\beta};$	starting value (5.9a),
compute $\hat{f}^{\ell''}_{j,\beta}$ from $\hat{f}^{\ell''-1}_{j,\beta}$ via (5.8);	see (5.9b),
$\omega_{\ell'',\alpha} := \sum_{\beta=0}^{1} \hat{f}_{\ell'',\beta} * \Gamma_{\ell'',(\alpha,\beta)}$	see (5.9c)
end;	

(5.10)

The limitation by $\ell'' \le \min\{L^\omega, L^g\}$ in line 2 is caused by the fact that for $\ell'' > L^\omega$ no $\omega_{\ell''}$ are required, while for $\ell'' > L^g$ the sequence $\Gamma_{\ell''}$ is not defined (i.e., formally $\Gamma_{\ell''} = 0$).

The sum $\hat{f}^{\ell''-1}_{j,\beta} + f^{\ell''-1}_{j,\beta}$ in the third line defines $\hat{f}^{\ell''-1}_{j,\beta}$ as coefficients of $\sum_{\ell'=0}^{\ell''-1} f_{\ell'} = \sum_{j,\beta} \hat{f}^{\ell''-1}_{j,\beta} \Phi^{\ell''-1}_{j,\beta}$. Therefore the next two lines consider all combinations of $\ell' < \ell''$. Since $\Gamma_{\ell''}$ contains all contributions from $\ell \ge \ell''$, $\omega_{\ell''}$ is the projection $P_{\ell''}(\sum_{\ell',\ell \text{ with } \ell'<\ell''\le\ell} f_{\ell'} * g_\ell)$.

5.3 Case C: $\ell' \le \ell < \ell''$

Now the step size $h_{\ell''}$ used in the projection is smaller than $h_{\ell'}$ and h_ℓ.

5.3.1 Explanations

The exact convolution $\omega_{\text{exact}}(x) := \int f_{\ell'}(y) g_\ell(x-y)\mathrm{d}y$ $(x \in \mathbb{R})$ is a piecewise cubic and globally continuous function with possible jumps of the derivative at the grid points νh_ℓ $(\nu \in \mathbb{Z})$ of the grid at level ℓ. The projection $P_{\ell''}\omega_{\text{exact}} = \sum_{i,\alpha} \omega^{\ell''}_{i,\alpha} \Phi^{\ell''}_{i,\alpha}$ involves all scalar products

$$\omega^{\ell''}_{i,\alpha} = \int \Phi^{\ell''}_{i,\alpha}(x)\omega_{\text{exact}}(x)\mathrm{d}x.$$

Note that the whole support of $\Phi^{\ell''}_{i,\alpha}$ belongs to one of the intervals $[\nu h_\ell, (\nu+1)h_\ell]$, where $\omega_{\text{exact}}(x)$ is a cubic function.

We define the point values and one-sided derivatives

$$\delta^\ell_\nu := \omega_{\text{exact}}(\nu h_\ell), \quad \delta^\ell_{\nu,+} := \omega'_{\text{exact}}(\nu h_\ell + 0), \quad \delta^\ell_{\nu,-} := \omega'_{\text{exact}}(\nu h_\ell - 0).$$

Then ω_{exact} can be represented in the interval $I_i^\ell \in \mathcal{M}$ by the cubic polynomial

$$\begin{aligned} &\delta_i^\ell + (x - ih_\ell)\,\frac{\delta_{i+1}^\ell - \delta_i^\ell}{h_\ell} \\ &- \frac{(x-ih_\ell)\,(x-(i+1)\,h_\ell)}{h_\ell}\left(\delta_{i,+}^\ell - (x-ih_\ell)\,\frac{\delta_{i,+}^\ell + \delta_{i+1,-}^\ell}{h_\ell} + \frac{2x-(2i+1)\,h_\ell}{h_\ell}\frac{\delta_{i+1}^\ell - \delta_i^\ell}{h_\ell}\right) \end{aligned} \tag{5.11}$$

Its values at the midpoint $x_{i+1/2}^\ell := (i+1/2)\,h_\ell$ are

$$\begin{aligned} \omega_{\text{exact}}(x_{i+1/2}^\ell) &= \tfrac{1}{2}\left(\delta_i^\ell + \delta_{i+1}^\ell\right) + \tfrac{h_\ell}{8}\left(\delta_{i,+}^\ell - \delta_{i+1,-}^\ell\right), \\ \omega_{\text{exact}}'(x_{i+1/2}^\ell \pm 0) &= \tfrac{3}{2h_\ell}\left(\delta_{i+1}^\ell - \delta_i^\ell\right) - \tfrac{1}{4}\left(\delta_{i,+}^\ell + \delta_{i+1,-}^\ell\right). \end{aligned} \tag{5.12}$$

5.3.2 Pointwise Evaluations

Let ω_{exact} be described in I_ν^ℓ by the data δ_ν^ℓ, $\delta_{\nu+1}^\ell$, $\delta_{\nu,+}^\ell$, $\delta_{\nu+1,-}^\ell$ (cf. (5.11)). Then

$$\omega_{i,\alpha}^\ell = \int \Phi_{i,\alpha}^\ell(x)\omega_{\text{exact}}(x)\mathrm{d}x = \begin{cases} \frac{\sqrt{h_\ell}}{2}\left(\delta_\nu^\ell + \delta_{\nu+1}^\ell\right) + \frac{h_\ell^{3/2}}{12}\left(\delta_{\nu,+}^\ell - \delta_{\nu+1,-}^\ell\right) & \text{for } \alpha = 0, \\ \frac{\sqrt{3h_\ell}}{5}\left(\delta_{\nu+1}^\ell - \delta_\nu^\ell\right) - \frac{\sqrt{3}h_\ell^{3/2}}{60}\left(\delta_{i,+}^\ell + \delta_{i+1,-}^\ell\right) & \text{for } \alpha = 1, \end{cases} \tag{5.13}$$

yields the coefficients of the projection. It remains to determine the values δ_ν^ℓ, $\delta_{\nu,\pm}^\ell$.

5.3.3 Computation of $\delta, \delta_\pm$

We define new γ-coefficients

$${}^0\gamma_{i,(j,\beta),(k,\varkappa)}^\ell := \int \Phi_{j,\beta}^\ell(y)\Phi_{k,\varkappa}^\ell(ih_\ell - y)\mathrm{d}y \qquad (i,j,k \in \mathbb{Z},\ \beta,\varkappa \in \{0,1\})$$

involving only one level ℓ. Simple substitutions yield

$${}^0\gamma_{i,(j,\beta),(k,\varkappa)}^\ell = {}^0\gamma_{k-i+j,(\beta,\varkappa)}^\ell$$

for the "simplified" γ-coefficient ${}^0\gamma_{\nu,(\beta,\varkappa)}^\ell := {}^0\gamma_{\nu,(0,\beta),(0,\varkappa)}^\ell$.

The δ-values of $f_\ell * g_\ell$ are

$$\begin{aligned} \delta_i^\ell = (f_\ell * g_\ell)\,(ih_\ell) &= \sum_{j,k\in\mathbb{Z}}\sum_{\beta,\varkappa=0}^{1} f_{j,\beta}^\ell\, g_{k,\varkappa}^\ell \int \Phi_{j,\beta}^\ell(y)\Phi_{k,\varkappa}^\ell(ih_\ell - y)\mathrm{d}y \\ &= \sum_{j,k\in\mathbb{Z}}\sum_{\beta,\varkappa=0}^{1} f_{j,\beta}^\ell\, g_{k,\varkappa}^\ell\, {}^0\gamma_{i,(j,\beta),(k,\varkappa)}^\ell = \sum_{j,k\in\mathbb{Z}}\sum_{\beta,\varkappa=0}^{1} f_{j,\beta}^\ell\, g_{k,\varkappa}^\ell\ {}^0\gamma_{k-i+j,(\beta,\varkappa)}^\ell \\ &= \sum_{\beta=0}^{1}\sum_{j\in\mathbb{Z}} f_{j,\beta}^\ell\ {}^0\Gamma_{i-j,\beta}^\ell\,, \qquad \text{where } {}^0\Gamma_{i,\beta}^\ell := \sum_{\varkappa=0}^{1}\sum_{k\in\mathbb{Z}} g_{k,\varkappa}^\ell\ {}^0\gamma_{k-i,(\beta,\varkappa)}^\ell\,. \end{aligned} \tag{5.14a}$$

Analogously, we set

$${}^\pm\gamma_{i,(j,\beta),(k,\varkappa)}^\ell = \lim_{\varepsilon\searrow 0}\frac{\mathrm{d}}{\mathrm{d}x}\int \Phi_{j,\beta}^\ell(y)\Phi_{k,\varkappa}^\ell(ih_\ell \pm \varepsilon - y)\mathrm{d}y$$

and obtain

$$\delta^\ell_{i,\pm} = \sum_{\beta=0}^{1}\sum_{j\in\mathbb{Z}} f^\ell_{j,\beta}\, {}^{\pm}\Gamma^\ell_{i-j,\beta}, \qquad \text{where } {}^{\pm}\Gamma^\ell_{i-j,\beta} := \sum_{\varkappa=0}^{1}\sum_{k\in\mathbb{Z}} g^\ell_{k,\varkappa}\, {}^{\pm}\gamma^\ell_{k-i+j,(\beta,\varkappa)}. \tag{5.14b}$$

Remark 5.2 *Coefficients not indicated below are zero:*

$$\begin{aligned}
{}^{0}\gamma^\ell_{-1,(0,0)} &= 1,\\
{}^{0}\gamma^\ell_{k,(1,0)} &= {}^{0}\gamma^\ell_{k,(0,1)} = 0 \quad \textit{for all } k\in\mathbb{Z},\\
{}^{0}\gamma^\ell_{-1,(1,1)} &= -1,
\end{aligned}$$

$$\begin{aligned}
{}^{+}\gamma^\ell_{0,(0,0)} &= 1/h_\ell, & {}^{+}\gamma^\ell_{-1,(0,0)} &= -1/h_\ell,\\
{}^{+}\gamma^\ell_{0,(1,0)} &= {}^{+}\gamma^\ell_{0,(0,1)} = -\sqrt{3}/h_\ell, & {}^{+}\gamma^\ell_{-1,(1,0)} &= {}^{+}\gamma^\ell_{-1,(0,1)} = \sqrt{3}/h_\ell,\\
{}^{+}\gamma^\ell_{0,(1,1)} &= {}^{+}\gamma^\ell_{-1,(1,1)} = 3/h_\ell,
\end{aligned}$$

$$\begin{aligned}
{}^{-}\gamma^\ell_{-1,(0,0)} &= 1/h_\ell, & {}^{-}\gamma^\ell_{-2,(0,0)} &= -1/h_\ell,\\
{}^{-}\gamma^\ell_{-1,(1,0)} &= {}^{-}\gamma^\ell_{-1,(0,1)} = \sqrt{3}/h_\ell, & {}^{-}\gamma^\ell_{-2,(1,0)} &= {}^{-}\gamma^\ell_{-2,(0,1)} = -\sqrt{3}/h_\ell,\\
{}^{-}\gamma^\ell_{-1,(1,1)} &= {}^{-}\gamma^\ell_{-2,(1,1)} = -3/h_\ell.
\end{aligned}$$

Proof. Use the descriptions of the functions $\Phi^\ell_{0,\beta} * \Phi^\ell_{0,\varkappa}$ in (4.7). ■

Conclusion 5.3 *Remark 5.2 implies*

$$\begin{aligned}
{}^{0}\Gamma^\ell_{i,0} &= g^\ell_{i-1,0}, \quad {}^{0}\Gamma^\ell_{i,1} = -g^\ell_{i-1,0}\,,\\
{}^{+}\Gamma^\ell_{i,0} &= \tfrac{1}{h_\ell}\left(g^\ell_{i,0} - g^\ell_{i-1,0} + \sqrt{3}\left(g^\ell_{i-1,1} - g^\ell_{i,1}\right)\right),\\
{}^{+}\Gamma^\ell_{i,1} &= \tfrac{1}{h_\ell}\left(\sqrt{3}\left(g^\ell_{i-1,0} - g^\ell_{i,0}\right) + 3\left(g^\ell_{i-1,1} + g^\ell_{i,1}\right)\right),\\
{}^{-}\Gamma^\ell_{i,0} &= \tfrac{1}{h_\ell}\left(g^\ell_{i-1,0} - g^\ell_{i-2,0} + \sqrt{3}\left(g^\ell_{i-1,1} - g^\ell_{i-2,1}\right)\right),\\
{}^{-}\Gamma^\ell_{i,1} &= \tfrac{1}{h_\ell}\left(\sqrt{3}\left(g^\ell_{i-1,0} - g^\ell_{i-2,0}\right) - 3\left(g^\ell_{i-1,1} + g^\ell_{i-2,1}\right)\right).
\end{aligned}$$

Since $f_\ell * g_\ell$ is cubic in $I^\ell_i = [ih_\ell, (i+1)h_\ell)$, the recursions derived from (5.12) are

$$\begin{aligned}
\delta^{\ell''}_{2i} &= \delta^{\ell''-1}_{i}, & \delta^{\ell''}_{2i+1} &= \tfrac{1}{2}\left(\delta^{\ell''-1}_{i} + \delta^{\ell''-1}_{i+1}\right) + \tfrac{h_{\ell''-1}}{8}\left(\delta^{\ell''-1}_{i,+} - \delta^{\ell''-1}_{i+1,-}\right),\\
\delta^{\ell''}_{2i,\pm} &= \delta^{\ell''-1}_{i,\pm}, & \delta^{\ell''}_{2i+1,\pm} &= \tfrac{3}{2h_{\ell''-1}}\left(\delta^{\ell''-1}_{i+1} - \delta^{\ell''-1}_{i}\right) - \tfrac{1}{4}\left(\delta^{\ell''-1}_{i,+} + \delta^{\ell''-1}_{i+1,-}\right)
\end{aligned} \tag{5.15}$$

for $\ell'' > \ell$.

5.3.4 Combined Computations for all $\ell' \le \ell < \ell''$

The data $\hat{f}^{\ell}_{i,\alpha}$ have the same meaning as in Case B. Similarly, $\hat{\delta}^{\ell}_{i}, \hat{\delta}^{\ell}_{i,\pm}$ collect all δ-data from $\ell' < \ell$.

$$
\boxed{
\begin{array}{l}
\hat{f}^{0}_{i,\alpha} := 0;\ \hat{\delta}^{0}_{i} := \hat{\delta}^{0}_{i,\pm} := 0;\\
\textbf{for } \ell := 0 \textbf{ to } L^{\omega} \textbf{ do}\\
\textbf{begin if } \ell > 0 \textbf{ then}\\
\quad \textbf{begin} \text{ compute } \hat{f}^{\ell}_{i,\alpha} \text{ from } \hat{f}^{\ell-1}_{i,\alpha} \text{ by (5.8);}\\
\qquad \text{compute } \hat{\delta}^{\ell}_{i}, \hat{\delta}^{\ell}_{i,\pm} \text{ from } \hat{\delta}^{\ell-1}_{i}, \hat{\delta}^{\ell-1}_{i,\pm} \text{ by (5.15);}\\
\qquad \text{compute } \omega^{\ell}_{i,\alpha} \text{ from } \hat{\delta}^{\ell}_{i} \text{ by (5.13)}\\
\quad \textbf{end};\\
\quad \textbf{if } \ell \le \min\{L^{f}, L^{\omega} - 1\} \textbf{ then } \hat{f}^{\ell}_{i,\alpha} := \hat{f}^{\ell}_{i,\alpha} + f^{\ell}_{i,\alpha};\\
\quad \textbf{if } \ell \le \min\{L^{g}, L^{\omega} - 1\} \textbf{ then}\\
\quad \textbf{begin} \text{ compute } \delta^{\ell}_{i}, \delta^{\ell}_{i,\pm} \text{ by the convolutions (5.14a,b);}\\
\qquad \hat{\delta}^{\ell}_{i} := \hat{\delta}^{\ell}_{i} + \delta^{\ell}_{i};\ \hat{\delta}^{\ell}_{i,\pm} := \hat{\delta}^{\ell}_{i,\pm} + \delta^{\ell}_{i,\pm}\\
\textbf{end end};
\end{array}
}
\tag{5.16}
$$

The quantities $\hat{f}^{\ell}_{i,\alpha}$ used in the lines 4-6 are the coefficients of $\sum_{\ell'=0}^{\ell-1} f_{\ell'} = \sum_{i,\alpha} \hat{f}^{\ell}_{i,\alpha} \Phi^{\ell}_{i,\alpha}$. The convolutions in (5.15) called at line 5 involve the Γ-sequences defined in Conclusion 5.3. The coefficients $\omega^{\ell}_{i,\alpha}$ in line 6 belong to the projection $P_{\ell''}\left(\sum_{\ell',\lambda \text{ with } 0 \le \ell' \le \lambda \le \ell} f_{\ell'} * g_{\lambda}\right)$ at level ℓ'', where ℓ'' is the actual value ℓ of the loop index.

5.4 Range of Products

In the previous subsections we have reduced the problem to a number of specific discrete convolutions (the first example is (5.3)). The resulting products are infinite sequences $(c_\nu)_{\nu\in\mathbb{Z}}$. The first reasonable reduction would be to determine $(c_\nu)_{\nu=\nu_1}^{\nu_2}$ only in the support $[\nu_1, \nu_2] \cap \mathbb{Z}$ of the sequence. But it is essential to go a step further. Even if we need the function $f_{\ell'} * g_{\ell}$ (see (3.4)) in the whole support $S := \operatorname{supp}(f_{\ell'} * g_{\ell})$, the projections $P_{\ell''}(f_{\ell'} * g_{\ell})$ are required in disjoint subsets $S_{\ell''} \subset S$. In terms of the sequences $(c_\nu)_{\nu\in\mathbb{Z}}$ this means that we are interested in the components c_ν in an index interval $[\nu_1', \nu_2'] \cap \mathbb{Z}$ which is possibly much smaller than the support $[\nu_1, \nu_2]$. The restriction to the minimal range of the discrete convolution is an essential part of the algorithm. The appropriate treatment of the fast discrete convolution is explained in [3].

6 Globally Continuous and Piecewise Linear Case

The space $\mathcal{S} = \mathcal{S}(\mathcal{M})$ consists of *discontinuous* piecewise linear functions. An alternative is the subspace

$$\mathcal{S}^1 := \mathcal{S}(\mathcal{M}) \cap C^0(\mathbb{R})$$

of *globally continuous* and piecewise linear functions. Next we consider the projection $\omega^{\mathcal{S}^1}$ of the convolution $\omega_{\text{exact}} := f * g$ into the space $\mathcal{S}^1$. The direct computation of $\omega^{\mathcal{S}^1} \in \mathcal{S}^1$

cannot follow the same lines as before, since the standard basis functions of $\mathcal{S}^1$ (the usual hat functions) are not orthogonal and any orthonormal basis has a larger support.

Nevertheless, there is a simple indirect way of computing $\omega^{\mathcal{S}^1}$. The inclusions

$$\mathcal{S}^1 \subset \mathcal{S} \subset L^2(\mathbb{R})$$

allow the following statement: Let $P_{\mathcal{S}} : L^2(\mathbb{R}) \to \mathcal{S}$ be the L^2-orthogonal projection onto $\mathcal{S}$ and $P_{\mathcal{S}^1} : \mathcal{S} \to \mathcal{S}^1$ the L^2-orthogonal projection onto $\mathcal{S}^1$. Then the product

$$P := P_{\mathcal{S}^1} \circ P_{\mathcal{S}} : L^2(\mathbb{R}) \to \mathcal{S}^1$$

is the L^2-orthogonal projection onto $\mathcal{S}^1$. This leads to the following algorithm.

Step 1: Let $f, g \in \mathcal{S}^1$. Because of $\mathcal{S}^1 \subset \mathcal{S}$, the data f, g can be used as input of the algorithm described in the previous part. The result is the projection $\omega = P_{\mathcal{S}}(f * g) \in \mathcal{S}$.

Step 2: The projection $\omega \mapsto \omega^{\mathcal{S}^1} = P_{\mathcal{S}^1}\omega \in \mathcal{S}^1$ can be computed by solving a tridiagonal system (the system matrix is the Gram matrix $\left(\int b_i b_j \mathrm{d}x\right)_{i,j=1,\ldots,n}$ generated by the piecewise linear hat functions b_i).

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