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signature (2,2) manifolds.

by

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**COMPLETENESS, RICCI BLOWUP, THE OSSERMAN AND THE
CONFORMAL OSSERMAN CONDITION FOR WALKER
SIGNATURE (2, 2) MANIFOLDS**

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ABSTRACT. Walker manifolds of signature (2, 2) have been used by many authors to provide examples of Osserman and of conformal Osserman manifolds of signature (2, 2). We study questions of geodesic completeness and Ricci blowup in this context.

1. INTRODUCTION

We adopt the following notational conventions. Let $\mathcal{M} := (M, g)$ be a *pseudo-Riemannian manifold* of signature (p, q) and dimension $m = p + q$, let ∇ be the *Levi-Civita connection* of \mathcal{M} , let $\mathcal{R}(x, y) := \nabla_x \nabla_y - \nabla_y \nabla_x - \nabla_{[x, y]}$ be the *curvature operator*, let $\mathcal{J}(x) : y \rightarrow \mathcal{R}(y, x)x$ be the *Jacobi operator*, and let $S^\pm(\mathcal{M})$ be the *pseudo-sphere bundles* of unit spacelike (+) and unit timelike (−) tangent vectors.

Relating properties of the spectrum of the Jacobi operator \mathcal{J} to the underlying geometry of the manifold is an important area of investigation in recent years; we refer to [13] for a fuller discussion than is possible here. One says that \mathcal{M} is *pointwise spacelike* (resp. *timelike*) *Osserman* if the spectrum of \mathcal{J} is constant on $S_P^+(\mathcal{M})$ (resp. on $S_P^-(\mathcal{M})$) for every point $P \in M$. If $p > 0$ and $q > 0$, these are equivalent concepts [16] so we shall simply speak of a *pointwise Osserman* manifold; in fact if $p > 0$ and if $q > 0$, one need only assume the spectrum of \mathcal{J} is bounded on $S^+(\mathcal{M})$ or on $S^-(\mathcal{M})$ to ensure \mathcal{M} is pointwise Osserman [1]. One replaces the word ‘pointwise’ by ‘globally’ if the spectrum does not in fact depend on P .

The field began with a question raised by Osserman [20] in the Riemannian setting. Let \mathcal{M} be a Riemannian 2-point homogeneous space, i.e. \mathcal{M} is either flat or is locally isometric to a rank 1-symmetric space. Osserman noted that the local isometries of \mathcal{M} act transitively on the bundle $S^+(\mathcal{M})$ and hence, in the notation adopted by subsequent authors, \mathcal{M} is globally Osserman. He wondered if the converse held: is any globally Osserman Riemannian manifold a local 2-point homogeneous space? This question has been called the *Osserman conjecture* and has been answered in the affirmative by the work of Chi [7] and of Nikolayevsky [19] except (possibly) in dimension 16 where the question is still open. There is a similar classification result in the Lorentzian setting. It is known [2, 15] that any locally Osserman Lorentzian manifold has constant sectional curvature. In the higher signature setting, such classification results fail. There are, for example, Osserman manifolds that are not even locally affine homogeneous [12, 18].

Let $\{e_i\}$ be a local frame for the tangent bundle and let $m = \dim(M)$. We set $g_{ij} := g(e_i, e_j)$ and let g^{ij} be the inverse matrix. The *Ricci operator* ρ , the associated *Ricci tensor* $\rho(\cdot, \cdot)$, the *scalar curvature* τ , the *Weyl conformal curvature*

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operator \mathcal{W} , and the *conformal Jacobi operator* \mathcal{J}_W are given by:

$$\begin{aligned} \rho x &:= \sum_{ij} g^{ij} \mathcal{R}(x, e_i) e_j, & \rho(x, y) &:= g(\rho x, y), & \tau &:= \sum_{ij} g^{ij} \rho(e_i, e_j), \\ \mathcal{W}(x, y) z &= \mathcal{R}(x, y) z + \frac{1}{(m-1)(m-2)} \tau \{g(y, z)x - g(x, z)y\} \\ &\quad - \frac{1}{m-2} \{g(x, z)\rho y + g(\rho x, z)y - g(y, z)\rho x - g(\rho y, z)x\}, \\ \mathcal{J}_W(x) &: y \rightarrow \mathcal{W}(y, x)x. \end{aligned}$$

One says that \mathcal{M} is *conformal Osserman* if for any P in M one has that \mathcal{J}_W has constant spectrum on $S^+(\mathcal{M}, P)$ for $p > 0$ or, equivalently, on $S^-(\mathcal{M}, P)$ for $q > 0$. This is a conformal notion [3]; if $\tilde{g} = e^\alpha g$ is conformally equivalent to g , then (M, g) is conformal Osserman if and only if (M, \tilde{g}) is conformal Osserman. Since any pointwise Osserman manifold is necessarily Einstein, any pointwise Osserman manifold is necessarily conformal Osserman.

We say that \mathcal{M} is *geodesically complete* if all geodesics exist for all time. We say that \mathcal{M} exhibits *Ricci blowup* if there exists a geodesic γ defined for $t \in [0, T)$ with $T < \infty$ and if $\lim_{t \rightarrow T} |\rho(\dot{\gamma}, \dot{\gamma})| = \infty$. Clearly if \mathcal{M} exhibits Ricci blowup, then it is geodesically incomplete and it can not be isometrically embedded in a geodesically complete manifold.

In this paper, we will concentrate on signature $(2, 2)$ where a great deal is known; the classification of Osserman algebraic curvature tensors of signature $(2, 2)$ is complete, see for example the discussion in [4]. Our focus will be to relate the Osserman condition and the conformal Osserman condition, which are purely algebraic conditions, to the global geometry of the manifold by studying questions of geodesic completeness and of Ricci blowup. Here is a brief outline to the paper. In Section 2, we introduce the family of Walker manifolds we shall be considering and give their geodesic equations. In Section 3, we show any strict Walker manifold is nilpotent Osserman and geodesically complete. In Section 4, we prove a result from the theory of ODEs which we will use subsequently to establish geodesic incompleteness and Ricci blowup.

In the indefinite setting, the spectrum of a self-adjoint operator does not determine the Jordan normal form of the operator. Let $k \in \mathbb{R}$. There is a family of Walker manifolds of signature $(2, 2)$ whose Jacobi operator has eigenvalues $\{0, 4k, k, k\}$ but whose Jacobi operator is not diagonalizable [10]. In Section 5 we show these manifolds exhibit *Ricci blowup*. In Section 6, we discuss examples of conformal Osserman manifolds which exhibit various eigenvalue structures following the discussion in [5]. We show that some of these manifolds are geodesically complete and others exhibit Ricci blowup. Throughout this paper, we shall focus on the global geometry of these manifolds and refer to previous results in the literature for the corresponding algebraic features of the curvature tensor.

2. SIGNATURE $(2, 2)$ WALKER MANIFOLDS

One says that a pseudo-Riemannian manifold \mathcal{M} of signature $(2, 2)$ is a *Walker manifold* if it admits a parallel totally isotropic 2-plane field. We refer to [6] for further details. Such a manifold is locally isometric to an example of the following form:

Definition 2.1. Let (x_1, x_2, x_3, x_4) be coordinates on \mathbb{R}^4 . Let $\psi_{ij}(\vec{x}) = \psi_{ji}(\vec{x})$ be smooth functions for $i, j = 3, 4$. Let $\partial_i := \partial_{x_i}$. Let $\mathcal{M} := (\mathbb{R}^4, g)$ where:

$$g(\partial_1, \partial_3) = g(\partial_2, \partial_4) = 1, \quad g(\partial_i, \partial_j) = \psi_{ij} \text{ for } i, j = 3, 4.$$

Lemma 2.2. Let $\psi_{ij/k} := \partial_k \psi_{ij}$ and let \mathcal{M} be as in Definition 2.1. Then the geodesic equations for \mathcal{M} are given by:

$$\begin{aligned} 0 &= \ddot{x}_1 + \dot{x}_1 \dot{x}_3 \psi_{33/1} + \dot{x}_1 \dot{x}_4 \psi_{34/1} + \dot{x}_2 \dot{x}_3 \psi_{33/2} + \dot{x}_2 \dot{x}_4 \psi_{34/2} \\ &\quad + \frac{1}{2} \dot{x}_3 \dot{x}_3 (\psi_{33/3} + \psi_{34} \psi_{33/2} + \psi_{33} \psi_{33/1}) \end{aligned}$$

$$\begin{aligned}
& +\dot{x}_3\dot{x}_4(\psi_{33/4} + \psi_{34}\psi_{34/2} + \psi_{33}\psi_{34/1}) \\
& +\frac{1}{2}\dot{x}_4\dot{x}_4(2\psi_{34/4} - \psi_{44/3} + \psi_{34}\psi_{44/2} + \psi_{33}\psi_{44/1}), \\
0 = & \ddot{x}_2 + \dot{x}_1\dot{x}_3\psi_{34/1} + \dot{x}_1\dot{x}_4\psi_{44/1} + \dot{x}_2\dot{x}_3\psi_{34/2} + \dot{x}_2\dot{x}_4\psi_{44/2} \\
& +\frac{1}{2}\dot{x}_3\dot{x}_3(2\psi_{34/3} - \psi_{33/4} + \psi_{44}\psi_{33/2} + \psi_{34}\psi_{33/1}) \\
& +\dot{x}_3\dot{x}_4(\psi_{44/3} + \psi_{44}\psi_{34/2} + \psi_{34}\psi_{34/1}) \\
& +\frac{1}{2}\dot{x}_4\dot{x}_4(\psi_{44/4} + \psi_{44}\psi_{44/2} + \psi_{34}\psi_{44/1}), \\
0 = & \ddot{x}_3 - \frac{1}{2}\dot{x}_3\dot{x}_3\psi_{33/1} - \dot{x}_3\dot{x}_4\psi_{34/1} - \frac{1}{2}\dot{x}_4\dot{x}_4\psi_{44/1}, \\
0 = & \ddot{x}_4 - \frac{1}{2}\dot{x}_3\dot{x}_3\psi_{33/2} - \dot{x}_3\dot{x}_4\psi_{34/2} - \frac{1}{2}\dot{x}_4\dot{x}_4\psi_{44/2}.
\end{aligned}$$

Proof. We use the discussion in [14] to determine the Christoffel symbols; the Lemma then follows:

$$\begin{aligned}
\nabla_{\partial_1}\partial_3 &= \frac{1}{2}\psi_{33/1}\partial_1 + \frac{1}{2}\psi_{34/1}\partial_2, & \nabla_{\partial_1}\partial_4 &= \frac{1}{2}\psi_{34/1}\partial_1 + \frac{1}{2}\psi_{44/1}\partial_2, \\
\nabla_{\partial_2}\partial_3 &= \frac{1}{2}\psi_{33/2}\partial_1 + \frac{1}{2}\psi_{34/2}\partial_2, & \nabla_{\partial_2}\partial_4 &= \frac{1}{2}\psi_{34/2}\partial_1 + \frac{1}{2}\psi_{44/2}\partial_2, \\
\nabla_{\partial_3}\partial_3 &= \frac{1}{2}(2\psi_{34/3} - \psi_{33/4} + \psi_{44}\psi_{33/2} + \psi_{34}\psi_{33/1})\partial_2 \\
& +\frac{1}{2}(\psi_{33/3} + \psi_{34}\psi_{33/2} + \psi_{33}\psi_{33/1})\partial_1 - \frac{1}{2}\psi_{33/1}\partial_3 - \frac{1}{2}\psi_{33/2}\partial_4, \\
\nabla_{\partial_3}\partial_4 &= \frac{1}{2}(\psi_{33/4} + \psi_{34}\psi_{34/2} + \psi_{33}\psi_{34/1})\partial_1 \\
& +\frac{1}{2}(\psi_{44/3} + \psi_{44}\psi_{34/2} + \psi_{34}\psi_{34/1})\partial_2 - \frac{1}{2}\psi_{34/1}\partial_3 - \frac{1}{2}\psi_{34/2}\partial_4, \\
\nabla_{\partial_4}\partial_4 &= \frac{1}{2}(2\psi_{34/4} - \psi_{44/3} + \psi_{34}\psi_{44/2} + \psi_{33}\psi_{44/1})\partial_1 \\
& +\frac{1}{2}(\psi_{44/4} + \psi_{44}\psi_{44/2} + \psi_{34}\psi_{44/1})\partial_2 - \frac{1}{2}\psi_{44/1}\partial_3 - \frac{1}{2}\psi_{44/2}\partial_4. \quad \square
\end{aligned}$$

We specialize this to the following situation. Assertion (1) in the following Lemma is a consequence of Lemma 2.2 while Assertion (2) in the following Lemma follows by specializing results of [11]:

Lemma 2.3. *Let $\mathcal{M} := (\mathbb{R}^4, g)$ where the non-zero components of g are given by $g(\partial_1, \partial_3) = g(\partial_2, \partial_4) = 1$ and $g(\partial_3, \partial_4) = \psi_{34}(x_1, x_2, x_3, x_4)$. Then:*

(1) *The geodesic equations are:*

$$\begin{aligned}
0 &= \ddot{x}_1 + \dot{x}_1\dot{x}_4\psi_{34/1} + \dot{x}_2\dot{x}_4\psi_{34/2} + \dot{x}_3\dot{x}_4\psi_{34}\psi_{34/2} + \dot{x}_4\dot{x}_4\psi_{34/4}, \\
0 &= \ddot{x}_2 + \dot{x}_1\dot{x}_3\psi_{34/1} + \dot{x}_2\dot{x}_3\psi_{34/2} + \dot{x}_3\dot{x}_3\psi_{34/3} + \dot{x}_3\dot{x}_4\psi_{34}\psi_{34/1}, \\
0 &= \ddot{x}_3 - \dot{x}_3\dot{x}_4\psi_{34/1}, & 0 &= \ddot{x}_4 - \dot{x}_3\dot{x}_4\psi_{34/2}.
\end{aligned}$$

(2) *The non-zero components of the Ricci tensor are:*

$$\begin{aligned}
\rho_{13} &= \rho_{24} = \frac{1}{2}\psi_{34/12}, & \rho_{14} &= \frac{1}{2}\psi_{34/11}, & \rho_{23} &= \frac{1}{2}\psi_{34/22}, \\
\rho_{33} &= \frac{1}{2}\{-\psi_{34/2}^2 + 2\psi_{34/23}\}, & \rho_{44} &= \frac{1}{2}\{-\psi_{34/1}^2 + 2\psi_{34/14}\}, \\
\rho_{34} &= \frac{1}{2}\{\psi_{34/1}\psi_{34/2} + 2\psi_{34}\psi_{34/12} - \psi_{34/13} - \psi_{34/24}\}.
\end{aligned}$$

3. STRICT WALKER MANIFOLDS

One says that \mathcal{M} is *nilpotent Osserman* if $\mathcal{J}(x)$ has only the eigenvalue 0 or, equivalently, if $\mathcal{J}(x)^{\dim(M)} = 0$ for any tangent vector x . One says that \mathcal{M} is a *strict Walker manifold* if $\psi_{ij} = \psi_{ij}(x_3, x_4)$. Our first result deals with such examples:

Theorem 3.1. *Let \mathcal{M} be as in Definition 2.1 with $\psi_{ij} = \psi_{ij}(x_3, x_4)$. Then \mathcal{M} is nilpotent Osserman and geodesically complete.*

Proof. By Lemma 2.2, the geodesic equations in this setting are:

$$\begin{aligned}
0 &= \ddot{x}_1 + \frac{1}{2}\dot{x}_3\dot{x}_3\psi_{33/3} + \frac{1}{2}\dot{x}_4\dot{x}_4(2\psi_{34/4} - \psi_{44/3}) + \dot{x}_3\dot{x}_4\psi_{33/4}, \\
0 &= \ddot{x}_2 + \frac{1}{2}\dot{x}_3\dot{x}_3(2\psi_{34/3} - \psi_{33/4}) + \frac{1}{2}\dot{x}_4\dot{x}_4\psi_{44/4} + \dot{x}_3\dot{x}_4\psi_{44/3}, \\
0 &= \ddot{x}_3, & 0 &= \ddot{x}_4.
\end{aligned}$$

The final two equations can be solved to yield $x_3 = a + bt$ and $x_4 = c + dt$. The first two equations then have the form $\ddot{x}_1 = f_1(t)$ and $\ddot{x}_2 = f_2(t)$ which can be solved. Thus geodesics extend for infinite time and \mathcal{M} is geodesically complete. One computes easily that $\mathcal{J}(x) : \text{Span}\{\partial_3, \partial_4\} \rightarrow \text{Span}\{\partial_1, \partial_2\} \rightarrow 0$ for any tangent

vector x and hence $\mathcal{J}(x)$ is nilpotent. This shows \mathcal{M} is nilpotent Osserman; we refer to the discussion in [17] for further details. \square

4. A RESULT FROM ODES

Before continuing our investigations further, we shall need the following result:

Lemma 4.1. *Let $f(t)$ satisfy $\ddot{f}(t) = \Xi(\dot{f}, f)$ with $f(0) = 1$ and $\dot{f}(0) = 1$ and maximal domain $[0, T)$. Assume $\Xi(x, y) \geq \varepsilon x^a y^b$ for $x \geq 1$ and $y \geq 1$ where $2a + b \geq 3$ and $\varepsilon > 0$. Then $T < \infty$, $\lim_{t \rightarrow T} \dot{f}(t) = \infty$, and $\lim_{t \rightarrow T} f(t) = \infty$.*

Proof. Since \ddot{f} is positive, f and \dot{f} are monotonically increasing. Suppose $T < \infty$ but $\lim_{t \rightarrow T} \dot{f} < \infty$. Then \dot{f} is bounded and hence f is bounded as well. Thus $\lim_{t \rightarrow T} \dot{f} = \dot{f}_T$ and $\lim_{t \rightarrow T} f = f_T$ exist and are finite and the fundamental theorem of ODEs shows $[0, T)$ is not the maximal domain of the function f . Thus if T is finite, $\lim_{t \rightarrow T} \dot{f}(t) = \infty$. Hence $\lim_{t \rightarrow T} \ddot{f}(t) = \infty$ as well and the Lemma holds.

To complete the proof, we suppose that $T = \infty$ and argue for a contradiction. Without loss of generality, we assume $\varepsilon < 1$. Let $t_1 := 0$ and let $t_{n+1} := t_n + \frac{3}{\varepsilon n^2}$. We wish to show $f(t_n) \geq n$ and $\dot{f}(t_n) \geq n^2$. As this holds for $n = 1$, we proceed by induction on n . Because \dot{f} is monotonically increasing,

$$f(t_{n+1}) \geq f(t_n) + \dot{f}(t_n) \frac{3}{\varepsilon n^2} \geq n + \frac{3n^2}{\varepsilon n^2} \geq n + 1.$$

To prove the second estimate, we use the mean value theorem to choose s with $s \in [t_n, t_{n+1}]$ so $\dot{f}(t_{n+1}) = \dot{f}(t_n) + \frac{3}{\varepsilon n^2} \ddot{f}(s)$. We may estimate that:

$$\ddot{f}(s) = \Xi(\dot{f}(s), f(s)) \geq \varepsilon \dot{f}(s)^a f(s)^b \geq \varepsilon \dot{f}(t_n)^a f(t_n)^b \geq \varepsilon n^{2a+b} \geq \varepsilon n^3.$$

Consequently $\dot{f}(t_{n+1}) \geq n^2 + \frac{3}{\varepsilon n^2} \varepsilon n^3 \geq n^2 + 3n \geq (n+1)^2$. The desired contradiction follows as $\lim_{n \rightarrow \infty} t_n < \infty$. \square

5. NON-DIAGONALIZABLE JACOBI OPERATORS

In signature $(2, 2)$, the eigenvalue structure of the Jacobi operator does not determine the operator up to conjugacy; one must instead consider the Jordan normal form. Theorem 3.1 shows any strict Walker manifold of signature $(2, 2)$ is nilpotent Osserman. However there are Walker manifolds of signature $(2, 2)$ which are Osserman but not nilpotent and whose Jacobi operators are not diagonalizable.

Theorem 5.1. *Let \mathcal{M} be given by Definition 2.1 with $\psi_{33} = 4kx_1^2 - \frac{1}{4k}f(x_4)^2$, $\psi_{44} = 4kx_2^2$, and $\psi_{34} = 4kx_1x_2 + x_2f(x_4) - \frac{1}{4k}\dot{f}(x_4)$ where $f = f(x_4)$ is non-constant and $k \neq 0$. Then \mathcal{M} is Osserman with eigenvalues $\{0, 4k, k, k\}$ and the Jacobi operators are diagonalizable at P if and only if*

$$24kf(x_4)\dot{f}(x_4)x_2 - 12k\ddot{f}(x_4)x_1 + 3f(x_4)\ddot{f}(x_4) + 4\dot{f}(x_4)^2 = 0.$$

Furthermore \mathcal{M} is geodesically incomplete and can not be embedded isometrically in a geodesically complete manifold.

Proof. We refer to [10] for the proof that \mathcal{M} is Osserman with the indicated Jordan normal form. We must show \mathcal{M} is geodesically incomplete. Since f is non-constant, we may choose ξ_4 so $f(\xi_4) \neq 0$ and $\dot{f}(\xi_4) \neq 0$. Choose ξ_1 so $16k^2\xi_1^2 = f(\xi_4)^2$; normalize the choice of sign so $k\xi_1 > 0$. As an ansatz, we set $x_1 = \xi_1$, $x_2 = 0$, and $x_4 = \xi_4$ to be constant. This implies $\psi_{33} = 0$. The geodesic equations in \ddot{x}_1 , \ddot{x}_2 , and \ddot{x}_4 given by Lemma 2.2 then become $\ddot{x}_1 = \ddot{x}_2 = \ddot{x}_4 = 0$ which are satisfied. The remaining geodesic equation is $0 = \ddot{x}_3 - 4k\xi_1\dot{x}_3\dot{x}_3$. We can solve this equation by setting

$$x_3 = -\frac{1}{4k\xi_1} \ln(1-t), \quad \dot{x}_3 = \frac{1}{4k\xi_1}(1-t)^{-1}, \quad \ddot{x}_3 = \frac{1}{4k\xi_1}(1-t)^{-2} = 4k\xi_1\dot{x}_3\dot{x}_3.$$

This is defined for $t \in (-\infty, 1)$ and we have $\lim_{t \rightarrow 1} 4k\xi_1x_3 = \infty$. In particular, \mathcal{M} is geodesically incomplete.

Since \mathcal{M} is Einstein, it does not exhibit Ricci blowup. Instead we use a different argument to show \mathcal{M} is essentially incomplete. Let $\{e_1, e_2, e_3, e_4\}$ be a parallel frame along γ with $e_i(0) = \partial_i$. The argument used to establish Lemma 2.2 shows:

$$\begin{aligned}\nabla_{\partial_3}\partial_1 &= 4k\xi_1\partial_1, & \nabla_{\partial_3}\partial_2 &= 4k\xi_1\partial_2, \\ \nabla_{\partial_3}\partial_3 &= -4k\xi_1\partial_3, & \nabla_{\partial_3}\partial_4 &= -2\xi_1\dot{f}(\xi_4)\partial_1 - 4k\xi_1\partial_4.\end{aligned}$$

Consequently

$$\begin{aligned}e_1(x_3) &= e^{-4k\xi_1x_3}\partial_1, & e_2(x_3) &= e^{-4k\xi_1x_3}\partial_2, & e_3(x_3) &= e^{4k\xi_1x_3}\partial_3, \\ e_4(x_3) &= \frac{1}{4k}\dot{f}(\xi_4)(e^{4k\xi_1x_3} - e^{-4k\xi_1x_3})\partial_1 + e^{4k\xi_1x_3}\partial_4.\end{aligned}$$

Since $R(\partial_1, \partial_3, \partial_3, \partial_4) = 0$, since $R(\partial_1, \partial_3, \partial_3, \partial_1) = 4k$, since $\dot{f}(\xi_4) \neq 0$, and since $4k\xi_1x_3(t) \rightarrow \infty$ as $t \rightarrow 1$, \mathcal{M} is seen to be essentially incomplete as:

$$\begin{aligned}\lim_{t \rightarrow 1} R(e_1, e_3, e_3, e_4) &= \lim_{t \rightarrow 1} \left\{ \frac{1}{4k}\dot{f}(\xi_4)(e^{4k\xi_1x_3} - e^{-4k\xi_1x_3}) \right\} e^{4k\xi_1x_3} 4k \\ &= \lim_{t \rightarrow 1} \dot{f}(\xi_4)(e^{8k\xi_1x_3} - 1) = \pm\infty. \quad \square\end{aligned}$$

6. CONFORMAL OSSERMAN MANIFOLDS

Let Spec_W denote the spectrum of the conformal Jacobi operator and let m_λ denote the minimal polynomial of the conformal Jacobi operator for a conformal Osserman manifold. We refer to [5] for the proof of:

Theorem 6.1. *Let \mathcal{M} be as in Definition 2.1 where $\psi_{33} = \psi_{44} = 0$. With the following choices of ψ_{34} , \mathcal{M} is conformal Osserman and:*

- (1) *The Jordan normal form does not change from point to point:*
 - (a) $\psi_{34} = x_1^2 - x_2^2 \Rightarrow m_\lambda = \lambda(\lambda^2 - \frac{1}{4})$ and $\text{Spec}_W = \{0, 0, \pm\frac{1}{2}\}$.
 - (b) $\psi_{34} = x_1^2 + x_2^2 \Rightarrow m_\lambda = \lambda(\lambda^2 + \frac{1}{4})$ and $\text{Spec}_W = \{0, 0, \pm\frac{\sqrt{-1}}{2}\}$.
 - (c) $\psi_{34} = x_1x_4 + x_3x_4 \Rightarrow m_\lambda = \lambda^2$ and $\text{Spec}_W = \{0\}$.
 - (d) $\psi_{34} = x_1^2 \Rightarrow m_\lambda = \lambda^3$ and $\text{Spec}_W = \{0\}$.
- (2) $\text{Spec}_W = \{0\}$ *but Jordan normal form changes from point to point.*
 - (a) $\psi_{34} = x_2x_4^2 + x_3^2x_4 \Rightarrow m_\lambda = \begin{cases} \lambda^3 & \text{if } x_4 \neq 0, \\ \lambda^2 & \text{if } x_4 = 0, x_3 \neq 0, \\ \lambda & \text{if } x_3 = x_4 = 0. \end{cases}$
 - (b) $\psi_{34} = x_2x_4^2 + x_3x_4 \Rightarrow m_\lambda = \begin{cases} \lambda^3 & \text{if } x_4 \neq 0, \\ \lambda^2 & \text{if } x_4 = 0. \end{cases}$
 - (c) $\psi_{34} = x_1x_3^2 \Rightarrow m_\lambda = \begin{cases} \lambda^3 & \text{if } x_3 \neq 0, \\ \lambda & \text{if } x_3 = 0. \end{cases}$
 - (d) $\psi_{34} = x_1x_3 + x_2x_4 \Rightarrow m_\lambda = \begin{cases} \lambda^2 & \text{if } x_1x_3 + x_2x_4 \neq 0, \\ \lambda & \text{if } x_1x_3 + x_2x_4 = 0. \end{cases}$
- (3) *The eigenvalues can change from point to point:*
 - (a) $\psi_{34} = x_1^4 + x_1^2 - x_2^4 - x_2^2 \Rightarrow \text{Spec}_W = \{0, 0, \pm\frac{1}{2}\sqrt{(6x_1^2 + 1)(6x_2^2 + 1)}\}$.
 - (b) $\psi_{34} = x_1^4 + x_1^2 + x_2^4 + x_2^2 \Rightarrow \text{Spec}_W = \{0, 0, \pm\frac{1}{2}\sqrt{-(6x_1^2 + 1)(6x_2^2 + 1)}\}$.
 - (c) $\psi_{34} = x_1^3 - x_2^3 \Rightarrow \text{Spec}_W = \{0, 0, \pm\frac{3}{2}\sqrt{x_1x_2}\}$.

Remark 6.2. In revisiting the manifolds of Theorem 6.1 whilst writing this paper, we made some geometrical observations that, although not directly in the focus of this paper, never the less illustrate why they form a rich geometrical family that it is important to study. Recall that a manifold is said to be *curvature homogeneous* if given $P, Q \in M$, there is an isometry $\phi : T_P M \rightarrow T_Q M$ so $\phi^* R_Q = R_P$ - i.e. the curvature tensor “looks the same at any point of the manifold”; we refer to [12] for further details. Of the manifolds in Theorem 6.1, only the manifold of 1-d) is curvature homogeneous. Let $R_\Lambda : \Lambda^2 T^* M \rightarrow \Lambda^2 T^* M$ be the curvature operator. If \mathcal{M} is the manifold of 1-a) or 1-b) above, then ρ^2 is a multiple of the identity at a point if and only if $x_1 = x_2 = 0$. If \mathcal{M} is as in 1-c), then \mathcal{M} is Ricci flat at a point

if and only if $x_4^2 = 2$; for 2-a) and 2-b), this happens if and only if $x_4 = 0$; for 2-c), this happens if and only if $x_3 = 0$. For 2-d), $R_\Lambda^2 = 0$ if and only if $x_3x_4 = 1$. The eigenvalues of R_Λ change from point to point for the manifolds of Theorem 3-a), 3-b), and 3-c). These observations show that none of these manifolds are curvature homogeneous. On the other hand, a rather more delicate argument shows Example 1-d) is curvature homogeneous. Further details concerning these matters will be forthcoming in a subsequent article. We note finally that Derdzinski [8] showed a 4-dimensional Riemannian manifold is curvature homogeneous if and only if R_Λ has constant eigenvalues; furthermore if such a manifold is Einstein, then it is locally symmetric. In Example 2-d), $\text{Spec}\{R_\Lambda\} = \{0\}$ but the manifold is not curvature homogeneous. Thus this result of Derdzinski fails in signature $(2, 2)$; we refer to Derdzinski [9] for additional results in this direction.

We study the global geometry of the manifolds of Theorem 6.1:

Theorem 6.3. *Of the manifolds in Theorem 6.1, only $\psi_{34} = x_1x_4 + x_3x_4$ defines a geodesically complete manifold; the remaining tensors ψ define manifolds which exhibit Ricci blowup and which therefore can not be embedded isometrically in a geodesically complete manifold.*

Proof. Suppose first that for $x_1 \geq 1$ one has:

$$\psi_{34/1} = p(x_1) \geq x_1, \quad \psi_{34/11} \geq 1, \quad \psi_{34/4} = 0.$$

This is the case for the warping functions of (1a), (1b), (1d), (3a), (3b), and (3c). We set $x_2(t) = 0$, $x_3(t) = 0$ and $x_4(t) = -t$. The geodesic equations given in Lemma 2.3 then become:

$$\ddot{x}_1 - \dot{x}_1 p(x_1) = 0, \quad \ddot{x}_2 = 0, \quad \ddot{x}_3 = 0, \quad \ddot{x}_4 = 0.$$

This yields a consistent set of equations with $\ddot{x}_1 = \dot{x}_1 p(x_1)$. By Lemma 4.1, $\lim_{t \rightarrow T} \dot{x}_1(t) = \infty$ for T finite. By Lemma 2.3,

$$\rho(\dot{\gamma}, \dot{\gamma}) = \dot{x}_1 \dot{x}_1 \rho_{11} + 2\dot{x}_1 \dot{x}_4 \rho_{14} + \dot{x}_4 \dot{x}_4 \rho_{44} = -\dot{x}_1 \psi_{34/11} - \frac{1}{2} \psi_{34/1}^2.$$

As $\psi_{34/11} \geq 1$, $\lim_{t \rightarrow T} \rho(\dot{\gamma}, \dot{\gamma}) = -\infty$ and these manifolds exhibit Ricci blowup.

Let $\psi_{34} = x_1x_3 + x_2x_4$ be as in (2d). The geodesic equations are:

$$\begin{aligned} 0 &= \ddot{x}_1 + \dot{x}_1 \dot{x}_4 x_3 + \dot{x}_2 \dot{x}_4 x_4 + \dot{x}_3 \dot{x}_4 (x_1 x_3 + x_2 x_4) x_4 + \dot{x}_4 \dot{x}_4 x_2, \\ 0 &= \ddot{x}_2 + \dot{x}_1 \dot{x}_3 x_3 + \dot{x}_2 \dot{x}_3 x_4 + \dot{x}_3 \dot{x}_3 x_1 + \dot{x}_3 \dot{x}_4 (x_1 x_3 + x_2 x_4) x_3, \\ 0 &= \ddot{x}_3 - \dot{x}_3 \dot{x}_4 x_3, \quad 0 = \ddot{x}_4 - \dot{x}_3 \dot{x}_4 x_4. \end{aligned}$$

We start with initial conditions $x_3(0) = \dot{x}_3(1) = x_4(1) = \dot{x}_4(1) = 1$. Symmetry implies that $x_3(t) = x_4(t) = h(t)$ where h satisfies $\dot{h}(t) = \dot{h}(t) \dot{h}(t) h(t)$. Lemma 4.1 now shows $\dot{h} \rightarrow \infty$ at finite time so \mathcal{M} is incomplete. Furthermore, we use Lemma 2.3 to see that $\rho_{ij} = 0$ for $i, j \neq 3, 4$ and thereby show \mathcal{M} exhibits Ricci blowup by computing:

$$\begin{aligned} \rho(\dot{\gamma}, \dot{\gamma}) &= \dot{x}_3^2 \rho_{33} + \dot{x}_4^2 \rho_{44} + 2\dot{x}_3 \dot{x}_4 \rho_{34} = \dot{x}_3^2 \{\rho_{33} + \rho_{44} + 2\rho_{34}\} \\ &= \dot{x}_3^2 \{-\frac{1}{2}x_4^2 - \frac{1}{2}x_3^2 + x_3x_4 - 2\} = -2\dot{x}_3^2. \end{aligned}$$

Let $\psi_{34} = x_1x_3^2$ be the warping function of (2c). The final two geodesic equations become $\ddot{x}_3 = \dot{x}_3 \dot{x}_4 x_3^2$ and $\ddot{x}_4 = 0$. Setting $x_4 = t$ then yields the equation $\ddot{x}_3 = \dot{x}_3 x_3^2$ and thus by Lemma 4.1, $\dot{x}_3(t) \rightarrow \infty$ as $t \rightarrow T$ for $T < \infty$. All the components of the Ricci tensor vanish except ρ_{34} and ρ_{44} . Since $x_3(t) \geq 1$, we show \mathcal{M} exhibits Ricci blowup by computing:

$$\lim_{t \rightarrow T} \rho(\dot{\gamma}, \dot{\gamma}) = \lim_{t \rightarrow T} \{-2\dot{x}_3 \dot{x}_4 x_3 - \frac{1}{2} \dot{x}_4 \dot{x}_4 x_3^4\} \leq \lim_{t \rightarrow T} \{-2\dot{x}_3\} = -\infty.$$

Suppose that $\psi_{34} = x_2x_4^2 + x_3^2x_4$ or that $\psi_{34} = x_2x_4^2 + x_3x_4$ are the warping functions of Theorem 6.1 (2a) and Theorem 6.1 (2b). The final two geodesic equations become $0 = \ddot{x}_3$ and $\ddot{x}_4 = \dot{x}_3 \dot{x}_4 x_4^2$. We take $x_3 = t$ so $\ddot{x}_4 = \dot{x}_4 x_4^2$. Let

$x_4(0) = \dot{x}_4(0) = 1$. Thus $\lim_{t \rightarrow T} \dot{x}_4 = \infty$ at some finite time and $x_4(t) \geq 1$ for all t . Only ρ_{33} and ρ_{34} are non-zero. Thus we may show \mathcal{M} exhibits Ricci blowup by computing:

$$\lim_{t \rightarrow T} \rho(\dot{\gamma}, \dot{\gamma}) = \lim_{t \rightarrow T} \left\{ -\frac{1}{2} \dot{x}_3 \dot{x}_3 x_4^4 - 2 \dot{x}_3 \dot{x}_4 x_4 \right\} \leq \lim_{t \rightarrow T} \{-2 \dot{x}_4\} = -\infty.$$

Finally let $\psi_{34} = x_1 x_4 + x_3 x_4$ be the warping function of (1c). The geodesic equations in the last two variables are $\ddot{x}_3 - \dot{x}_3 \dot{x}_4 x_4 = 0$ and $\ddot{x}_4 = 0$ so

$$x_4 = a + bt \quad \text{and} \quad \dot{x}_3 = ce^{b(at + b\frac{1}{2}t^2)}.$$

We integrate this equation to determine x_3 . As the equation for \ddot{x}_2 takes the form $\ddot{x}_2 + F(x_1, x_3, x_4, \dot{x}_1, \dot{x}_3, \dot{x}_4) = 0$, it poses no difficulty and only task is to determine x_1 . The equation for x_1 takes the form:

$$\ddot{x}_1 + \dot{x}_1 \dot{x}_4 x_4 + \dot{x}_4 \dot{x}_4 x_1 + \dot{x}_4 \dot{x}_4 x_3 = 0.$$

By rescaling the geodesic parameter, we see that there are really only two cases to be considered. These are $x_4 = a$ and $x_4 = t$. If $x_4 = a$, we get the equation $\ddot{x}_1 = 0$ which has linear solutions. If $x_4 = t$, we get the equation

$$\ddot{x}_1 + t \dot{x}_1 + x_1 = \alpha(t)$$

for suitably chosen α . We set $x_1 := fe^{-\frac{1}{2}t^2}$ to reduce the order of the equation:

$$\begin{aligned} \dot{x}_1 &= (\dot{f} - tf)e^{-\frac{1}{2}t^2}, \quad \ddot{x}_1 = (\ddot{f} - 2t\dot{f} + t^2f - f)e^{-\frac{1}{2}t^2}, \\ \ddot{x}_1 + t\dot{x}_1 + x_1 &= (\ddot{f} - 2t\dot{f} + t^2f - f + t\dot{f} - t^2f + f)e^{-\frac{1}{2}t^2} \\ &= (\ddot{f} - t\dot{f})e^{-\frac{1}{2}t^2} = \alpha(t). \end{aligned}$$

Setting $f_1 := \dot{f}$ then leads to an equation of the form $\dot{f}_1 - tf_1 = \alpha_1(t)$ for suitably chosen α_1 . Setting $f_1 = f_2 e^{\frac{1}{2}t^2}$ then yields

$$\dot{f}_1 - tf_1 = (\dot{f}_2 + tf_2 - tf_2)e^{\frac{1}{2}t^2} = \alpha_1(t)$$

which leads to the equation $\dot{f}_2 = \alpha(t)$. This equation can be solved for all time; it now follows that \mathcal{M} is geodesically complete. \square

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