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Abstract

We investigate synchronization of linearly coupled map lattices with asymmetric and irreducible coupling matrices. In terms of graph theory, the coupling matrix represents a directed graph. In case that the uncoupled map satisfies Lipschitz conditions, a criterion of global synchronization of the coupled system is derived. With this criterion, we investigate how synchronizability depends on the coupling matrix as well as graph topology. In [16], the author proved that for linearly coupled continuous networks, chaos can be synchronized if and only if the graph contains a spanning tree. In this paper, we show that this conclusion also holds for linearly coupled map lattices.

Index Terms

Synchronization, linearly coupled map lattices, diagonally stable, directed graph, spanning tree.

I. Introduction

Recently, synchronization of linearly coupled map lattices (LCMLs) has attracted increasing attention (see [1]-[13]). LCMLs, which was firstly introduced in [2], is a large class of dynamical systems with discrete space and time as well as continuous state. This class of dynamical systems have been investigated as theoretical models of spatiotemporal phenomena in a variety

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of problems in nonlinear systems and computation studies (for example, see [1], [2]). In general, LCMLs can be written as:

$$x^i(t+1) = f(x^i(t)) + \epsilon \sum_{j=1}^{m} l_{ij} f(x^j(t)),\ i = 1, 2, \cdots, m \tag{1}$$

where $x^i(t) = (x^i_1(t), \cdots, x^i_n(t))^\top \in R^n$ is the state variable of the $i-th$ node, t is the discrete time, $f : R^n \to R^n$ is a continuous map, $L = (l_{ij}) \in R^{m,m}$ is the coupling matrix, which is determined by the topological structure of the network and satisfies $l_{ij} \geq 0$ for all $i \neq j$, and $\sum_{j=1}^{m} l_{ij} = 0$ for all $i = 1, \cdots, m$, and ϵ is the coupling strength.

In this paper, synchronization of the LCMLs is defined as $\lim_{t\to\infty} \|x^i(t) - x^j(t)\| = 0$, for $i = 1, 2, \cdots, m$. In [6], [8], [9], [12], [13], linear stability analysis of the synchronization manifold was proposed and transverse Lyapunov exponents were used to analyze the influence of topological structure on the synchronization of the coupled networks. If the uncoupled map $f(\cdot)$ is Lipschitz continuous, Lyapunov direct method can be used to study global synchronization [3], [6], [11], [15]. Most of these papers focused on the situation when the coupling configuration in the model (1) is symmetric and the coupling matrix is D-symmetrizable. In [3], [4], asymmetric irreducible coupling configuration was considered. In [11], the authors studied the general model (1) for asymmetric and reducible coupling by a distance between the collective states and the synchronization manifold defined in [14].

It is well known that the coupling of LCMLs can be regarded as a graph. We define a directed graph $G = [N, E]$, where $N = \{1, 2, \cdots, m\}$ denotes the node set, $E = \{(i, j) : l_{ij} > 0\}$ denotes the edge set. If L is asymmetric or reducible, then the corresponding G is directed. So, the coupling matrix L can be regarded as the general Laplacian matrix of the graph G. In [16], with this viewpoint, the author indicated that for linearly coupled ordinary differential systems, chaos synchronization can be obtained if and only if the graph contains a spanning tree. It should be pointed out that our definition of graph corresponding the coupling matrix in this paper is right the reverse to that in [16].

In this paper, the coupling matrix L can be asymmetric and reducible. Correspondingly, graph G can be asymmetric and not strongly connected. In the case that the coupled map $f(\cdot)$ is generally Lipschitz continuous, a sufficient condition guaranteeing global synchronization is derived by introducing a weighted average of all state variables of nodes which can be regarded as certain projection on the synchronization manifold. This condition is equivalent to the diagonal stability of certain matrix related to the Laplacian. By some linear algebra, we conclude that for some special coupling matrix L, this condition means that some function of the eigenvalues of the coupling matrix L satisfies an inequality. Furthermore, by the viewpoint of directed graph, we investigate what kind of topological structure can synchronize a map with Lipschitz constant greater than 1. In particular, the uncoupled system is chaotic. Similar as in [16], we prove that the graph can synchronize some chaotic maps if and only if a graph contains a spanning directed tree.

II. PRELIMINARIES

In this section, we introduce some notations, definitions and a lemma, which will be used in the following sections. Throughout the paper, the following notations are needed. e denotes the synchronization direction vector of which all components are 1. I_m denotes the identity matrix of dimension m. $\top$ denotes the transpose of a matrix or vector. $A > 0$ denotes that A is positive definite, i.e., $x^\top Ax > 0$, for all $x \neq 0$. So it is with $A \geq 0$, $A < 0$, and $A \leq 0$. $|A|$ denotes the matrix with each entry being the absolute value of that of A. $\lambda_k(A)$ denotes the k-th largest eigenvalue of the symmetric matrix A. $\|A\|_2$ denotes the 2-norm of matrix A by $\|A\|_2 = \sqrt{\lambda_1(A^\top A)}$. For any matrices $A \in R^{m,m}$ and $B \in R^{n,n}$, $A \otimes B \in R^{mn,mn}$ denotes the Kronecker product. We also denote the class of the diagonal positive definite matrices in $R^{m,m}$ by PD and the inner product induced by some $P \in PD$ by $\langle \cdot, \cdot \rangle_P$: $\langle x, y \rangle_P = x^\top Py$.

Definition 1: A map $f : R^n \to R^n$ is said to be (globally) Lipschitz continuous with Lipschitz constant $\kappa > 0$, if there exists a positive definite matrix T such that

$$[f(x) - f(y)]^\top T[f(x) - f(y)] \leq \kappa^2 (x - y)^\top T(x - y)$$

holds for any $x, y \in R^n$. The class of (globally) Lipschitz continuous maps with Lipschitz constant $\kappa > 0$ is denoted by $Glip(\kappa)$.

It can be seen that if $\kappa < 1$, then each solution of the uncoupled system $s(t + 1) = f(s(t))$ is stable. Therefore, when investigating chaos synchronization, we always assume $\kappa > 1$.

Definition 2: A matrix $A \in R^{m,m}$ is said to be (Schur) diagonally stable if there exists $R \in PD$ such that $A^\top RA - R < 0$.

Definition 3: (see[18]) A matrix $A \in R^{m,m}$ is said to be D-symmetrizable if there exists a nonsingular diagonal matrix T such that $T^{-1}AT$ is symmetric.

It is easy to see that A is D-symmetrizable, if and only if there exists $D \in PD$ such that DA is symmetric.

Definition 4: (see [18]) A matrix $A \in R^{m,m}$ is said to be (Schur) stable if all eigenvalues of A are located in the unit disc.

Definition 5: [19], [16] A graph is said to contain a spanning directed tree if there exists a node called $root$ such that for each other node j there must exist at least one directed path from $root$ to node j.

The following lemma characterizes the directed graph containing a spanning tree with the coupling matrix.

Lemma 1: [16] A graph G contains a spanning directed tree if and only if with proper permutation, its Laplacian matrix L can be reduced to the Frobenius form

$$L = \begin{bmatrix} L_{11} & L_{12} & \cdots & L_{1k} \\ 0 & L_{22} & \cdots & L_{2k} \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & L_{kk} \end{bmatrix}, \tag{2}$$

such that L_{jj}, $i = 1, 2, \cdots, k-1$ are irreducible, each L_{jj} has at least one row with positive row sum, $j = 1, 2, \cdots, k-1$, L_{kk} is irreducible or is zero matrix with dimension 1.

III. Global synchronization and synchronizability

In this section, we give a sufficient condition for global synchronization of the LCMLs (1) under assumption that the map $f(\cdot)$ satisfies Lipschitz condition.

It is clear that the matrix L has a right eigenvector e corresponding to eigenvalue 0. Moreover, by Perron Frobenius theory [19], its left eigenvector $w = [w_1, w_2, \cdots, w_m]^\top$ corresponding to eigenvalue 0 satisfies $w_i \geq 0$, for all $i = 1, 2, \cdots, m$. We normalize it by $\sum_{j=1}^{m} w_i = 1$.

In the sequel, we denote $\bar{x}(t) = \sum_{j=1}^{m} w_i x^i(t)$, $\delta x^i(t) = x^i(t) - \bar{x}(t)$, and $\Xi = [w^\top, \cdots, w^\top]^\top$. Then, we have

$$\begin{aligned} &\delta x^i(t+1) = x^i(t+1) - \bar{x}(t+1) \\ &= f(x^i(t)) + \epsilon \sum_{j=1}^{m} L_{ij} f(x^j(t)) - \sum_{k=1}^{m} w_k \Big[f(x^k(t)) + \epsilon \sum_{l=1}^{m} L_{kl} f(x^l(t)) \Big] \\ &= f(x^i(t)) - f(\bar{x}(t)) - \sum_{k=1}^{m} w_k \Big[f(x^k(t)) - f(\bar{x}(t)) \Big] + \epsilon \sum_{j=1}^{m} L_{ij} \Big[f(x^j(t)) - f(\bar{x}(t)) \Big]. \end{aligned}$$

Synchronization is equivalent to that $\lim_{t\to\infty}\delta x^i(t)=0$ holds for $i=1,2,\cdots,m$.

Let $x(t)=[x^{1^\top}(t),\cdots,x^{m^\top}(t)]^\top$, $F(x(t))=[f^\top(x^1(t)),\cdots,f^\top(x^m(t))]^\top$, $\mathbf{L}=L\otimes I_n,\ \mathbf{\Xi}=\Xi\otimes I_n,\ \mathbf{I}=I_m\otimes I_n$. And $\delta x(t)=[\delta x^{1^\top}(t),\cdots,\delta x^{m^\top}(t)]^\top$,
$\delta F(t)=[f^\top(x^1(t))-f^\top(\bar{x}(t)),\cdots,f^\top(x^m(t))-f^\top(\bar{x}(t))]^\top$. Thus, the LCMLs (1) can be rewritten in matrix form:

$$\delta x(t+1)=\Big[\mathbf{I}-\mathbf{\Xi}+\epsilon\mathbf{L}\Big]\delta F(t).$$

Then, we have

Theorem 1: Suppose $f(\cdot)\in Glip(\kappa)$. If $\kappa(I_m-\Xi+\epsilon L)$ is diagonally stable, then the LCMLs (1) can be globally synchronized.

The proof is given in Appendices. Define the following Rayleigh quotient

$$\gamma(\epsilon)=\max_{P\in PD}\min_{x\neq 0,x\in R^m}\frac{\langle x,x\rangle_P}{\langle y(x),y(x)\rangle_P}$$

where $y(x)=(I_m-\Xi+\epsilon L)x$. If $\gamma(\epsilon)>\kappa$, then by Theorem 1, the LCMLs (1) can be globally synchronized. Thus, we can define $\eta(L)=\sup_{\epsilon>0}\gamma(\epsilon)$ as a quantity to measure synchronizability for the coupled system with the coupling matrix L. If $f\in Glip(\kappa)$ and $\eta(L)>\kappa$, then the coupled system (1) can be globally synchronized for some properly chosen coupling strength ϵ.

Remark 1: In theorem 1, $f(\cdot)$ is required to be globally Lipschitz continuous. However, for those maps which are not globally Lipschitz, if there exists a globally attracting region where $f(\cdot)$ is Lipschitz continuous, the theorem is also viable.

Remark 2: In [11], the authors also presented conditions guaranteeing global synchronization by searching an irreducible symmetric matrix with nonpositive off-diagonal elements and zero row sums (see [16], [11] for details). In this paper, we use a different approach by defining a projection on the diagonal synchronization manifold and proving the difference between the states and this projection converge to zero.

Let $0=\lambda_1,\lambda_2,\cdots,\lambda_m$ be the eigenvalues of L. If they are all real numbers, then order them by $0=\lambda_1\geq\lambda_2\geq\cdots\geq\lambda_m$. Thus, we have the following proposition.

Proposition 1:

1) $\gamma(\epsilon)\leq\frac{1}{\max_{i\geq 2}|1+\epsilon\lambda_i|}$;
2) If L is either symmetric, D-symmetrizable, triangular, or dimension 2, then $\gamma(\epsilon)=\frac{1}{\max_{i\geq 2}|1+\epsilon\lambda_i|}$ and $\eta(L)=\frac{|\lambda_m|+|\lambda_2|}{|\lambda_m|-|\lambda_2|}$; (if $m=2$, $\eta(L)=+\infty$).

See the proof in appendices.

Remark 3: Laplacian matrix L of a undirected graph is defined as follows (see [17]): For any pair (i,j), $A_{ij} > 0$ denote an edge from node j to node i with weight A_{ij}. Otherwise, $A_{ij} = 0$. $K = \text{diag}[k_1, k_2, \cdots, k_m]$ where k_i is the connection degree of the node i, i.e., $k_i = \sum_{j=1}^{m} A_{ij}$. Then, $L = K - A$, $L = I_m - K^{-1/2}AK^{-1/2}$, or $L = I_m - K^{-1}A$. It is clear that each form of the Laplacian matrix is D-symmetrizable. In this case, we have $\gamma(\epsilon) = \frac{1}{\max_{i\geq 2}|1+\epsilon\lambda_i|}$.

IV. Directed graph as a chaos synchronizer

As mentioned in Introduction, a coupling matrix L can be regarded as a Laplacian matrix of a connecting graph G. Thus, we denote $\eta(L)$ by $\eta(G)$. In this section, we will show that with some specific configurations, the graph G can synchronize certain chaotic map $f \in Glip(\kappa)$ with $\kappa > 1$, i.e., $\eta(G) > 1$.

First, we consider that G is strongly connected, i.e., for each node pair (i,j), there must exist a directed path from node i to j. In this case, L is irreducible. Let $w = [w_1, \cdots, w_m]$ be the left eigenvector of L with respect to eigenvalue 0 satisfying $w_i > 0$ for all $i = 1, 2, \cdots, m$ according to Perron Frobenius theory [19].

Proposition 2: Let $W = \text{diag}[w_1, \cdots, w_m]$, $\Sigma = W^{1/2}LW^{-1/2}$, $\beta = \|\Sigma\|_2$, and $\rho = \frac{1}{2}\lambda_2(\Sigma^\top + \Sigma)$. If G is strongly connected, then

$$\eta(G) \geq \sqrt{\frac{1}{1 - \frac{\rho^2}{\beta^2}}} > 1,$$

From proposition 2, it can be seen that $\eta(G) > 1$, i.e., any strongly connected graph can synchronize Lipschitz maps with Lipschitz constants greater than 1.

If G is not strongly connected, then with proper permutation, the Laplacian matrix L is k-reducible and has the Frobenius form (2). Thus, we have the following main result.

Theorem 2: A graph G contains a spanning directed tree if and only if $\eta(G) > 1$.

See the proof in appendices. From the proof, it can be seen that $\eta(G)$ has the following estimation:

$$\eta(G) \geq \sqrt{\frac{1}{\min_{\epsilon>0} \max_{i=1,\cdots,k} \left\{1 + 2\epsilon\rho_i + \epsilon^2\beta_i^2\right\}}},$$

where ρ_j and β_j are defined in the proof of Theorem 2, $j = 1, \cdots, k$, and this estimation is determined by the diagonal block of the Frobenius form of matrix L: L_{jj}, for $j = 1, \cdots, k$.

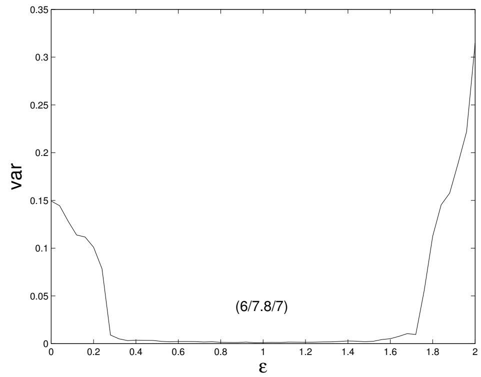

Fig. 1. Illustration of synchronization. The initial data are randomly picked in $[-5, 5]$ and synchronization is measured by the variance: var $= \langle \frac{1}{m-1} \sum_{i=1}^{m} \|x^j(t) - \bar{x}(t)\|_2^2 \rangle$, where $\langle \cdot \rangle$ denotes the time average.

Namely, the synchronization region of the coupling strength ϵ has the following estimation: if

$$\frac{-\rho_i - \sqrt{\rho_i^2 - \beta_i^2(1 - \frac{1}{\kappa^2})}}{\beta_i^2} < \epsilon < \frac{-\rho_i + \sqrt{\rho_i^2 - \beta_i^2(1 - \frac{1}{\kappa^2})}}{\beta_i^2} \quad (3)$$

holds for all $i = 1, \cdots, k$, then the LCMLs (1) can be globally synchronized. We present the following example to illustrate the synchronization of the LCMLs (1) via a directed graph with the following Laplacian: $L = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 1 & 0 & -1 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & -1 \end{bmatrix}$, which corresponds a directed and reducible graph. The coupled map is a two-dimensional neural networks as described in [11]:

$$f(x) = kx + \Delta t \Big\{ \begin{bmatrix} -1 & -0.5 \\ -0.5 & -2 \end{bmatrix} \tanh(125x) + 0.8 \Big\}$$

with $\Delta t = 0.02$, $k = 1 - \Delta t = 0.98$, from which $\kappa < 7$ can be concluded. By the estimation above and simple algebra, we can conclude that if $\frac{6}{7} < \epsilon < \frac{8}{7}$, then the coupled system (1) can be synchronized. Fig. 1 indicates that the interval $(\frac{6}{7}, \frac{8}{7})$ is located in the synchronization region of the coupling strength despite that this region estimation seems inaccurate because Theorem 1 just presents a sufficient condition for synchronization.

Remark 4: In [16], the author proved that a directed graph can be a chaos synchronizer in continuous-time coupled systems if and only if it has a spanning tree. Here, we have proved the same conclusion for the discrete-time coupled networks. However, we obtain the result by a different way from in [16]. Also, he pointed out that the synchronizability of graph in continuous-time coupled system should be related to the algebraic connectivity of this graph. The algebraic connectivity was studied in [22] and in [23], [24] the author generalized the Fieldler's definition [25] to directed graphs by the second largest eigenvalue of some matrix related in the Laplacian of the directed graph, which can be used to measure the synchronizability of a directed graph. From the estimation (3), we also can see that similar to $\tilde{\alpha}(G)$ the definition 1 in [24], the nonzero eigenvalues of some matrices depended on the Laplcian, namely, $\rho_j = \frac{1}{2}\lambda_2(\Sigma_j^{\top} + \Sigma_j)$, $j = 1, \cdots, k$, play the important role for the estimation of the synchronizability of directed graphs in discrete-time coupled systems. Namely, a larger ρ_j and a smaller β_j implies a larger synchronization region estimation of ϵ . Moreover, we can prove $\rho(G) \leq \tilde{\alpha}(G)$ holds for a strongly connected graph.

V. Conclusions

In this paper, we focus on global synchronization. We present a sufficient condition guaranteeing global synchronization of the LCMLs (1). Based on this condition, we study the synchronizability of the coupling configuration. Furthermore, we also discuss the synchronizability with viewpoint of graph. By regarding the coupling matrix as the Laplacian matrix of a directed graph, synchronizability of the coupling matrix is also that of the graph. We prove that the synchronizability $\eta(G) > 1$, which means that certain chaotic maps with Lipschtiz constant greater than 1 can be synchronized globally, if and only if graph G has a spanning directed tree.

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APPENDICES

Proof of theorem 1:Under the assumptions given in Theorem 1, there must exist a positive definite diagonal matrix $P \in R^{m,m}$, a positive definite matrix $T \in R^{n,n}$, and $\gamma_1 > \kappa$ such that

$$[f(x) - f(y)]^\top T[f(x) - f(y)] \leq \kappa^2 (x-y)^\top T(x-y),$$

$$\gamma_1^2(I_m - \Xi + \epsilon L)^\top P(I_m - \Xi + \epsilon L) - P < 0.$$

Define $V(\delta x(t)) = \delta x(t)^\top \Big\{ P \otimes T \Big\} \delta x(t)$. Then, we have

$$\begin{aligned} & V(\delta x(t+1)) = \delta x(t+1)^\top \Big\{ P \otimes T \Big\} \delta x(t+1) \\ = \; & \delta F(t)^\top \Big\{ \Big[\mathbf{I} - \mathbf{\Xi} + \epsilon \mathbf{L} \Big]^\top (P \otimes T) \Big[\mathbf{I} - \mathbf{\Xi} + \epsilon \mathbf{L} \Big] \Big\} \delta F(t) \\ = \; & \delta F(t)^\top \Big\{ \Big[(I_m - \Xi + \epsilon L)^\top P (I_m - \Xi + \epsilon L) \Big] \otimes T \Big\} \delta F(t) \\ \leq \; & \frac{1}{\gamma_1^2} \delta F(t)^\top \Big\{ P \otimes T \Big\} \delta F(t) \leq \Big(\frac{\kappa}{\gamma_1} \Big)^2 V(\delta x(t)). \end{aligned}$$

Therefore, $V(\delta x(t)) \leq V(\delta x(0)) \Big(\frac{\kappa}{\gamma_1} \Big)^{2t}$. The theorem is proved.

Proof of proposition 1: Let u_i be the right eigenvector of L corresponding to distinct eigenvalue λ_i with $u_1 = e$, which implies $w^\top u_i = 0$ holds for all $i \geq 2$. Therefore, for any $\gamma > 0$,

$$\begin{aligned} & \frac{\gamma^2 u_i^*(I_m - \Xi + \epsilon L)^\top P (I_m - \Xi + \epsilon L) u_i}{u_i^* P u_i} \\ = \; & \gamma^2 |1 + \epsilon \lambda_i|^2 \frac{u_i^* P u_i}{u_i^* P u_i} = \gamma^2 |1 + \epsilon \lambda_i|^2 \end{aligned}$$

holds for all $i \geq 2$ and $P \in PD$, where u_i^* denotes the conjugate transport of u_i. This implies that $\gamma(\epsilon) \leq \frac{1}{\max\limits_{i \geq 2} |1 + \epsilon \lambda_i|}$ holds. The first claim of proposition 1 is proved.

For the second claim, we only need to prove $\gamma(\epsilon) \geq \frac{1}{\max\limits_{i \geq 2} |1 + \epsilon \lambda_i|}$

Case 1: L is D-symmetrizable (noting that symmetry is a special kind of D-symmetry). In this case, we have $L = K^{-1} \tilde{L}$, where $K = \text{diag}\{k_1, \cdots, k_m\}$ is a diagonal nonsingular matrix and $\tilde{L}$ is symmetric. Let λ_i, u_i, $i = 1, \cdots, m$ be eigenvalues and corresponding eigenvectors of L and $P = \text{diag}\{k_1^2, \cdots, k_m^2\}$. Therefore, $PL = L^\top P$, $u_i^\top P L u_j = \lambda_j u_i^\top P u_j$, and $u_i^\top P L u_j = u_i^\top L^\top P u_j = \lambda_i u_i^\top P u_j$, which means that $u_i^\top P u_j = 0$, if $\lambda_i \neq \lambda_j$. Also, $u_i^\top P u_1 = 0$ for all $i \geq 2$. If some λ_i has multiplicity greater than one, by proper choice of basis $u_1, \cdots, u_m$ in the eigen-space corresponding eigenvalue λ_i, we can conclude that $u_i P u_j = 0$ for all $i \neq j$.

For each $x \in R^m$, letting $x = \sum_{i=1}^{m} x_i u_i$, we have

$$\begin{aligned}
& x^\top (I - \Xi + \epsilon L)^\top P (I - \Xi + \epsilon L) x \\
= & \sum_{i,j=1}^{m} x_i x_j u_i^\top (I - \Xi + \epsilon L)^\top P (I - \Xi + \epsilon L) u_j \\
= & \sum_{j \geq 2} x_j^2 |1 + \epsilon \lambda_j|^2 u_j^\top P u_j \leq \max_{j \geq 2} |1 + \lambda_j|^2 \sum_{j \geq 2} u_j^\top P u_j \\
= & \max_{j \geq 2} |1 + \lambda_j|^2 x^\top P x.
\end{aligned}$$

Then, $\gamma(\epsilon) \geq \frac{1}{\max_{j \geq 2} |1+\lambda_j|}$.

Case 2: L is triangular. Without loss of generality, suppose that $L = (l_{ij})$ is upper triangular, i.e., $l_{ij} = 0$, if $i > j$. In this case, $l_{nn} = 0$ and $l_{ii} < 0$ for all $i \leq n-1$, and $\Xi = [0, e]$. Then, $Y = I_m - \Xi + \epsilon L$ is also upper triangular and $Y_{nn} = 0$. In fact, if $\gamma_1 |I_m - \Xi + \epsilon L|$ is stable, i.e., $\gamma_1 |1 + \epsilon l_{ii}| \leq 1$, for all $i \geq 2$, then $I_m - \gamma_1 |Y|$ is an M-matrix [20]. So, $\gamma_1 Y$ is diagonally stable (see theorem 2 in [21]). Therefore,

$$\gamma(\epsilon) \geq \gamma_1 \geq \frac{1}{\max_{j \geq 2} |1 + \epsilon L_{ii}|}.$$

Since a two dimensional matrix should be either D-symmetrizable or triangular and the latter item in claim 2 can be concluded by the first item according to Theorem 4 in [11], the proof is completed.

Proof of Proposition 2: Define a new inner-product $\langle x, y \rangle = x^\top W y$. Then, for each x with $\langle x, e \rangle_W = 0$, we have

$$\begin{aligned}
& \frac{\langle (I_m - \Xi + \epsilon L)x, (I_m - \Xi + \epsilon L)x \rangle}{\langle x, x \rangle} \\
= & \frac{\langle (I_m + \epsilon L)x, (I_m + \epsilon L)x \rangle}{\langle x, x \rangle} \\
= & \frac{\langle x, x \rangle + 2\epsilon \langle x, Lx \rangle + \epsilon^2 \langle Lx, Lx \rangle}{\langle x, x \rangle} \\
\leq & 1 + \epsilon \lambda_2 (\Sigma + \Sigma^\top) + \epsilon^2 \|\Sigma\|_2^2 \\
= & 1 + 2\rho\epsilon + \epsilon^2 \beta^2.
\end{aligned}$$

Noting that $\rho < 0$ and $(I_m - \Xi + \epsilon L)e = 0$, we pick $\epsilon = -\frac{\rho}{\beta^2}$ and have

$$\begin{aligned}
\frac{1}{\gamma^2(\epsilon)} \leq & \max_{x \neq 0} \frac{\langle (I_m - \Xi + \epsilon L)x, (I_m - \Xi + \epsilon L)x \rangle}{\langle x, x \rangle} \\
\leq & 1 - \frac{\rho^2}{\beta^2},
\end{aligned}$$

which implies

$$\eta(G) \geq \sqrt{\frac{1}{1-\frac{\rho^2}{\beta^2}}}.$$

Proposition 2 is proved.

Proof of theorem 2: If a graph G does not have a spanning directed tree, then there must exist two disjoint subsets of nodes N_1 and N_2 [16]. Without loss of generality, by proper permutation, matrix L can be written as

$$L = \begin{bmatrix} \tilde{L}_{11} & \tilde{L}_{12} & \tilde{L}_{13} \\ 0 & L_{j_0j_0} & 0 \\ 0 & 0 & \tilde{L}_{22} \end{bmatrix}.$$

Let $w = [0, 0, w_2^\top]^\top$ with $w_2^\top \tilde{L}_{22} = 0$, which implies $w^\top L = 0$. Pick $e_1 = [0, e_2^\top, 0]^\top$ with $L_{j_0j_0}e_2 = 0$ and $e_2 \neq 0$. Thus,

$$(I + \epsilon L)e_1 = \begin{bmatrix} \epsilon\tilde{L}_{12}e_2 \\ e_2 \\ 0 \end{bmatrix}.$$

Then, for any positive diagonal matrix P, we have

$$e_1^\top (I + \epsilon L)^\top P (I + \epsilon L)e_1 \geq e_1^\top P e_1, \tag{4}$$

which leads to $\eta(G) \leq 1$.

On the other hand, suppose that the graph G has a spanning directed tree. By Lemma 1, the corresponding Laplacian matrix L has the Frobenius form (2), where $L_{jj} \in R^{m_j \times m_j}$ and $L_{jj} = -D_j + V_j$, $j = 1, \cdots, k$, D_j is a nonnegative definite diagonal matrix and the irreducible matrix V_j has nonnegative off-diagonal elements with zero row sums.

Let $w_j \in R^{m_j}$ be the left eigenvector of V_j corresponding to 0. $P_j = \omega_j \text{diag}\{w_j\}$, where $\omega_j > 0$ will be defined later, $j = 1, 2, \cdots, k$. $P = \text{diag}\{P_1, P_2, \cdots, P_k\}$. Moreover, denote $\Sigma_j = P_j^{1/2} L_{jj} P_j^{-1/2}$, $\rho_j = \frac{1}{2}\lambda_1(\Sigma_j + \Sigma_j^\top)$, $j = 1, \cdots, k-1$, $\rho_k = \frac{1}{2}\lambda_2(\Sigma_k + \Sigma_k^\top)$, and $\beta_j = \|\Sigma_j\|_2$, $j = 1, 2, \cdots, k$. From lemma 1, it can be seen that all $-(\Sigma_j + \Sigma_j^\top)$, $j = 1, 2, \cdots, k-1$, are M-matrices, which implies that $\lambda_1(\Sigma_j^\top + \Sigma_j) < 0$ holds for $j = 1, 2, \cdots, k-1$. And, since $[\Sigma_k + \Sigma_k^\top]w_k = 0$, $\Sigma_k + \Sigma_k^\top$ is semi-diagonal dominant. Combined with the fact that $\Sigma_k + \Sigma_k^\top$ is irreducible, we have $\lambda_2(\Sigma_k + \Sigma_k^\top) < 0$.

Here, we only prove $\eta(G) > 1$ for $k = 2$. The proof for case $k \geq 3$ can be given by similar argument inductively.

Let $N(x) = x^\top (I_{m_1+m_2} - \Xi + \epsilon L)^\top P(I_{m_1+m_2} - \Xi + \epsilon L)x$ and $x = [\tilde{x}_1, \tilde{x}_2]^\top$, $\tilde{x}_j \in R^{m_j}$, $j = 1, 2$. Pick $\omega_2 = 1$. Noting

$$2\epsilon \tilde{x}_2^\top L_{12}^\top P_1 (I + \epsilon L_{11})\tilde{x}_1 \leq \delta_1 \tilde{x}_1^\top (I + \epsilon L_{11})^\top P_1 (I + \epsilon L_{11})\tilde{x}_1$$
$$+\frac{\epsilon^2}{\delta_1}\tilde{x}_2^\top P_2^{1/2} P_2^{-1/2} L_{12}^\top P_1^{1/2} P_1^{1/2} L_{12} P_2^{-1/2} P_2^{1/2} \tilde{x}_2,$$

we obtain

$$N(x) \leq \omega_1 (1 + \delta_1)(1 + 2\epsilon\rho_1 + \epsilon^2 \beta_1^2)\tilde{x}_1^\top P_1 \tilde{x}_1$$
$$+(1 + 2\epsilon\rho_2 + \epsilon^2\beta_2^2)\tilde{x}_2^\top P_2 \tilde{x}_2 + \omega_1 \epsilon^2 (1 + \delta_1^{-1})\|P_1^{1/2} L_{12} P_2^{-1/2}\|_2^2 \tilde{x}_2^\top P_2 \tilde{x}_2. \quad (5)$$

Since $\rho_i < 0$, we can conclude that there exists $\epsilon > 0$ and $\delta > 0$ such that $1 + \epsilon\rho_i + \epsilon^2\beta_i^2$ holds for all $i = 1, 2$. This implies that there exists $\omega_1 > 0$ and $\delta_1 > 0$ sufficient small such that

$$N(x) \leq (1 - \delta)\Big(\omega_1 \tilde{x}_1^\top P_1 \tilde{x}_1 + \omega_2 \tilde{x}_2^\top P_2 \tilde{x}_2\Big) = (1 - \delta)x^\top P x$$

holds, which implies $\eta(G) \geq \sqrt{\frac{1}{1-\delta}} > 1$. Theorem 2 is proved.