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Elduque superalgebras

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# Cartan matrices and presentations of Cunha and Elduque superalgebras

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## Abstract

All inequivalent Cartan matrices (in other words, inequivalent systems of simple roots) of the ten simple exceptional finite dimensional Lie superalgebras in characteristic 3, recently identified by Cunha and Elduque as constituents of Elduque's superization of the Freudenthal Magic Square, are listed together with defining relations between analogs of their Chevalley generators.

## §1 Introduction

Recently Strade had published a monograph [S] summarizing the description of newly classified simple finite dimensional Lie algebras over the algebraically closed fields  $\mathbb{K}$  of characteristic  $p > 3$ , and also gave an overview of the examples (due to Brown, Frank, Ermolaev and Skryabin) of simple finite dimensional Lie algebras for  $p = 3$  with no counterparts for  $p > 3$ . Several researchers started afresh to work on the cases where  $p = 2$  and 3, and, after a period of quietness after [KL], new examples of simple Lie algebras with no counterparts for  $p \neq 2, 3$  started to appear ([J1, GL4, GG, Le1]). The  $\mathbb{Z}$ -graded of the above mentioned exceptional examples of simple Lie algebras for  $p = 3$  (Brown, Frank, Ermolaev and Skryabin algebras) were interpreted as algebras of vector fields preserving certain distributions ([GL4]).

Among the main points made in [KL], one was indication to similarity between modular Lie algebras and Lie superalgebras (even over  $\mathbb{C}$ ).

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Classification of simple Lie *superalgebras* for  $p > 0$  and the study of their representations is, however, of independent interest. A conjectural list of simple finite dimensional Lie superalgebras over an algebraically closed field of characteristic  $p > 5$ , known for some time, was recently cited in [BjL]: the idea is, as in the Kostrikin–Shafarevich conjecture, for the two classes of simple complex Lie superalgebras  $\mathfrak{g}$  (finite dimensional and vectorial), select a  $\mathbb{Z}$ -form  $\mathfrak{g}_{\mathbb{Z}}$  of  $\mathfrak{g}$ , if any such exists, and take  $\mathfrak{g}_{\mathbb{K}} := \mathfrak{g}_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{K}$ , take a simple finite dimensional subquotient  $\mathfrak{si}(\mathfrak{g}_{\mathbb{K}})$  of  $\mathfrak{g}_{\mathbb{K}}$ ; we call such examples *KSh-type algebras*. Finally, take deformations<sup>1</sup> thereof if any exist.

Alberto Elduque worked lately on getting examples of simple Lie superalgebras for  $p = 3, 5$  different from KSh-type. His approach to superization of the Freudenthal Magic Square in terms of *symmetric compositions algebras* led him and his Ph.D. student, Isabel Cunha, to discovery of several new simple finite dimensional Lie superalgebras with Cartan matrix, which, conjecturally, are indigenous to  $p = 3$ , cf. [CE, E11, CE2].

Interestingly, the exceptional simple Lie superalgebras  $\mathfrak{ag}(2)$  and  $\mathfrak{ab}(3)$  do not appear in the superizations of Freudenthal’s magic square for  $p = 0$  but their analogs do appear after their integer structure constants with respect to the Chevalley basis are being reduced modulo 3: the simple pieces of the superalgebras thus obtained appear on Elduque’s super square as  $\mathfrak{g}(2, 3)$  and  $\mathfrak{g}(2, 6)$  in notations described below. Just compare the Cartan matrices 3) of  $\mathfrak{ag}(2)$  and 6) of  $\mathfrak{ab}(3)$  in [GL1] with the Cartan matrices 3) of  $\mathfrak{g}(2, 3)$  and 6) of  $\mathfrak{g}(2, 6)$ , respectively: They are identical. Observe that, for any field  $\mathbb{K}$  of characteristic 3, the superdimensions (and structures) of  $\mathfrak{g}(2, 3)$  and  $\mathfrak{ag}(2)_{\mathbb{K}}$  are distinct as well as those of  $\mathfrak{g}(2, 6)$  and  $\mathfrak{ab}(3)_{\mathbb{K}}$ .

Elduque and Cunha gave only one Cartan matrix for each Lie superalgebra that possess Cartan matrix, and here we list all possible inequivalent Cartan matrices (in other words, inequivalent systems of simple roots). Among all inequivalent systems of simple roots, some have properties particularly useful in certain applications (e.g., the two extremes: with the least number of odd roots and with the maximal number of odd roots).

We also need the list of all systems of simple roots as a step towards classification of prolongations of the non-positive components of the Lie superalgebras we consider here in their various gradings (for details of this approach, see [GL4]).

To interpret these new Elduque-and-Cunha Lie superalgebras as Lie superalgebras preserving a tensor is still a task to be performed. This

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<sup>1</sup>It is not clear, actually, if the conventional notion of deformation can always be applied if  $p > 0$  (for the arguments, see [LL]); to give the correct notion is an open problem, but let it pass for the moment.

interpretation is not, however, always possible; in the sequel to this paper we intend to interpret these Lie superalgebras as preserving not just a tensor but certain distributions, as in [GL4], and get new examples of simple Lie superalgebras, as in [BjL]. Here, with the aid of the **SuperLie** package, we describe presentations of those Elduque-and-Cunha Lie superalgebras that possess Cartan matrix.

In terms of the Chevalley generators of the Lie superalgebras with Cartan matrix, there are two types of defining relations: Serre-type ones and non-Serre type ones (over  $\mathbb{C}$ , all the relations are listed in [GL1]). Sometimes some of the Serre-type relations are redundant but this does not matter in practical calculations. At the moment, the problem how to encode the non-Serre type relations in terms of Cartan matrix is open. Some relations are so complicated that we conjecture that there is no *general* encoding procedure. This is why our list of relations is of practical interest.

In the arXiv version of this paper, we also give the coefficients of linear dependence over  $\mathbb{Z}$  of the corresponding maximal roots with respect to the simple ones and the inverses of Cartan matrices whenever the latter are invertible.

## §2 Background

**2.1. Notations.** The ground field  $\mathbb{K}$  is assumed to be algebraically closed; its characteristic is denoted by  $p$ . We follow Bourbaki's convention: if  $G$  is a Lie group, then its Lie algebra is designated  $\mathfrak{g}$ , although the modern tradition does not favor Gothic font in characteristic  $p > 0$ . Let  $\mathfrak{g}' := [\mathfrak{g}, \mathfrak{g}]$ .

**2.2. What  $\mathfrak{g}(A)$  is.** First, recall, how to construct a Lie superalgebra from a Cartan matrix ([Se, vdL, FLS, Eg, FSS]). Let  $A = (A_{ij})$  be an arbitrary  $n \times n$  matrix of rank  $l$  with entries in  $\mathbb{K}$ . Fix a vector space  $\mathfrak{h}$  of dimension  $2n - l$  and its dual  $\mathfrak{h}^*$ , select  $n$  linearly independent vectors  $h_i \in \mathfrak{h}$  and let  $\alpha_j \in \mathfrak{h}^*$  be such that  $\alpha_i(h_j) = A_{ij}$ .

Let  $I = \{i_1, \dots, i_n\} \subset (\mathbb{Z}/2\mathbb{Z})^n$ ; consider the free Lie superalgebra  $\tilde{\mathfrak{g}}(A, I)$  with generators  $e_1^\pm, \dots, e_n^\pm$ , where  $p(e_j^\pm) = i_j$ , and defining relations (for convenience, we set  $h_i := [e_i^+, e_i^-]$ ):

$$[e_i^+, e_j^-] = \delta_{ij} h_j; \quad [h_i, e_j^\pm] = \pm A_{ij} e_j; \quad [h_i, h_j] = 0. \quad (1)$$

The following statements over  $\mathbb{C}$  is well known for the Lie algebras; for Lie superalgebras over  $\mathbb{C}$  it is due to Serganova and van de Leur [Se, vdL]; for for Lie superalgebras over  $p \neq 2$ , see [CE2]. (The proof does not seem to depend on the super structure or  $p$  if  $p \neq 2$ .)

**Statement.** a) Let  $\tilde{\mathfrak{n}}_+$  and  $\tilde{\mathfrak{n}}_-$  be the superalgebras in  $\tilde{\mathfrak{g}}(A, I)$  generated by  $e_1^\pm, \dots, e_n^\pm$ ; then  $\tilde{\mathfrak{g}}(A, I) \cong \tilde{\mathfrak{n}}_+ \oplus \mathfrak{h} \oplus \tilde{\mathfrak{n}}_-$ , as vector superspaces.

b) Among the ideals of  $\tilde{\mathfrak{g}}(A, I)$  whose intersection with  $\mathfrak{h}$  is zero, there exists a maximal ideal  $\mathfrak{r}$  such that  $\mathfrak{r}$  is the direct sum of ideals  $\mathfrak{r} \cap \tilde{\mathfrak{n}}_+$  and  $\mathfrak{r} \cap \tilde{\mathfrak{n}}_-$ .

Set  $\mathfrak{g}(A, I) = \tilde{\mathfrak{g}}(A, I)/\mathfrak{r}$ . Both  $\mathfrak{g}(A, I)$  and  $\mathfrak{g}'(A, I)$  may contain a center. As proved in [Se, vdL], the centers  $\mathfrak{c}$  of  $\mathfrak{g}(A, I)$  and  $\mathfrak{c}'$  of  $\mathfrak{g}'(A, I)$  consist of all  $h \in \mathfrak{h}$  such that  $\alpha_i(h) = 0$  for all  $i = 1, \dots, n$ ; this is also true for  $p > 0$ .

Clearly,

$$\text{the rescaling } e_i^\pm \mapsto \sqrt{\lambda_i} e_i^\pm, \text{ sends } A \text{ to } \text{diag}(\lambda_1, \dots, \lambda_n) \cdot A. \quad (2)$$

Two pairs  $(A, I)$  and  $(A', I')$  are said to be *equivalent* if  $(A', I')$  is obtained from  $(A, I)$  by a composition of a permutation of indices and a rescaling  $A' = \text{diag}(\lambda_1, \dots, \lambda_n) \cdot A$ , where  $\lambda_1 \dots \lambda_n \neq 0$ . Clearly, equivalent pairs determine isomorphic Lie superalgebras.

The matrix  $A$  (more precisely, the pair  $(A, I)$ ) is said to be a *Cartan matrix* of the Lie superalgebra  $\mathfrak{g}(A, I)$  and also of  $\tilde{\mathfrak{g}}(A, I)$ ,  $\mathfrak{g}'(A, I)$ , as well as of  $\mathfrak{g}(A, I)/\mathfrak{c}$  and  $\mathfrak{g}'(A, I)/\mathfrak{c}'$ . We tirelessly repeat:

The “relatives” (such as nontrivial central extensions and algebras of derivations) of simple Lie (super)algebras are no less important than the simple algebras themselves.

In particular, in Elduque’s superization of the Freudenthal Magic Square for  $p = 3$ , there naturally appear  $\mathfrak{g}(2, 8) = \mathfrak{e}(6)$ ,  $\mathfrak{g}(2, 3)$  and  $\mathfrak{g}(2, 6)$  that have 1-dimensional centers.

**2.3. Systems of simple roots.** Since  $\mathfrak{g}(A, I)$ , where  $A$  is of size  $n \times n$ , is naturally  $\mathbb{Z}^n$ -graded we can consider the root system  $R$  of  $\mathfrak{g}(A, I)$  as a subset of  $\mathbb{Z}^n$ . For any subset  $B = \{\sigma_1, \dots, \sigma_n\} \subset R$ , we set (we denote by  $\mathbb{Z}_+$  the set of non-negative integers):

$$R_B^\pm = \{\alpha \in R \mid \alpha = \pm \sum n_i \sigma_i, \quad n_i \in \mathbb{Z}_+\}.$$

The set  $B$  is called a *system of simple roots* of  $R$  (or  $\mathfrak{g}$ ) if  $\sigma_1, \dots, \sigma_n$  are linearly independent and there exist  $\tilde{e}_1^\pm \in \mathfrak{g}_{\pm\sigma_1}, \dots, \tilde{e}_n^\pm \in \mathfrak{g}_{\pm\sigma_n}$  such that:

$$\mathfrak{g} = \mathfrak{g}_B^- \oplus \mathfrak{h} \oplus \mathfrak{g}_B^+,$$

where  $\mathfrak{g}_B^\pm$  is the superalgebra generated by  $\tilde{e}_1^\pm, \dots, \tilde{e}_n^\pm$ .

Let  $B$  be a system of simple roots and  $\tilde{e}_1^\pm, \dots, \tilde{e}_n^\pm$  the corresponding elements of  $\mathfrak{g}(A, I)$ . Set  $\tilde{h}_i = [\tilde{e}_i^+, \tilde{e}_i^-]$ ,  $A_B = (A_{ij})$ , where  $A_{ij} = \sigma_i(\tilde{h}_j)$  and

$I_B = \{p(\sigma_1), \dots, p(\sigma_n)\}$ . The Cartan matrix  $(A_B, I_B)$  thus constructed does not have to coincide with the initial Cartan matrix  $(A, I)$ .

Two systems of simple roots  $B_1$  and  $B_2$  are said to be *equivalent* if the pairs  $(A_{B_1}, I_{B_1})$  and  $(A_{B_2}, I_{B_2})$  are equivalent.

Hereafter  $\mathfrak{g} = \mathfrak{g}(A, I)$ . The following statement is subject to a direct verification:

**Proposition.** *Let  $B$  be a system of simple roots of  $\mathfrak{g}$ ,  $\tilde{e}_i^\pm$  for  $i = 1, \dots, n$  the corresponding set of generators and  $A_B = (A_{ij})$  the Cartan matrix. Fix an  $i$ . Then:*

a) *If  $p(\sigma_i) = \bar{0}$  and  $A_{ii} \neq 0$ , then the Lie subalgebra generated by the  $e_i^\pm$  is isomorphic to  $\mathfrak{sl}(2)$ .*

b) *If  $p(\sigma_i) = \bar{0}$  and  $A_{ii} = 0$ , then the Lie subalgebra generated by the  $e_i^\pm$  is isomorphic an analog of the Heisenberg Lie algebra; we denote this Lie algebra  $\mathfrak{hei}(2; p; \underline{N})$ , where  $N \in \mathbb{N}$ . Its natural representation is realized in the Fock space of functions  $\mathcal{O}(1; \underline{N})$ ; it is indecomposable<sup>2</sup> for  $\underline{N} > 1$  and irreducible for  $\underline{N} = 1$ .*

c) *If  $p(\sigma_i) = \bar{1}$  and  $A_{ii} = 0$ , then  $2\sigma_i \notin R$  and the subsuperalgebra generated by the  $e_i^\pm$  is isomorphic to  $\mathfrak{sl}(1|1)$ .*

d) *If  $p(\sigma_i) = \bar{1}$  and  $A_{ii} \neq 0$ , then  $3\sigma_i \notin R$  and the subsuperalgebra generated by the  $e_i^\pm$  is isomorphic to  $\mathfrak{osp}(1|2)$ .*

**2.4. Chevalley generators and odd reflections.** How many Cartan matrices correspond to the same Lie superalgebra  $\mathfrak{g}$ ? To answer this question, let us multiply  $A_B$  by a diagonal matrix as in (2), so that in the cases a), d) of Proposition 2.3 the diagonal elements of  $A_B$  become 2 or 1, respectively. The row with a 0 on the main diagonal can be multiplied by any nonzero factor; we usually multiply it so that  $A_B$  be symmetric if possible, and additionally, if  $p = 0$ , so as to make the off-diagonal elements negative (this is needed in relations (12)). Such a matrix is said to be *normalized*.

For each simple finite dimensional Lie algebras (and, conjecturally, for Lie superalgebras) for  $p > 3$  the normalized Cartan matrix might be not symmetric, but it is symmetrizable. We only consider normalized matrices, unless we need them in a symmetric form. A symmetrized matrix is not a Cartan matrix but is also useful: it gives the values of the inner products of simple roots and is needed to pass from one system of simple roots to another.

A usual way to represent simple Lie algebras over  $\mathbb{C}$  with integer Cartan matrices is via graphs called, in the finite dimensional case, *Dynkin*

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<sup>2</sup>At least, under the conventional definition of representations of Lie algebras for  $p > 0$ ; for conjectural other versions, see [LL].

*diagrams.* The Cartan matrices of Lie *superalgebras*  $\mathfrak{g}$  (even over  $\mathbb{C}$ ) can be non-symmetrizable or (for any  $p$  in the super case and for  $p > 0$  in the non-super case) have entries belonging to the ground field  $\mathbb{K}$ . Still, it is always possible to assign an analog of the Dynkin diagram to both Lie superalgebras of polynomial growth, and to finite dimensional Lie algebras (if these (super)algebras possess Cartan matrices). Perhaps, the edges and nodes of the graph should be rigged with an extra information. Namely, the *Dynkin–Kac diagram* of the matrix  $(A, I)$  is a set of  $n$  nodes connected by multiple edges, perhaps endowed with an arrow, according to the usual rules ([K]) or their modification, most naturally formulated by Serganova [Se, FLS].

To be able to distinguish the case of the even root from the odd one when  $A_{ii} = 0$ , we write  $A_{ii} = \emptyset$  if the root is even. The nodes are of four types: *To every simple root there corresponds*

$$\left\{ \begin{array}{ll} \text{a node } \otimes & \text{if } p(\alpha_i) = \bar{1} \text{ and } A_{ii} = 0, \\ \text{a node } \bullet & \text{if } p(\alpha_i) = \bar{1} \text{ and } A_{ii} = 1; \\ \text{a node } \circ & \text{if } p(\alpha_i) = \bar{0}, \text{ and } A_{ii} = 2 \\ \text{a node } * & \text{if } p(\alpha_i) = \bar{0}, \text{ and } A_{ii} = \emptyset. \end{array} \right.$$

*A posteriori* (from the classification of simple Lie superalgebras with Cartan matrix and of polynomial growth for  $p = 0$ ) we find out that the roots  $*$  can only occur if  $\mathfrak{g}(A, I)$  grows faster than polynomially. Thanks to classification again, if  $\dim \mathfrak{g} < \infty$ , the roots  $*$  can not occur if  $p > 3$ ; whereas for  $p = 3$ , the Brown Lie algebras are examples of  $\mathfrak{g}(A)$  with a simple root of type roots  $*$ , see [GL4].

To more graphically express normalized Cartan matrices, we apply Serganova’s rules even for  $p > 0$ . Although the analog of the Dynkin graph is now uniquely recovered from the Cartan matrix only if there are no two odd “grey” nodes  $\otimes$  connected, this analog helps to grasp the geometry of the system of simple roots.

We often denote the set of generators corresponding to a normalized matrix by  $X_1^\pm, \dots, X_n^\pm$  instead of  $e_1^\pm, \dots, e_n^\pm$ ; and call them the *Chevalley generators*.

Let  $\alpha$  be a root of  $\mathfrak{g}(A, I)$  with the entries of  $A$  in  $\mathbb{Z}$  or  $\mathbb{Z}/p$ . Define the reflections  $r_\alpha$  acting on the root lattice over  $\mathbb{Z}$  by the formulas

$$r_{\alpha_i}(\alpha_j) = \begin{cases} -\alpha_j & \text{for } i = j \\ \alpha_j - A_{ij}\alpha_i & \text{for } i \neq j \text{ and } A_{ii} = 2 \\ \alpha_j - 2A_{ij}\alpha_i & \text{for } i \neq j \text{ and } A_{ii} = 1 \\ \alpha_j + \alpha_i & \text{for } i \neq j \text{ and } A_{ii} = 0, A_{ji} \neq 0 \\ \alpha_j & \text{for } i \neq j \text{ and } A_{ii} = A_{ji} = 0 \end{cases} \quad (3)$$



The reflection in the  $i$ th root sends one set of Chevalley generators into the new one:

$$\tilde{X}_i^\pm = X_i^\mp; \quad \tilde{X}_j^\pm = \begin{cases} [X_i^\pm, X_j^\pm] & \text{if } A_{ij} \neq 0 \\ X_j^\pm & \text{otherwise.} \end{cases} \quad (4)$$

The reflections in the odd roots are referred to as *odd reflections*. For  $p = 0$ , there is always a chain of odd reflections connecting every two systems of simple roots  $B_1$  and  $B_2$ . Serganova's proof of this fact [Se] is applicable without modifications for  $p \neq 2$ .

**2.5. Defining relations for Lie superalgebras  $\mathfrak{g} = \mathfrak{g}(A, I)$ .** The simple Lie superalgebras of the form  $\mathfrak{g} = \mathfrak{g}(A, I)$  have several quite distinct sets of relations (cf. [Sa] and refs therein) but usually they are given by their *Chevalley generators*  $X_i^\pm$  of degree  $\pm 1$  to which the elements  $H_i = [X_i^+, X_i^-]$  are added for convenience. These generators satisfy the following relations (hereafter in similar occasions either all superscripts  $\pm$  are  $+$  or all are  $-$ )

$$[X_i^+, X_j^-] = \delta_{ij} H_i, \quad [H_i, H_j] = 0, \quad [H_i, X_j^\pm] = \pm A_{ij} X_j^\pm, \quad (5)$$

and additional relations  $R_i = 0$  whose left sides are implicitly described, for a general Cartan matrix with entries in  $\mathbb{K}$ , as ([K])

$$\begin{aligned} & \text{“the } R_i \text{ that generate the maximal ideal } I \text{ such that} \\ & I \cap \text{Span}(H_i \mid 1 \leq i \leq n) = 0.” \end{aligned} \quad (6)$$

For  $p = 0$  and normalized Cartan matrices of simple finite dimensional Lie algebras, there is only one (*Chevalley*) basis in which all structure constants are integer, cf. [Er]. (Having normalized the Cartan matrix of  $\mathfrak{o}(2n+1)$  so that  $A_{nn} = 1$  we get another basis with integer structure constants.)

We conjecture that if  $p > 2$ , then for Lie algebra and Lie superalgebras with Cartan matrix with entries in  $\mathbb{Z}/p$ , there also exists only one analog of the Chevalley basis. At least, one such basis definitely exists: we use it.

**2.5.1 Serre relations** Let  $\mathfrak{g} = \mathfrak{g}(A, I)$ ; let  $X_i^\pm$ , where  $1 \leq i \leq n$ , be root vectors corresponding to simple roots  $\pm\sigma_i$ . Clearly, the  $X_i^\pm$  generate  $\mathfrak{n}^\pm$ . We find the defining relations by induction on  $n$  with the help of the Hochschild–Serre spectral sequence (for its description for Lie superalgebras, which has certain subtleties, see [Po]). For the basis of induction consider the following cases:

- or ● no relations, i.e.,  $\mathfrak{n}^\pm$  are free Lie superalgebras if  $p \neq 3$ ;
- $(\text{ad}_{X^\pm})^2(X^\pm) = 0$  if  $p = 3$ ;
- ⊗  $[X^\pm, X^\pm] = 0$ .

Set  $\deg X_i^\pm = 0$  for  $1 \leq i \leq n-1$  and  $\deg X_n^\pm = \pm 1$ . Let  $\mathfrak{n}^\pm = \bigoplus \mathfrak{n}_i^\pm$  and  $\mathfrak{g} = \bigoplus \mathfrak{g}_i$  be the corresponding  $\mathbb{Z}$ -gradings. Set  $\mathfrak{n}_\pm = \mathfrak{n}^\pm / \mathfrak{n}_0^\pm$ . From the Hochschild–Serre spectral sequence for the pair  $\mathfrak{n}_0^\pm \subset \mathfrak{n}^\pm$  we get:

$$H_2(\mathfrak{n}_0^\pm) \oplus H_1(\mathfrak{n}_0^\pm; H_1(\mathfrak{n}_\pm)) \oplus H_0(\mathfrak{n}_0^\pm; H_2(\mathfrak{n}_\pm)). \quad (8)$$

It is clear that

$$H_1(\mathfrak{n}_\pm) = \mathfrak{n}_1^\pm, \quad H_2(\mathfrak{n}_\pm) = E^2(\mathfrak{n}_1^\pm) / \mathfrak{n}_2^\pm, \quad (9)$$

where  $E^2$  is the functor of the second exterior power. So, the second summand in (8) provides us with relations of the form:

$$\begin{aligned} (\mathrm{ad}_{X_n^\pm})^{k_{ni}}(X_i^\pm) &= 0 && \text{if the } n\text{-th root is not } \otimes \\ [X_n, X_n] &= 0 && \text{if the } n\text{-th root is } \otimes. \end{aligned} \quad (10)$$

while the third summand in (8) is spanned by the  $\mathfrak{n}_0^\pm$ -lowest vectors in

$$E^2(\mathfrak{n}_1^\pm) / (\mathfrak{n}_2^\pm + \mathfrak{n}^\pm E^2(\mathfrak{n}_1^\pm)). \quad (11)$$

Let the matrix  $B = (b_{ij})$  be obtained from the Cartan matrix  $A = (A_{ij})$  by replacing all nonzero elements in the row with  $A_{ii} = 0$  by  $-1$  and multiplying the row with  $A_{ii} = 1$  by  $2$ . The following proposition, whose proof is straightforward, illustrates the usefulness of our normalization of Cartan matrices as compared with other options:

**Proposition.** *The numbers  $k_{in}$  and  $k_{ni}$  in (10) are expressed in terms of  $(b_{ij})$  as follows:*

$$\begin{aligned} (\mathrm{ad}_{X_i^\pm})^{1-b_{ij}}(X_j^\pm) &= 0 && \text{for } i \neq j \\ [X_i^\pm, X_i^\pm] &= 0 && \text{if } A_{ii} = 0. \end{aligned} \quad (12)$$

The relations (5) and (12) will be called *Serre relations* for Lie superalgebra  $\mathfrak{g}(A, I)$ . If  $p = 3$ , then the relation

$$[X_i^\pm, [X_i^\pm, X_i^\pm]] = 0 \text{ for } X_i^\pm \text{ odd and } A_{ii} = 1 \quad (13)$$

is not a consequence of the Jacobi identity; for simplicity, we will incorporate it to the set of Serre relations.

**2.5.2 Non-Serre relations** They correspond to the third summand in (8). Let us consider the simplest case:  $\mathfrak{sl}(m|n)$  in the realization with the system of simple roots

$$\bigcirc - \dots - \bigcirc - \otimes - \bigcirc - \dots - \bigcirc \quad (14)$$

Then  $H_2(\mathfrak{n}_\pm)$  from the third summand in (8) is just  $E^2(\mathfrak{n}_\pm)$ . For simplicity, confine ourselves to the positive roots. Let  $X_1, \dots, X_{m-1}$  and  $Y_1, \dots, Y_{n-1}$  be the root vectors corresponding to even roots separated by the root vector  $Z$  corresponding to the root  $\otimes$ .

If  $n = 1$  or  $m = 1$ , then  $E^2(\mathfrak{n})$  is an irreducible  $\mathfrak{n}_0$ -module and there are no non-Serre relations. If  $n \neq 1$  and  $m \neq 1$ , then  $E^2(\mathfrak{n})$  splits into 2 irreducible  $\mathfrak{n}_0$ -modules. The lowest component of one of them corresponds to the relation  $[Z, Z] = 0$ , the other one corresponds to the non-Serre-type relation

$$[[X_{m-1}, Z], [Y_1, Z]] = 0. \quad (15)$$

If, instead of  $\mathfrak{sl}(m|n)$ , we would have considered the Lie algebra  $\mathfrak{sl}(m+n)$ , the same argument would have led us to the two relations:  $\text{ad}_Z^2(X_{m-1}) = 0$  and  $\text{ad}_Z^2(Y_1) = 0$  both of Serre type.

Although we have found all the *basic* relations, the ones that generate the ideal of relations, in what follows, we only list non-Serre relations assuming that all Serre relations are satisfied despite the fact that some of the Serre relations turn out to be consequences of these basic relations. Such redundancies are rare, to single them out is a boring and not rewarding task.

### §3 Elduque and Cunha superalgebras: Systems of simple roots

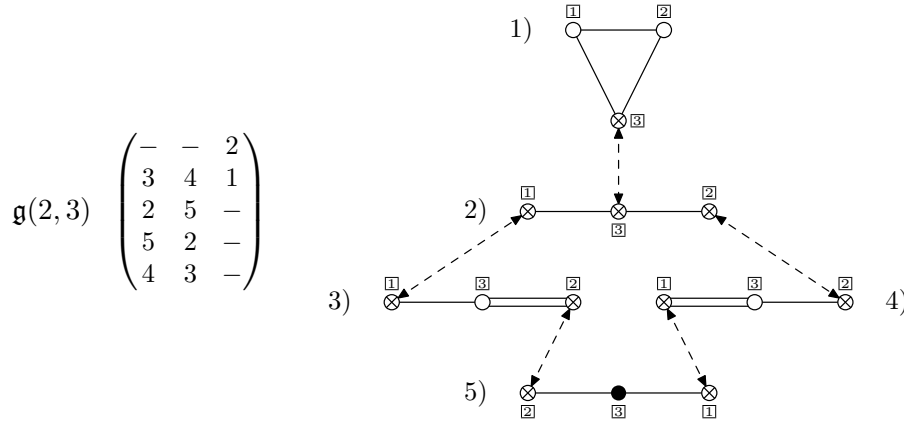
For details of description of Elduque and Cunha superalgebras in terms of symmetric composition algebras, see [E11, CE, CE2]. Here we consider the simple Elduque and Cunha superalgebras with Cartan matrix for  $p = 3$ . In what follows, we list them using somewhat shorter notations as compared with the original ones: here  $\mathfrak{g}(A, B)$  denotes the superalgebra occupying  $(A, B)$ th slot in the Elduque–Freudenthal magic super square; the first Cartan matrix is usually the one given in [CE], the other matrices are obtained from the first matrix by means of odd reflections. Accordingly,  $\mathfrak{g}(A, B)^c$  is the shorthand for the realization of  $\mathfrak{g}(A, B)$  by means of Cartan matrix  $c$ .

The table at the beginning of each case shows the result of odd reflections (the number of the row is the number of the matrix in the list below, the number of the column is the the number of the root in which reflection

is made; the cells contain the results of reflections (the number of the matrix obtained) or a “-” if the reflection is not appropriate because  $A_{ii} \neq 0$ . Some of the matrices thus obtained are equivalent; we did not eliminate a few redundancies.

If the diagram of  $\mathfrak{g}$  is symmetric, it gives rise to an outer automorphism whose fixed points constitute a Lie superalgebra. The examples below did not lead to any new simple Lie superalgebra.

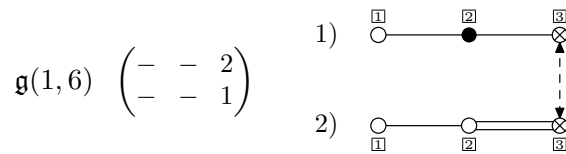
The numbers of matrices with the maximal number of even roots are boxed, those with the maximal number of add roots are underlined. The nodes are numbered by small boxed numbers; the dashed lines with arrows depict odd reflections.



$$\mathfrak{g}(2,3) \begin{pmatrix} - & - & 2 \\ 3 & 4 & 1 \\ 2 & 5 & - \\ 5 & 2 & - \\ 4 & 3 & - \end{pmatrix}$$

$$\boxed{1)} \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 0 \end{pmatrix} \quad \underline{2)} \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & -1 \\ -1 & -1 & 0 \end{pmatrix} \quad 3) \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & -2 \\ -1 & -2 & 2 \end{pmatrix}$$

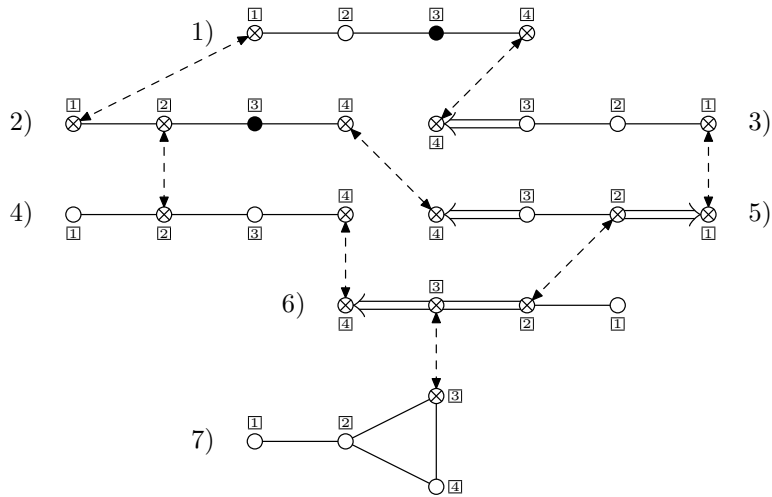
$$4) \begin{pmatrix} 0 & 0 & -2 \\ 0 & 0 & -1 \\ -2 & -1 & 2 \end{pmatrix} \quad \underline{5)} \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & -1 \\ -1 & -1 & 1 \end{pmatrix}$$



$$\mathfrak{g}(1,6) \begin{pmatrix} - & - & 2 \\ - & - & 1 \end{pmatrix}$$

$$1) \begin{pmatrix} 2 & -1 & 0 \\ -1 & 1 & -1 \\ 0 & -1 & 0 \end{pmatrix} \quad 2) \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -2 \\ 0 & -2 & 0 \end{pmatrix}$$

$$\mathfrak{g}(3,6) \begin{pmatrix} 2 & - & - & 3 \\ 1 & 4 & - & 5 \\ 5 & - & - & 1 \\ - & 2 & - & 6 \\ 3 & 6 & - & 2 \\ - & 5 & 7 & 4 \\ - & - & 6 & - \end{pmatrix}$$



$$1) \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 1 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \quad 2) \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & -1 & 0 \\ 0 & -1 & 1 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \quad 3) \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -2 \\ 0 & 0 & -1 & 0 \end{pmatrix}$$

$$4) \begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 0 & -2 & 0 \\ 0 & -2 & 2 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \quad 5) \begin{pmatrix} 0 & -1 & 0 & 0 \\ -2 & 0 & -1 & 0 \\ 0 & -1 & 2 & -2 \\ 0 & 0 & -1 & 0 \end{pmatrix}$$

$$6) \begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 0 & -2 & 0 \\ 0 & -2 & 0 & -2 \\ 0 & 0 & -1 & 0 \end{pmatrix} \quad \boxed{7) \begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & -1 \\ 0 & -1 & 0 & -1 \\ 0 & -1 & -1 & 2 \end{pmatrix}}$$



$$1) \begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -2 & 2 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \quad 2) \begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \quad 3) \begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 0 & -2 & -2 \\ 0 & -2 & 0 & -2 \\ 0 & -1 & -1 & 2 \end{pmatrix}$$

$$4) \begin{pmatrix} 0 & -1 & 0 & 0 \\ -2 & 0 & -1 & -1 \\ 0 & -1 & 2 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \quad 5) \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 2 & -1 & -1 \\ 0 & -1 & 2 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \quad 6) \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 2 & -2 & -1 \\ 0 & -1 & 2 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}$$

$$7) \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & -1 & -2 \\ 0 & -1 & 2 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \quad 8) \begin{pmatrix} 2 & -1 & -1 & 0 \\ -2 & 0 & -2 & -1 \\ -1 & -1 & 0 & 0 \\ 0 & -1 & 0 & 2 \end{pmatrix}$$

$$9) \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 2 & -1 & -1 \\ -1 & -1 & 0 & 0 \\ 0 & -1 & 0 & 2 \end{pmatrix} \quad 10) \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 2 & -1 & -1 \\ -1 & -2 & 2 & 0 \\ 0 & -1 & 0 & 2 \end{pmatrix}$$

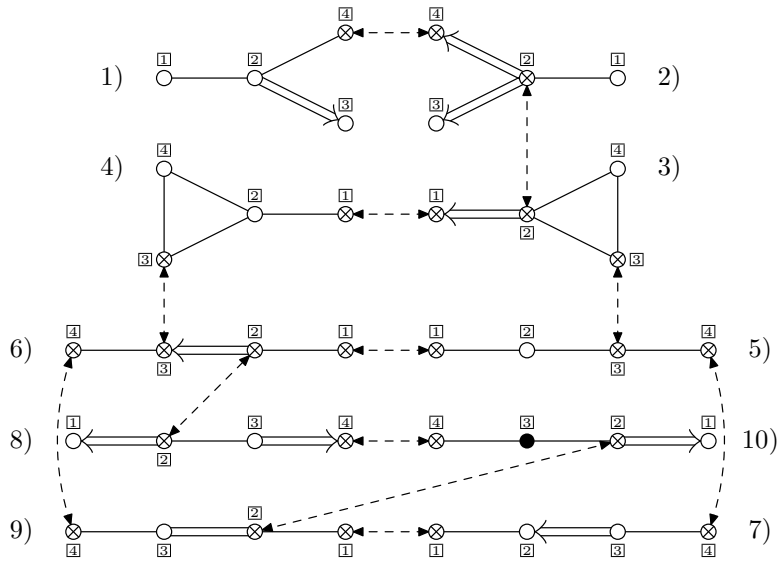
$$\mathfrak{g}(4,3) \begin{pmatrix} - & - & - & 2 \\ - & 3 & - & 1 \\ 4 & 2 & 5 & - \\ 3 & - & 6 & - \\ 6 & - & 3 & 7 \\ 5 & 8 & 4 & 9 \\ 9 & - & - & 5 \\ - & 6 & - & 10 \\ 7 & 10 & - & 6 \\ - & 9 & - & 8 \end{pmatrix}$$

$$1) \begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -2 & -1 \\ 0 & -1 & 2 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \quad 2) \begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 0 & -2 & -2 \\ 0 & -1 & 2 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \quad 3) \begin{pmatrix} 0 & -1 & 0 & 0 \\ -2 & 0 & -1 & -1 \\ 0 & -1 & 0 & -1 \\ 0 & -1 & -1 & 2 \end{pmatrix}$$

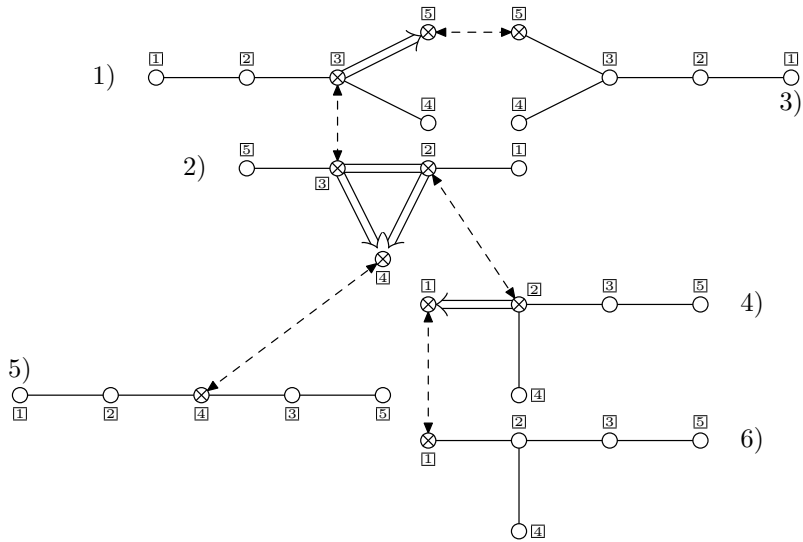
$$4) \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 2 & -1 & -1 \\ 0 & -1 & 0 & -1 \\ 0 & -1 & -1 & 2 \end{pmatrix} \quad 5) \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \quad 6) \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & -2 & 0 \\ 0 & -1 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix}$$

$$7) \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -2 & 2 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \quad 8) \begin{pmatrix} 2 & -1 & 0 & 0 \\ -2 & 0 & -1 & 0 \\ 0 & -1 & 2 & -2 \\ 0 & 0 & -1 & 0 \end{pmatrix}$$

$$9) \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & -2 & 0 \\ 0 & -2 & 2 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \quad 10) \begin{pmatrix} 2 & -1 & 0 & 0 \\ -2 & 0 & -1 & 0 \\ 0 & -1 & 1 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix}$$



$$g(2, 6) \begin{pmatrix} - & - & 2 & - & 3 \\ - & 4 & 1 & 5 & - \\ - & - & - & - & 1 \\ 6 & 2 & - & - & - \\ - & - & - & 2 & - \\ 4 & - & - & - & - \end{pmatrix}$$

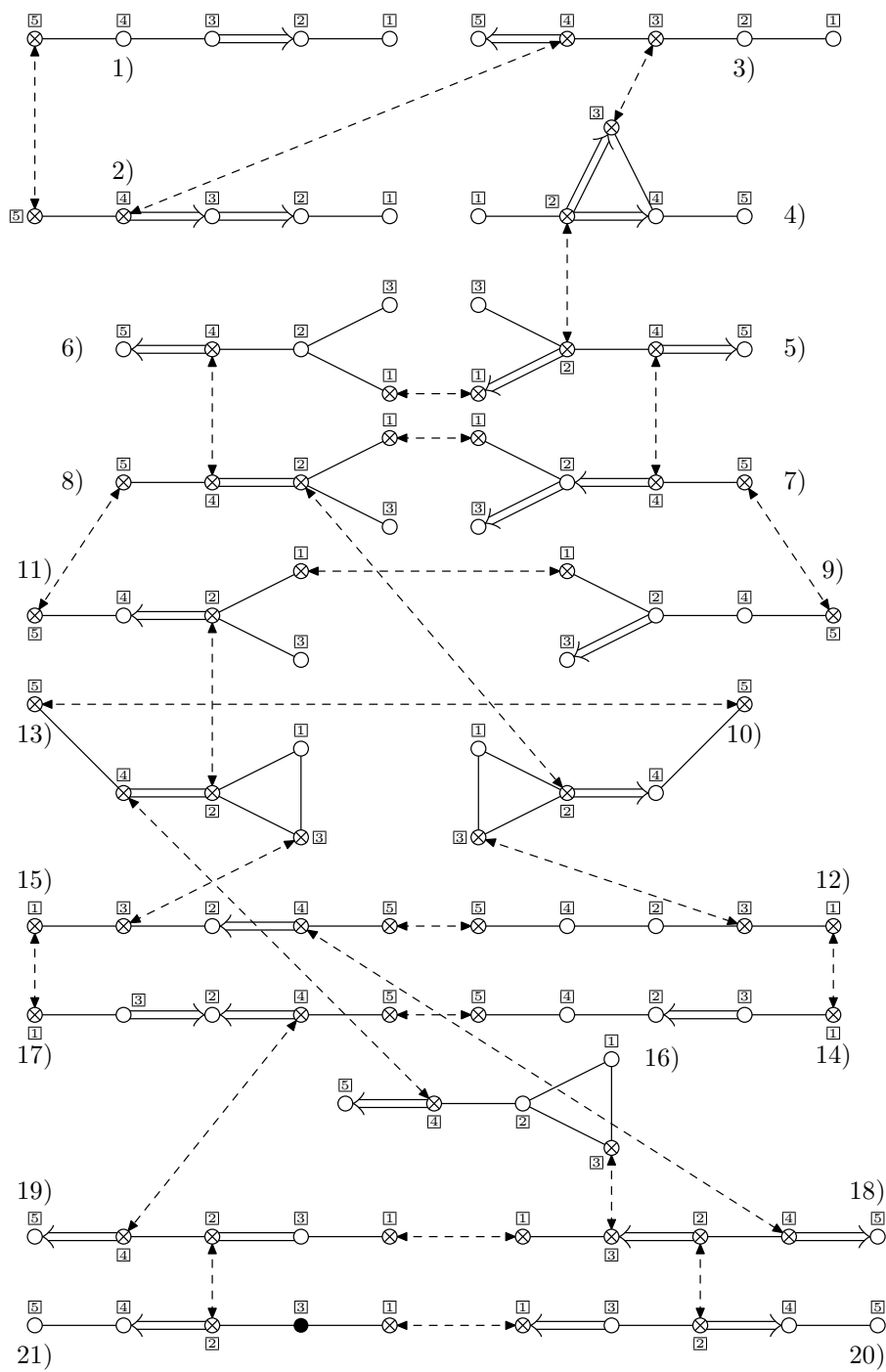




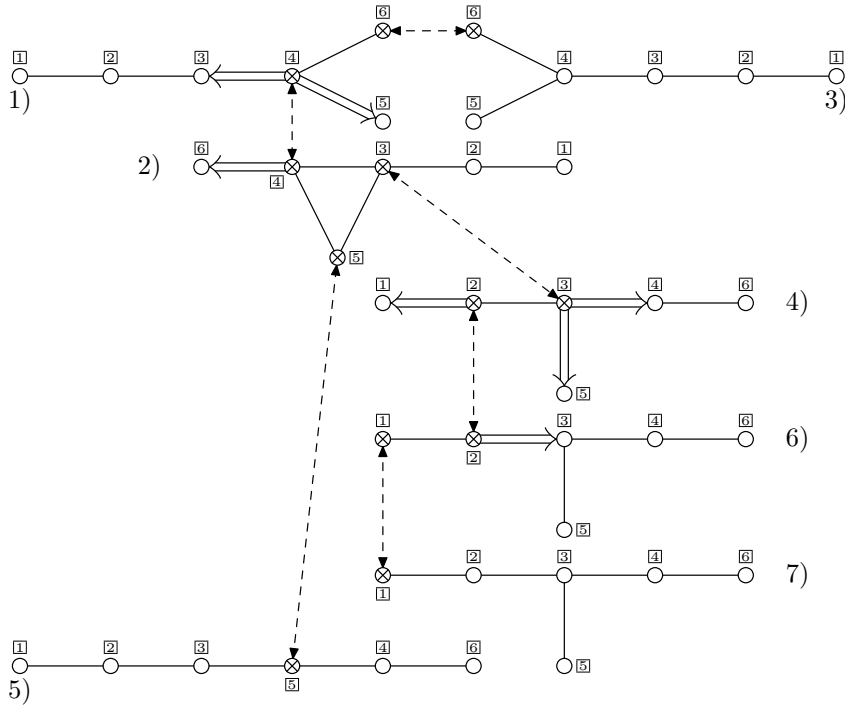
$$1) \begin{pmatrix} 2 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 0 & -1 & -2 \\ 0 & 0 & -1 & 2 & 0 \\ 0 & 0 & -1 & 0 & 0 \end{pmatrix} \quad 2) \begin{pmatrix} 2 & -1 & 0 & 0 & 0 \\ -1 & 0 & -2 & -2 & 0 \\ 0 & -2 & 0 & -2 & -1 \\ 0 & -1 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 2 \end{pmatrix} \quad 3) \begin{pmatrix} 2 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & -1 \\ 0 & 0 & -1 & 2 & 0 \\ 0 & 0 & -1 & 0 & 0 \end{pmatrix}$$

$$4) \begin{pmatrix} 0 & -1 & 0 & 0 & 0 \\ -2 & 0 & -1 & -1 & 0 \\ 0 & -1 & 2 & 0 & -1 \\ 0 & -1 & 0 & 2 & 0 \\ 0 & 0 & -1 & 0 & 2 \end{pmatrix} \quad 5) \begin{pmatrix} 2 & -1 & 0 & 0 & 0 \\ -1 & 2 & 0 & -1 & 0 \\ 0 & 0 & 2 & -1 & -1 \\ 0 & -1 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 2 \end{pmatrix} \quad 6) \begin{pmatrix} 0 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & -1 & 0 \\ 0 & -1 & 2 & 0 & -1 \\ 0 & -1 & 0 & 2 & 0 \\ 0 & 0 & -1 & 0 & 2 \end{pmatrix}$$

$$\mathfrak{g}(8,3) \begin{pmatrix} - & - & - & - & 2 \\ - & - & - & 3 & 1 \\ - & - & 4 & 2 & - \\ - & 5 & 3 & - & - \\ 6 & 4 & - & 7 & - \\ 5 & - & - & 8 & - \\ 8 & - & - & 5 & 9 \\ 7 & 10 & - & 6 & 11 \\ 11 & - & - & - & 7 \\ - & 8 & 12 & - & 13 \\ 9 & 13 & - & - & 8 \\ 14 & - & 10 & - & 15 \\ - & 11 & 15 & 16 & 10 \\ 12 & - & - & - & 17 \\ 17 & - & 13 & 18 & 12 \\ - & - & 18 & 13 & - \\ 15 & - & - & 19 & 14 \\ 19 & 20 & 16 & 15 & - \\ 18 & 21 & - & 17 & - \\ 21 & 18 & - & - & - \\ 20 & 19 & - & - & - \end{pmatrix}$$





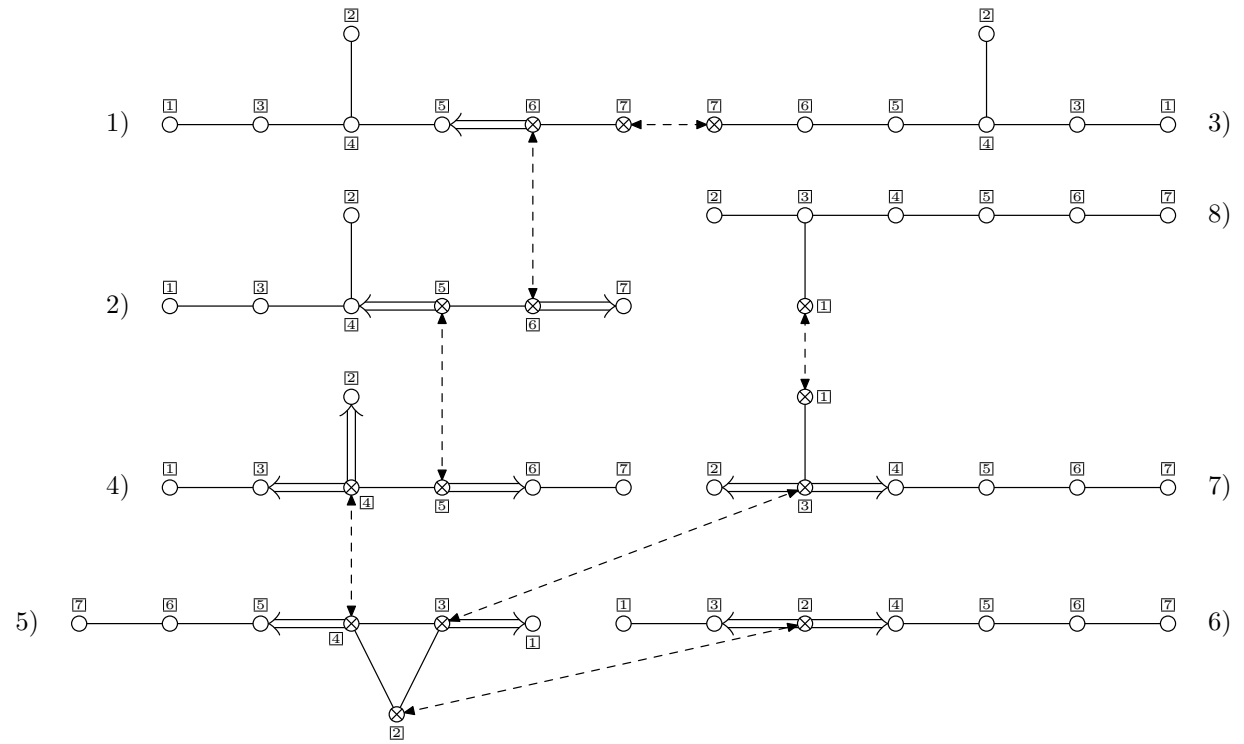


$$\begin{array}{l}
 1) \begin{pmatrix} 2 & -1 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & -2 & 0 & -2 & -1 \\ 0 & 0 & 0 & -1 & 2 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \end{pmatrix} \quad
 2) \begin{pmatrix} 2 & -1 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & -2 & 0 & -1 & -1 & 0 \\ 0 & 0 & -1 & 0 & -1 & -2 \\ 0 & 0 & -1 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 2 \end{pmatrix} \quad
 3) \begin{pmatrix} 2 & -1 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & -1 \\ 0 & 0 & 0 & -1 & 2 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \end{pmatrix} \\
 4) \begin{pmatrix} 2 & -1 & 0 & 0 & 0 & 0 \\ -2 & 0 & -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & -2 & -2 & 0 \\ 0 & 0 & -1 & 2 & 0 & -1 \\ 0 & 0 & -1 & 0 & 2 & 0 \\ 0 & 0 & 0 & -1 & 0 & 2 \end{pmatrix} \quad
 5) \begin{pmatrix} 2 & -1 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & -1 & 2 & 0 & -1 & 0 \\ 0 & 0 & 0 & 2 & -1 & -1 \\ 0 & 0 & -1 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 2 \end{pmatrix} \\
 6) \begin{pmatrix} 0 & -1 & 0 & 0 & 0 & 0 \\ -1 & 0 & -2 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & -1 & 0 \\ 0 & 0 & -1 & 2 & 0 & -1 \\ 0 & 0 & -1 & 0 & 2 & 0 \\ 0 & 0 & 0 & -1 & 0 & 2 \end{pmatrix} \quad
 7) \begin{pmatrix} 0 & -1 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & -1 & 0 \\ 0 & 0 & -1 & 2 & 0 & -1 \\ 0 & 0 & -1 & 0 & 2 & 0 \\ 0 & 0 & 0 & -1 & 0 & 2 \end{pmatrix}
 \end{array}$$

$$\mathfrak{g}(6,6) \begin{pmatrix} 2 & 3 & - & 4 & - & 5 \\ 1 & - & - & 6 & - & 7 \\ - & 1 & 8 & 9 & - & 10 \\ 6 & 9 & 11 & 1 & 12 & - \\ 7 & 10 & - & - & - & 1 \\ 4 & - & 13 & 2 & 14 & - \\ 5 & - & - & - & - & 2 \\ - & - & 3 & - & - & 15 \\ - & 4 & - & 3 & 16 & - \\ - & 5 & 15 & - & - & 3 \\ 13 & - & 4 & - & - & - \\ 14 & 16 & - & - & 4 & - \\ 11 & 17 & 6 & - & - & - \\ 12 & - & - & - & 6 & - \\ - & - & 10 & 18 & - & 8 \\ - & 12 & 19 & - & 9 & - \\ - & 13 & - & - & - & - \\ - & - & - & 15 & 20 & - \\ - & - & 16 & - & - & - \\ - & - & - & - & 18 & 21 \\ - & - & - & - & - & 20 \end{pmatrix}$$



$$\mathfrak{g}(8,6) \begin{pmatrix} - & - & - & - & - & 2 & 3 \\ - & - & - & - & 4 & 1 & - \\ - & - & - & - & - & - & 1 \\ - & - & - & 5 & 2 & - & - \\ - & 6 & 7 & 4 & - & - & - \\ - & 5 & - & - & - & - & - \\ 8 & - & 5 & - & - & - & - \\ 7 & - & - & - & - & - & - \end{pmatrix}$$





$$\begin{array}{l}
1) \begin{pmatrix} 2 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & -1 & 0 & 0 & 0 \\ -1 & 0 & 2 & -1 & 0 & 0 & 0 \\ 0 & -1 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & -2 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 \end{pmatrix} \quad 2) \begin{pmatrix} 2 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & -1 & 0 & 0 & 0 \\ -1 & 0 & 2 & -1 & 0 & 0 & 0 \\ 0 & -1 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & -2 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & -2 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 \end{pmatrix} \\
3) \begin{pmatrix} 2 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & -1 & 0 & 0 & 0 \\ -1 & 0 & 2 & -1 & 0 & 0 & 0 \\ 0 & -1 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 \end{pmatrix} \quad 4) \begin{pmatrix} 2 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & -1 & 0 & 0 & 0 \\ -1 & 0 & 2 & -1 & 0 & 0 & 0 \\ 0 & -2 & -2 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 \end{pmatrix} \\
5) \begin{pmatrix} 2 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & -1 & 0 & 0 & 0 \\ -2 & -1 & 0 & -1 & 0 & 0 & 0 \\ 0 & -1 & -1 & 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 \end{pmatrix} \quad \boxed{6)} \begin{pmatrix} 2 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -2 & -2 & 0 & 0 & 0 \\ -1 & -1 & 2 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 \end{pmatrix} \\
7) \begin{pmatrix} 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 2 & -1 & 0 & 0 & 0 & 0 \\ -1 & -2 & 0 & -2 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 \end{pmatrix} \quad \boxed{8)} \begin{pmatrix} 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 2 & -1 & 0 & 0 & 0 & 0 \\ -1 & -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 \end{pmatrix}
\end{array}$$

#### §4 Elduque and Cunha superalgebras: Defining relations

To save space, in what follows we omit indicating the Serre relations; their fulfilment is assumed.

$$\underline{\mathfrak{g}(2, 3)}, \text{ sdim} = (11|14)$$

- 1)  $[x_2, [x_1, x_3]] - 2[x_3, [x_1, x_2]] = 0;$   
 $(\text{ad}_{[x_1, x_2]})^2([x_1, x_3]) = 0; \quad (\text{ad}_{[x_1, x_2]})^2([x_2, x_3]) = 0.$
- 2)  $(\text{ad}_{[x_1, x_3]})^2([x_2, x_3]) = 0; \quad (\text{ad}_{[x_2, x_3]})^2([x_1, x_3]) = 0;$   
 $[[x_2, [x_1, x_3]], [[x_1, x_3], [x_2, [x_1, x_3]]]] = 0;$   
 $[[x_2, [x_1, x_3]], [[x_2, x_3], [x_2, [x_1, x_3]]]] = 0.$

- 3)  $[[x_1, x_3], [x_3, [x_2, x_3]]] = 0; \quad (\text{ad}_{[x_2, x_3]})^2(x_3) = 0;$   
 $[[x_2, [x_1, x_3]], [[x_1, x_3], [x_2, x_3]]] = 0;$   
 $[[x_2, [x_1, x_3]], [[x_2, x_3], [x_2, x_3]]] = 0.$
- 4)  $(\text{ad}_{[x_1, x_3]})^2(x_3) = 0; \quad [[x_2, x_3], [x_3, [x_1, x_3]]] = 0;$   
 $[[x_2, [x_1, x_3]], [[x_1, x_3], [x_1, x_3]]] = 0;$   
 $[[x_2, [x_1, x_3]], [[x_1, x_3], [x_2, x_3]]] = 0.$
- 5)  $[x_3, [x_2, [x_1, x_3]]] - [[x_1, x_3], [x_2, x_3]] = 0;$   
 $[[x_1, x_3], [x_3, x_3]] = 0; \quad [[x_2, x_3], [x_3, x_3]] = 0;$   
 $[[x_2, [x_1, x_3]], [x_3, [x_1, x_3]]] = 0; \quad [[x_2, [x_1, x_3]], [x_3, [x_2, x_3]]] = 0.$

$\mathfrak{g}(1, 6)$ ,  $\text{sdim} = (21|14)$

- 1)  $[x_3, [x_2, [x_1, x_2]]] - [[x_1, x_2], [x_2, x_3]] = 0;$   
 $[[x_1, x_2], [x_2, x_2]] = 0; \quad [[x_2, x_2], [x_2, x_3]] = 0,$   
 $[[x_3, [x_1, x_2]], [x_3, [x_2, x_2]]] = 0; \quad [[x_3, [x_1, x_2]], [[x_1, x_2], [x_1, x_2]]] = 0.$
- 2)  $[[x_1, x_2], [x_2, [x_2, x_3]]] = 0; \quad [[x_2, x_3], [x_2, [x_2, x_3]]] = 0;$   
 $[[x_3, [x_1, x_2]], [x_3, [x_2, [x_2, x_3]]]] = 0;$   
 $[[x_3, [x_1, x_2]], [[x_1, x_2], [x_3, [x_1, x_2]]]] = 0.$

$\mathfrak{g}(3, 6)$ ,  $\text{sdim} = (36|40)$

- 1)  $[x_4, [x_3, [x_2, x_3]]] - [[x_2, x_3], [x_3, x_4]] = 0; \quad [[x_2, x_3], [x_3, x_3]] = 0;$   
 $[[x_3, x_3], [x_3, x_4]] = 0; \quad [[x_4, [x_2, x_3]], [x_4, [x_3, x_3]]] = 0;$   
 $[[x_4, [x_2, x_3]], [[x_2, x_3], [x_2, x_3]]] = 0;$   
 $[[x_3, [x_2, x_3]], [[x_1, x_2], [x_3, x_4]]] - [[x_4, [x_2, x_3]], [[x_1, x_2], [x_3, x_3]]] = 0.$
- 2)  $[x_4, [x_3, [x_2, x_3]]] - [[x_2, x_3], [x_3, x_4]] = 0;$   
 $[[x_1, x_2], [x_2, x_3]] = 0; \quad [[x_2, x_3], [x_3, x_3]] = 0; \quad [[x_3, x_3], [x_3, x_4]] = 0;$   
 $[[x_3, [x_2, x_3]], [x_4, [x_2, x_3]]] = 0; \quad [[x_4, [x_2, x_3]], [x_4, [x_3, x_3]]] = 0;$   
 $[[x_1, x_2], [x_3, x_4]], [[x_1, x_2], [[x_1, x_2], [x_3, x_3]]]] = 0.$

- 3)  $[[x_2, x_3], [x_3, [x_3, x_4]]] = 0; \quad (\text{ad}_{[x_3, x_4]})^2(x_3) = 0;$   
 $[[x_4, [x_2, x_3]], [x_4, [x_3, [x_3, x_4]]] = 0;$   
 $[[x_4, [x_2, x_3]], [[x_2, x_3], [x_4, [x_2, x_3]]] = 0;$   
 $[[x_4, [x_2, x_3]], [[x_3, x_4], [x_3, [x_1, x_2]]] -$   
 $2[[x_1, x_2], [x_3, x_4]], [[x_2, x_3], [x_3, x_4]] = 0.$
- 5)  $[[x_1, x_2], [x_2, x_3]] = 0; \quad [[x_2, x_3], [x_3, [x_2, x_3]]] = 0;$   
 $[[x_3, x_4], [x_3, [x_2, x_3]]] = 0; \quad [[x_4, [x_2, x_3]], [[x_2, x_3], [x_2, x_3]]] = 0;$   
 $[[x_4, [x_2, x_3]], [[x_2, x_3], [x_3, x_4]]] = 0;$   
 $[[x_1, x_2], [x_3, x_4]], [[x_1, x_2], [x_3, [x_3, [x_1, x_2]]]] = 0.$
- 4)  $(\text{ad}_{[x_3, x_4]})^2(x_3) = 0; \quad [[x_1, x_2], [x_2, x_3]] = 0; \quad [[x_2, x_3], [x_3, [x_3, x_4]]] = 0;$   
 $[[x_4, [x_2, x_3]], [x_4, [x_3, [x_3, x_4]]] = 0;$   
 $[[x_4, [x_2, x_3]], [[x_2, x_3], [x_3, x_4]]] = 0;$   
 $[[x_1, x_2], [x_3, x_4]], [[x_3, [x_1, x_2]], [[x_1, x_2], [x_3, x_4]]] = 0.$
- 6)  $[[x_1, x_2], [x_2, x_3]] = 0; \quad [[x_2, x_3], [[x_2, x_3], [x_3, x_4]]] = 0;$   
 $[[x_3, x_4], [[x_2, x_3], [x_3, x_4]]] = 0;$   
 $[[x_4, [x_2, x_3]], [[x_2, x_3], [x_4, [x_2, x_3]]] = 0;$   
 $[[x_4, [x_2, x_3]], [[x_3, x_4], [x_4, [x_2, x_3]]] = 0;$   
 $[[x_1, x_2], [x_3, x_4]], [[x_3, [x_1, x_2]], [[x_1, x_2], [x_3, x_4]]] = 0.$
- 7)  $(\text{ad}_{[x_2, x_4]})^2([x_2, x_3]) = 0; \quad (\text{ad}_{[x_2, x_4]})^2([x_3, x_4]) = 0;$   
 $[x_3, [x_2, x_4]] - 2[x_4, [x_2, x_3]] = 0;$   
 $[[x_4, [x_1, x_2]], [[x_3, [x_1, x_2]], [x_4, [x_1, x_2]]] = 0.$

$\mathfrak{g}(3, 3)$ ,  $\text{sdim} = (21|16)$

**SuperLie** tells us that  $\text{sdim} = (22|16)$ : indeed the Cartan matrices below determine a nontrivial central extension of a simple Lie superalgebra. The relations below serve also the simple quotient algebras.

- 1)  $[[x_3, x_4], [x_3, [x_2, x_3]]] = 0; \quad [[x_3, [x_1, x_2]], [x_3, [x_2, x_3]]] = 0;$   
 $[[x_1, x_2], [x_3, x_4]], [[x_2, x_3], [x_3, x_4]] = 0.$
- 2)  $(\text{ad}_{[x_3, x_4]})^2([x_2, x_3]) = 0; \quad [[x_3, [x_1, x_2]], [[x_2, x_3], [x_3, x_4]]] = 0;$   
 $[[x_1, x_2], [x_3, x_4]], [[x_3, x_4], [x_4, [x_2, x_3]]] = 0.$

- 3)  $[x_3, [x_2, x_4]] - [x_4, [x_2, x_3]] = 0$ ;  $[[x_1, x_2], [x_2, x_3]] = 0$ ;  
 $[[x_1, x_2], [x_2, x_4]] = 0$ ;  $[[x_4, [x_1, x_2]], [[x_2, x_4], [x_3, x_4]]] = 0$ .
- 4)  $[[x_1, x_2], [x_2, x_3]] = 0$ ;  $[[x_1, x_2], [x_2, x_4]] = 0$ ;  
 $[[x_2, x_4], [[x_2, x_3], [x_2, x_4]]] = 0$ ;  
 $[[x_4, [x_1, x_2]], [[x_2, x_4], [x_4, [x_2, x_3]]]] = 0$ .
- 5)  $[[x_4, [x_1, x_2]], [[x_3, [x_1, x_2]], [x_4, [x_1, x_2]]]] = 0$ ;  
 $[[x_4, [x_1, x_2]], [[x_4, [x_1, x_2]], [x_4, [x_2, x_3]]]] = 0$ .
- 6)  $[[x_1, x_2], [x_2, [x_2, x_3]]] = 0$ ;  $[[x_2, x_4], [x_2, [x_2, x_3]]] = 0$ ;  
 $[[x_2, x_3], [x_4, [x_1, x_2]]] - 2[[x_2, x_4], [x_3, [x_1, x_2]]] = 0$ .
- 7)  $[[x_1, x_2], [x_2, x_4]] = 0$ ;  $[[x_4, [x_1, x_2]], [[x_1, x_2], [x_3, [x_1, x_2]]]] = 0$ ;  
 $[[x_2, x_3], [x_2, x_4]] = 0$ ;  $(\text{ad}_{[x_1, x_2]})^2([x_2, x_3]) = 0$ ;
- 8)  $[x_2, [x_1, x_3]] - [x_3, [x_1, x_2]] = 0$ ;  $[[x_1, x_2], [x_2, x_4]] = 0$ ;  
 $[[x_2, x_3], [x_2, x_4]] = 0$ ;  $[[x_4, [x_1, x_2]], [[x_1, x_2], [x_1, x_3]]] = 0$ .
- 9)  $[[x_1, x_3], [[x_1, x_3], [x_2, x_3]]] = 0$ ;  $[[x_4, [x_2, x_3]], [[x_1, x_3], [x_2, x_3]]] = 0$ ;  
 $[[x_1, x_3], [x_2, x_4]], [[x_1, x_3], [x_2, [x_1, x_3]]] = 0$ .
- 10)  $[[x_1, x_3], [x_3, [x_2, x_3]]] = 0$ ,  $[[x_3, [x_2, x_3]], [x_4, [x_2, x_3]]] = 0$ ;  
 $[[x_1, x_3], [x_2, x_3]], [[x_1, x_3], [x_2, x_4]] = 0$

$\mathfrak{g}(4, 3)$ ,  $\text{sdim} = (24|26)$

- 1)  $[[x_1, x_2], [x_2, [x_2, x_3]]] = 0$ ;  
 $[[x_2, x_3], [x_4, [x_1, x_2]]] - 2[[x_2, x_4], [x_3, [x_1, x_2]]] = 0$ ;  
 $[[x_2, x_4], [x_2, [x_2, x_3]]] = 0$ ;  
 $[[x_4, [x_1, x_2]], [[x_1, x_2], [x_3, [x_1, x_2]]]] = 0$ .
- 2)  $[[x_1, x_2], [x_2, x_3]] = 0$ ;  $[[x_2, x_3], [x_2, x_4]] = 0$ ;  
 $[[x_2, x_4], [[x_1, x_2], [x_2, x_4]]] = 0$ ;  
 $[[x_4, [x_2, x_3]], [[x_2, x_4], [x_4, [x_1, x_2]]]] = 0$ ;  
 $[[x_4, [x_2, x_3]], [[x_3, [x_1, x_2]], [x_4, [x_2, x_3]]]] = 0$ .

- 3)  $[x_3, [x_2, x_4]] - [x_4, [x_2, x_3]] = 0$ ;  $[[x_1, x_2], [x_2, x_3]] = 0$ ;  
 $[[x_1, x_2], [x_2, x_4]] = 0$ ;  $[[x_4, [x_1, x_2]], [[x_2, x_4], [x_3, x_4]]] = 0$ ;  
 $[[x_4, [x_1, x_2]], [[x_3, [x_1, x_2]], [x_4, [x_1, x_2]]]] = 0$ .
- 4)  $[x_3, [x_2, x_4]] - [x_4, [x_2, x_3]] = 0$ ;  $[[x_1, x_2], [x_2, x_3]] = 0$ ;  
 $[[x_1, x_2], [x_2, x_4]] = 0$ ;  $[[x_4, [x_1, x_2]], [[x_2, x_4], [x_3, x_4]]] = 0$ ;  
 $[[x_4, [x_1, x_2]], [[x_3, [x_1, x_2]], [x_4, [x_1, x_2]]]] = 0$ .
- 5)  $[[x_3, [x_1, x_2]], [[x_2, x_3], [x_3, x_4]]] = 0$ ;  
 $[[[x_1, x_2], [x_3, x_4]], [[x_3, x_4], [x_4, [x_2, x_3]]]] = 0$ ;  
 $[[[x_1, x_2], [x_3, x_4]], [[x_3, [x_1, x_2]], [[x_1, x_2], [x_3, x_4]]]] = 0$ .
- 6)  $[[x_1, x_2], [x_2, x_3]] = 0$ ;  $(\text{ad}_{[x_2, x_3]})^2([x_3, x_4]) = 0$ ;  
 $[[x_3, x_4], [[x_2, x_3], [x_3, x_4]]] = 0$ ;  
 $[[x_4, [x_2, x_3]], [[x_3, x_4], [x_4, [x_2, x_3]]]] = 0$ ;  
 $[[x_4, [x_2, x_3]], [[x_2, x_3], [x_4, [x_2, x_3]]]] = 0$ .
- 7)  $[[x_3, x_4], [x_3, [x_2, x_3]]] = 0$ ;  $[[x_3, [x_1, x_2]], [x_3, [x_2, x_3]]] = 0$ ;  
 $[[[x_1, x_2], [x_3, x_4]], [[x_2, x_3], [x_3, x_4]]] = 0$ ;  
 $[[[x_1, x_2], [x_3, x_4]], [[x_1, x_2], [x_3, [x_3, [x_1, x_2]]]]] = 0$ .
- 8)  $[[x_1, x_2], [x_2, x_3]] = 0$ ;  $[[x_2, x_3], [x_3, [x_3, x_4]]] = 0$ ;  
 $(\text{ad}_{[x_3, x_4]})^2(x_3) = 0$ ;  $[[x_4, [x_2, x_3]], [x_4, [x_3, [x_3, x_4]]]] = 0$ ;  
 $[[x_4, [x_2, x_3]], [[x_2, x_3], [x_3, x_4]]] = 0$ .
- 9)  $[[x_1, x_2], [x_2, x_3]] = 0$ ;  $(\text{ad}_{[x_2, x_3]})^2(x_3) = 0$ ;  
 $[[x_3, x_4], [x_3, [x_2, x_3]]] = 0$ ;  $[[x_4, [x_2, x_3]], [[x_2, x_3], [x_2, x_3]]] = 0$ ;  
 $[[x_4, [x_2, x_3]], [[x_2, x_3], [x_3, x_4]]] = 0$ .
- 10)  $[x_3, [x_3, x_3]] = 0$ ;  $[x_4, [x_3, [x_2, x_3]]] - [[x_2, x_3], [x_3, x_4]] = 0$ ;  
 $[[x_1, x_2], [x_2, x_3]] = 0$ ;  $[[x_2, x_3], [x_3, x_3]] = 0$ ;  
 $[[x_3, x_3], [x_3, x_4]] = 0$ ;  $[[x_3, [x_2, x_3]], [x_4, [x_2, x_3]]] = 0$ ;  
 $[[x_4, [x_2, x_3]], [x_4, [x_3, x_3]]] = 0$ .

$\mathfrak{g}(2, 6)$ ,  $\text{sdim} = (35|20)$

- 1)  $[[x_2, x_3], [x_3, x_5]] = 0; \quad [[x_3, x_4], [x_3, x_5]] = 0;$   
 $[[x_3, [x_1, x_2]], [[x_2, x_3], [x_3, x_4]]] = 0.$
- 2)  $[x_3, [x_2, x_4]] - [x_4, [x_2, x_3]] = 0; \quad [[x_1, x_2], [x_2, x_3]] = 0;$   
 $[[x_1, x_2], [x_2, x_4]] = 0; \quad [[x_2, x_3], [x_3, x_5]] = 0; \quad [[x_3, x_4], [x_3, x_5]] = 0.$
- 3)  $[[[x_1, x_2], [x_3, x_5]], [[x_5, [x_2, x_3]], [x_5, [x_3, x_4]]]] = 0.$
- 4)  $[[x_1, x_2], [x_2, x_3]] = 0; \quad [[x_1, x_2], [x_2, x_4]] = 0;$   
 $[[x_5, [x_2, x_3]], [[x_2, x_3], [x_2, x_4]]] = 0.$
- 5)  $[[x_4, [x_1, x_2]], [[x_2, x_4], [x_3, x_4]]] = 0;$   
 $[[x_5, [x_3, x_4]], [[x_2, x_4], [x_3, x_4]]] = 0.$
- 6)  $[[[x_1, x_2], [x_3, x_5]], [[x_3, [x_1, x_2]], [x_4, [x_1, x_2]]]] = 0.$

$\mathfrak{g}(8, 3)$ ,  $\text{sdim} = (55|50)$

- 1)  $[x_3, [x_3, [x_2, x_3]]] = 0; \quad [[x_3, x_4], [x_3, [x_2, x_3]]] = 0;$   
 $[[x_3, [x_1, x_2]], [x_3, [x_2, x_3]]] = 0;$   
 $[[[x_1, x_2], [x_3, x_4]], [[x_3, x_4], [x_4, [x_2, x_3]]]] = 0;$   
 $[[[x_1, x_2], [x_3, x_4]], [[x_4, x_5], [x_3, [x_2, x_3]]]] -$   
 $[[[x_2, x_3], [x_4, x_5]], [[x_3, x_4], [x_3, [x_1, x_2]]]] = 0.$
- 2)  $[[x_3, x_4], [x_4, x_5]] = 0; \quad [[x_3, x_4], [x_3, [x_2, x_3]]] = 0;$   
 $[[x_3, [x_1, x_2]], [x_3, [x_2, x_3]]] = 0;$   
 $[[[x_1, x_2], [x_3, x_4]], [[x_2, x_3], [x_3, x_4]]] = 0;$   
 $[[[x_4, x_5], [x_3, [x_1, x_2]]], [[x_5, [x_3, x_4]], [[x_2, x_3], [x_4, x_5]]]] = 0.$
- 3)  $[[x_3, x_4], [x_4, x_5]] = 0; \quad [[x_3, x_4], [[x_2, x_3], [x_3, x_4]]] = 0;$   
 $[[x_3, [x_1, x_2]], [[x_2, x_3], [x_3, x_4]]] = 0;$   
 $[[[x_1, x_2], [x_3, x_4]], [[x_3, x_4], [x_4, [x_2, x_3]]]] = 0;$   
 $[[[x_4, x_5], [x_3, [x_1, x_2]]], [[x_5, [x_3, x_4]], [[x_2, x_3], [x_4, x_5]]]] = 0.$
- 4)  $[x_3, [x_2, x_4]] - [x_4, [x_2, x_3]] = 0; \quad [[x_1, x_2], [x_2, x_3]] = 0;$   
 $[[x_1, x_2], [x_2, x_4]] = 0; \quad [[x_4, [x_1, x_2]], [[x_2, x_4], [x_3, x_4]]] = 0;$   
 $[[[x_1, x_2], [x_4, x_5]], [[x_5, [x_2, x_4]], [x_5, [x_3, x_4]]]] = 0.$

- 5)  $[[x_1, x_2], [x_2, x_3]] = 0; \quad [[x_1, x_2], [x_2, x_4]] = 0;$   
 $[[x_2, x_4], [x_4, x_5]] = 0; \quad (\text{ad}_{[x_2, x_4]})^2([x_2, x_3]) = 0;$   
 $[[x_4, [x_1, x_2]], [[x_2, x_4], [x_4, [x_2, x_3]]]] = 0;$   
 $[[[x_1, x_2], [x_4, x_5]], [[x_5, [x_2, x_4]], [[x_2, x_3], [x_4, x_5]]]] = 0.$
- 6)  $[[x_2, x_4], [x_4, x_5]] = 0; \quad [[x_4, [x_1, x_2]], [[x_3, [x_1, x_2]], [x_4, [x_1, x_2]]]] = 0;$   
 $[[x_4, [x_1, x_2]], [[x_4, [x_1, x_2]], [x_4, [x_2, x_3]]]] = 0;$   
 $[[[x_1, x_2], [x_4, x_5]], [[x_1, x_2], [x_4, x_5]], [[x_2, x_3], [x_4, x_5]]]] = 0.$
- 7)  $[[x_2, x_4], [x_4, x_5]] = 0; \quad [[x_1, x_2], [x_2, [x_2, x_3]]] = 0;$   
 $[[x_2, x_3], [x_4, [x_1, x_2]]] - 2[[x_2, x_4], [x_3, [x_1, x_2]]] = 0;$   
 $[[x_2, x_4], [x_2, [x_2, x_3]]] = 0;$   
 $[[[x_1, x_2], [x_4, x_5]], [[x_5, [x_2, x_4]], [[x_2, x_3], [x_4, x_5]]]] = 0.$
- 8)  $[[x_1, x_2], [x_2, x_4]] = 0; \quad [[x_2, x_3], [x_2, x_4]] = 0;$   
 $[[x_2, x_4], [x_4, x_5]] = 0; \quad [[x_1, x_2], [[x_1, x_2], [x_2, x_3]]] = 0;$   
 $[[x_4, [x_1, x_2]], [[x_1, x_2], [x_3, [x_1, x_2]]]] = 0;$   
 $[[[x_1, x_2], [x_4, x_5]], [[x_1, x_2], [x_4, x_5]], [[x_2, x_3], [x_4, x_5]]]] = 0.$
- 9)  $[[x_1, x_2], [x_2, [x_2, x_3]]] = 0;$   
 $[[x_2, x_3], [x_4, [x_1, x_2]]] - 2[[x_2, x_4], [x_3, [x_1, x_2]]] = 0;$   
 $[[x_2, x_4], [x_2, [x_2, x_3]]] = 0; \quad [[x_4, [x_1, x_2]], [[x_2, x_4], [x_4, [x_2, x_3]]]] = 0.$
- 10)  $[x_2, [x_1, x_3]] - [x_3, [x_1, x_2]] = 0; \quad [[x_1, x_2], [x_2, x_4]] = 0;$   
 $[[x_4, [x_1, x_2]], [[x_1, x_2], [x_1, x_3]]] = 0; \quad [[x_2, x_3], [x_2, x_4]] = 0;$   
 $[[[x_1, x_2], [x_4, x_5]], [[x_1, x_2], [x_4, x_5]], [[x_2, x_3], [x_4, x_5]]]] = 0.$
- 11)  $[[x_1, x_2], [x_2, x_4]] = 0; \quad [[x_2, x_3], [x_2, x_4]] = 0;$   
 $(\text{ad}_{[x_1, x_2]})^2([x_2, x_3]) = 0; \quad [[x_4, [x_1, x_2]], [[x_1, x_2], [x_3, [x_1, x_2]]]] = 0;$   
 $[[x_4, [x_1, x_2]], [[x_4, [x_1, x_2]], [x_4, [x_2, x_3]]]] = 0.$
- 12)  $[[x_1, x_3], [[x_1, x_3], [x_2, x_3]]] = 0; \quad [[x_4, [x_2, x_3]], [[x_1, x_3], [x_2, x_3]]] = 0;$   
 $[[[x_1, x_3], [x_2, x_4]], [[x_1, x_3], [x_2, [x_1, x_3]]]] = 0;$   
 $[[[x_4, x_5], [x_2, [x_1, x_3]]], [[x_2, x_3], [x_4, x_5]], [[x_4, x_5], [x_2, [x_1, x_3]]]] = 0.$

- 13)  $[x_2, [x_1, x_3]] - [x_3, [x_1, x_2]] = 0$ ;  $[[x_1, x_2], [x_2, x_4]] = 0$ ;  
 $[[x_2, x_3], [x_2, x_4]] = 0$ ;  $[[x_2, x_4], [x_4, x_5]] = 0$ ;  
 $[[x_4, [x_1, x_2]], [[x_1, x_2], [x_1, x_3]]] = 0$ ;  
 $[[x_4, [x_1, x_2]], [[x_4, [x_1, x_2]], [x_4, [x_2, x_3]]]] = 0$ .
- 14)  $[[x_1, x_3], [x_3, [x_2, x_3]]] = 0$ ;  $[[x_3, [x_2, x_3]], [x_4, [x_2, x_3]]] = 0$ ;  
 $[[[x_1, x_3], [x_2, x_3]], [[x_1, x_3], [x_2, x_4]]] = 0$ ;  
 $[[[x_4, x_5], [x_2, [x_1, x_3]]], [[x_5, [x_2, x_4]], [[x_4, x_5], [x_3, [x_2, x_3]]]]] = 0$ .
- 15)  $[[x_2, x_4], [x_4, x_5]] = 0$ ;  $[[x_1, x_3], [[x_1, x_3], [x_2, x_3]]] = 0$ ;  
 $[[x_4, [x_2, x_3]], [[x_1, x_3], [x_2, x_3]]] = 0$ ;  
 $[[[x_1, x_3], [x_2, x_4]], [[x_1, x_3], [x_2, [x_1, x_3]]]] = 0$ ;  
 $[[[x_1, x_3], [x_2, x_4]], [[x_4, [x_2, x_3]], [[x_1, x_3], [x_2, x_4]]]] = 0$ .
- 16)  $[x_2, [x_1, x_3]] - 2[x_3, [x_1, x_2]] = 0$ ;  $[[x_2, x_4], [x_4, x_5]] = 0$ ;  
 $(\text{ad}_{[x_1, x_2]})^2([x_1, x_3]) = 0$ ;  $(\text{ad}_{[x_1, x_2]})^2([x_2, x_3]) = 0$ .
- 17)  $[[x_2, x_4], [x_4, x_5]] = 0$ ;  $[[x_1, x_3], [x_3, [x_2, x_3]]] = 0$ ;  
 $[[x_3, [x_2, x_3]], [x_4, [x_2, x_3]]] = 0$ ;  
 $[[[x_1, x_3], [x_2, x_3]], [[x_1, x_3], [x_2, x_4]]] = 0$ ;  
 $[[[x_1, x_3], [x_2, x_4]], [[x_2, x_4], [x_4, [x_3, [x_2, x_3]]]]] = 0$ .
- 18)  $[[x_2, x_3], [x_2, x_4]] = 0$ ;  $[[x_2, x_4], [x_4, x_5]] = 0$ ;  
 $(\text{ad}_{[x_1, x_3]})^2([x_2, x_3]) = 0$ ;  $(\text{ad}_{[x_2, x_3]})^2([x_1, x_3]) = 0$ ;  
 $[[x_2, [x_1, x_3]], [[x_1, x_3], [x_2, [x_1, x_3]]]] = 0$ ;  
 $[[x_2, [x_1, x_3]], [[x_2, x_3], [x_2, [x_1, x_3]]]] = 0$ .
- 19)  $[[x_2, x_3], [x_2, x_4]] = 0$ ;  $[[x_2, x_4], [x_4, x_5]] = 0$ ;  
 $[[x_1, x_3], [x_3, [x_2, x_3]]] = 0$ ;  $(\text{ad}_{[x_2, x_3]})^2(x_3) = 0$ ;  
 $[[x_2, [x_1, x_3]], [[x_1, x_3], [x_2, x_3]]] = 0$ ;  
 $[[x_2, [x_1, x_3]], [[x_2, x_3], [x_2, x_3]]] = 0$ .
- 20)  $[[x_2, x_3], [x_2, x_4]] = 0$ ;  $(\text{ad}_{[x_1, x_3]})^2(x_3) = 0$ ;  
 $[[x_2, x_3], [x_3, [x_1, x_3]]] = 0$ ;  $[[x_2, [x_1, x_3]], [[x_1, x_3], [x_1, x_3]]] = 0$ ;  
 $[[x_2, [x_1, x_3]], [[x_1, x_3], [x_2, x_3]]] = 0$ .



$$\begin{aligned}
21) \quad & [x_3, [x_3, x_3]] = 0; \quad [x_3, [x_2, [x_1, x_3]]] - [[x_1, x_3], [x_2, x_3]] = 0; \\
& [[x_1, x_3], [x_3, x_3]] = 0; \quad [[x_2, x_3], [x_2, x_4]] = 0; \quad [[x_2, x_3], [x_3, x_3]] = 0; \\
& [[x_2, [x_1, x_3]], [x_3, [x_1, x_3]]] = 0; \quad [[x_2, [x_1, x_3]], [x_3, [x_2, x_3]]] = 0.
\end{aligned}$$

$\mathfrak{g}(4, 6)$ ,  $\text{sdim} = (66|32)$

$$\begin{aligned}
1) \quad & [[x_3, x_4], [x_4, x_6]] = 0; \quad [[x_4, x_5], [x_4, x_6]] = 0; \\
& [[x_4, [x_2, x_3]], [[x_3, x_4], [x_4, x_5]]] = 0. \\
2) \quad & [x_4, [x_3, x_5]] - [x_5, [x_3, x_4]] = 0; \quad [[x_2, x_3], [x_3, x_4]] = 0; \\
& [[x_2, x_3], [x_3, x_5]] = 0; \quad [[x_3, x_4], [x_4, x_6]] = 0; \quad [[x_4, x_5], [x_4, x_6]] = 0. \\
3) \quad & [[[x_2, x_3], [x_4, x_6]], [[x_6, [x_3, x_4]], [x_6, [x_4, x_5]]] = 0. \\
4) \quad & [[x_1, x_2], [x_2, x_3]] = 0; \quad [[x_2, x_3], [x_3, x_4]] = 0; \quad [[x_2, x_3], [x_3, x_5]] = 0; \\
& [[x_6, [x_3, x_4]], [[x_3, x_4], [x_3, x_5]]] = 0. \\
5) \quad & [[x_5, [x_2, x_3]], [[x_3, x_5], [x_4, x_5]]] = 0; \\
& [[x_6, [x_4, x_5]], [[x_3, x_5], [x_4, x_5]]] = 0. \\
6) \quad & [[x_1, x_2], [x_2, x_3]] = 0; \\
& [[[x_2, x_3], [x_4, x_6]], [[x_4, [x_2, x_3]], [x_5, [x_2, x_3]]] = 0. \\
7) \quad & [[[x_4, x_6], [x_3, [x_1, x_2]]], [[[x_1, x_2], [x_3, x_4]], [[x_1, x_2], [x_3, x_5]]] = 0.
\end{aligned}$$

$\mathfrak{g}(6, 6)$ ,  $\text{sdim} = (78|64)$

$$\begin{aligned}
1) \quad & [[x_1, x_2], [x_2, x_3]] = 0; \quad [[x_3, x_4], [x_4, x_6]] = 0; \quad [[x_4, x_5], [x_4, x_6]] = 0; \\
& [[x_4, [x_2, x_3]], [[x_3, x_4], [x_4, x_5]]] = 0; \\
& [[[x_2, x_3], [x_4, x_6]], [[x_4, [x_2, x_3]], [[x_2, x_3], [x_4, x_5]]] = 0. \\
2) \quad & [[x_3, x_4], [x_4, x_6]] = 0; \quad [[x_4, x_5], [x_4, x_6]] = 0; \\
& [[x_4, [x_2, x_3]], [[x_3, x_4], [x_4, x_5]]] = 0; \\
& [[[x_4, x_6], [x_3, [x_1, x_2]]], [[[x_1, x_2], [x_3, x_4]], [[x_4, x_5], [x_3, [x_1, x_2]]] = 0. \\
7) \quad & [[[x_2, x_3], [x_4, x_6]], [[x_6, [x_3, x_4]], [x_6, [x_4, x_5]]] = 0; \\
& [[[x_4, x_6], [x_3, [x_1, x_2]]], [[[x_4, x_5], [x_3, [x_1, x_2]]], \\
& \quad \quad \quad [[x_4, x_6], [x_3, [x_1, x_2]]] = 0.
\end{aligned}$$

- 5)  $[[x_1, x_2], [x_2, x_3]] = 0$ ;  
 $[[[x_2, x_3], [x_4, x_6]], [[x_6, [x_3, x_4]], [x_6, [x_4, x_5]]]] = 0$ ;  
 $[[[x_2, x_3], [x_4, x_6]], [[[x_2, x_3], [x_4, x_5]], [[x_2, x_3], [x_4, x_6]]]] = 0$ .
- 6)  $[x_4, [x_3, x_5]] - [x_5, [x_3, x_4]] = 0$ ;  $[[x_2, x_3], [x_3, x_4]] = 0$ ;  
 $[[x_2, x_3], [x_3, x_5]] = 0$ ;  $[[x_3, x_4], [x_4, x_6]] = 0$ ;  $[[x_4, x_5], [x_4, x_6]] = 0$ ;  
 $[[[x_4, x_6], [x_3, [x_1, x_2]]], [[[x_1, x_2], [x_3, x_4]], [[x_1, x_2], [x_3, x_5]]]] = 0$ .
- 3)  $[[x_1, x_2], [x_2, x_3]] = 0$ ;  $[[x_2, x_3], [x_3, x_4]] = 0$ ;  $[[x_3, x_4], [x_4, x_6]] = 0$ ;  
 $[[x_4, x_5], [x_4, x_6]] = 0$ ;  $[[x_3, x_4], [[x_3, x_4], [x_4, x_5]]] = 0$ ;  
 $[[x_6, [x_3, x_4]], [[x_3, x_4], [x_5, [x_3, x_4]]]] = 0$ .
- 4)  $[x_4, [x_3, x_5]] - [x_5, [x_3, x_4]] = 0$ ;  $[[x_1, x_2], [x_2, x_3]] = 0$ ;  
 $[[x_2, x_3], [x_3, x_4]] = 0$ ;  $[[x_2, x_3], [x_3, x_5]] = 0$ ;  
 $[[x_3, x_4], [x_4, x_6]] = 0$ ;  $[[x_4, x_5], [x_4, x_6]] = 0$ ;  
 $[[[x_2, x_3], [x_4, x_6]], [[x_4, [x_2, x_3]], [x_5, [x_2, x_3]]]] = 0$ .
- 8)  $[[x_2, x_3], [x_3, x_4]] = 0$ ;  $[[x_3, x_4], [x_4, [x_4, x_5]]] = 0$ ;  
 $[[x_4, x_6], [x_4, [x_4, x_5]]] = 0$ ;  
 $[[x_4, x_5], [x_6, [x_3, x_4]]] - 2[[x_4, x_6], [x_5, [x_3, x_4]]] = 0$ .
- 9)  $[x_4, [x_3, x_5]] - [x_5, [x_3, x_4]] = 0$ ;  
 $[[x_1, x_2], [x_2, x_3]] = 0$ ;  $[[x_3, x_4], [x_4, x_6]] = 0$ ;  
 $[[x_4, x_5], [x_4, x_6]] = 0$ ;  $[[x_6, [x_3, x_4]], [[x_3, x_4], [x_3, x_5]]] = 0$ .
- 10)  $[[x_1, x_2], [x_2, x_3]] = 0$ ;  $[[x_2, x_3], [x_3, x_4]] = 0$ ;  
 $[[x_6, [x_3, x_4]], [[x_5, [x_3, x_4]], [x_6, [x_3, x_4]]]] = 0$ ;  
 $[[x_6, [x_3, x_4]], [[x_6, [x_3, x_4]], [x_6, [x_4, x_5]]]] = 0$ .
- 11)  $[[x_2, x_3], [x_3, x_4]] = 0$ ;  $[[x_2, x_3], [x_3, x_5]] = 0$ ;  
 $[[x_6, [x_3, x_4]], [[x_3, x_4], [x_3, x_5]]] = 0$ ;  
 $[[[x_2, x_3], [x_4, x_6]], [[x_4, [x_2, x_3]], [x_5, [x_2, x_3]]]] = 0$ .
- 12)  $[[x_1, x_2], [x_2, x_3]] = 0$ ;  $[[x_5, [x_2, x_3]], [[x_3, x_5], [x_4, x_5]]] = 0$ ;  
 $[[x_6, [x_4, x_5]], [[x_3, x_5], [x_4, x_5]]] = 0$ ;  
 $[[[x_4, x_6], [x_5, [x_2, x_3]]], [[x_5, [x_2, x_3]], [[x_2, x_3], [x_4, x_5]]]] = 0$ .

- 13)  $[[x_1, x_2], [x_2, x_3]] = 0$ ;  $[[x_2, x_3], [x_3, x_4]] = 0$ ;  
 $[[x_2, x_3], [x_3, x_5]] = 0$ ;  $[[x_6, [x_3, x_4]], [[x_3, x_4], [x_3, x_5]]] = 0$ ;  
 $[[[x_4, x_6], [x_3, [x_1, x_2]]], [[[x_1, x_2], [x_3, x_4]], [[x_1, x_2], [x_3, x_5]]]] = 0$ .
- 14)  $[[x_5, [x_2, x_3]], [[x_3, x_5], [x_4, x_5]]] = 0$ ;  
 $[[x_6, [x_4, x_5]], [[x_3, x_5], [x_4, x_5]]] = 0$ ;  
 $[[[x_3, [x_1, x_2]], [x_6, [x_4, x_5]]], [[[x_1, x_2], [x_3, x_5]],$   
 $[[x_4, x_5], [x_3, [x_1, x_2]]]] = 0$ .
- 15)  $[[x_2, x_3], [x_3, x_4]] = 0$ ;  $[[x_3, x_4], [x_4, x_5]] = 0$ ;  
 $[[x_3, x_4], [x_4, x_6]] = 0$ ;  $[[x_4, x_6], [[x_4, x_5], [x_4, x_6]]] = 0$ ;  
 $[[x_6, [x_3, x_4]], [[x_4, x_6], [x_6, [x_4, x_5]]]] = 0$ .
- 16)  $[[x_1, x_2], [x_2, x_3]] = 0$ ;  $[[x_2, x_3], [x_3, x_5]] = 0$ ;  
 $[[x_3, x_5], [[x_3, x_5], [x_4, x_5]]] = 0$ ;  $[[x_6, [x_4, x_5]], [[x_3, x_5], [x_4, x_5]]] = 0$ ;  
 $[[[x_3, x_5], [x_4, x_6]], [[x_3, x_5], [x_4, [x_3, x_5]]]] = 0$ .
- 17)  $[[x_1, x_2], [x_2, x_3]] = 0$ ;  
 $[[[x_2, x_3], [x_4, x_6]], [[x_4, [x_2, x_3]], [x_5, [x_2, x_3]]]] = 0$ ;  
 $[[[x_4, x_6], [x_3, [x_1, x_2]]], [[[x_1, x_2], [x_3, x_4]], [[x_1, x_2], [x_3, x_5]]]] = 0$ .
- 18)  $[x_5, [x_4, x_6]] - [x_6, [x_4, x_5]] = 0$ ;  $[[x_3, x_4], [x_4, x_5]] = 0$ ;  
 $[[x_3, x_4], [x_4, x_6]] = 0$ ;  $[[x_6, [x_3, x_4]], [[x_4, x_6], [x_5, x_6]]] = 0$ .
- 19)  $[[x_2, x_3], [x_3, x_5]] = 0$ ;  $[[x_3, x_5], [x_5, [x_4, x_5]]] = 0$ ;  
 $[[x_5, [x_4, x_5]], [x_6, [x_4, x_5]]] = 0$ ;  
 $[[[x_3, x_5], [x_4, x_5]], [[x_3, x_5], [x_4, x_6]]] = 0$ .
- 20)  $[[x_5, x_6], [[x_4, x_5], [x_5, x_6]]] = 0$ ;  $[[x_5, [x_3, x_4]], [[x_4, x_5], [x_5, x_6]]] = 0$ ;  
 $[[[x_3, x_4], [x_5, x_6]], [[x_5, x_6], [x_6, [x_4, x_5]]]] = 0$ .
- 21)  $[[x_5, x_6], [x_5, [x_4, x_5]]] = 0$ ;  $[[x_5, [x_3, x_4]], [x_5, [x_4, x_5]]] = 0$ ;  
 $[[[x_3, x_4], [x_5, x_6]], [[x_4, x_5], [x_5, x_6]]] = 0$ .

$\mathfrak{g}(8, 6)$ ,  $\text{sdim} = (133|56)$

- 1)  $[[x_5, x_6], [x_6, x_7]] = 0$ ;  
 $[[[x_5, x_6], [x_4, [x_1, x_3]]], [[[x_2, x_4], [x_5, x_6]], [[x_3, x_4], [x_5, x_6]]]] = 0$ .

- 2)  $[[x_4, x_5], [x_5, x_6]] = 0; \quad [[x_5, x_6], [x_6, x_7]] = 0;$   
 $[[[x_1, x_3], [x_4, x_5]], [[x_5, [x_2, x_4]], [x_5, [x_3, x_4]]]] = 0.$
- 3)  $[[[x_4, [x_1, x_3]], [x_7, [x_5, x_6]]], [[[x_6, x_7], [x_5, [x_2, x_4]]],$   
 $[[x_6, x_7], [x_5, [x_3, x_4]]]] = 0.$
- 4)  $[[x_2, x_4], [x_4, x_5]] = 0; \quad [[x_3, x_4], [x_4, x_5]] = 0; \quad [[x_4, x_5], [x_5, x_6]] = 0;$   
 $[[x_4, [x_1, x_3]], [[x_2, x_4], [x_3, x_4]]] = 0.$
- 5)  $[x_3, [x_2, x_4]] - [x_4, [x_2, x_3]] = 0; \quad [[x_1, x_3], [x_2, x_3]] = 0;$   
 $[[x_1, x_3], [x_3, x_4]] = 0; \quad [[x_2, x_4], [x_4, x_5]] = 0; \quad [[x_3, x_4], [x_4, x_5]] = 0.$
- 6)  $[[x_2, [x_1, x_3]], [[x_2, x_3], [x_2, x_4]]] = 0;$   
 $[[x_5, [x_2, x_4]], [[x_2, x_3], [x_2, x_4]]] = 0.$
- 7)  $[[x_1, x_3], [x_2, x_3]] = 0; \quad [[x_1, x_3], [x_3, x_4]] = 0;$   
 $[[x_5, [x_3, x_4]], [[x_2, x_3], [x_3, x_4]]] = 0.$
- 8)  $[[[x_1, x_3], [x_4, x_5]], [[x_2, [x_1, x_3]], [x_4, [x_1, x_3]]]] = 0.$

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## Appendix A The maximal root and the coefficients of linear dependence over $\mathbb{Z}$ with respect to the simple roots

$\mathfrak{g}(2, 3)$ ,  $\text{sdim} = (11|14)$

- 1)  $x_{11} = [[x_2, x_3], [[x_1, x_2], [x_1, x_3]]]$ ;  $\text{weight}(x_{11}) = (0, 0, 2)$ ;  $(2, 2, 2)$ .
- 2)  $x_{11} = [[x_2, [x_1, x_3]], [[x_1, x_3], [x_2, x_3]]]$ ;  $\text{weight}(x_{11}) = (0, 0, 2)$ ;  $(2, 2, 3)$ .
- 3)  $x_{11} = [[x_2, [x_1, x_3]], [x_3, [x_2, x_3]]]$ ;  $\text{weight}(x_{11}) = (0, 0, 1)$ ;  $(1, 2, 3)$ .
- 4)  $x_{11} = [[x_2, [x_1, x_3]], [x_3, [x_1, x_3]]]$ ;  $\text{weight}(x_{11}) = (0, 0, 1)$ ;  $(2, 1, 3)$ .
- 5)  $x_{11} = [[x_3, x_3], [x_2, [x_1, x_3]]]$ ;  $\text{weight}(x_{11}) = (0, 0, 1)$ ;  $(1, 1, 3)$ .

$\mathfrak{g}(1, 6)$ ,  $\text{sdim} = (21|14)$

- 1)  $x_{16} = [[[x_3, [x_2, x_2]], [[x_1, x_2], [x_3, [x_1, x_2]]]]]$ ;  $\text{weight}(x_{16}) = (0, 0, 1)$ ;  $(2, 4, 2)$ .
- 2)  $x_{16} = ( [[[x_1, x_2], [x_2, x_3]], [[x_2, x_3], [x_3, [x_1, x_2]]]] ]]$ ;  $\text{weight}(x_{16}) = (0, 0, -1)$ ;  $(2, 4, 3)$ .

$\mathfrak{g}(3, 6)$ ,  $\text{sdim} = (36|40)$

- 1)  $x_{36} = [[[[x_3, [x_1, x_2]], [x_4, [x_3, x_3]]], [[[x_1, x_2], [x_3, x_4]], [[x_2, x_3], [x_2, x_3]]]]]$ ;  $\text{weight}(x_{36}) = (1, 0, 0, 0)$ ;  $(2, 4, 6, 2)$ .
- 2)  $x_{36} = [[[[x_4, [x_2, x_3]], [[x_1, x_2], [x_3, x_3]]], [[[x_1, x_2], [x_3, x_3]], [[x_1, x_2], [x_3, x_4]]]]]$ ;  $\text{weight}(x_{36}) = (2, 0, 0, 0)$ ;  $(3, 4, 6, 2)$ .
- 3)  $x_{36} = [[[[x_4, [x_2, x_3]], [[x_1, x_2], [x_3, x_4]]], [[[x_2, x_3], [x_3, x_4]], [[x_3, x_4], [x_3, [x_1, x_2]]]]]]]$ ;  $\text{weight}(x_{36}) = (2, 0, 0, 0)$ ;  $(2, 4, 6, 4)$ .
- 4)  $x_{36} = [[[[x_3, [x_3, [x_1, x_2]]], [[x_1, x_2], [x_3, x_4]]], [[[x_1, x_2], [x_3, x_4]], [[x_2, x_3], [x_2, x_3]]]]]$ ;  $\text{weight}(x_{36}) = (1, 0, 0, 0)$ ;  $(3, 5, 6, 2)$ .
- 5)  $x_{36} = [[[[[x_1, x_2], [x_3, x_4]], [[x_2, x_3], [x_3, x_4]]], [[[x_1, x_2], [x_3, x_4]], [[x_3, x_4], [x_3, [x_1, x_2]]]]]]]$ ;  $\text{weight}(x_{36}) = (2, 0, 0, 0)$ ;  $(3, 4, 6, 4)$ .
- 6)  $x_{36} = [[[[[x_1, x_2], [x_3, x_4]], [[x_2, x_3], [x_4, [x_2, x_3]]]]], [[[x_1, x_2], [x_3, x_4]], [[x_3, x_4], [x_3, [x_1, x_2]]]]]]]$ ;  $\text{weight}(x_{36}) = (1, 0, 0, 0)$ ;  $(3, 5, 6, 4)$ .
- 7)  $x_{36} = [[[[x_4, [x_2, x_3]], [[x_1, x_2], [x_3, x_4]]], [[x_4, [x_1, x_2]], [[x_2, x_4], [x_3, [x_1, x_2]]]]]]]$ ;  $\text{weight}(x_{36}) = (1, 0, 0, 0)$ ;  $(3, 5, 3, 4)$ .

$\mathfrak{g}(3, 3)$ ,  $\text{sdim} = (21|16)$ 

- 1)  $x_{17} = [[x_4, [x_2, x_3]], [x_3, [x_3, [x_1, x_2]]]]$ ;  $\text{weight}(x_{17}) = (0, 0, 1, 0)$ ;  $(1, 2, 3, 1)$ .
- 2)  $x_{17} = [[[x_1, x_2], [x_3, x_4]], [[x_2, x_3], [x_3, x_4]]]$ ;  $\text{weight}(x_{17}) = (0, 0, 1, 0)$ ;  $(1, 2, 3, 2)$ .
- 3)  $x_{17} = [[x_4, [x_2, x_3]], [[x_1, x_2], [x_3, x_4]]]$ ;  $\text{weight}(x_{17}) = (0, 0, 2, 0)$ ;  $(1, 2, 2, 2)$ .
- 4)  $x_{17} = [[x_4, [x_3, [x_1, x_2]]], [[x_2, x_3], [x_2, x_4]]]$ ;  $\text{weight}(x_{17}) = (0, 0, 1, 0)$ ;  $(1, 3, 2, 2)$ .
- 5)  $x_{17} = [[x_4, [x_3, [x_1, x_2]]], [[x_2, x_4], [x_3, [x_1, x_2]]]]$ ;  $\text{weight}(x_{17}) = (0, 0, 1, 0)$ ;  $(2, 3, 2, 2)$ .
- 6)  $x_{17} = [[x_4, [x_2, x_3]], [[x_1, x_2], [x_2, x_3]]]$ ;  $\text{weight}(x_{17}) = (0, 0, 1, 0)$ ;  $(1, 3, 2, 1)$ .
- 7)  $x_{17} = [[x_4, [x_3, [x_1, x_2]]], [[x_1, x_2], [x_2, x_3]]]$ ;  $\text{weight}(x_{17}) = (0, 0, 1, 0)$ ;  $(2, 3, 2, 1)$ .
- 8)  $x_{17} = [[x_4, [x_2, x_3]], [[x_1, x_2], [x_1, x_3]]]$ ;  $\text{weight}(x_{17}) = (0, 0, -1, 0)$ ;  $(2, 2, 2, 1)$ .
- 9)  $x_{17} = [[[x_1, x_3], [x_2, x_3]], [[x_1, x_3], [x_2, x_4]]]$ ;  $\text{weight}(x_{17}) = (0, 0, -1, 0)$ ;  $(2, 2, 3, 1)$ .
- 10)  $x_{17} = [[x_4, [x_2, x_3]], [[x_1, x_3], [x_2, x_3]]]$ ;  $\text{weight}(x_{17}) = (0, 0, 1, 0)$ ;  $(1, 2, 3, 1)$ .

 $\mathfrak{g}(4, 3)$ ,  $\text{sdim} = (24|26)$ 

- 1)  $x_{23} = [[[x_2, x_3], [x_2, x_4]], [[x_3, [x_1, x_2]], [x_4, [x_1, x_2]]]]$ ;  $\text{weight}(x_{23}) = (0, 0, 0, 1)$ ;  $(2, 4, 2, 2)$ .
- 2)  $x_{23} = [[[x_2, x_4], [x_4, [x_1, x_2]]], [[x_3, [x_1, x_2]], [x_4, [x_2, x_3]]]]$ ;  $\text{weight}(x_{23}) = (0, 0, 0, 1)$ ;  $(2, 4, 2, 3)$ .
- 3)  $x_{23} = [[[x_1, x_4], [x_4, [x_2, x_3]]], [[x_2, x_4], [x_3, [x_1, x_2]]]]$ ;  $\text{weight}(x_{23}) = (0, 0, 0, 1)$ ;  $(2, 3, 2, 3)$ .
- 4)  $x_{23} = [[[x_2, x_4], [x_3, [x_1, x_2]]], [[x_3, x_4], [x_4, [x_1, x_2]]]]$ ;  $\text{weight}(x_{23}) = (0, 0, 0, 1)$ ;  $(2, 3, 2, 3)$ .
- 5)  $x_{23} = [[[x_3, x_4], [[x_1, x_2], [x_3, x_4]]], [[x_3, [x_1, x_2]], [x_4, [x_2, x_3]]]]$ ;  $\text{weight}(x_{23}) = (0, 0, 0, 0, -1)$ ;  $(2, 3, 4, 3)$ .
- 6)  $x_{23} = [[[x_3, x_4], [x_4, [x_2, x_3]]], [[x_3, [x_1, x_2]], [x_4, [x_2, x_3]]]]$ ;  $\text{weight}(x_{23}) = (0, 0, 0, 0, -1)$ ;  $(1, 3, 4, 3)$ .
- 7)  $x_{23} = [[[x_3, x_4], [x_3, [x_1, x_2]]], [[x_3, [x_1, x_2]], [x_4, [x_2, x_3]]]]$ ;  $\text{weight}(x_{23}) = (0, 0, 0, -1)$ ;  $(2, 3, 4, 2)$ .
- 8)  $x_{23} = [[[x_3, x_4], [x_3, [x_1, x_2]]], [[x_3, x_4], [x_4, [x_2, x_3]]]]$ ;  $\text{weight}(x_{23}) = (0, 0, 0, -1)$ ;  $(1, 2, 4, 3)$ .
- 9)  $x_{23} = [[[x_2, x_3], [x_4, [x_2, x_3]]], [[x_3, x_4], [x_3, [x_1, x_2]]]]$ ;  $\text{weight}(x_{23}) = (0, 0, 0, -1)$ ;  $(1, 3, 4, 2)$ .
- 10)  $x_{23} = [[[x_2, x_3], [x_3, x_4]], [[x_3, x_4], [x_3, [x_1, x_2]]]]$ ;  $\text{weight}(x_{23}) = (0, 0, 0, -1)$ ;  $(1, 2, 4, 2)$ .



$\mathfrak{g}(2, 6)$ ,  $\text{sdim} = (35|20)$ 

- 1)  $x_{25} = [[[x_2, x_3], [x_3, x_4]], [x_1, x_2], [x_5, [x_3, x_4]]]; \quad \text{weight}(x_{25}) = (0, 0, 0, 1, 0); \quad (1, 2, 3, 2, 1).$
- 2)  $x_{25} = [[[x_1, x_2], [x_3, x_4]], [x_2, x_4], [x_3, x_5]]; \quad \text{weight}(x_{25}) = (0, 0, 0, -1, 0); \quad (1, 2, 2, 2, 1).$
- 3)  $x_{25} = [[[x_1, x_2], [x_5, [x_3, x_4]]], [x_3, x_5], [x_4, [x_2, x_3]]]; \quad \text{weight}(x_{25}) = (0, 0, 0, -1, 0); \quad (1, 2, 3, 2, 2).$
- 4)  $x_{25} = [[[x_2, x_4], [x_3, x_5]], [x_2, x_4], [x_3, [x_1, x_2]]]; \quad \text{weight}(x_{25}) = (0, 0, 0, 1, 0); \quad (1, 3, 2, 2, 1).$
- 5)  $x_{25} = [[[x_2, x_4], [x_3, x_5]], [x_3, x_4], [x_4, [x_1, x_2]]]; \quad \text{weight}(x_{25}) = (0, 0, 0, -1, 0); \quad (1, 2, 2, 3, 1).$
- 6)  $x_{25} = [[[x_2, x_4], [x_3, [x_1, x_2]]], [x_3, x_5], [x_4, [x_1, x_2]]]; \quad \text{weight}(x_{25}) = (0, 0, 0, 1, 0); \quad (2, 3, 2, 2, 1).$

 $\mathfrak{g}(8, 3)$ ,  $\text{sdim} = (55|50)$ 

- 1)  $x_{50} = [[[x_1, x_2], [x_3, x_4]], [x_2, x_3], [x_4, x_5]], [[[x_3, x_4], [x_3, [x_1, x_2]]], [x_4, x_5], [x_3, [x_2, x_3]]];$   
 $\text{weight}(x_{50}) = (0, 0, 0, 0, 1); \quad (2, 4, 6, 4, 2).$
- 2)  $x_{50} = [[[x_2, x_3], [x_4, x_5]], [x_4, x_5], [x_3, [x_1, x_2]]], [[[x_3, x_4], [x_3, [x_1, x_2]]], [x_4, x_5], [x_3, [x_2, x_3]]];$   
 $\text{weight}(x_{50}) = (0, 0, 0, 0, 2); \quad (2, 4, 6, 4, 3).$
- 3)  $x_{50} = [[[x_2, x_3], [x_5, [x_3, x_4]]], [x_4, x_5], [x_3, [x_1, x_2]]], [[[x_3, x_4], [x_4, [x_2, x_3]]], [x_4, x_5], [x_3, [x_1, x_2]]];$   
 $\text{weight}(x_{50}) = (0, 0, 0, 0, 1); \quad (2, 4, 6, 5, 5, 3).$
- 4)  $x_{50} = [[[x_1, x_2], [x_4, x_5]], [x_2, x_4], [x_3, x_4]], [[[x_2, x_3], [x_4, x_5]], [x_4, x_5], [x_3, [x_1, x_2]]];$   
 $\text{weight}(x_{50}) = (0, 0, 0, 0, 1); \quad (2, 4, 3, 5, 3).$
- 5)  $x_{50} = [[[x_2, x_3], [x_4, x_5]], [x_2, x_4], [x_3, [x_1, x_2]]], [[[x_1, x_2], [x_4, x_5]], [x_4, [x_2, x_3]], [x_5, [x_2, x_4]]];$   
 $\text{weight}(x_{50}) = (0, 0, 0, 0, 1); \quad (2, 6, 3, 5, 3).$
- 6)  $x_{50} = [[[x_2, x_4], [x_3, [x_1, x_2]]], [x_4, x_5], [x_3, [x_1, x_2]]], [[[x_1, x_2], [x_4, x_5]], [x_5, [x_2, x_4]], [x_4, [x_3, [x_1, x_2]]];$   
 $\text{weight}(x_{50}) = (0, 0, 0, 0, 1); \quad (4, 6, 3, 5, 3).$
- 7)  $x_{50} = [[[x_1, x_2], [x_4, x_5]], [x_4, x_5], [x_2, [x_2, x_3]]], [[[x_2, x_3], [x_4, x_5]], [x_2, x_4], [x_3, [x_1, x_2]]];$   
 $\text{weight}(x_{50}) = (0, 0, 0, 0, 2); \quad (2, 6, 3, 4, 3).$
- 8)  $x_{50} = [[[x_1, x_2], [x_4, x_5]], [x_3, [x_1, x_2]], [x_5, [x_2, x_4]]], [[[x_2, x_4], [x_3, [x_1, x_2]]], [x_4, x_5], [x_3, [x_1, x_2]]];$   
 $\text{weight}(x_{50}) = (0, 0, 0, 0, 2); \quad (4, 6, 3, 4, 3).$

- 9)  $x_{50} = [[[[x_1, x_2], [x_4, x_5]], [[x_2, x_3], [x_2, x_4]]], [[x_2, x_3], [x_4, x_5]], [[x_2, x_4], [x_3, [x_1, x_2]]]]$   
 $\text{weight}(x_{50}) = (0, 0, 0, 0, 2); \quad (2, 6, 3, 4, 2).$
- 10)  $x_{50} = [[[[x_2, x_3], [x_4, x_5]], [[x_1, x_3], [x_4, [x_1, x_2]]], [[x_1, x_2], [x_4, x_5]], [[x_3, [x_1, x_2]], [x_5, [x_2, x_4]]]]]$   
 $\text{weight}(x_{50}) = (0, 0, 0, 0, 2); \quad (4, 5, 3, 4, 3).$
- 11)  $x_{50} = [[[[x_2, x_3], [x_4, x_5]], [[x_1, x_2], [x_3, [x_1, x_2]]], [[x_1, x_2], [x_4, x_5]], [[x_4, [x_1, x_2]], [x_4, [x_2, x_3]]]]]$   
 $\text{weight}(x_{50}) = (0, 0, 0, 0, 2); \quad (4, 6, 3, 4, 2).$
- 12)  $x_{50} = [[[[x_4, x_5], [x_2, [x_1, x_3]]], [[x_4, x_5], [x_1, x_3], [x_2, x_3]]],$   
 $[[[x_4, x_5], [x_2, [x_1, x_3]]], [[x_2, [x_1, x_3]], [x_4, [x_2, x_3]]]], \text{weight}(x_{50}) = (0, 0, 0, 0, 2); \quad (4, 5, 6, 4, 3).$
- 13)  $x_{50} = [[[[x_1, x_2], [x_4, x_5]], [[x_2, x_4], [x_3, [x_1, x_2]]], [[x_2, x_3], [x_4, x_5]], [[x_1, x_3], [x_4, [x_1, x_2]]]]]$   
 $\text{weight}(x_{50}) = (0, 0, 0, 0, 2); \quad (4, 5, 3, 4, 2).$
- 14)  $x_{50} = [[[[x_2, x_4], [x_3, [x_2, x_3]]], [[x_4, x_5], [x_2, [x_1, x_3]]], [[x_4, x_5], [x_2, [x_1, x_3]]], [[x_4, x_5], [x_3, [x_2, x_3]]]]]$   
 $\text{weight}(x_{50}) = (0, 0, 0, 0, 2); \quad (2, 5, 6, 4, 3).$
- 15)  $x_{50} = [[[[x_1, x_3], [x_4, [x_2, x_3]]], [[x_4, x_5], [x_2, [x_1, x_3]]],$   
 $[[[x_4, x_5], [x_2, [x_1, x_3]]], [[x_2, [x_1, x_3]], [x_4, [x_2, x_3]]]], \text{weight}(x_{50}) = (0, 0, 0, 0, 2); \quad (4, 5, 6, 4, 2).$
- 16)  $x_{50} = [[[[x_1, x_2], [x_4, x_5]], [[x_1, x_3], [x_2, x_4]]], [[x_2, x_3], [x_4, x_5]], [[x_1, x_2], [x_3, [x_1, x_2]]]]]$   
 $\text{weight}(x_{50}) = (0, 0, 0, 0, 1); \quad (4, 5, 3, 3, 2).$
- 17)  $x_{50} = [[[[x_2, x_3], [x_4, x_5]], [[x_1, x_3], [x_4, [x_2, x_3]]], [[x_2, x_4], [x_3, [x_2, x_3]]], [[x_4, x_5], [x_2, [x_1, x_3]]]]]$   
 $\text{weight}(x_{50}) = (0, 0, 0, 0, 2); \quad (2, 5, 6, 4, 2).$
- 18)  $x_{50} = [[[[x_1, x_3], [x_4, [x_2, x_3]]], [[x_4, x_5], [x_2, [x_1, x_3]]], [[x_2, x_3], [x_2, [x_1, x_3]]], [[x_4, x_5], [x_2, [x_1, x_3]]]]]$   
 $\text{weight}(x_{50}) = (0, 0, 0, 0, 1); \quad (4, 5, 6, 3, 2).$
- 19)  $x_{50} = [[[[x_2, x_3], [x_4, x_5]], [[x_1, x_3], [x_4, [x_2, x_3]]], [[x_2, x_3], [x_4, x_5]], [[x_2, x_3], [x_2, [x_1, x_3]]]]]$   
 $\text{weight}(x_{50}) = (0, 0, 0, 0, 1); \quad (2, 5, 6, 3, 2).$
- 20)  $x_{50} = [[[[x_2, x_3], [x_4, x_5]], [[x_1, x_3], [x_2, [x_1, x_3]]], [[x_2, x_4], [x_3, [x_1, x_3]]], [[x_4, x_5], [x_2, [x_1, x_3]]]]]$   
 $\text{weight}(x_{50}) = (0, 0, 0, 0, 1); \quad (4, 4, 6, 3, 2).$
- 21)  $x_{50} = [[[[x_1, x_3], [x_2, x_4]], [[x_2, x_3], [x_4, x_5]]], [[x_2, x_3], [x_4, x_5]], [[x_3, x_3], [x_2, [x_1, x_3]]]]]$   
 $\text{weight}(x_{50}) = (0, 0, 0, 0, 1); \quad (2, 4, 6, 3, 2).$

$\mathfrak{g}(4, 6)$ ,  $\text{sdim} = (66|32)$

- 1)  $x_{46} = [[[x_6, [x_3, x_4]], [x_2, x_3], [x_4, x_5]], [[x_6, [x_4, x_5]], [x_1, x_2], [x_3, x_4]]];$   
 $\text{weight}(x_{46}) = (0, 0, 0, 0, 0, 2); \quad (1, 2, 3, 4, 2, 2).$
- 2)  $x_{46} = [[[x_5, [x_2, x_3]], [x_6, [x_3, x_4]], [[x_6, [x_4, x_5]], [x_1, x_2], [x_3, x_4]]];$   
 $\text{weight}(x_{46}) = (0, 0, 0, 0, 0, 1); \quad (1, 2, 3, 3, 2, 2).$
- 3)  $x_{46} = [[[x_6, [x_4, x_5]], [x_2, x_3], [x_4, x_6]], [[x_6, [x_5, [x_3, x_4]], [x_1, x_2], [x_3, x_4]]];$   
 $\text{weight}(x_{46}) = (0, 0, 0, 0, 0, 2); \quad (1, 2, 3, 4, 2, 3).$
- 4)  $x_{46} = [[[x_6, [x_3, x_4]], [x_5, [x_4, [x_2, x_3]]], [[x_6, [x_3, x_4]], [x_1, x_2], [x_3, x_5]]];$   
 $\text{weight}(x_{46}) = (0, 0, 0, 0, 0, 1); \quad (1, 2, 4, 3, 2, 2).$
- 5)  $x_{46} = [[[x_6, [x_4, x_5]], [x_2, x_3], [x_4, x_5]], [[[x_1, x_2], [x_3, x_5]], [x_3, x_5], [x_4, x_6]]];$   
 $\text{weight}(x_{46}) = (0, 0, 0, 0, 0, 1); \quad (1, 2, 3, 3, 4, 2).$
- 6)  $x_{46} = [[[x_6, [x_3, x_4]], [x_1, x_2], [x_3, x_5]], [[x_5, [x_4, [x_2, x_3]], [x_2, x_3], [x_4, x_6]]];$   
 $\text{weight}(x_{46}) = (0, 0, 0, 0, 0, 1); \quad (1, 3, 4, 3, 2, 2).$
- 7)  $x_{46} = [[[[x_1, x_2], [x_3, x_4]], [x_3, x_5], [x_4, x_6]], [[[x_1, x_2], [x_3, x_5]], [x_2, x_3], [x_4, x_6]]];$   
 $\text{weight}(x_{46}) = (0, 0, 0, 0, 0, 1); \quad (2, 3, 4, 3, 2, 2).$

$\mathfrak{g}(6, 6)$ ,  $\text{sdim} = (78|64)$

- 1)  $x_{68} = [ [[[x_4, x_5], [x_4, [x_2, x_3]]], [[x_3, [x_1, x_2]], [x_6, [x_4, x_5]]], [ [x_4, x_6], [x_3, [x_1, x_2]] ], [[x_4, [x_2, x_3]], [x_5, [x_3, x_4]]] ] ] ]$ ;  $\text{weight}(x_{68}) = (1, 0, 0, 0, 0, 0)$ ;  $(2, 4, 5, 6, 3, 2)$ .
- 2)  $x_{68} = [ [[[x_4, x_6], [x_3, [x_1, x_2]]], [[x_4, [x_2, x_3]], [x_5, [x_3, x_4]]], [ [x_4, x_5], [x_1, x_2], [x_3, x_4] ], [[x_3, [x_1, x_2]], [x_6, [x_4, x_5]]] ] ] ]$ ;  $\text{weight}(x_{68}) = (2, 0, 0, 0, 0, 0)$ ;  $(3, 4, 5, 6, 3, 2)$ .
- 7)  $x_{68} = [ [[[x_3, [x_1, x_2]], [x_6, [x_4, x_5]]], [[x_4, [x_2, x_3]], [x_6, [x_4, x_5]]], [ [x_4, x_6], [x_3, [x_1, x_2]] ], [[x_6, [x_5, [x_3, x_4]]], [[x_1, x_2], [x_3, x_4]]] ] ] ]$ ;  $\text{weight}(x_{68}) = (2, 0, 0, 0, 0, 0)$ ;  $(3, 4, 5, 6, 3, 4)$ .
- 5)  $x_{68} = [ [[[x_4, x_6], [x_3, [x_1, x_2]]], [[x_6, [x_3, x_4]], [[x_2, x_3], [x_4, x_5]]], [ [x_3, [x_1, x_2]], [x_6, [x_4, x_5]] ], [[x_4, [x_2, x_3]], [x_6, [x_4, x_5]]] ] ] ]$ ;  $\text{weight}(x_{68}) = (1, 0, 0, 0, 0, 0)$ ;  $(2, 4, 5, 6, 3, 4)$ .
- 6)  $x_{68} = [ [[[x_4, x_5], [x_3, [x_1, x_2]]], [[x_4, x_6], [x_5, [x_2, x_3]]], [ [x_4, x_6], [x_3, [x_1, x_2]] ], [[x_3, [x_1, x_2]], [x_5, [x_3, x_4]]] ] ] ]$ ;  $\text{weight}(x_{68}) = (2, 0, 0, 0, 0, 0)$ ;  $(3, 4, 5, 4, 3, 2)$ .
- 3)  $x_{68} = [ [[[x_4, x_5], [x_4, [x_2, x_3]]], [[x_4, x_6], [x_3, [x_1, x_2]]], [ [x_4, x_5], [x_3, [x_1, x_2]] ], [[x_5, [x_3, x_4]], [x_6, [x_3, x_4]]] ] ] ]$ ;  $\text{weight}(x_{68}) = (1, 0, 0, 0, 0, 0)$ ;  $(2, 3, 5, 6, 3, 2)$ .
- 4)  $x_{68} = [ [[[x_3, x_5], [x_4, [x_2, x_3]]], [[x_4, x_6], [x_3, [x_1, x_2]]], [ [x_4, x_5], [x_3, [x_1, x_2]] ], [[x_4, x_6], [x_5, [x_2, x_3]]] ] ] ]$ ;  $\text{weight}(x_{68}) = (2, 0, 0, 0, 0, 0)$ ;  $(2, 4, 5, 4, 3, 2)$ .
- 8)  $x_{68} = [ [[[x_4, x_5], [x_3, [x_1, x_2]]], [[x_4, x_6], [x_5, [x_3, x_4]]], [ [x_4, x_5], [x_4, [x_2, x_3]] ], [[x_4, x_6], [x_3, [x_1, x_2]]] ] ] ]$ ;  $\text{weight}(x_{68}) = (1, 0, 0, 0, 0, 0)$ ;  $(2, 3, 4, 6, 3, 2)$ .
- 9)  $x_{68} = [ [[[x_3, x_5], [x_4, x_6]], [[x_4, x_5], [x_3, [x_1, x_2]]], [ [x_3, x_5], [x_4, [x_2, x_3]] ], [[x_4, x_6], [x_3, [x_1, x_2]]] ] ] ]$ ;  $\text{weight}(x_{68}) = (1, 0, 0, 0, 0, 0)$ ;  $(2, 3, 5, 4, 3, 2)$ .
- 10)  $x_{68} = [ [[[x_4, x_6], [x_5, [x_3, x_4]]], [[x_3, [x_1, x_2]], [x_6, [x_4, x_5]]], [ [x_4, x_6], [x_3, [x_1, x_2]] ], [[x_6, [x_3, x_4]], [[x_2, x_3], [x_4, x_5]]] ] ] ]$ ;  $\text{weight}(x_{68}) = (1, 0, 0, 0, 0, 0)$ ;  $(2, 3, 5, 6, 3, 4)$ .

- 11)  $x_{68} = [[[[x_3, x_5], [x_4, [x_2, x_3]], [[x_4, x_6], [x_3, [x_1, x_2]]], [[x_4, x_6], [x_5, [x_2, x_3]], [[x_3, [x_1, x_2]], [x_5, [x_3, x_4]]]]]; \text{ weight}(x_{68}) = (2, 0, 0, 0, 0, 0); \quad (2, 4, 6, 4, 3, 2).$
- 12)  $x_{68} = [[[[x_4, x_6], [x_5, [x_2, x_3]], [[x_4, x_5], [[x_1, x_2], [x_3, x_5]]], [[x_3, [x_1, x_2]], [x_6, [x_4, x_5]], [[x_4, [x_3, x_5]], [x_5, [x_2, x_3]]]]]; \text{ weight}(x_{68}) = (2, 0, 0, 0, 0, 0); \quad (2, 4, 5, 4, 6, 2).$
- 13)  $x_{68} = [[[[x_4, x_6], [x_3, [x_1, x_2]], [[x_3, [x_1, x_2]], [x_5, [x_3, x_4]]], [[x_4, x_6], [x_5, [x_2, x_3]], [[x_3, [x_1, x_2]], [x_5, [x_3, x_4]]]]]; \text{ weight}(x_{68}) = (2, 0, 0, 0, 0, 0) \quad (3, 4, 6, 4, 3, 2).$
- 14)  $x_{68} = [[[[x_4, x_5], [[x_1, x_2], [x_3, x_5]], [[x_3, [x_1, x_2]], [x_6, [x_4, x_5]]], [[x_3, [x_1, x_2]], [x_6, [x_4, x_5]], [[x_3, [x_1, x_2]], [x_6, [x_4, x_5]]], [[x_4, [x_3, x_5]], [x_5, [x_2, x_3]]]]]; \text{ weight}(x_{68}) = (2, 0, 0, 0, 0, 0); \quad (3, 4, 5, 4, 6, 2).$
- 15)  $x_{68} = [[[[x_4, x_6], [x_3, [x_1, x_2]], [[x_4, [x_2, x_3]], [x_6, [x_4, x_5]]], [[x_4, x_6], [x_5, [x_3, x_4]], [[x_3, [x_1, x_2]], [x_6, [x_4, x_5]]]]]; \text{ weight}(x_{68}) = (1, 0, 0, 0, 0, 0); \quad (2, 3, 4, 6, 3, 4).$
- 16)  $x_{68} = [[[[x_4, x_5], [x_3, [x_1, x_2]], [[x_3, x_5], [[x_3, x_5], [x_4, x_6]]], [[x_4, x_6], [x_5, [x_2, x_3]], [[x_4, x_5], [[x_1, x_2], [x_3, x_5]]]]]; \text{ weight}(x_{68}) = (1, 0, 0, 0, 0, 0); \quad (2, 3, 5, 4, 6, 2).$
- 17)  $x_{68} = [[[[x_4, x_6], [x_5, [x_2, x_3]], [[x_3, [x_1, x_2]], [x_5, [x_3, x_4]]], [[x_4, x_6], [x_3, [x_1, x_2]], [[x_5, [x_2, x_3]], [[x_1, x_2], [x_3, x_4]]]]]; \text{ weight}(x_{68}) = (1, 0, 0, 0, 0, 0); \quad (3, 5, 6, 4, 3, 2).$
- 18)  $x_{68} = [[[[x_4, x_6], [x_3, [x_1, x_2]], [[x_5, x_6], [x_4, [x_2, x_3]]], [[x_4, x_6], [x_5, [x_3, x_4]], [[x_3, [x_1, x_2]], [x_6, [x_4, x_5]]]]]; \text{ weight}(x_{68}) = (1, 0, 0, 0, 0, 0); \quad (2, 3, 4, 5, 3, 4).$
- 19)  $x_{68} = [[[[x_4, x_5], [x_3, [x_1, x_2]], [[x_4, x_6], [x_5, [x_2, x_3]]], [[x_3, x_5], [x_6, [x_4, x_5]], [[x_3, [x_1, x_2]], [x_5, [x_4, x_5]]]]]; \text{ weight}(x_{68}) = (1, 0, 0, 0, 0, 0); \quad (2, 3, 4, 4, 6, 2).$
- 20)  $x_{68} = [[[[x_5, x_6], [[x_2, x_3], [x_4, x_5]], [[x_3, [x_1, x_2]], [x_6, [x_4, x_5]]], [[x_3, [x_1, x_2]], [x_6, [x_4, x_5]], [[x_5, [x_3, x_4]], [x_6, [x_4, x_5]]]]]; \text{ weight}(x_{68}) = (1, 0, 0, 0, 0, 0); \quad (2, 3, 4, 5, 6, 3).$
- 21)  $x_{68} = [[[[x_4, x_5], [x_5, [x_3, x_4]], [[x_3, [x_1, x_2]], [x_6, [x_4, x_5]]], [[x_5, x_6], [x_4, [x_2, x_3]], [[x_3, [x_1, x_2]], [x_5, [x_4, x_5]]]]]; \text{ weight}(x_{68}) = (1, 0, 0, 0, 0, 0); \quad (2, 3, 4, 5, 6, 2).$

- $\mathfrak{g}(8, 6)$ ,  $\text{sdim} = (133|56)$
- 1)  $x_{91} = [[[[x_3, [x_2, x_4]], [x_7, [x_5, x_6]], [[x_6, [x_4, x_5]], [[x_1, x_3], [x_2, x_4]]], [[x_4, [x_1, x_3]], [x_7, [x_5, x_6]], [[x_6, [x_4, x_5]], [x_5, [x_3, [x_2, x_4]]]]]]]; \text{weight}(x_{91}) = (0, 0, 0, 0, 0, 0, 2); \quad (2, 3, 4, 6, 5, 4, 2).$
  - 3)  $x_{91} = [[[[x_4, [x_1, x_3]], [x_7, [x_5, x_6]], [[x_6, [x_4, x_5]], [x_5, [x_3, [x_2, x_4]]], [[x_3, [x_2, x_4]], [[x_4, x_5], [x_6, x_7]], [[x_7, [x_5, x_6]], [[x_1, x_3], [x_2, x_4]]]]]]]; \text{weight}(x_{91}) = (0, 0, 0, 0, 0, 0, 2); \quad (2, 3, 4, 6, 5, 4, 3).$
  - 2)  $x_{91} = [[[[x_4, [x_1, x_3]], [x_7, [x_5, x_6]], [[x_5, [x_2, x_4]], [x_5, [x_3, x_4]]], [[x_3, [x_2, x_4]], [x_7, [x_5, x_6]], [[x_6, [x_4, x_5]], [[x_1, x_3], [x_2, x_4]]]]]]]; \text{weight}(x_{91}) = (0, 0, 0, 0, 0, 0, 1); \quad (2, 3, 4, 6, 5, 3, 2).$
  - 4)  $x_{91} = [[[[x_3, [x_2, x_4]], [x_6, [x_4, x_5]], [[x_4, [x_1, x_3]], [x_7, [x_5, x_6]]], [[x_3, [x_2, x_4]], [x_7, [x_5, x_6]], [[x_4, [x_1, x_3]], [x_5, [x_2, x_4]]]]]]]; \text{weight}(x_{91}) = (0, 0, 0, 0, 0, 0, 1); \quad (2, 3, 4, 6, 4, 3, 2).$
  - 5)  $x_{91} = [[[[x_6, x_7], [x_5, [x_3, x_4]], [[x_4, [x_1, x_3]], [x_5, [x_2, x_4]]], [[x_2, [x_1, x_3]], [x_6, [x_4, x_5]], [[x_4, [x_2, x_3]], [x_7, [x_5, x_6]]]]]]]; \text{weight}(x_{91}) = (0, 0, 0, 0, 0, 0, 1); \quad (2, 3, 4, 5, 4, 3, 2).$
  - 6)  $x_{91} = [[[[x_4, [x_2, x_3]], [x_7, [x_5, x_6]], [[x_2, [x_1, x_3]], [[x_2, x_4], [x_5, x_6]]], [[x_4, [x_2, x_3]], [x_7, [x_5, x_6]], [[x_5, [x_2, x_4]], [[x_1, x_3], [x_2, x_4]]]]]]]; \text{weight}(x_{91}) = (0, 0, 0, 0, 0, 0, 1); \quad (2, 6, 4, 5, 4, 3, 2).$
  - 7)  $x_{91} = [[[[x_4, [x_1, x_3]], [x_7, [x_5, x_6]], [[x_4, [x_2, x_3]], [x_5, [x_3, x_4]]], [[x_4, [x_2, x_3]], [x_7, [x_5, x_6]], [[x_2, [x_1, x_3]], [[x_3, x_4], [x_5, x_6]]]]]]]; \text{weight}(x_{91}) = (0, 0, 0, 0, 0, 0, 1); \quad (2, 3, 6, 5, 4, 3, 2).$
  - 8)  $x_{91} = [[[[x_4, [x_2, x_3]], [x_7, [x_5, x_6]], [[x_2, [x_1, x_3]], [[x_1, x_3], [x_4, x_5]]], [[x_4, [x_1, x_3]], [x_7, [x_5, x_6]], [[x_4, [x_2, [x_1, x_3]]], [[x_3, x_4], [x_5, x_6]]]]]]]; \text{weight}(x_{91}) = (0, 0, 0, 0, 0, 0, 1); \quad (4, 3, 6, 5, 4, 3, 2).$

## Appendix B The inverse matrices of the Cartan matrices

In what follows the numbering of inverse matrices that of the Cartan matrices. For the algebras not listed here the Cartan matrices are degenerate.

$$\underline{\mathfrak{g}(1,6)} \quad 1) \begin{pmatrix} 2 & 0 & 2 \\ 0 & 0 & 1 \\ -2 & -1 & 2 \end{pmatrix}; \quad 2) \begin{pmatrix} -1 & 0 & 1 \\ 0 & 0 & -1 \\ -1 & 1 & 0 \end{pmatrix}$$

$$\underline{\mathfrak{g}(3,6)} \quad 1) \begin{pmatrix} 2 & -1 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & -1 & 1 \end{pmatrix}; \quad 2) \begin{pmatrix} 0 & -2 & 0 & 2 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & -1 & -1 \end{pmatrix}$$

$$3) \begin{pmatrix} -2 & -1 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 2 & 0 & -2 & -1 \end{pmatrix} \quad 4) \begin{pmatrix} 0 & -1 & 0 & 2 \\ -1 & -2 & 0 & 1 \\ 0 & 0 & 0 & -1 \\ 2 & 1 & -1 & -1 \end{pmatrix}; \quad 5) \begin{pmatrix} 0 & -2 & 0 & 2 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 2 & 0 & -2 & -1 \end{pmatrix}$$

$$6) \begin{pmatrix} 0 & -1 & 0 & 2 \\ -1 & -2 & 0 & 1 \\ 0 & 0 & 0 & -1 \\ 1 & -1 & 1 & 2 \end{pmatrix}; \quad 7) \begin{pmatrix} 0 & 2 & 0 & -2 \\ 2 & 1 & 0 & -1 \\ 0 & 0 & -2 & -1 \\ -2 & -1 & -1 & 1 \end{pmatrix}$$

$$\underline{\mathfrak{g}(4,3)} \quad 1) \begin{pmatrix} 2 & 0 & 0 & 2 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 2 & 2 \\ -2 & -1 & -1 & 2 \end{pmatrix}; \quad 2) \begin{pmatrix} 2 & 0 & 0 & 2 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 2 & 2 \\ -2 & -1 & -1 & 0 \end{pmatrix}$$

$$3) \begin{pmatrix} -1 & -2 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & -2 & -1 \\ 1 & 0 & -1 & 0 \end{pmatrix} \quad 4) \begin{pmatrix} 0 & -1 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & -2 & -1 \\ 1 & 0 & -1 & 0 \end{pmatrix}; \quad 5) \begin{pmatrix} -2 & -1 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & -1 & 0 \end{pmatrix}$$

$$6) \begin{pmatrix} 0 & -1 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & -1 & 0 \end{pmatrix} \quad 7) \begin{pmatrix} -2 & -1 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 2 & 0 & -1 & -2 \end{pmatrix}$$





$$\begin{array}{l}
1) \begin{pmatrix} 0 & 2 & 1 & 0 & 0 & -1 \\ 2 & 1 & 2 & 0 & 0 & -2 \\ 1 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 2 & -2 \\ 1 & 2 & 0 & -1 & 2 & -2 \end{pmatrix}; \quad 2) \begin{pmatrix} \mathbf{g(4,6)} \\ 0 & 2 & 1 & 2 & 0 & 1 \\ 2 & 1 & 2 & 1 & 0 & 2 \\ 1 & 2 & 0 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 2 & 2 \\ 1 & 2 & 0 & 0 & 2 & 2 \end{pmatrix} \quad 3) \begin{pmatrix} 0 & 2 & 1 & 0 & 0 & 2 \\ 2 & 1 & 2 & 0 & 0 & 1 \\ 1 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 2 & 1 \\ 2 & 1 & 0 & 2 & 1 & 0 \end{pmatrix}; \\
4) \begin{pmatrix} 0 & -2 & 0 & 2 & 0 & 1 \\ -1 & -1 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & -1 & 0 & -2 \\ 2 & 2 & -2 & 0 & -2 & 0 \\ 0 & 0 & 0 & -2 & 2 & -1 \\ 1 & 1 & -1 & 0 & -1 & 2 \end{pmatrix} \quad 5) \begin{pmatrix} 0 & 2 & 1 & -1 & 0 & -2 \\ 2 & 1 & 2 & -2 & 0 & -1 \\ 1 & 2 & 0 & 0 & 0 & 0 \\ -1 & -2 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 & -2 \\ -2 & -1 & 0 & 0 & -2 & 2 \end{pmatrix}; \\
6) \begin{pmatrix} 0 & -1 & 0 & -1 & 0 & -2 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 1 \\ -2 & 0 & 2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 2 & 2 \\ -1 & 0 & 1 & 0 & 2 & 2 \end{pmatrix} \quad 7) \begin{pmatrix} -2 & -1 & 0 & -2 & 0 & -1 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 1 \\ -2 & 0 & 2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 2 & 2 \\ -1 & 0 & 1 & 0 & 2 & 2 \end{pmatrix}
\end{array}$$



$\mathfrak{g}(8,6)$

$$1) \begin{pmatrix} 1 & 2 & 1 & 1 & 2 & 0 & -2 \\ 2 & 2 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 2 & 2 & 1 & 0 & -1 \\ 1 & 0 & 2 & 0 & 0 & 0 & 0 \\ 2 & 0 & 1 & 0 & 2 & 0 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 2 & 0 & 1 & 0 & 2 & -1 & -2 \end{pmatrix};$$

$$3) \begin{pmatrix} 1 & 2 & 1 & 1 & 2 & 0 & -2 \\ 2 & 2 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 2 & 2 & 1 & 0 & -1 \\ 1 & 0 & 2 & 0 & 0 & 0 & 0 \\ 2 & 0 & 1 & 0 & 2 & 0 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ -2 & 0 & -1 & 0 & -2 & -1 & 0 \end{pmatrix}$$

$$2) \begin{pmatrix} 1 & -1 & 1 & -2 & -2 & 2 & 2 \\ -1 & 2 & -2 & 0 & 0 & 0 & 0 \\ 1 & -2 & 2 & -1 & -1 & 1 & 1 \\ -2 & 0 & -1 & 0 & 0 & 0 & 0 \\ -1 & 0 & -2 & 0 & -2 & 2 & 2 \\ 1 & 0 & 2 & 0 & 2 & 0 & 0 \\ 2 & 0 & 1 & 0 & 1 & 0 & -1 \end{pmatrix};$$

$$4) \begin{pmatrix} 1 & -1 & 1 & -1 & 0 & 1 & 2 \\ -1 & 2 & -2 & 0 & 0 & 0 & 0 \\ 1 & -2 & 2 & -2 & 0 & 2 & 1 \\ -2 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & -2 \\ 1 & 0 & 2 & 0 & -2 & 0 & 0 \\ 2 & 0 & 1 & 0 & -1 & 0 & 2 \end{pmatrix}$$

$$5) \begin{pmatrix} -2 & -2 & -1 & 1 & 0 & 1 & 2 \\ -1 & 1 & -1 & 0 & 0 & 0 & 0 \\ -2 & -1 & -2 & 2 & 0 & 2 & 1 \\ 2 & 0 & 2 & -2 & 0 & -2 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 & -2 \\ 1 & 0 & 1 & -1 & -1 & 0 & 0 \\ 2 & 0 & 2 & -2 & -2 & 0 & 2 \end{pmatrix};$$

$$6) \begin{pmatrix} 1 & -1 & 1 & -1 & 0 & -2 & -1 \\ -2 & 0 & -1 & 0 & 0 & 0 & 0 \\ 1 & -2 & 2 & -2 & 0 & -1 & -2 \\ -1 & 0 & -2 & 2 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 2 & 1 \\ -2 & 0 & -1 & 1 & 2 & 0 & 0 \\ -1 & 0 & -2 & 2 & 1 & 0 & 2 \end{pmatrix}$$

$$7) \begin{pmatrix} 2 & -1 & -1 & -1 & 0 & -2 & -1 \\ -2 & 2 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -2 & 0 & 0 & 2 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 2 & 1 \\ -1 & 0 & 0 & 1 & 2 & 0 & 0 \\ -2 & 0 & 0 & 2 & 1 & 0 & 2 \end{pmatrix};$$

$$8) \begin{pmatrix} -1 & -2 & -1 & -2 & 0 & -1 & -2 \\ -2 & 2 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -2 & 0 & 0 & 2 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 2 & 1 \\ -1 & 0 & 0 & 1 & 2 & 0 & 0 \\ -2 & 0 & 0 & 2 & 1 & 0 & 2 \end{pmatrix}.$$

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