

Max-Planck-Institut
für Mathematik
in den Naturwissenschaften
Leipzig

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exceptional simple Elduque Lie superalgebra

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Preprint no.: 125

2006



Cartan matrices and presentations of the exceptional simple Elduque Lie superalgebra

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Abstract

Recently Alberto Elduque listed all simple and graded modulo 2 finite dimensional Lie algebras and superalgebras whose odd component is the spinor representation of the orthogonal Lie algebra equal to the even component, and discovered one exceptional such Lie superalgebra in characteristic 5. For this Lie superalgebra all inequivalent Cartan matrices (in other words, inequivalent systems of simple roots) are listed together with defining relations between analogs of its Chevalley generators.

§1 Introduction

In [BGL], for the ten exceptional simple finite dimensional Lie superalgebras of Elduque and Cunha over an algebraically closed field of characteristic 3, we have listed all inequivalent Cartan matrices (in other words, inequivalent systems of simple roots) and their inverse, if invertible, defining relations between analogs of the Chevalley generators, and the coefficients of linear dependence over \mathbb{Z} of the maximal roots with respect to simple roots.

Here we provide with the same type of information for the only for $p = 5$ exceptional (new) simple finite dimensional Lie superalgebra \mathfrak{el} among the Lie superalgebras whose odd component is the spinor representation of the orthogonal Lie algebra equal to the even component. A. Elduque described \mathfrak{el} in components: $\mathfrak{el}_{\bar{0}} = \mathfrak{o}(11)$, $\mathfrak{el}_{\bar{1}} = \Gamma_5$ (in notations of [FH], p. 376; i.e., the spinor representation), so $\text{sdim } \mathfrak{el} = (55|32)$. To determine \mathfrak{el} , we have to explicitly define the bracket: $[\cdot, \cdot] : S^2(\mathfrak{el}_{\bar{1}}) \longrightarrow \mathfrak{el}_{\bar{0}}$, which is not that easy; for details, see [E12]. Since the bracket is not easy to describe in these

Key words and phrases. Cartan matrix, Elduque superalgebra, Lie superalgebra.

2000 Mathematics Subject Classification. 17B50, 70F25.

*DL is thankful to MPIMiS, Leipzig, for financial support and most creative environment; to A. Elduque for comments; to O. Shirokova for a \TeX pert help and to A. Protopopov for a help with the graphics, see [Pro].

terms, it was not clear if \mathfrak{el} possesses a Cartan matrix (we found out that it does).

Not every simple finite dimensional Lie superalgebra \mathfrak{g} possesses a Cartan matrix, but if it does, it is often more convenient to express \mathfrak{g} in terms of its Cartan matrices; for their definition and description of the analogs of Dynkin graphs, see [GL1, BGL]. With the help of **SuperLie** package [Gr] we list all inequivalent Cartan matrices (hence, systems of simple roots) of \mathfrak{el} and describe the defining relations corresponding to these matrices.

For Lie superalgebras with Cartan matrix generated by Chevalley generators, there are two types of defining relations: Serre-type ones and non-Serre type ones (over \mathbb{C} , all these relations are listed in [GL1]). Sometimes some of the Serre-type relations are redundant but this does not matter in practical calculations. At the moment, the problem how to encode the non-Serre type relations in terms of Cartan matrix is open for $p > 0$. Some relations (for any p) are so complicated that we conjecture that there is no *general* encoding procedure. This is why our list of relations is of practical interest.

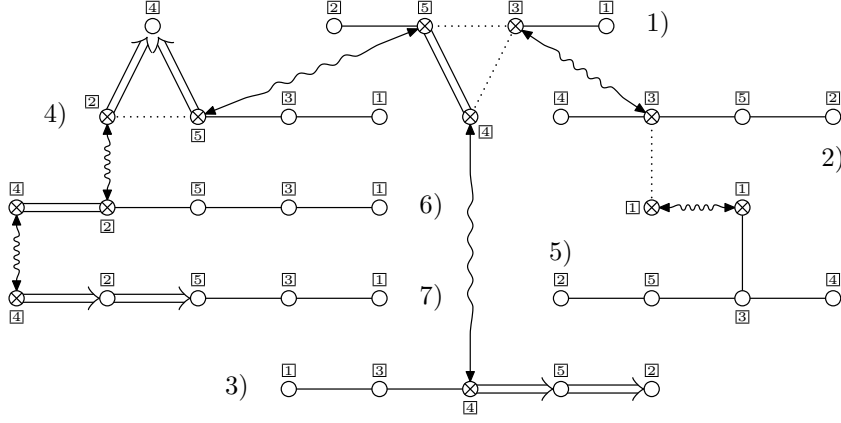
For the background on Cartan matrices and “odd reflections” that connect inequivalent matrices, see [BGL].

§2 The Elduque superalgebra: Systems of simple roots

Having found out one Cartan matrix of \mathfrak{el} , we list them all. We denote by $\mathfrak{el}^{(n)}$ the realization of \mathfrak{el} by means of the n th Cartan matrix.

The table 8) shows the result of odd reflections (the number n of the row is the number of the matrix n) in the list below, the number of the column is the the number of the root in which reflection is made; the cells contain the results of reflections (the number of the matrix obtained) or a “-” if the reflection is not appropriate because $A_{ii} \neq 0$. The nodes are numbered by small boxed numbers; instead of joining nodes with four segments in the cases where $A_{ij} = A_{ji} = 1 \equiv -4 \pmod{5}$ we use one dotted segment; curly lines with arrows depict odd reflections.

$$\begin{array}{ccc}
 1) \begin{pmatrix} 2 & 0 & -1 & 0 & 0 \\ 0 & 2 & 0 & 0 & -1 \\ -1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & -2 \\ 0 & -1 & 1 & -2 & 0 \end{pmatrix} & 2) \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 & -1 \\ 1 & 0 & 0 & -1 & -1 \\ 0 & 0 & -1 & 2 & 0 \\ 0 & -1 & -1 & 0 & 2 \end{pmatrix} & \boxed{3) \begin{pmatrix} 2 & 0 & -1 & 0 & 0 \\ 0 & 2 & 0 & 0 & -1 \\ -1 & 0 & 2 & -1 & 0 \\ 0 & 0 & -1 & 0 & 2 \\ 0 & -2 & 0 & -1 & 2 \end{pmatrix} \\
 4) \begin{pmatrix} 2 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 2 & 1 \\ -1 & 0 & 2 & 0 & -1 \\ 0 & -1 & 0 & 2 & -1 \\ 0 & 1 & -1 & 2 & 0 \end{pmatrix} & \boxed{5) \begin{pmatrix} 0 & 0 & -1 & 0 & 0 \\ 0 & 2 & 0 & 0 & -1 \\ -1 & 0 & 2 & -1 & -1 \\ 0 & 0 & -1 & 2 & 0 \\ 0 & -1 & -1 & 0 & 2 \end{pmatrix} & 6) \begin{pmatrix} 2 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -2 & -1 \\ -1 & 0 & 2 & 0 & -1 \\ 0 & -2 & 0 & 0 & 0 \\ 0 & -1 & -1 & 0 & 2 \end{pmatrix}
 \end{array}$$



$$\begin{array}{c}
 \boxed{7)} \\
 \begin{pmatrix}
 2 & 0 & -1 & 0 & 0 \\
 0 & 2 & 0 & -1 & -2 \\
 -1 & 0 & 2 & 0 & -1 \\
 0 & 2 & 0 & 0 & 0 \\
 0 & -1 & -1 & 0 & 2
 \end{pmatrix}
 \end{array}
 \quad
 \begin{array}{c}
 \boxed{8)} \\
 \begin{pmatrix}
 - & - & 2 & 3 & 4 \\
 5 & - & 1 & - & - \\
 - & - & - & 1 & - \\
 - & 6 & - & - & 1 \\
 2 & - & - & - & - \\
 - & 4 & - & 7 & - \\
 - & - & - & 6 & -
 \end{pmatrix}
 \end{array}$$

§3 The Elduque superalgebra: Defining relations

To save space, we omit indicating the Serre relations in what follows; their fulfilment is assumed.

- 1) $[x_4, [x_3, x_5]] - [x_5, [x_3, x_4]] = 0$; $[[x_1, x_3], [x_3, x_4]] = 0$; $[[x_1, x_3], [x_3, x_5]] = 0$;
 $[[x_2, x_5], [x_3, x_5]] = 0$; $[[x_4, x_5], [[x_2, x_5], [x_4, x_5]]] = 0$
- 2) $[[x_1, x_3], [x_3, x_4]] = 0$; $[[x_1, x_3], [x_3, x_5]] = 0$;
 $[[[x_3, x_4], [x_3, x_5]], [[x_3, [x_2, x_5]], [[x_3, x_4], [x_3, x_5]]]] = 0$
- 3) $[[x_3, x_4], [[x_2, x_5], [x_4, x_5]]] - 4 [[x_4, [x_2, x_5]], [x_5, [x_3, x_4]]] = 0$;
 $[[x_4, [x_1, x_3]], [[x_3, x_4], [x_4, x_5]]] = 0$
- 4) $[x_4, [x_2, x_5]] - 3 [x_5, [x_2, x_4]] = 0$; $[[x_2, x_5], [x_3, x_5]] = 0$;
 $[[x_5, [x_1, x_3]], [[x_3, x_5], [x_4, x_5]]] = 0$
- 5) $[[[x_4, [x_1, x_3]], [x_5, [x_1, x_3]]], [[[x_1, x_3], [x_2, x_5]], [[x_4, [x_1, x_3]], [x_5, [x_1, x_3]]]] = 0$
- 6) $[[x_2, x_4], [[x_2, x_4], [x_2, x_5]]] = 0$;
 $[[[x_1, x_3], [x_2, x_5]], [[x_3, [x_2, x_5]], [[x_2, x_4], [x_2, x_5]]]] = 0$
- 7) $[x_2, [x_2, [x_2, x_5]]] = 0$;
 $[[[x_2, x_4], [x_5, [x_1, x_3]]], [[x_5, [x_2, x_4]], [x_3, [x_2, [x_2, x_5]]]]]$
 $- 2 [[[x_2, x_4], [[x_1, x_3], [x_2, x_5]]], [[x_3, [x_2, x_5]], [x_5, [x_2, x_4]]]] = 0$

3.1. The maximal roots. The maximal roots are of height 41 and their weights are equal to $(1, 0, 0, 0, 0)$, except root 5) whose weight is equal to

$(4, 0, 0, 0, 0)$. The coefficients of their decomposition with respect to simple roots (over \mathbb{Z}) are as follows:

- 1) $(2, 2, 3, 3, 4)$, 2) $(2, 2, 6, 3, 4)$, 3) $(2, 2, 3, 4, 4)$, 4) $(2, 2, 3, 3, 4)$,
 5) $(5, 2, 6, 3, 4)$, 6) $(2, 5, 3, 3, 4)$, 7) $(2, 5, 3, 2, 4)$

§4 The inverse matrices of the Cartan matrices

(Their numbering matches that of the Cartan matrices.)

$$\begin{array}{l}
 1) \begin{pmatrix} 2 & 2 & 3 & 3 & 4 \\ 2 & 4 & 4 & 0 & 2 \\ 3 & 4 & 1 & 1 & 3 \\ 3 & 0 & 1 & 3 & 0 \\ 4 & 2 & 3 & 0 & 4 \end{pmatrix} \quad 2) \begin{pmatrix} 2 & 2 & 1 & 3 & 4 \\ 2 & 4 & 0 & 0 & 2 \\ 1 & 0 & 0 & 0 & 0 \\ 3 & 0 & 0 & 3 & 0 \\ 4 & 2 & 0 & 0 & 4 \end{pmatrix} \quad 3) \begin{pmatrix} 2 & 2 & 3 & 4 & 2 \\ 2 & 4 & 4 & 1 & 1 \\ 3 & 4 & 1 & 3 & 4 \\ 4 & 1 & 3 & 2 & 1 \\ 4 & 2 & 3 & 2 & 2 \end{pmatrix} \quad 4) \begin{pmatrix} 2 & 2 & 3 & 4 & 4 \\ 2 & 4 & 4 & 0 & 1 \\ 3 & 4 & 1 & 3 & 3 \\ 3 & 0 & 1 & 4 & 4 \\ 4 & 1 & 3 & 2 & 2 \end{pmatrix} \\
 5) \begin{pmatrix} 0 & 3 & 4 & 2 & 1 \\ 3 & 4 & 0 & 0 & 2 \\ 4 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 3 & 0 \\ 1 & 2 & 0 & 0 & 4 \end{pmatrix} \quad 6) \begin{pmatrix} 2 & 0 & 3 & 3 & 4 \\ 0 & 0 & 0 & 2 & 0 \\ 3 & 0 & 1 & 1 & 3 \\ 3 & 2 & 1 & 3 & 4 \\ 4 & 0 & 3 & 4 & 2 \end{pmatrix} \quad 7) \begin{pmatrix} 2 & 0 & 3 & 2 & 4 \\ 0 & 0 & 0 & 3 & 0 \\ 3 & 0 & 1 & 4 & 3 \\ 2 & 4 & 4 & 4 & 1 \\ 4 & 0 & 3 & 1 & 2 \end{pmatrix}
 \end{array}$$

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