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A Parabolic Free Boundary Problem with Bernoulli Type Condition on the Free Boundary

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A PARABOLIC FREE BOUNDARY PROBLEM WITH BERNOULLI TYPE CONDITION ON THE FREE BOUNDARY

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Abstract. Consider the parabolic free boundary problem

$$\Delta u - \partial_t u = 0 \text{ in } \{u > 0\}\, ,\ |\nabla u| = 1 \text{ on } \partial\{u > 0\}\ .$$

For a realistic class of solutions, containing for example *all* limits of the singular perturbation problem

$$\Delta u_\varepsilon - \partial_t u_\varepsilon = \beta_\varepsilon(u_\varepsilon) \ \text{ as } \varepsilon \to 0,$$

we prove that one-sided flatness of the free boundary implies regularity.
In particular, we show that the topological free boundary $\partial\{u > 0\}$ can be decomposed into an *open* regular set (relative to $\partial\{u > 0\}$) which is locally a surface with Hölder-continuous space normal, and a closed singular set.
Our result extends the main theorem in the paper by H.W. Alt-L.A. Caffarelli (1981) to more general solutions as well as the time-dependent case. Our proof uses methods developed in H.W. Alt-L.A. Caffarelli (1981), however we replace the core of that paper, which relies on non-positive mean curvature at singular points, by an argument based on scaling discrepancies, which promises to be applicable to more general free boundary or free discontinuity problems.

1. Introduction

The parabolic free boundary problem

$$\Delta u - \partial_t u = 0 \text{ in } \{u > 0\}\, ,\ |\nabla u| = 1 \text{ on } \partial\{u > 0\} \tag{1.1}$$

has originally been derived as singular limit from a model for the propagation of equidiffusional premixed flames with high activation energy ([3]); here $u = \lambda(T_c - T)$, T_c is the flame temperature, which is assumed to be constant, T is the temperature outside the flame and λ is a normalization factor.
Let us shortly summarize the mathematical results directly relevant in this context, beginning with the limit problem (1.1): in the brilliant paper [1], H.W. Alt and L.A.

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Caffarelli proved via minimization of the energy $\int(|\nabla u|^2 + \chi_{\{u>0\}})$ – here $\chi_{\{u>0\}}$ denotes the characteristic function of the set $\{u > 0\}$ – existence of a stationary solution of (1.1) in the sense of distributions. They also derived regularity of the free boundary $\partial\{u > 0\}$ up to a set of vanishing $n-1$-dimensional Hausdorff measure. By [12] existence of singular minimizers implies the existence of singular minimizing cones. L.A. Caffarelli-D. Jerison-C. Kenig showed that singular minimizing cones do not exist in dimension 3 ([6]). Moreover it is known that singular minimizing cones exist for $n \geq 7$ ([8]). *Non-minimizing* singular cones appear already for $n = 3$ (see [1, example 2.7]). Moreover it is known, that solutions of the Dirichlet problem in two space dimensions are not unique (see [1, example 2.6]).
For the time-dependent (1.1), both "trivial non-uniqueness" (the positive solution of the heat equation is always another solution of (1.1)) and "non-trivial uniqueness" (see [10]) occur. Even for flawless initial data, classical solutions of (1.1) develop singularities after a finite time span; consider e.g. the example of two colliding traveling waves

$$u(t,x) = \chi_{\{x+t>1\}}(\exp(x+t-1)-1) + \chi_{\{-x+t>1\}}(\exp(-x+t-1)-1) \text{ for } t \in [0,1) \tag{1.2}$$

(see Figure 1).

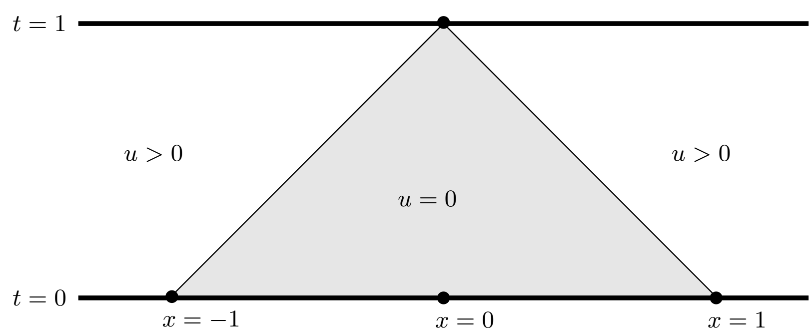

FIGURE 1. Colliding traveling waves

There are several approaches concerning the construction of a solution of the time-dependent problem, all of which are based in some form on the convergence of the solution u_ε of the reaction-diffusion equation

$$\Delta u_\varepsilon - \partial_t u_\varepsilon = \beta_\varepsilon(u_\varepsilon) \tag{1.3}$$

to (1.1) as $\varepsilon \to 0$; here $\beta_\varepsilon(z) = \frac{1}{\varepsilon}\beta(\frac{z}{\varepsilon})$, $\beta \in C_0^1([0,1])$, $\beta > 0$ in $(0,1)$ and $\int \beta = \frac{1}{2}$. L.A. Caffarelli and J.L. Vazquez proved in [7] uniform estimates for (1.3) and a convergence result: for initial data u^0 that are strictly mean concave in the interior of their support, a sequence of ε-solutions converges to a solution of (1.1) in the sense of distributions.

Let us also mention several results on the corresponding two-phase problem, which are relevant as solutions of the one-phase problem are automatically solutions of the corresponding two-phase problem. In [5] and [4], L.A. Caffarelli, C. Lederman and N. Wolanski prove convergence to a barrier solution in the case that the limit function satisfies $\{u = 0\}^\circ = \emptyset$.

Then, there is the convergence to a solution in the sense of domain variations [11] which seems to contain more information than the barrier solutions in [5] and [4]. For more general two-phase problems see [13]. Domain variation solutions play an important rule in this paper and will be discussed in more detail in Section 3.

Here let it suffice to say that domain variation solutions are pairs (u, χ) where the order parameter χ shares many properties with the characteristic function $\chi_{\{u>0\}}$ but does not necessarily coincide with it. By [11], *all limits* of the singular perturbation problem (1.3) are domain variation solutions, so all results in the present paper hold for all limits of (1.3).

Our main result Theorem 8.4 states – leaving out inessential assumptions – that if $(0, \rho^2)$ is a point on the topological free boundary and if the set $\{\chi > 0\}$ is flat enough, i.e.

$$\chi(x,t) = 0 \text{ when } (x,t) \in Q_\rho \text{ and } x_n \geq \sigma\rho,$$

for some $\sigma \leq \sigma_0$ (see Figure 2), then the free boundary $Q_{\rho/4} \cap \partial\{u > 0\}$ is a surface with Hölder-continuous space normal.

As a consequence we obtain that the regular set is open relative to $\partial\{u > 0\}$

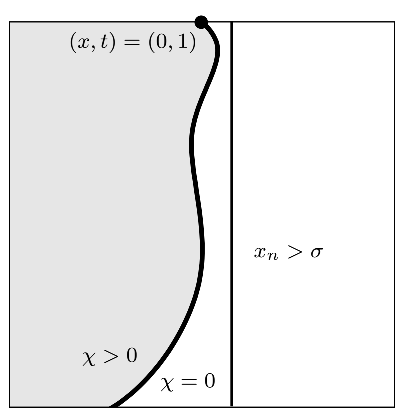

FIGURE 2. One-sided flatness in the case $\rho = 1$

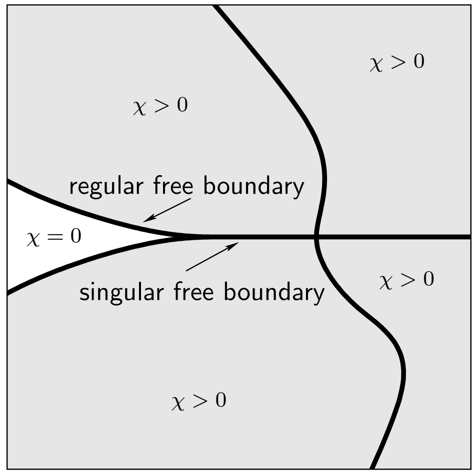

FIGURE 3. Example of the set of regular free boundary points (stationary)

(Corollary 8.5, cf. Figure 3).
Note that even in the stationary case our result extends the result in [1] as our assumptions *do not exclude* degenerate points or cusps close to the origin (excluded by the definition of weak solutions [1, 5.1]), *our result does that*.
In the proof of our result we use ingenious tools developed in [1]: We prove that flatness on the side of $\{\chi = 0\}$ implies flatness on the side of $\{\chi > 0\}$ which in turn yields uniform convergence of an inhomogeneously scaled sequence of free boundaries.
However we replace the core in the method of H.W. Alt-L.A. Caffarelli, relying on non-positive mean curvature of $\partial\{u > 0\}$ at singularities, by a method based on *scaling discrepancies* (Proposition 7.1). This original component gives hope that the method may now be applicable to more general free boundary or free discontinuity problems, in particular two-phase free boundary problems.

2. Notation

Throughout this article $\mathbf{R}^n$ will be equipped with the Euclidean inner product $x \cdot y$ and the induced norm $|x|$, $B_r(x_0)$ will denote the open n-dimensional ball of center x_0, radius r and volume $r^n\,\omega_n$, $B'_r(0)$ the open $n-1$-dimensional ball of center 0 and radius r, and e_i the i-th unit vector in $\mathbf{R}^n$. We define $Q_r(x_0,t_0) := B_r(x_0)\times(t_0-r^2,t_0+r^2)$ to be the cylinder of radius r and height $2r^2$, $Q_r^-(x_0,t_0) := B_r(x_0)\times(t_0-r^2,t_0)$ its "negative part" and $T_r^-(t_0) := \mathbf{R}^n \times (t_0-4r^2, t_0-r^2)$ the horizontal layer from t_0-4r^2 to t_0-r^2. Let us also introduce the parabolic distance $\mathrm{pardist}((t,x),A) := \inf_{(s,y)\in A}\sqrt{|x-y|^2+|t-s|}$. Considering a function $\phi \in H^{1,2}_{\mathrm{loc}}(\mathbf{R}^n;\mathbf{R}^n)$ we denote by $\mathrm{div}\,\phi := \sum_{i=1}^n \partial_i\phi_i$ the space divergence and by

$$D\phi := \begin{pmatrix} \partial_1\phi_1 & \dots & \partial_n\phi_1 \\ & \dots & \\ \partial_1\phi_n & \dots & \partial_n\phi_n \end{pmatrix}$$

the matrix of the spatial partial derivatives.
Given a set $A\subset\mathbf{R}^n$, we denote its interior by A° and its characteristic function by χ_A . In the text we use the n-dimensional Lebesgue-measure $\mathcal{L}^n$ and the m-dimensional Hausdorff measure $\mathcal{H}^m$. When considering a given set $A\subset\mathbf{R}^n$, let

$$\partial_M A := \{x\in\mathbf{R}^n\ :\ \limsup_{r\to 0}\frac{\mathcal{L}^n(B_r(x)\cap A)}{\mathcal{L}^n(B_r)} > 0 \text{ and } \limsup_{r\to 0}\frac{\mathcal{L}^n(B_r(x)-A)}{\mathcal{L}^n(B_r)} > 0\}$$

be the measure-theoretic boundary of A, let $\partial^* A := \{x\in\mathbf{R}^n\ :\ \text{ there is } \nu(x)\in\partial B_1(0) \text{ such that } r^{-n}\int_{B_r(x)}|\chi_A - \chi_{\{y:(y-x)\cdot\nu(x)<0\}}|\to 0 \text{ as } r\to 0\}$ (by [14, Corollary 5.6.8] $\partial^* A$ coincides $\mathcal{H}^{n-1}$-a.e. with the reduced boundary of a set of finite perimeter defined in [14, Definition 5.5.1]), and let $\nu:\partial^* A\to\partial B_1(0)$ denote this measure theoretic outward normal to ∂A. We shall often use abbreviations for inverse images like $\{u>0\} := \{x\in\Omega\ :\ u(x)>0\}$, $\{x_n>0\} := \{x\in\mathbf{R}^n\ :\ x_n > 0\}$, $\{s=t\} := \{(s,y)\in\mathbf{R}^{n+1}\ :\ s=t\}$ etc. as well as $A(t) := A\cap\{s=t\}$ for a set $A\subset\mathbf{R}^{n+1}$, and occasionally we employ the decomposition $x=(x',x_n)$ of a vector $x\in\mathbf{R}^n$ as well as the corresponding decompositions of the gradient and the Laplace operator,

$$\nabla u = (\nabla' u, \partial_n u) \text{ and } \Delta u = \Delta' u\ +\ \partial_{nn} u\ .$$

Finally, $\mathbf{C}^{\beta,\mu} := \mathbf{H}^{\mu,\beta}$ denotes the parabolic Hölder-space defined in [9].

3. Notion of solution and Preliminaries

In this section we gather some results from [11]. As degenerate points are unavoidable in the parabolic problem (see the introduction of [11] for examples), an extension of the *weak solutions* in [1] does not seem to be the right choice. Instead we use the solutions of [11, Definition 6.1], which are, roughly speaking, solutions in the sense of domain variations. The advantage is that the class of solutions defined in [11, Definition 6.1] is closed under the blow-up process. Moreover, *all* limits of the singular perturbation problem discussed in [7] *are* domain variation solutions and satisfy [11, Definition 6.1] (see [11, Section 6]). Let us recall the definition of solutions and the monotonicity formula used therein:

Theorem 3.1 (Monotonicity Formula, cf. [11, Theorem 5.2])**.** *Let* $(x_0, t_0) \in \mathbf{R}^n \times (0,\infty)$, $T_r^-(t_0) = \mathbf{R}^n \times (t_0 - 4r^2, t_0 - r^2)$, $0 < \rho < \sigma < \frac{\sqrt{t_0}}{2}$ *and*

$$G_{(x_0,t_0)}(x,t) = 4\pi(t_0 - t)\,|4\pi(t_0 - t)|^{-\frac{n}{2}-1}\ \exp\left(-\frac{|x - x_0|^2}{4(t_0 - t)}\right) \quad .$$

Then

$$\Psi_{(x_0,t_0)}(r) \ = \ r^{-2} \int_{T_r^-(t_0)} \left(|\nabla u|^2 \ + \ \chi\right) G_{(x_0,t_0)} - \ \tfrac{1}{2}\, r^{-2} \int_{T_r^-(t_0)} \tfrac{1}{t_0 - t}\, u^2\, G_{(x_0,t_0)}$$

satisfies the monotonicity formula

$$\Psi_{(x_0,t_0)}(\sigma) \ - \ \Psi_{(x_0,t_0)}(\rho)$$

$$\geq \ \int_\rho^\sigma r^{-1-2} \int_{T_r^-(t_0)} \frac{1}{t_0 - t} \left(\nabla u \cdot (x - x_0) \ - \ 2(t_0 - t)\partial_t u \ - \ u\right)^2 G_{(x_0,t_0)}\, dr \ \geq \ 0 \quad .$$

Definition 3.2 (cf. [11, Definition 6.1])**.** We call (u, χ) a solution in $\Omega_0 := \mathbf{R}^n \times (0,\infty)$ (in which case we set $\tau := 0$) or $\Omega_1 := \mathbf{R}^n \times (-\infty,\infty)$ (in which case we set $\tau := 1$), if:

1) $u \in \mathbf{C}^{1,\frac{1}{2}}_{\mathrm{loc}}(\Omega_\tau) \cap C^2(\Omega_\tau \cap \{u > 0\}) \cap H^{1,2}_{\mathrm{loc}}(\Omega_\tau)$ and $\chi \in L^1((-\tau R, R); BV(B_R(0)))$ for each $R \in (0,\infty)$. For each $R \in (0,\infty)$ and $\delta \in (0,1)$ there exists $C_1 < \infty$ such that for $Q_r(x_0,t_0) \subset \Omega_\tau \cap Q_R(0)$

$$\int_{Q_r(x_0,t_0)} |\nabla \chi| \ \leq \ C_1\, r^{n+1},$$

$$\int_{Q_r(x_0,t_0)} |\partial_t u|^2 \ \leq \ C_1\, r^n, \ \text{and}$$

$$\int_{B_r(x_0)\times(t_0+S_1 r^2, t_0+S_2 r^2)} |\partial_t(|\nabla u|^2 \ + \ \chi) * \phi_{r\delta}| \ \leq \ C_1 \sqrt{S_2 - S_1}\, r^n$$

for $0 < S_1 < S_2 < \infty$; here the mollifier $(\phi_\delta)_{\delta\in(0,1)}$ should be non-negative and satisfy $\phi_\delta(\cdot) = \frac{1}{\delta^n}\phi(\frac{\cdot}{\delta}), \phi \in C_0^{0,1}(\mathbf{R}^n)$, $\int \phi = 1$ and supp $\phi \subset B_1(0)$.

Moreover, $\chi \in \{0,1\}$ a.e. in Ω_τ and $\chi_{\{u>0\}} \leq \chi$ a.e. in Ω_τ .

2) The solution u satisfies the monotonicity formula Theorem 3.1 (in the case of $\tau = 1$ for $(x_0,t_0) \in \mathbf{R}^{n+1}$ and $\sigma \in (0,\infty)$).

$$3)\ 0 \ = \ \int_{-\infty}^{\infty} \int_{\mathbf{R}^n} [-2\partial_t u\, \nabla u \cdot \xi \ + \ (|\nabla u|^2 \ + \ \chi)\,\mathrm{div}\, \xi \ - \ 2\nabla u D\xi \nabla u]$$

for every $\xi \in C_0^{0,1}(\Omega_\tau; \mathbf{R}^n)$.

4) The solution u is non-negative.

5) The solution u attains the initial data $u^0 \in C_0^{0,1}(\mathbf{R}^n)$ in $L^2_{\mathrm{loc}}(\mathbf{R}^n)$ in the case that $\tau = 0$.

6) For each $\kappa > 0$ there is $\delta > 0$ such that $Q_r(x_0,t_0) \subset \Omega_\tau$ and $\|\frac{u(x_0+rx, t_0+r^2 t)}{r} - \theta |x_n|\|_{C^0(Q_1(0))} < \delta$ imply $\theta < 1 + \kappa$.

7) For $\delta \in (0,1)$, $\psi_\delta \in C_0^{0,1}(\{|y|^2 + s^2 < \delta^2\})$, $u_r(y,s) := \frac{u(t_0+r^2 s, x_0+ry)}{r}$ and $\chi_r(y,s) := \chi(x_0+ry, t_0+r^2 s)$ the following holds:

$$\text{a)} \quad \int_{Q_\rho(x_1,t_1)} |(\nabla \chi_r \cdot x \ + \ 2t\partial_t \chi_r) * \psi_\delta|$$

$$\leq \ C(\delta, Z, T, S, \rho) \left(\Psi_{(x_0,t_0)}(r\sqrt{\frac{-t_1+\delta+\rho^2}{2}}) \ - \ \Psi_{(x_0,t_0)}(r\sqrt{\frac{-t_1-\delta-\rho^2}{2}}) \right)$$

for $-S \leq t_1 \leq -T < 0$, $\delta + \rho^2 \leq \frac{T}{2}$, $|x_1| \leq Z$ and, in the case of $\tau = 0$, $t_0 - 2r^2(-t_1+\rho^2+\delta) > 0$.

$$\text{b)} \quad \int_{Q_\rho(t_1,x_1)} |(\nabla \chi_r \cdot \xi) * \psi_\delta| \ \leq \ C(\delta) \int_{Q_{\sqrt{\delta}+\rho}(t_1,x_1)} |\nabla u_r \cdot \xi|$$

for $\xi \in \partial B_1(0)$, $t_1 < 0$ and, in the case of $\tau = 0$, $t_0 - r^2(-t_1 + (\rho+\sqrt{\delta})^2) > 0$.

$$\text{c)} \quad \int_{t_1}^{t_2} \partial_t((|\nabla u_r|^2 + \chi_r) * \phi_\delta)(t,x_0) \ \leq \ \int_{t_1}^{t_2} \int_{\mathbf{R}} 2\partial_t u_r(t,z) \nabla u_r(t,z) \cdot \nabla \phi_\delta(x_0 - z)\, dz$$

for $-\infty < t_1 < t_2 < \infty$ and, in the case of $\tau = 0$, $t_0 + r^2 t_1 > 0$.

Remark 3.3. As the function χ is defined only almost everywhere, all pointwise equalities/inequalities involving χ should be understood as equalities/inequalities that hold almost everywhere with respect to the Lebesgue measure.

The reader may wonder whether a solution in the sense of distributions (possibly defined by the identity in [11, Lemma 11.3]) would not be good enough for the purposes of this paper. It turns however out that the information yielded by the order parameter χ in Definition 3.2 carries information that is essential in what follows. Incidentally, χ may be different from $\chi_{\{u>0\}}$ (see [11, Remark 4.1]).

Lemma 3.4. *Let (u,χ) be a solution in the sense of Definition 3.2 and suppose that for some (x_0,t_0) in the set of definition and for some sequence $r_m \to 0, m \to \infty$*

$$u_{r_m}(y,s) := \frac{u(x_0 + r_m y, t_0 + {r_m}^2 s)}{r_m} \to 0 \textit{ locally in } \{y_n < 0\} \times (-\infty, 0) \textit{ as } m \to \infty$$

and

$$\chi_{r_m}(y,s) := \chi(x_0 + r_m y, t_0 + {r_m}^2 s) \to 0 \textit{ a.e. in } \{y_n > 0\} \times (-\infty, 0) \textit{ as } m \to \infty \,.$$

Then for some $\delta > 0$, u is caloric in $Q_\delta(x_0,t_0)$ and satisfies

$$u = 0 \textit{ in } Q_\delta^-(x_0,t_0)\,.$$

Proof. The assumptions imply by Definition 3.2 1) that

$$u_{r_m} \to 0 \text{ a.e. in } \{x_n > 0\} \times (-\infty, 0) \text{ as } m \to \infty \,.$$

Moreover, they imply by [11, Proposition 10.1 2)] that the density

$$\Psi_{(x_0,t_0)}(0+) \in \{0\} \cup \{H_n\} \,,$$

where H_n is the energy of the half-plane solution defined in [11, Section 10]. In the case

$$\Psi_{(x_0,t_0)}(0+) = 0$$

we obtain from [11, Proposition 10.1 2)] immediately the statement of the lemma. In the case

$$\Psi_{(x_0,t_0)}(0+) = H_n$$

it follows from [11, Proposition 10.1 1)] that the limit of $u_{r_m}(y,s)$ as $m \to \infty$ must after rotation be the half-plane solution $\max(-x_n, 0)$, a contradiction to the limit of u_{r_m} being 0. □

4. Flatness Classes

Definition 4.1. Let $0 < \sigma_+, \sigma_- < 1$ and $\tau \geq 0$. We say that

$$u \in F(\sigma_+, \sigma_-, \tau) \quad \text{in } Q_\rho \quad \text{in direction } e_n$$

if

(1) (u, χ) is a solution in the sense of Definition 3.2 in a domain containing Q_ρ.

(2)

$$(0, \rho^2) \in \partial\{u > 0\},$$

$$u(x,t) = \chi(x,t) = 0 \text{ when } (x,t) \in Q_\rho \text{ and } x_n \geq \sigma_+\rho,$$

$$\chi(x,t) = 1 \text{ and } u(x,t) \geq -(x_n + \sigma_-\rho) \text{ when } (x,t) \in Q_\rho \text{ and } x_n \leq -\sigma_-\rho\ .$$

(3)

$$|\nabla u| \leq 1 + \tau \text{ in } Q_\rho.$$

When the origin is replaced by (x_0, t_0) and the flatness direction e_n is replaced by ν then we define u to belong to the flatness class $F(\sigma_+, \sigma_-, \tau)$ in $Q_\rho(x_0, t_0)$ in direction ν.

5. Flatness on the side of $\{\chi = 0\}$ implies flatness on the side of $\{\chi > 0\}$

The aim of this and the following sections is to draw information from properties of an inhomogeneous blow-up limit. One of the central problems when using blow-up arguments is "*not-strong convergence*" or "*energy loss*" in the limit. Here we avoid those problems by working with *uniform convergence* (not some Sobolev norm). The approach is based on a powerful idea by H.W. Alt-L.A. Caffarelli who used "flatness on the side of $\{u = 0\}$ implies flatness on the side of $\{u > 0\}$" to prove uniform convergence to an inhomogeneous blow-up limit (cf [1, Section 7]). In this section we extend their result to a weaker class of solutions and to the parabolic case, using results in [11].
The following Lemma is the parabolic version of [1, Lemma 4.10].

Lemma 5.1. *Let (u,χ) be a solution in the sense of Definition 3.2 in a domain containing the closure of a non-empty open ball $B = \{(y,s) : |(y,s)-(y_0,s_0)| < c\}$ such that $B \subset \{\chi = 0\}$ and B touches the set $\{u > 0\}$ at the origin.*
Then

$$\limsup_{\{u>0\}\ni(x,t)\to 0} \frac{u(x,t)}{\mathrm{pardist}((x,t),B)} = 1.$$

Proof. Let $Y_k = (y_k, s_k) \in \mathbf{R}^{n+1}$ be a sequence such that

$$\ell = \limsup_{\{u>0\}\ni(x,t)\to 0} \frac{u(x,t)}{\mathrm{pardist}((x,t),B)} = \lim_{k\to\infty} \frac{u(Y_k)}{\mathrm{pardist}(Y_k,B)}.$$

Set $d_k := \mathrm{pardist}(Y_k,B)$ and let $(x_k,t_k) = X_k \in \partial B$ be such that $\mathrm{pardist}(Y_k,X_k) = d_k$.
We consider the blow-up sequence

$$u_k(x,t) = \frac{u(d_k x + x_k, d_k^2 t + t_k)}{d_k}, \chi_k(x,t) = \chi(d_k x + x_k, d_k^2 t + t_k).$$

We know that, passing to a subsequence if necessary, $u_k \to u_0$ locally uniformly in $\mathbf{R}^{n+1}$ and $\chi_k \rightharpoonup \chi_0$ weakly-* in $L^\infty_{\mathrm{loc}}(\mathbf{R}^{n+1})$ as $k \to \infty$. Also, after a rotation and translation, the scaled B converges to $\{x_n > 0\}$ and $\big((y_k - x_k)/d_k, (s_k - t_k)/d_k^2\big) \to (\xi,\tau) \in \partial Q_1(0)$ as $k \to \infty$. The limit function u_0 satisfies

$$\begin{aligned} &\Delta u_0 - \partial_t u_0 = 0 \quad &&\text{in } \mathbf{R}^{n+1} \cap \{u_0 > 0\}\ , \\ &u_0(\xi,\tau) = \ell &&\text{and} \\ &u_0(x,t) = 0 &&\text{in } \{x_n > 0\}. \end{aligned}$$

By the definition of the limit superior we know also that

$$u_0(x,t) \le -\ell x_n \quad \text{ in } \{x_n < 0\}.$$

The strong maximum principle (applied to $u_0(x,t) + \ell x_n$) tells us therefore that $u_0(x,t) = \ell \max(-x_n, 0)$ for $t < \tau$. We have to show that $\ell = 1$.
In the case $\ell > 0$ we obtain from the fact that (u,χ) is a solution in the sense of Definition 3.2, that $\chi_0 = 1$ in $\{x_n < 0\} \cap \{t < \tau\}$. Furthermore, we infer from the assumption that $\chi_0 = 0$ in $\{x_n > 0\}$. But then $(u_0(\cdot, t+\tau), \chi_0(\cdot, t+\tau))$ is in $\{t < 0\}$ a solution in the sense of Definition 3.2 whose energy $M(u_0(\cdot, t+\tau), \chi_0(\cdot, t+\tau)) = H_n$ (cf. [11, Section 10]), whence [11, Proposition 10.1] implies that $\ell = 1$.
In the case $\ell = 0$ we apply Lemma 3.4 to obtain for some $\delta > 0$ that u is caloric in Q_δ and satisfies

$$u = 0 \text{ in } Q_\delta^- \ .$$

As $\{u = 0\}$ contains B, u being caloric in Q_δ and therefore analytic with respect to the space variables implies

$$u = 0 \text{ in } Q_{\delta_1}$$

for some $\delta_1 > 0$. This is a contradiction in view of the origin being a free boundary point. □

The following theorem extends [1, Lemma 7.2].

Theorem 5.2. *There exists a constant $C \in (0, +\infty)$ depending only on the space dimension n such that if $u \in F(\sigma, 1, \sigma)$ in Q_ρ then $u \in F(C\sigma, C\sigma, \sigma)$ in $Q_{\rho/2}(0, y_n, 0))$ for some $|y_n| \leq C\sigma$.*

Proof. The idea is to touch the boundary $\partial\{\chi = 0\}$ with the graph of a C^2-function, to apply Lemma 5.1 and to proceed then with a Harnack inequality argument.

Step 1 (Touching $\partial\{\chi = 0\}$ with a smooth surface):

Rescaling $u_\rho(x,t) := \frac{u(\rho x, \rho^2 t)}{\rho}, \chi_\rho(x,t) := \chi(\rho x, \rho^2 t)$ we may assume that $\rho = 1$. Let

$$\eta(x',t) = \begin{cases} \exp(\frac{16(|x'|^2+|t-1|)}{1-16(|x'|^2+|t-1|)}), & |x'|^2 + |t-1| < 1/16, \\ 0, & \text{else} \end{cases}$$

and let s be the largest constant such that

$$Q_1 \cap \{u > 0\} \subset \{(x,t) \in Q_1 : \ x_n < \sigma - s\eta(x',t)\} =: D.$$

This implies that there exists a point $(x_0, t_0) := Z \in \partial D \cap \partial\{u > 0\} \cap \{t \geq 15/16\}$. As $(0, 1)$ is a free boundary point, we know furthermore that $s \leq \sigma$.

Let us also define the barrier function v by

$$\begin{aligned} &\Delta v - \partial_t v = 0 && \text{in } D\ , \\ &v = 0 && \text{on } \partial D \cap Q_1 \text{ and} \\ &v = 2\sigma - x_n && \text{on } \partial D \cap \partial' Q_1. \end{aligned}$$

Note that this implies that $-\sigma \leq v + x_n \leq 2\sigma$.

Since $|\nabla u| \leq 1 + \sigma$ we also obtain that $v \geq u$ on ∂D and thus, by the maximum principle, that $v \geq u$ in D. As η is a C^2-function, the assumptions of Lemma 5.1 are satisfied at Z. Therefore

$$1 \leq \limsup_{(x,t)\to Z} \frac{u(x,t)}{\text{pardist}((x,t), B)} \leq -\partial_\nu v(Z), \tag{5.1}$$

where ν is the outward space normal to ∂D at Z. In order to obtain an estimate from above we define

$$F(x,t) = 2\sigma - x_n - v(x,t).$$

F is caloric in D and satisfies $0 \leq F \leq \sigma$. Since D is a regular parabolic domain, we know from standard regularity theory for parabolic equations that $\sup_D |\nabla F| \leq C_1\sigma$. Therefore

$$-\partial_n v(Z) = 1 + \partial_n F(Z) \leq 1 + C_1\sigma.$$

By the flatness assumption we know that ν is close to e_n. More precisely,

$$|\nu - e_n| = \Big|\frac{(-s\nabla\eta, 1 - \sqrt{s^2|\nabla\eta|^2 + 1})}{\sqrt{s^2|\nabla\eta|^2+1}}\Big| \leq \sqrt{10}|\nabla\eta|s.$$

Thus

$$-\partial_\nu v(Z) = -\nabla v(Z)\cdot(\nu - e_n) - \partial_n v(Z) \leq 1 + C_1\sigma + \sqrt{10}|\nabla\eta||\nabla v(Z)|s \leq 1 + C_2\sigma\ .$$

From inequality (5.1) we infer that

$$1 \le -\partial_\nu v(Z) \le 1 + C_2\sigma. \tag{5.2}$$

Step 2 (Harnack inequality argument):
As we know already that v is σ-close to $-x_n$, it is sufficient to show that u is σ-close to v on the set $\{(x,t):\ x_n = -3/4,\ |x'| \le 1/2,\ t \le 3/4\}$. Once this is done, we may integrate u in the x_n-direction to establish the lemma.
In order to prove the σ-closeness we define for $\xi = (\gamma, \tau)$, $\tau \in (-1, 3/4)$, $|\gamma'| \le 1/2$ and $\gamma_n = -3/4$ the function ω_ξ by

$$\begin{aligned} &\Delta\omega_\xi - \partial_t\omega_\xi = 0 && \text{in } D \cap \{t > \tau\} \\ &\omega_\xi = -x_n && \text{on } B_{1/8}(\gamma) \times \{t = \tau\} \\ &\omega_\xi = 0 && \text{on the remainder of the parabolic boundary of } D \cap \{t > \tau\}. \end{aligned}$$

By the Hopf lemma we have

$$\partial_\nu \omega_\xi(Z) \le -\alpha < 0$$

uniformly in ξ.
We would like to show that $u \ge v - C_4\sigma x_n$. The trick is to compare u to $v - K\sigma\omega_\xi$ on the set $B_{1/8}(\gamma) \times \{t = \tau\}$ and to use the information on the normal derivative of u at Z to prove that if K is large, then $u > v - K\sigma\omega_\xi$ for at least one point in $B_{1/8}(\gamma) \times \{t = \tau\}$. More precisely:
Assume that $u \le v - K\sigma\omega_\xi$ in $B_{1/8}(\gamma) \times \{t = \tau\}$. Then $u \le v - K\sigma\omega_\xi$ in $D \cap \{t > \tau\}$. Consequently, we obtain from inequalities (5.1) and (5.2) that

$$1 \le -\partial_\nu v(Z) + K\sigma\partial_\nu\omega_\xi(Z) \le 1 + C_2\sigma - K\alpha\sigma.$$

This yields a contradiction when K is large enough, say $K = 2C_2/\alpha$. Thus $u(X_\xi) > v(X_\xi) - K\sigma\omega_\xi(X_\xi)$ for at least one point $X_\xi \in B_{1/8}(\gamma) \times \{t = \tau\}$.
On the other hand, $v - u \ge 0$. Therefore we can apply the Harnack inequality and deduce that

$$(v-u)(\tilde\xi) \le C_3 \inf_{Q_{1/8}(\tilde\xi + (0,1/32))} (v-u) \le C_4\sigma,$$

for every $\tilde\xi \in \{(x', -3/4, t):\ |x'| < 1/2,\ -1 \le t \le 1/2\}$.
This implies that $u(x', -3/4, t) \ge 3/4 - C_5\sigma$ in the above region. Integrating in the e_n direction and using the assumption $|\nabla u| \le 1 + \sigma$ yields the estimate

$$u \ge -(x_n + C_6\sigma) \text{ in } \{-3/4 \le x_n \le -\sigma\} \times Q'_{1/2}$$

By our initial assumption we also know that $u = 0$ in $\{3/4 \ge x_n \ge \sigma\} \cap Q'_{1/2}$. Translating (u, χ) in the e_n direction so that the point $(0, 1/4) \in \partial\{u > 0\}$ and using $\chi \ge \chi_{\{u>0\}}$ of Definition 3.2 1) we obtain the statement of our theorem. □

6. Inhomogeneous Blow-up

In this section we consider inhomogeneous scaling of the solution and the free boundary. The following lemma is our version of [1, Lemma 7.3]

Lemma 6.1. *Suppose that* $u_k \in F(\sigma_k, \sigma_k, \tau_k)$ *in* Q_{ρ_k}*, that* $\sigma_k \to 0$ *and that* $\tau_k/\sigma_k^2 \to 0$*, and define*

$$f_k^+(x', t) := \sup\{h : \limsup_{r\to 0} r^{-n-2} \int_{Q_r(\rho_k x', \sigma_k \rho_k h, \rho_k^2 t)} \chi > 0\},$$

$$f_k^-(x', t) := \inf\{h : \limsup_{r\to 0} r^{-n-2} \int_{Q_r(\rho_k x', \sigma_k \rho_k h, \rho_k^2 t)} \chi > 0\}.$$

Then, as a subsequence $k \to \infty$*,* f_k^+ *and* f_k^- *converge in* $L^\infty_{\rm loc}(Q_1')$ *to some function* f*, and* f *is continuous in* Q_1'*.*

Proof. Rescaling as before we may assume that $\rho_k = 1$. Let

$$D_k := \{(y', h, t) : \limsup_{r\to 0} r^{-n-2} \int_{Q_r(y', \sigma_k h, t)} \chi > 0\} .$$

We may assume – passing if necessary to a subsequence – that D_k converges with respect to the usual (not the parabolic) Hausdorff distance as $k \to \infty$. Let us define

$$f(x', t) := \limsup_{(y',s)\to(x',t), k\to\infty} f_k^+(y', s),$$

where we take the limit superior with respect to the above subsequence. For every (y_0', t_0) there exists then a sequence $(y_k', t_k) \to (y_0', t_0)$ such that $f_k^+(y_k', t_k) \to f(y_0', t_0)$ as $k \to \infty$. By definition f is upper semi-continuous. Therefore we obtain for $\varepsilon > 0$ and sufficiently large k that

$$\big(\overline{Q_\varepsilon'(y_k', t_k)} \times [f_k^+(y_k', t_k) + \delta, \infty)\big) \cap \bar{D}_k = \emptyset.$$

Consequently $u_k \in F(\sigma_k \frac{\delta}{\varepsilon}, 1, \tau_k)$ in $Q_\varepsilon(y_k, \sigma_k f_k^+(y_k', t_k), t_k)$. Applying Theorem 5.2 to u_k we deduce that

$$u_k(x, t) \geq -(x_n + C\sigma_k \delta/2) \text{ for } (x, t) \in Q_{\varepsilon/2}(y_k', \sigma_k f_k^+(y_k', t_k), t_k) .$$

In terms of f_k^+ and f_k^- this yields $f_k^-(y', t) \geq f_k^+(y_k', t_k) - C\delta$ in $Q_{\varepsilon/4}'(y_k', t_k)$. It follows that $\lim_{k\to\infty} f_k^-(y', t) = f(y', t)$, that f_k^+ and f_k^- converge locally uniformly and that f is continuous. □

The next Proposition follows the lines of [2, Lemma 5.7].

Proposition 6.2. *Suppose that the assumptions of Lemma 6.1 are satisfied and that* k *is the subsequence of Lemma 6.1. Then*

$$w_k(x', h, t) = \frac{u_k(\rho_k x', \rho_k h, \rho_k^2 t) + \rho_k h}{\sigma_k}$$

is for each $\delta \in (0,1)$ bounded in $Q_{1-\delta} \cap \{x_n < 0\}$ (by a constant depending only on δ and n) and converges on compact subsets of Q_1^- in C^2 to a caloric function w. Moreover, $w(x', h, t)$ is non-decreasing in the h-variable in Q_1^- and

$$\lim_{Q_1^- \ni (y,s) \to (x',0,t) \in Q_1', k\to\infty} w_k(y,s) = f(x',t) \, ;$$

here f is the function defined in Lemma 6.1.

Proof. Rescaling as before we may assume that $\rho_k = 1$.
The function w_k is caloric in $Q_1 \cap \{h < -\sigma_k\}$. Using Definition 4.1 3), we obtain that

$$u_k \le -x_n + 2\sigma_k \text{ in } Q_1 \cap \{x_n \le 0\} \, ,$$

implying that $w_k \le 2$. From Theorem 5.2 and Definition 4.1 3) we infer furthermore that $u_k(x,t) \ge -(x_n + C_\delta \sigma_k)$ for $(x', x_n, t) \in Q_{1-\delta} \cap \{x_n \le 0\}$, implying that $w_k \ge -C_\delta$ in $Q_{1-\delta} \cap \{x_n \le 0\}$.
By Definition 4.1 3) and the assumptions, $|\nabla u_k| \le 1 + o(\sigma_k^2)$. Consequently,

$$-\partial_h w_k \le \frac{|\nabla u_k| - 1}{\sigma_k} \le \frac{\tau_k}{\sigma_k} \to 0 \text{ as } k \to \infty \, . \tag{6.1}$$

In the remainder of the proof we will show that w attains the boundary data f as $h \to 0$. First, we show that for fixed $L \in (1, +\infty)$

$$w_k(x', \sigma_k h, t) - f_k^+(x',t) \to 0 \qquad \text{uniformly in } Q'_{1-\delta} \times \{-L \le h < 0\} \tag{6.2}$$

as $k \to \infty$. An estimate from above can be obtained easily from inequality (6.1):

$$w_k(x', h\sigma_k, t) - f_k^+(x',t) \le w_k(x', \sigma_k f_k^+(x',t), t) - f_k^+(x',t) + (f_k^+(x',t) - h)\frac{\tau_k}{\sigma_k}$$
$$\le (1+L)\frac{\tau_k}{\sigma_k} \to 0 \text{ as } k \to \infty \, .$$

This establishes an estimate from above. In order to derive an estimate from below we use Theorem 5.2: Consider a sequence of points $(x'_k, t_k) \in Q'_{1-\delta}$ and fixed $S \in (4, +\infty)$. Then

$$u_k \in F(\tilde\sigma_k, 1, \tau_k) \text{ in } Q_{S\sigma_k}(x'_k, \sigma_k f_k^+(x'_k, t_k), t_k)$$

for

$$\tilde\sigma_k = \frac{1}{S} \sup_{(x',t)\in Q'_{S\sigma_k}} (f_k^+(x',t) - f_k^+(x'_k, t_k)).$$

From the uniform convergence of f_k^+ to the continuous function f, we infer that $\tilde\sigma_k \to 0$ as $k \to \infty$. Now by Theorem 5.2,

$$u_k \in F(C\bar\sigma_k, C\bar\sigma_k, \tau_k) \qquad \text{in } Q_{S\sigma_k/2}(x'_k, \sigma_k f_k^+(x'_k, t_k) + CS\bar\sigma_k\theta/2, t_k),$$

where $\bar\sigma_k = \max(\tilde\sigma_k, \tau_k)$ and $\theta \in [0,1]$.
Thus for $h \in (\max(-L, -S/4), 0)$

$$u_k(x_k + h\sigma_k e_n, t_k) \ge -\sigma_k \left(h - f_k^+(x'_k, t_k) + C\bar\sigma_k S\right).$$

Consequently

$$w_k(x_k + h\sigma_k e_n, t_k) = \frac{u_k(x_k + h\sigma_k e_n, t_k) + h\sigma_k}{\sigma_k} \geq f_k^+(x_k', t_k) - C\bar{\sigma}_k S ,$$

and (6.2) holds.

To establish $\lim_{Q_1^- \ni (y,s) \to (x',0,t) \in Q_1', k\to\infty} w_k(y,s) = f(x',t)$, we need to extend the convergence (6.2) to larger values of h. To this end, we define the barrier function z_ε by

$$\begin{aligned} &\Delta z_\varepsilon - \partial_t z_\varepsilon = 0 && \text{in } Q_{1-\delta}^- ,\\ &z_\varepsilon = g_\varepsilon && \text{on } \partial' Q_{1-\delta} \cap \{h = 0\} ,\\ &z_\varepsilon = \inf_k \inf_{Q_{1-\delta}^-} w_k && \text{on } \partial' Q_{1-\delta} \cap \{h < 0\} , \end{aligned}$$

where $g_\varepsilon \in C^\infty$ and $f - 2\varepsilon \leq g_\varepsilon \leq f - \varepsilon$. By (6.2) we know that $w_k \geq z_\varepsilon$ on $\partial'(Q_{1-\delta} \cap \{h \leq -L\sigma_k\})$. From the comparison principle it follows that $w_k \geq z_\varepsilon$ in $Q_{1-\delta}^- \cap \{h \leq -L\sigma_k\}$. Thus, by local boundary regularity for solutions of the heat equation, $\liminf_{Q_{1-2\delta}^- \ni (y,s) \to (x',0,t), k\to\infty} w_k(y,s) \geq g_\varepsilon(x',t) \geq f(x',t) - 2\varepsilon$.

The opposite inequality follows from a similar argument, comparing w_k to the upper barrier $\tilde{z}$ defined by

$$\begin{aligned} &\Delta \tilde{z}_\varepsilon - \partial_t \tilde{z}_\varepsilon = 0 && \text{in } Q_{1-\delta}^- ,\\ &\tilde{z}_\varepsilon = \tilde{g}_\varepsilon && \text{on } \partial' Q_{1-\delta} \cap \{h = 0\} ,\\ &\tilde{z}_\varepsilon = \sup_k \sup_{Q_{1-\delta}^-} w_k && \text{on } \partial' Q_{1-\delta} \cap \{h < 0\} , \end{aligned}$$

where $\tilde{g}_\varepsilon \in C^\infty$ and $f + 2\varepsilon \geq \tilde{g}_\varepsilon \geq f + \varepsilon$. ☐

7. Scaling discrepancy and C^∞-regularity of blow-up limits

In order to obtain "better-than-Lipschitz"-regularity of the inhomogeneous blow-up limit f, H.W. Alt-L.A. Caffarelli used the non-positive mean curvature of $\partial\{u > 0\}$ at singularities. The analogue of the non-positive mean curvature property can still be proved in the time-dependent case, however that path leads to problems in the sequel. Therefore we replace it by a scaling discrepancy argument which gives hope to be applicable in more general situations. We obtain C^∞-regularity of f.

Proposition 7.1. *Suppose that the assumptions of Lemma 6.1 are satisfied and that k is the subsequence of Lemma 6.1. Then $\partial_n w = 0$ on $Q_{1/2}'$ in the sense of distributions.*

Proof. Rescaling as before we may assume that $\rho_k = 1$.

In what follows, $g(x',t) = 8(|x'|^2 + |t|) - 4$. Note that $f \geq g$ in $Q_{1/2}'$. Let us introduce the following notation: Z shall be the set $\{(x', x_n, t) : (x',t) \in Q_1', x_n \in \mathbf{R}\}$. Given a function $\phi : Q_1' \to \mathbf{R}$, we divide Z into the three parts

$$Z^+(\phi) = \{(x,t) \in Z : \ x_n > \phi(x',t)\},$$

$$Z^-(\phi) = \{(x,t) \in Z : \ x_n < \phi(x',t)\},$$

$$Z^0(\phi) = \{(x,t) \in Z : \ x_n = \phi(x',t)\}.$$

Moreover let μ be defined by $\mu(A) := \int_{-\infty}^{\infty} \mathcal{H}^{n-1}(A \cap \{s = t\})\, dt$ for any Borel set $A \subset \mathbf{R}^{n+1}$. Adding an arbitrarily small constant to the function g, we may assume that $\mu(Z^0(\sigma_k g) \cap R_k) = 0$ for all k; here R_k is the regular part of the free boundary $\partial\{u_k > 0\}$ introduced in [11, Proposition 9.1], i.e.

$$R_k(t) := \{x \in \partial\{u_k(t) > 0\} \,:\, \text{there is } \nu_{R_k}(x,t) \in \partial B_1(0) \text{ such that } v_r(y,s) = \frac{u_k(x + ry, t + r^2 s)}{r} \to \max(-y \cdot \nu_{R_k}(x,t), 0) \text{ locally uniformly in } (y,s) \in \mathbf{R}^{n+1} \text{ as } r \to 0\}\ .$$

Last, we define $E_k := \{u_k > 0\} \cap Z^-(\sigma_k g)$ and $\Sigma_k := \{(x',t) : (x', \sigma_k g(x',t), t) \in \{u_k > 0\} \cap Z\}$. By the choice of g we know that the limit inferior of the sets Σ_k contains $Q'_{1/2}$.

We will deduce the result from the following three claims.

Claim 1:

$$\mu(Z^+(\sigma_k g) \cap R_k) \le -\int_{\Sigma_k} (\partial_n u_k + 1) dx' dt + \mathcal{L}^n(\Sigma_k) + C_1 \sigma_k^2.$$

Claim 2:

$$\mathcal{L}^n(\Sigma_k) - C_2 \sigma_k^2 \le \mu(Z^+(\sigma_k g) \cap R_k).$$

Claim 3:

$$\int_{\Sigma_k} |\partial_n w_k(x', \sigma_k g(x',t), t)| \ \to\ 0 \text{ as } k \to \infty\ .$$

Proof of Claim 1: By the representation theorem [11, Lemma 11.3] we know that for non-negative $\phi \in C_0^\infty$,

$$\int_{-\infty}^{\infty} \int_{R_k(t)} \phi\, d\mathcal{H}^{n-1}\, dt \le -\int_{\{u_k > 0\}} (\nabla u_k \cdot \nabla \phi + \partial_t u_k \phi)\ dx\, dt\ . \tag{7.1}$$

Letting $\phi \to \chi_{Z^+(\sigma_k g)} \chi_{Q_2}$ the inequality (7.1) becomes

$$\mu(Z^+(\sigma_k g) \cap R_k) = \int_{-\infty}^{\infty} \int_{R_k(t) \cap Z^+(\sigma_k g)} d\mathcal{H}^{n-1}\, dt \tag{7.2}$$

$$\le \int_{\{u_k > 0\} \cap Z^0(\sigma_k g)} \nabla u_k \cdot \nu\, dx\, dt - \int_{\{u_k > 0\} \cap Z^+(\sigma_k g)} \partial_t u_k\, dx\, dt\ ,$$

where ν is the outward unit space normal on $\partial Z^+(\sigma_k g)$. Since

$$\nu = \frac{1}{\sqrt{1 + |\sigma_k \nabla' g|^2}} (\sigma_k \nabla' g, -1)\ ,$$

we obtain

$$\mu(Z^+(\sigma_k g) \cap R_k) \ \le\ \int_{\Sigma_k} (\nabla u_k)(x', \sigma_k g(x',t), t) \cdot (\sigma_k \nabla' g(x',t), -1)\, dx\, dt - \int_{\{u_k > 0\} \cap Z^+(\sigma_k g)} \partial_t u_k\, dx\, dt\ .$$

Let us rewrite the integral

$$\int_{\Sigma_k} (\nabla u_k)(x', \sigma_k g(x',t), t) \cdot (\sigma_k \nabla' g(x',t), -1)\, dx\, dt$$

$$= \int_{\Sigma_k} \sigma_k(\nabla' u_k)(x',\sigma_k g(x',t),t)\cdot\nabla' g(x',t) - (\partial_n u_k(x',\sigma_k g(x',t),t)+1) dx' dt + \mathcal{L}^n(\Sigma_k)$$

$$= \int_{\Sigma_k} -\sigma_k u_k(x',\sigma_k g(x',t),t)\Delta' g(x',t) \; - \; {\sigma_k}^2 \partial_n u_k(x',\sigma_k g(x',t),t)|\nabla' g(x',t)|^2$$

$$- (\partial_n u_k(x',\sigma_k g(x',t),t) + 1)\, dx'\, dt + \mathcal{L}^n(\Sigma_k)$$

$$+ \int_{\partial\Sigma_k} \sigma_k u_k(x',\sigma_k g(x',t),t)\partial_\eta g(x',t)\, d\mathcal{H}^{n-2}\, dt \; ,$$

where η is the outward space normal on $\partial\Sigma_k$. Since $u_k = 0$ on $\partial\Sigma_k$, the last integral is 0.

Moreover, $\Delta' g = 16$ and $u_k \le C_3\sigma_k$ on $(x', g(x',t), t)$, implying that

$$\int_{\Sigma_k} (\nabla u_k)(x',\sigma_k g(x',t),t)\cdot(\sigma_k \nabla' g(x',t), -1)\, dx\, dt$$

$$= \; - \int_{\Sigma_k} (\partial_n u_k(x',\sigma_k g(x',t),t)+1) dx' dt + \mathcal{L}^n(\Sigma_k) + C_4\sigma_k^2 \; .$$

By the definition of w_k this tells us also that

$$\int_{\Sigma_k} (\nabla w_k)(x',\sigma_k g(x',t),t)\cdot(\sigma_k \nabla' g(x',t), 0)\, dx\, dt \;\to\; 0 \text{ as } k\to\infty \; , \tag{7.3}$$

a fact that will be used later on.

Last, integration by parts of the last term in (7.2) with respect to the time variable yields

$$- \int_{\{u_k>0\}\cap Z^+(\sigma_k g)} \partial_t u_k\, dx\, dt \;\le\; C_5 {\sigma_k}^2 \; .$$

Combining the above estimates we obtain Claim 1.

Proof of claim 2: With the outward space normal on the boundary of $Z^-(\sigma_k g)$

$$\nu_{g_k} = \frac{1}{\sqrt{1+\sigma_k^2|\nabla' g|^2}}(-\sigma_k\nabla' g, 1)$$

and with the outward space normal ν_{R_k} on the regular boundary of E_k we compute

$$\mu(Z^+(\sigma_k g)\cap R_k) \ge \tag{7.4}$$

$$\int_{-1}^1 \int_{Z^+(\sigma_k g)\cap R_k(t)} \nu_{g_k}\cdot\nu_{R_k}\, d\mathcal{H}^{n-1}\, dt$$

$$= \int_{-1}^1 \int_{E_k\cap Z^+(\sigma_k g)} \operatorname{div}\, \nu_{g_k}\, d\mathcal{H}^{n-1}\, dt$$

$$+ \int_{-1}^1 \int_{\partial Z^+(\sigma_k g)\cap E_k} \nu_{g_k}\cdot\nu_{g_k} d\mathcal{H}^{n-1}\, dt.$$

The normal ν_{g_k} satisfies

$$\operatorname{div}\, \nu_{g_k} \ge \frac{-\sigma_k\Delta g}{\sqrt{1+\sigma_k^2|\nabla' g|^2}} \ge -C_6\sigma_k.$$

Inserting this estimate for the divergence into (7.4) yields

$$\mu(Z^+(\sigma_k g)\cap R_k) \ge \mu(\partial Z^+(\sigma_k g)\cap E_k) \tag{7.5}$$

$$
\begin{aligned}
& - \int_{-1}^{1} \int_{E_k \cap Z^+(\sigma_k g)} C_6 \sigma_k \, d\mathcal{H}^{n-1} \, dt \\
& \geq \mu(\partial Z^+(\sigma_k g) \cap E_k) - C_7 \sigma_k^2 ;
\end{aligned}
$$

the last inequality follows from the fact that the width of the set E_k is of order $O(\sigma_k)$. As the area of $\partial Z^+(\sigma_k g) \cap E_k)$ is greater than that of Σ_k, the statement of Claim 2 holds.

Proof of Claim 3: From *Claim 1* and *Claim 2* we infer that

$$-C_8 \sigma_k^2 \leq - \int_{\Sigma_k} (\partial_n u_k(x', \sigma_k g(x', t), t) + 1) \, dx' \, dt \, .$$

But since $u_k \in F(\sigma_k, \sigma_k, \tau_k)$ and $\tau_k / \sigma_k^2 \to 0$ as $k \to \infty$, it follows that

$$\partial_n u_k + 1 \geq -|\nabla u_k| + 1 \ \geq \ -o(\sigma_k^2).$$

Consequently

$$
\begin{aligned}
\int_{\Sigma_k} |\partial_n w_k(x', \sigma_k g(x', t), t)| \ &= \ \int_{\Sigma_k} \left| \frac{\partial_n u_k(x', \sigma_k g(x', t), t) + 1}{\sigma_k} \right| \\
\leq \ \int_{\Sigma_k} 2 \max \left(- \frac{\partial_n u_k(x', \sigma_k g(x', t), t) + 1}{\sigma_k}, 0 \right) \ &+ \ \int_{\Sigma_k} \frac{\partial_n u_k(x', \sigma_k g(x', t), t) + 1}{\sigma_k} \\
&\leq C_9 \sigma_k \to 0 \text{ as } k \to \infty \ ,
\end{aligned}
$$

and *Claim 3* is proved.

Proof of the Proposition: Let $\zeta \in C_0^1(Q_{1/2})$. From Claim 3, from the fact that w_k is caloric in $Z^-(\sigma_k g)$, from (7.3) and from a standard energy estimate for caloric functions we infer now that

$$
\begin{aligned}
o(1) \ &= \ \int_{\Sigma_k} \zeta \, \partial_n w_k(x', \sigma_k g(x', t), t) \nu_n \\
&= \ \int_{Z^-(\sigma_k g)} (\partial_n \zeta \, \partial_n w_k \ - \ \zeta \, \Delta' w_k \ + \ \zeta \, \partial_t w_k) \\
&= \ o(1) \ + \ \int_{Z^-(\sigma_k g)} (\partial_n \zeta \, \partial_n w_k \ + \ \nabla' \zeta \cdot \nabla' w_k \ - \ w_k \, \partial_t \zeta) \\
&\to \ \int_{Q_1^+} (\partial_n \zeta \, \partial_n w \ + \ \nabla' \zeta \cdot \nabla' w \ - \ w \, \partial_t \zeta) \text{ as } k \to \infty \ ;
\end{aligned}
$$

here ν is the outward unit space normal on $\partial Z^-(\sigma_k g)$. It follows that $\partial_n w = 0$ on $Q'_{1/2}$ in the sense of distributions. □

Corollary 7.2. *Suppose that the assumptions of Lemma 6.1 are satisfied and that k is the subsequence of Lemma 6.1. Then $f \in C^\infty(Q_{1/2})$; moreover,*

$$\left| \frac{\partial^{\alpha + k} f}{\partial x^\alpha \partial t^k} \right| \leq C(n, |\alpha|, k)$$

in $Q_{1/4}$ for any $k \in \mathbf{N}$ and multi-index $\alpha \in \mathbf{N}^n$.

Proof. Since $\partial_n w = 0$ on $Q'_{1/2}$ in the sense of distributions we may reflect w to a caloric function in $Q_{1/2}$. As $f = w|_{Q'_1}$ and $\|w\|_{L^\infty(Q_{3/4})} \leq C(n)$ (see Proposition 6.2), the result follows from standard regularity theory of caloric functions. □

8. Flatness improvement and regularity

Concluding regularity is then a standard procedure. The following Lemma 8.1, Lemma 8.3 and Theorem 8.4 extend Lemma 7.9, Lemma 7.10 and Theorem 8.1 in [1]. Finally, we apply Theorem 8.4 to regular free boundary points, i.e. points in the set R defined in [11, Proposition 9.1] (or the proof of Proposition 7.1) to obtain that R is open relative to $\partial\{u > 0\}$.

Lemma 8.1. *Let $\theta \in (0,1)$. Then there exists a constant $\sigma_\theta > 0$ depending only on θ and the dimension n such that if $\sigma < \sigma_\theta$, $\tau \le \sigma_\theta \sigma^2$ and $u \in F(\sigma, \sigma, \tau)$ in Q_ρ in direction η, then*

$$u \in F(\theta\sigma, 1, \tau) \quad \textit{in } Q_{c(n)\theta\rho}(\vartheta\eta, 0)$$

in direction $\overline{\eta}$ for some $\vartheta \in [-\sigma, \sigma]$ and some $\overline{\eta}$ satisfying $|\overline{\eta} - \eta| \le C(n)\sigma$. Here $c(n) > 0$ and $C(n) < +\infty$ are constants depending only on the dimension n.

Proof. We may rotate the coordinate system so $\eta = e_n$, and we may assume that $\rho = 1$. By a contradiction argument, it is sufficient to prove the statement of the lemma for u_k as in Lemma 6.1 and every large k.
First, observe that by Corollary 7.2,

$$f(x', t) \le f(0,0) + \ell \cdot x' + C(|x'|^2 + |t|) \text{ in } Q'_{1/4},$$

where ℓ is the space gradient of f, $|\ell| \le C$ and C depends only on the dimension n. Thus

$$f(x', t) \le f(0,0) + \ell \cdot x' + \frac{\theta}{4}\frac{\theta}{4C} \text{ in } Q_{\theta/(4C)} \; .$$

It follows that for large k the function f_k^+ in Lemma 6.1 satisfies

$$f_k^+(x', t) \le f(0,0) + \ell \cdot x' + \theta\frac{\theta}{4C} \quad \text{ in } Q_{\theta/(4C)} \; .$$

This means that $u_k \in F(\theta\sigma, 1, \tau)$ in $Q_{\theta/(4C)}(0, f(0,0), 0)$ in the direction $\bar{\eta}$, where

$$\bar{\eta} = \frac{(-\sigma_k \ell, 1)}{\sqrt{1 + |\sigma_k \ell|^2}}.$$

The lemma follows. □

Lemma 8.2. *Let u be a solution in the sense of Definition 3.2. Then*

$$\max(|\nabla u|^2 - 1, 0)(x,t) \to 0 \; \textit{as } 0 < \operatorname{pardist}((x,t), \{u = 0\}) \to 0 \; .$$

Proof. Consider a sequence $\{u > 0\} \ni (x_k, t_k) \to (x_0, t_0)$ such that

$$1 < \ell := \limsup_{\{u>0\} \ni (x,t) \to (x_0,t_0)} |\nabla u(x,t)|^2 = \lim_{k \to \infty} |\nabla u(x_k, t_k)|^2 \; .$$

Setting $r_k := \operatorname{pardist}((x_k, t_k), \{u = 0\})$, the blow-up sequence

$$u_k(y, s) := \frac{u(x_k + r_k y, t_k + {r_k}^2 s)}{r_k}, \chi_k(y, s) := \chi(x_k + r_k y, t_k + {r_k}^2 s)$$

converges to a solution (u_0, χ_0) in the sense of Definition 3.2 satisfying $u_0 > 0$ in Q_1, $|\nabla u_0|^2 \le \ell$ and $|\nabla u_0(0)|^2 = \ell$. The strong maximum principle implies that $u_0(y,s) = \ell \max(y \cdot e, 0)$ in $\{y \cdot e > 0\} \cap \{s < 0\}$ for some $e \in \partial B_1$.
From [11, Theorem 11.1] we infer that

$$\{y \cdot e = 0\} \cap \{s < 0\} \subset \Sigma_{**}$$

up to a set of vanishing $\mathcal{L}^{n-1}$-measure, where

$$\Sigma_{**}(t) := \{x \in \partial\{u_0(t) > 0\} \,:\, \text{there is } \theta(x,t) \in (0,1] \text{ and } \xi(x,t) \in \partial B_1(0) \text{ such that } \frac{u_0(x+ry, t+r^2 s)}{r} \to \theta(x,t)|y \cdot \xi(x,t)| \text{ locally uniformly in } (y,s) \in \mathbf{R}^{n+1} \text{ as } r \to 0\}\ .$$

However $\theta(x,t) \in (0,1]$ contradicts $\ell > 1$. □

Lemma 8.3. *For every $\theta \in (0,1)$ there exist $\sigma_\theta > 0$ and $c_\theta \in (0,1/2)$ depending only on θ and the dimension n such that if $u \in F(\sigma, 1, \tau)$ in Q_ρ in direction η with $\sigma \le \sigma_\theta$ and $\tau \le \sigma_\theta \sigma^2$ then $u \in F(\theta\sigma, \theta\sigma, \theta^2\tau)$ in $Q_{c_\theta \rho}(\bar{y}, 0)$ in the direction $\bar{\eta}$ for some $\bar{y}, \bar{\eta}$ satisfying $|\bar{\eta} - \eta| \le C(n)\sigma$ and $|\bar{y}| \le C(n)\sigma$. Here $C(n)$ depends only on the dimension n.*

Proof. We may assume that $\rho = 1$.
From Lemma 5.2 we infer that $u \in F(C\sigma, C\sigma, \tau)$ in $Q_{1/2}(y,0)$ in direction η for some $y \in B_{C\sigma}$. Consequently we may apply Lemma 8.1 to deduce that for some θ_1 to be determined later, $u \in F(C\theta_1\sigma, 1, \tau)$ in $Q_{c(n)\theta_1}(\tilde{y}, 0)$ in the direction $\bar{\eta}$ such that $|\eta - \bar{\eta}| \le C\sigma$ and $|\tilde{y} - \bar{y}| \le (C+1)\sigma < 1/2$, provided that σ_θ has been chosen small enough in terms of θ_1.
In order to be able to continue we need to show improvement with respect to the τ-variable. To that end, observe that $U = \max(|\nabla u| - 1, 0)$ is by Lemma 8.2 a continuous subcaloric function in Q_1 with boundary values less than $\tau\chi_{\{u>0\}} \le \tau\chi_{\{x_n \le \sigma\}}$. We may therefore compare U to the caloric function with boundary values $\tau\chi_{\{x_n \le \sigma\}}$. It follows that $0 \le U \le (1-c_1)\tau$ in $Q_{1/2}$ for some $c_1 > 0$ depending only on the dimension n. Thus $u \in F(C\theta_1\sigma, 1, (1-c_1)\tau)$ in $Q_{c(n)\theta_1}(\tilde{y}, 0)$ in the direction $\bar{\eta}$. Choosing $\theta_0 := \sqrt{1-c_1}$ and $\theta_1 := \theta_0/C$ we obtain $u \in F(\theta_0\sigma, 1, \theta_0^2\tau)$ in $Q_{c_2\theta_0}(y,0)$ in the direction $\bar{\eta}$ such that $|\eta - \bar{\eta}| \le C\sigma$, where $c_2 \in (0,1)$ depends only on the dimension n.
Iterating this process we see that

$$u \in F(\theta_0^m \sigma, 1, \theta_0^{2m}\tau) \text{ in } Q_{(c_2\theta_0)^m}(y_m, 0) \text{ in the direction } \bar{\eta}_m$$

where $|\eta - \bar{\eta}_m| \le C(n)\sigma \sum_{j=0}^{m-1} \theta_0^j$ and $|y_m| \le C(n)\sigma \sum_{j=0}^{m-1} (c_2\theta_0)^j$.
Applying once more Lemma 5.2 and choosing $\theta_0 := \theta^{\frac{1}{m}}/C$ we obtain the statement of the lemma. □

Theorem 8.4. *There exists a constant* $\sigma_0 > 0$ *such that if* $u \in F(\sigma, 1, \tau)$ *in* $Q_\rho(t_0, x_0)$, $\sigma \leq \sigma_0$ *and* $\tau \leq \sigma_0 \sigma^2$, *then the topological free boundary* $\partial\{u > 0\}$ *is in* $Q_{\rho/4}(t_0, x_0)$ *the graph of a* $\mathbf{C}^{1+\alpha,\alpha}$*-function; in particular the space normal is Hölder continuous in* $Q_{\rho/4}(t_0, x_0)$.

Proof. Using Lemma 8.3 inductively we see that
(8.1)

$u \in F(\theta^k\sigma, \theta^k\sigma, \theta^{2k}\tau)$ in $Q_{c_{\frac{\theta}{2}}{}^k\rho}(y,s)$ in the direction $\overline{\eta}^k$

where $|\overline{\eta}^k - \eta| \leq C(n)\sigma \sum_{j=0}^{k-1}(2\theta)^j$ and $|\overline{y}^k - y| \leq C(n)\sigma \sum_{j=0}^{k-1}(2c_{\theta/2}\theta)^j$,

provided that $(y, s) \in Q_{1/2}(t_0, x_0) \cap \partial\{u > 0\}$, $\theta < 1/4$ and

$$\sigma_0 < \min(1/(4C(n)), \sigma_{\theta/2}/2) ;$$

here we sacrificed some flatness in order to keep the original free boundary point (y, s). We obtain existence of the outward space normal ν on $Q_{1/2}(t_0, x_0)$. Moreover, ν satisfies by (8.1)

$$\mathrm{osc}_{Q_{c_{\theta/2}{}^k\rho}(y,s)}\nu \leq C(n,\theta)\theta^k\sigma ,$$

which implies Hölder-continuity of ν. □

Corollary 8.5. *For each point* (x_0, t_0) *of the set R, the topological free boundary* $\partial\{u > 0\}$ *is in an open neighborhood of* (x_0, t_0) *the graph of a* $\mathbf{C}^{1+\alpha,\alpha}$*-function; in particular, the space normal is Hölder continuous in an open space-time neighborhood of* (x_0, t_0).

Proof. The Corollary follows from [11, Proposition 9.1] and Lemma 8.2. □

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