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$C^{1,\alpha}$ Regularity for Solutions to the p-Harmonic thin Obstacle Problem.

by

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Abstract

We prove $C^{1,\alpha}$ regularity for a thin obstacle problem for the p-laplace equation. Due to the nonlinearity of the p-laplace operator we can not use the same methods used for the Laplace case, instead we use techniques developed by E. de Giorgi.

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1 Introduction

In this article we are interested in the minimisers of, for $1 < p < \infty$,

$$J(u) = \int_{B_1^+} |\nabla u|^p dx,$$

for $1 < p < \infty$,over the set $\{u \in W^{1,p}(B_1^+);\ u = f \text{ on } \partial B_1 \text{ and } u \geq 0 \text{ on } \Pi\}$, where $\Pi = B_1 \cap \{x_n = 0\}$. The equality $u = f$ on ∂B_1 is understood in the trace sense and f is assumed to be a function in C^∞. Weaker assumptions on f is possible, but since we are interested in the regularity of u in $\overline{B_{1/2}^+}$ so the exact assumptions on f is not of vital importance.

The Euler equations associated with this problem is

$$\begin{array}{ll} \mathrm{div}(|\nabla u|^{p-2}\nabla u) = 0 & \text{in } B_1^+ \\ u = f & \text{on } \partial B_1 \setminus \Pi \\ u \geq 0 & \text{on } \Pi \cap B_1 \\ \frac{\partial u}{\partial x_n} = 0 & \text{on } \pi \cap \{u = 0\}. \end{array}$$

This problem is known as the thin obstacle problem. In the case $p = 2$ this problem have been investigated by several authors, most recently in [1]. There L.A. Caffarelli and I. Athanasopoulos proves the optimal regularity for minimisers. Using the linear structure of the Laplace equation and monotonicity formulas they deduce that solutions are in $C^{1,1/2}$.

To the authors knowledge, nothing was known of the regularity of minimisers in the case $p \neq 2$, prior to this publication. In this paper we prove that minimisers are $C^{1,\alpha}$ for some $\alpha > 0$. In proving the $C^{1,\alpha}$ regularity we run in to considerate difficulties and the proofs in [1] are in general not applicable in the p-harmonic setting. Instead we use a modification of the regularity theory of E. de Giorgi. This have enough strength to deduce our regularity theorem. However, lacking a monotonicity formula, we have not been able to deduce the optimal regularity. This is not very surprising since the optimal regularity is not known in the interior for the p-laplace equation.

The theory on the interior regularity of p-harmonic functions is vast. The reader interested in that theory is refered to [2] and [5] for the interior regularity and [6] for regularity for the p-harmonic (thick) obstacle problem.

The structure of this paper is as follows. In the next section we prove weak regularity results. In section 3 we work through the regularity theory of de Giorgi in our setting and in the final section we state and prove our main regularity result.

Notation. Throughout this article we will try to follow the notation as established in [3]. However in this section we remind the reader of the most basic notations that we will use. $x = (x_1, x_2, ..., x_n)$ will denote a points in the n-dimensional space of real numbers $\mathbf{R}^n$. For an open ball centred at x^0 with radius r we write $B_r(x^0)$, we will also use $B_r(x^0)^+$ to indicate the ball intersected with the upper half space $\{x_n > 0\}$. The centre of the ball will in general not be indicated if it is given by context or if it is the origin. $W^{k,p}(A)$ will denote the usual sobolev space of functions defined on A whose distributional derivatives up to order k belongings to the usual Lebesgue space $L^p(A)$.

2 Weak Regularity of the Solution.

In this section we prove two weak regularity results. The proof of the first lemma follows the proof in [1].

Lemma 2.1. *Let u be a minimiser then $u \in W^{1,\infty}(B^+_{1/2})$.*

Proof: Let w be the solution to $div(|\nabla w|^{p-2}\nabla w) = 0$ and $w = \inf_{\partial B_1} u$ on ∂B_1 and $w = 0$ on $\{x_n = 0\}$ then $w \in W^{1,\infty}(B_{1/2})$, say $|\nabla w| \le C$ and $u \ge w$. In particular u, reflected in $\{x_n = 0\}$, is a solution to the obstacle problem with obstacle w.

That gives a bound on the growth of u from below. If $x^0 \in \Pi \cap \{u = 0\}$ then $u(x) \ge -C|x - x^0|$. Therefore we will have $u(x) + 2Cr \ge 0$ in $B_{2r}(x^0)$. Also $u(x^0) + 2Cr = 2Cr$ and u is a super-solution. By the Harnack inequality it follows that $u(x) + 2Cr \le C_0(u(x^0) + 2Cr) \le 2CC_0 r$ in $B_r(x^0)$. So u grows away from the contact set in a Lipschitz manner. The Lemma follows by standard techniques and the known interior $C^{1,\alpha}$ regularity of u. □

The following proposition is a standard difference quotient proof of almost $W^{2,2}$-regularity.

Proposition 2.2. *Let u be a minimiser, then $|\nabla u|^{\frac{p-2}{2}}\nabla u \in W^{1,2}(\overline{B}^+_{1/2})$.*

Proof: For each $h \in \Pi$ and small $t > 0$ the function $(1-t)u(x) + t\phi(x) = (1-t)u + t\xi^p(u_h - u)$, with $u_h(x) = u(x+h)$, is a competitor for minimality, here $\xi \in C_0^\infty(B_{3/4})$ and $\xi = 1$ on $B_{1/2}$ and $|\nabla \xi| \le 8$. Therefore

$$0 \le \int_{B_1^+} |\nabla u|^{p-2}\nabla u \cdot \nabla \phi = \int_{B_1^+} \xi^p |\nabla u|^{p-2}\nabla u \cdot \nabla(u_h - u) + (u_h - u)|\nabla u|^{p-2}\nabla u \nabla \phi^p.$$

Also u_h is a solution to the thin obstacle problem in B^+_{1-h} therefore we may test u_h with $(1-t)u_h + t\eta$ where $\eta = \xi^p(u - u_h)$ and deduce

$$0 \le \int_{B_1^+} |\nabla u_h|^{p-2}\nabla u_h \cdot \nabla \eta = \int_{B_1^+} \xi^p |\nabla u_h|^{p-2}\nabla u_h \cdot \nabla(u - u_h) + (u - u_h)|\nabla u_h|^{p-2}\nabla u_h \nabla \xi^p.$$

Adding the two inequalities results and we will get the following estimate

$$\int_{B_1^+} \xi^p \big(|\nabla u_h|^{p-2}\nabla u_h - |\nabla u|^{p-2}\nabla u\big) \cdot \nabla(u_h - u) \le$$

$$\int_{B_1^+} (u_h - u)\big(|\nabla u_h|^{p-2}\nabla u_h - |\nabla u|^{p-2}\nabla u\big) \cdot \nabla \xi^p \le$$

$$C \int_{B_1^+} |u_h - u||\nabla \xi| \big||\nabla u_h|^{\frac{p-2}{2}}\nabla u_h - |\nabla u|^{\frac{p-2}{2}}\nabla u\big| \xi^{\frac{p}{2}} \big(|\nabla u_h|^p + |\nabla u|^p\big)^{\frac{p-2}{2p}} \xi^{\frac{p-2}{2}} \le$$

$$C\bigg(\int_{B_1^+} \xi^p \big(|\nabla u_h|^{\frac{p-2}{2}}\nabla u_h - |\nabla u|^{\frac{p-2}{2}}\nabla u\big)^2\bigg)^{\frac{1}{2}} \bigg(\int_{B_1^+} \xi^p \big(|\nabla u_h|^p + |\nabla u|^p\big)\bigg)^{\frac{p-2}{2p}} \bigg(\int_{B_1^+} |u_h - u|^p |\nabla \xi|^p\bigg)^{\frac{1}{p}}.$$

Rewriting the left hand side using the inequality, for $p \ge 2$

$$||a|^{\frac{p-2}{2}}a - |b|^{\frac{p-2}{2}}b| \le \frac{p^2}{4}(|a|^{\frac{p-2}{2}}a - |b|^{\frac{p-2}{2}}b) \cdot (a - b),$$

and for $1 < p < 2$

$$(p-1)|a-b|(|a|^2+|b|^2)^{\frac{p-2}{2}} \leq |a|^{p-2}a - |b|^{p-2}b.$$

If we let $h \to 0$ we will get

$$\int_{B^+_{1/2}} \Big(\frac{\partial}{\partial h}|\nabla u|^{\frac{p-2}{2}}\nabla u\Big)^2 \leq C\|\nabla u\|^p_{L^p(B_1)}.$$

This proves the tangential derivatives of $|\nabla u|^{\frac{p-2}{2}}\nabla u$ are in L^2. The regularity of the n derivatives follows, by standard methods, using that u is a solution of an elliptic equation. □

3 Reminder of de Giorgi's Regularity Theory.

In this section we remind the reader of some classical regularity Lemmas of E. de Giorgi as well as adapt the Lemmas to our needs. Our presentation and proofs are very similar to the ones presented in [4].

3.1 Two preliminary Lemmas.

In this sub-section we will recall two simple lemmas that will be needed in the regularity theory.

Lemma 3.1. *Let $Z(t)$ be a bounded non negative function in $[\rho, R]$ and assume that for $\rho \leq t \leq s \leq R$we have*

$$Z(t) \leq [A(s-t)^{-\alpha} + B(s-t)^{-\beta}C] + \eta Z(s) \tag{3.1}$$

with $A, B, C \geq 0$, $\alpha > \beta > 0$ and $0 < \eta < 1$. Then

$$Z(\rho) \leq c[A(R-\rho)^{-\alpha} + B(R-\rho)^{-\beta} + C],$$

for a constant c depending only on α and η.

Proof: Consider the sequence t_i, $t_0 = \rho$ and

$$t_{i+1} - t_i = (1-\lambda)\lambda^i(R-\rho),$$

$0 < \lambda < 1$ and $\eta\lambda^{-\alpha} \leq 1$. Then the Lemma follows by induction using equation (3.1). □

Lemma 3.2. *Let $\alpha > 0$ and let $t_i \geq 0$ such that*

$$t_{i+1} \leq CK^i t_i^{1+\alpha},$$

with $C > 0$ and $K > 1$. Then if $t_0 \leq C^{-\frac{1}{\alpha}}K^{-\frac{1}{\alpha^2}}$ we have

$$t_i \leq K^{-\frac{i}{\alpha}}t_0.$$

Proof: This follows by an easy induction argument. □

3.2 de Giorgi's Regularity Lemmas.

Taking a directional derivative, in direction η, of

$$\operatorname{div}(|\nabla u|^{p-2}\nabla u) = 0$$

we will get

$$(a^{ij}|\nabla u|^{p-2}u_{\eta i})_j = 0,$$

with

$$a^{ij} = \delta_{ij} + (p-2)\frac{u_i u_j}{|\nabla v|^2}.$$

In particular the directional derivatives of v solves an elliptic partial differential equation in divergence form. Moreover this equation is uniformly elliptic whenever $v_\eta > k > 0$. Therefore it is natural to investigate the Hölder theory for elliptic equations. In this section we will work through the de Giorgi's regularity theory for divergence type equations in our setting. Most of the results in this section are slight variations of the theory found in [4]

The right boundary value problem when $\eta = e_k$, and $v = u_k$, is

$$\begin{array}{ll} (a^{ij}|\nabla u|^{p-2}v_j)_i = 0 & \text{in } B_1^+ \\ v = 0 & \text{on } \Pi\cap\Gamma \\ \frac{\partial v}{\partial x_n} = 0 & \text{on } \Pi\cap\Sigma \end{array} \tag{3.2}$$

where $\lambda|\xi|^2 \le a^{ij}\xi_i\xi_j \le \Lambda|\xi|^2$ for some $0<\lambda<\Lambda<\infty$, $\Sigma\cup\Gamma = \Pi$ and $\Sigma\cap\Gamma = \emptyset$.

We will also use the following notation;

$$A(k) = \{x\in B_1^+;\ v(x)\ge k\}.$$

Our first lemma states that the directional derivatives of a solution are in the de Giorgi classes.

Lemma 3.3. *Let v be a solution to (3.2), then for $k>0$*

$$\int_{B_\rho^+\cap A(k)} |\nabla v|^2 \le \frac{C}{k^{(p-2)}\rho^2}\int_{B_{2\rho}^+\cap A(k)} (v-k)_+^2.$$

Proof: Let ϕ be a smooth test function vanishing on $\partial B_1\cup\Gamma$ then, using that v is a weak solution (3.2),

$$0 = \int_{B_{2\rho}^+} (|\nabla u|^{p-2}a^{ij}v_j)_i\phi = -\int_{2B_\rho^+} a^{ij}v_j\phi_i + \int_\Gamma \phi a^{nj}|\nabla u|^{p-2}v_j\phi.$$

Now we choose $\phi = \eta(v-k)^+$ for $\eta\in C^\infty(B_{2\rho}^+)$, $\eta = 1$ in B_ρ^+, $\eta = 0$ on $\partial B_{2\rho}^2$ and $|\nabla\eta|\le 2/\rho$. This gives

$$0 = \int_{B_{2\rho}^+} v_j|\nabla u|^{p-2}a^{ij}\eta_i(v-k)^2 + \int_{B_{2\rho}^+} v_j|\nabla u|^{p-2}a^{ij}(v-k)_i^2\eta =$$

$$\int_{B_{2\rho}^+} v_j |\nabla u|^{p-2} a^{ij} \eta_i (v-k)^2 + \int_{B_{2\rho\cap A(k)}^+} v_j |\nabla u|^{p-2} a^{ij} v_i \eta.$$

Rewrite the above equality, using A for the matrix $|\nabla u|^{p-2}a^{ij}$, ellipticity and that $|\nabla u| \geq k$ in $A(k)$,

$$\int_{B_\rho^+ \cap A(k)} |\nabla v|^2 \leq -\frac{C}{k^{p-2}} \int_{B_{2\rho}^+} (\nabla v A \nabla \eta)(v-k)^+ \leq \frac{C}{k^{p-2}} \left(\int_{B_{2\rho} \setminus B_\rho} |\nabla v|^2 + \frac{((v-k)^+)^2}{\rho^2} \right).$$

Now we add

$$\frac{C}{k^{p-2}} \int_{B_\rho^+ \cap A(k)} |\nabla v|^2$$

to both sides and divide by $C+1$, the result then follows from Lemma 3.1. $\square$

Next we need control over the set where a solution is large.

Lemma 3.4. *If $v \leq 1$, for a solution v of equation (3.2), and $|A(k_0) \cap B_R^+| \leq \gamma |B_R^+|$ for $\gamma < 1$ then for every large constant C there exists a constant $k < 1$ such that*

$$|A(k) \cap B_R^+| < \frac{|B_R^+|}{C}.$$

Proof: To avoid unnecessary complicated notation we will assume that $k_0 = 1/2$. Define the following function

$$w = \begin{cases} k - h & v \geq k \\ u - h & h < v < k \\ 0 & v \leq h, \end{cases}$$

for two constants $1/2 < h < k < 1$. Then

$$w = 0 \quad \text{in } B_R^+ \setminus A(1/2)$$

and $|B_R^+ \setminus A(1/2)| \geq (1-\gamma)|B_R^+|$ so we may use Sobolev's inequality and deduce

$$\left(\int_{B_R^+} w^{n/(n-1)} \right)^{(n-1)/n} \leq \int_S |\nabla w| = \int_S |\nabla v| \leq C|S|^{1/2} \left(\int_{A(h) \cap B_R^+} |\nabla v|^2 \right)^{1/2},$$

where $S = (A(h) \setminus A(k)) \cap B_R^+$. By the definition of w we have $w = k-h$ on $A(k) \cap B_R^+$ which implies

$$(k-h)|A(k) \cap B_R^+|^{(n-1)/n} \leq C|S|^{1/2} \left(\int_{A(h) \cap B_R^+} |\nabla v|^2 \right)^{1/2}.$$

Now we use Lemma 3.3 and deduce

$$(k-h)|A(k) \cap B_R^+|^{(n-1)/n} \leq C\frac{|S|^{1/2}}{R} \left(\int_{A(h) \cap B_R^+} ((v-k)^+)^2 \right)^{1/2} \leq C|S|^{1/2} R^{(n-1)/2}(1-h). \tag{3.3}$$

Substitute $h = h_i = 1 - 2^{-i}$ and $k = k_i = 1 - 2^{-i-1}$ in equation (3.3) and we get (with S_i being the S corresponding to our choice of h_i and k_i)

$$|A(1-2^{-i-1}) \cap B_R^+|^{\frac{n-1}{n}} \leq C|S_i|R^{\frac{n-1}{2}}. \tag{3.4}$$

We finish this proof by an argument of contradiction. If the Lemma is false then, for all i,

$$|A(1-2^{-i-1}) \cap B_R^+| > \frac{R^n}{C_0},$$

where C_0 is a large constant to be determined later. Insert this in equation (3.4),

$$\frac{R^{n-1}}{C_0} \leq C|S_i|R^{\frac{n-2}{n}}.$$

That is

$$\frac{R^n}{(CC_0)^2} \leq |S_i|.$$

But by the definition of $|S_i|$ we have $\cup_i S_i \subset B_R^+$ this gives our contradiction since the last sum in the next equation diverges

$$|B_R^+| \geq \sum_{i=1}^{\infty} |S_i| \geq \sum_{i=1}^{\infty} \frac{R^n}{(CC_0)^2}.$$

□

The final contribution we need is an estimate controlling the supremum of a solution to (3.2).

Lemma 3.5. *Let v be a solution of (3.2), then for $k_0 \geq 1/2$*

$$sup_{B^+_{\rho/2}} v \leq C\left(\frac{1}{\rho^n}\int_{B_\rho^+}((v-k_0)^+)^2\right)^{1/2}\left(\frac{|A(k_0)\cap B_\rho^+|}{\rho^n}\right)^{\frac{\alpha}{2}} + k_0,$$

where $\alpha^2 + \alpha = 2/n$.

Proof: Assume that $\rho = 1$, $k_0 = 0$ and let $1/2 < \sigma < \tau < 1$, also set $w = \eta(u-k)^+$ for $k > 0$, with $\eta \in C^\infty(B^+_{(\sigma+\tau)/2})$, $\eta = 1$ in B_σ^+, $\eta = 0$ on $\partial B_{(\sigma+\tau)/2}$ and $|\nabla \eta| \leq 4/(\tau - \sigma)$, then

$$\int_{A(k)\cap B_\sigma^+}((v-k)^+)^2 \leq \int_{A(k)\cap B^+_{(\sigma+\tau)/2}} w^2 \leq$$

$$\left(\int_{A(k)\cap B^+_{(\sigma+\tau)/2}} w^{\frac{2n}{n-2}}\right)^{\frac{n-2}{n}} |A(k)\cap B^+_{(\sigma+\tau)/2}|^{\frac{2}{n}} \leq \left(\int_{A(k)\cap B^+_{(\sigma+\tau)/2}} |\nabla w|^2\right)|A(k)\cap B^+_{(\sigma+\tau)/2}|^{\frac{2}{n}}.$$

Using the definition of w and Lemma 3.3 we may deduce

$$\int_{A(k)\cap B_\sigma^+}((v-k)^+)^2 \leq \frac{C|A(k)\cap B^+_{(\sigma+\tau)/2}|^{\frac{2}{n}}}{(\tau-\sigma)^2}\int_{A(k)\cap B^+_{(\sigma+\tau)/2}}(v-k)^2. \tag{3.5}$$

To continue we need to estimate the term $|A(k)\cap B_\tau^+|^{\frac{2}{n}}$ from above;

$$|A(k)\cap B_\tau^+| \le \frac{1}{(k-h)^2}\int_{A(h)\cap B_\tau^+}(v-h)^2,$$

for $h<k$. Taking this to the power α, to be determined later, and multiplying this inequality with the respective sides of equation 3.5 we will get;

$$|A(k)\cap B_\tau^+|^\alpha \int_{A(k)\cap B_\sigma^+}((v-k)^+)^2 \le \frac{C|A(k)\cap B_\tau^+|^{\frac{2}{n}}}{(\tau-\sigma)^2}\left(\int_{A(h)\cap B_\tau^+}(v-h)^2\right)^{1+\alpha}. \quad (3.6)$$

To conclude the proof we choose, for a d to be determined later,

$$k_i = 2d(1-2^{-i-1}),$$

$$\sigma_i = \frac{1}{2}(1+2^{-i}).$$

Then equation (3.6) implies, with $\sigma=\sigma_{i+1}$, $\tau=\sigma_i$, $k=k_{i+1}$, $h=k_i$, $\alpha^2+\alpha=2/n$ and

$$\Psi_i = |A(k_{i+1}\cap B_{\sigma_i}|^\alpha \int_{A(k_{i+1})\cap B_{\sigma_i}^+}(u-k_{i+1})^2,$$

that

$$\Psi_{i+1} \le C2^{2i}\Psi_i^{\frac{n+2}{n}}.$$

If we choose, for a large constant C,

$$d \ge C\Psi_0^{\frac{1}{2}}$$

then we can apply Lemma 3.2 and deduce

$$\lim_{i\to\infty}\Psi_i = 0.$$

This proves the Lemma for $k_0=0$, the general case follows by considering $v-k_0$. $\square$

4 The Main Regularity Theorem.

In the previous section we laid the foundation of the regularity proof, however we need some more information on the set where the partial derivatives are small to use Lemma 3.4 and 3.5. This can not be done without using the particular structure of our minimisation problem. As a matter of fact, solutions of (3.2) will not have any apriori Hölder estimates without any information on Γ. We establish control of the measure of the sets where the partial derivatives are small in the next Lemma.

Lemma 4.1. *Assume that we have a minimiser in* B_1^+ *and that* $sup_{B_1^+}|\nabla u| = 1$ *and that the origin is a boundary point of the contact set, that is* $0 \in \partial(\Pi \cap \{u = 0\})$*. Then if* $supu_i = M_i \geq 1/\sqrt{n}$ *for* $i = 1...n$ *there exists a constant* $\epsilon > 0$ *such that*

$$\int_{B_1^+} |u_i - M_i|^2 \geq \epsilon.$$

The same result is true for $-u$*.*

Proof: Let us first prove the statement in the case when $i = 1...n-1$. If the statement is not true then there exists a sequence of solutions u_j, with the origin at the boundary of the contact set, such that for at least one $i = 1...n-1$

$$\frac{\partial u_j}{\partial x_i} = M_i^j \geq \frac{1}{\sqrt{n}}$$

and

$$\int_{B_1^+} \left|\frac{\partial u_j}{\partial x_i} - M_i^j\right|^2 \to 0.$$

Using the weak regularity theory in section 2 we can conclude that, for a subsequence, $u_j \to u_0$ in $W^{1,q}$ (for any $q < \infty$), $M_i^j \to M_i^0$, and that

$$\frac{\partial u_0}{\partial x_i} = M_i^0.$$

The Lipschitz regularity, the possitivity of u on Π, and that the origin is a contact point contradicts this. Thus no such sequence exists and the result follows, in the case $i = 1...n-1$.

To prove the statement in the case $i = n$ we argue similarly. Let $\partial u_j/\partial x_n \geq 1/\sqrt{n}$ and

$$\int_{B_1^+} \left|\frac{\partial u_j}{\partial x_n} - sup_{B_1^+}\frac{\partial u_j}{\partial x_n}\right|^2 \to 0.$$

Then from the weak regularity theory of section 2 it follows that, for a subsequence, $u_j \to u_0$ in $W^{1,q}$ $(q < \infty)$ and that

$$|\nabla u_j|^{(p-2)/2}\frac{\partial u_j}{\partial x_n} \rightharpoonup |\nabla u_0|^{(p-2)/2}\frac{\partial u_0}{\partial x_n}$$

in $W^{1,2}$, also

$$\frac{\partial u_0}{\partial x_n} = M \geq \frac{1}{\sqrt{n}}. \tag{4.7}$$

Therefore $u_0 = 0$ on Π, in particular $u_0 = -x_n$. So, by the uniform convergence $|u_j + x_n| \leq \delta << 1$ when j is large, but this contradicts that the origin is on the boundary of the contact set. To see this we only need to consider the solution with boundary data $-x_n + \delta$ as barrier.

The same proof can be used for $-u$. □

Theorem 4.2. *If u is a minimiser then $u \in C^{1,\alpha}(\overline{B^+_{1/2}})$ for some $\alpha > 0$ depending only on p, $\|u\|_{L^\infty(B_1^+)}$ and n.*

Proof: We will show that if we have a solution in B_1^+ with the origin on the boundary of the contact set and with $\sup_{B_1^+}|\nabla u| = 1$ then $\sup_{B^+_{1/2}} u_i \le (1-\lambda)\sup_{B_1^+} u_i$ for all i such that $\sup_{B_1^+} u_i \ge 1/\sqrt{n}$. The theorem follows by rescaling and renormalising.

Assume that $\sup_{B_1^+} u_i \ge 1/\sqrt{n}$, then by Lemma 4.1 we have

$$\int_{B_1^+} |u_i - M_i|^2 \ge \epsilon.$$

Therefore $|A(k_0)| \le \gamma |B_1^+|$ for γ and k_0 close enough to 1 and M_i respectively. By Lemma 3.4 there is a k such that $|A(k, u_i)| \le |B_1^+|/C_0$ for a large constant C_0 to be chosen soon, here we have indicated that $A(k)$ is taken with respect to u_i. Finally by Lemma 3.5 we have

$$\sup_{B^+_{1/2}} u_i \le C\left(\int_{B_1^+}(u_i-k)^2\right)^{1/2} |A(k)\cap B_1^+|^{\frac{\alpha}{2}} + k.$$

If C_0 is chosen large enough (which changes k, however $k < 1$ for all C_0) we may deduce

$$\sup_{B_{1/2}} u_i \le \frac{C}{C_0^{\frac{\alpha}{2}}}(M_i - k) + k \le (1-\lambda)M_i,$$

for a universal $\lambda > 0$. The theorem follows by iterating this process, see for instance [2]. □

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