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Adjunction of monoids

by

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Adjunctions in Monoids

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Let M and N be monoids considered as categories with the only object.

Let

$$f\colon M \longrightarrow N, \quad g\colon N \longrightarrow M \tag{1}$$

be morphisms of monoids, considered as functors. Let the functor f is left adjoint to the functor g.

Is it true then that f (or, what is the same, g) is always an isomorphism?

In [1], p.136, this question was posed as an open question. Here I answer this question and the answer is *no*.[1]

To prove this, I will construct a Birkhoff variety of algebras, which is naturally equivalent to the category of adjunctions in monoids, and consider its initial object which is a monoid generated by $2 \times \mathbb{N}$ free variables subject to a certain set of relations. An application of M. H. A. Newman's reduction theorem ([4], cited by [3]) permits one to describe the canonical form of elements in the monoid and, in particular, to negatively answer the question posed.

Let

$$\varphi\colon N \longrightarrow M, \tag{2}$$

be an isomorphism of *sets* such that the triple $(f, g, \varphi)\colon M \rightharpoonup N$ is an adjunction, which means that the identity

$$g(n)\varphi(n')m = \varphi(nn'f(m)) \tag{3}$$

holds for every $n, n' \in N$ and $m \in M$ (see, e.g., [2], p.78).

The identity (3) together with the fact that φ is an iso implies, evidently, that

$$\varphi^{-1}\colon M \longrightarrow N \tag{2°}$$

satisfies the "dual" identity:

$$n\varphi^{-1}(m')f(m) = \varphi^{-1}(g(n)m'm) \tag{3°}$$

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[1] This result was obtained actually somewhere at beginning of 90^{th}; at the end of 1999 I asked Prof. Manes in an e-mail about the status of the question. He answered that "to his knowledge the question is still open".

(to get from some identity its " dual" replace n's with m's and vice versa; $f \leftrightarrow g$, $\varphi \leftrightarrow \varphi^{-1}$ and, finally, invert the order of all compositions).

Let, further, $\eta \in M$ and $\varepsilon \in N$ be defined as

$$\eta := \varphi(1), \quad \varepsilon := \varphi^{-1}(1). \tag{4}$$

Then by setting $n' = m = 1$ in (3) we get:

$$\varphi(n) = g(n)\eta \tag{5}$$

and, dually:

$$\varphi^{-1}(m) = \varepsilon f(m) \tag{5°}$$

i.e., φ and φ^{-1} are factorized as follows:

$$\varphi = \mathrm{R}_\eta \circ g \quad (\varphi\colon N \xrightarrow{g} M \xrightarrow{\mathrm{R}_\eta} M) \tag{6}$$

$$\varphi^{-1} = \mathrm{L}_\varepsilon \circ f \quad (\varphi^{-1}\colon M \xrightarrow{f} N \xrightarrow{\mathrm{L}_\varepsilon} N), \tag{6°}$$

where R_η (resp. L_ε) is the right shift by η (resp. the left shift by ε) in the monoid M (resp. N).

The equalities (6)-(6°) together with the fact that φ and φ^{-1} are iso's imply

Proposition 1 *Both f and g are injective morphisms of monoids, whereas R_η and L_ε are surjective maps of sets.*

Setting now $n = n' = 1$ in (3) one gets

$$(\varphi \circ f)(m) = \eta m \tag{7}$$

and, dually,

$$(\varphi^{-1} \circ g)(n) = n\varepsilon \tag{7°}$$

which, together with Prop.1, implies that both L_η and R_ε are *injective*, so that if there exists an adjunction with f (resp. g) non-iso, then M (resp. N) is, at least, non-commutative monoid; it may not be a group as well.

Let us now reinterpret η and ε as natural transformations:

$$\eta\colon 1_M \longrightarrow g \circ f, \quad \varepsilon\colon f \circ g \longrightarrow 1_N \tag{8}$$

(unit and counit of the adjunction (f, g, φ)).

Indeed, given two morphisms of monoids $f_1, f_2\colon M \rightrightarrows M'$, a natural transformation $\mu\colon f_1 \longrightarrow f_2$ is uniquely determined by an element $\mu' \in M'$ such that the identity

$$f_2(m)\mu' = \mu' f_1(m) \tag{9}$$

holds for every $m \in M$; in this case μ itself can be identified with the triple (f_1, μ', f_2) or, by abuse of notations, with μ' itself.

But identities (5)-(5°) and (7)-(7°) together give:

$$((g\circ f)(m))\eta = \eta m \quad \text{for any } m \in M \tag{10}$$

and, dually,

$$\varepsilon((f\circ g)(n)) = n\varepsilon \quad \text{for any } n \in N \tag{10°}$$

which exactly states that η and ε define natural transformations (8).

Moreover, one must have the identities:

$$1_M = g(\varepsilon)\eta \tag{11}$$

$$1_N = \varepsilon f(\eta). \tag{11°}$$

Finally, the adjunction $(f, g, \varphi)\colon M \rightharpoonup N$ can be described by the data

$$(f\colon M \to N, g\colon N \to M, \eta \in M, \varepsilon \in N)$$

satisfying the identities (10)-(11°) (see [2], p.81); φ and φ^{-1} are then *defined* by eqs. (5-5°).

Define now the category **AdMon** such that its objects are just all adjunctions $(f, g, \eta, \varepsilon)\colon M \rightharpoonup N$ and, given another adjunction $(f', g', \eta', \varepsilon')\colon M' \rightharpoonup N'$, a pair of monoid morphisms $(l\colon M \to M',\ r\colon N \to N')$ is a morphism $(f, g, \eta, \varepsilon) \xrightarrow{(l,r)} (f', g', \eta', \varepsilon')$ in **AdMon** if the diagram

$$\begin{array}{ccccc} M & \xrightarrow{f} & N & \xrightarrow{g} & M \\ {\scriptstyle l}\downarrow & & {\scriptstyle r}\downarrow & & {\scriptstyle l}\downarrow \\ M' & \xrightarrow{f'} & N' & \xrightarrow{g'} & M' \end{array} \tag{12}$$

is commutative and, besides, if

$$r(\varepsilon) = \varepsilon', \quad l(\eta) = \eta'. \tag{13}$$

Denote by **Mon** the category of monoids. One immediately sees that there are two forgetful functors

$$\mathrm{L}, \mathrm{R}\colon \mathbf{AdMon} \longrightarrow \mathbf{Mon} \tag{14}$$

defined as follows:

$$\mathrm{L}(l, r) = l; \quad \mathrm{R}(l, r) = r. \tag{15}$$

Given now an adjunction $(f, g, \eta, \varepsilon)\colon M \rightharpoonup N$ and considering the commutative diagram

$$\begin{array}{ccccc} M & \xrightarrow{f} & N & \xrightarrow{g} & M \\ {\scriptstyle 1_M}\downarrow & & {\scriptstyle I}\downarrow & & {\scriptstyle 1_M}\downarrow \\ M & \longrightarrow & \mathrm{Im}(g) & \xrightarrow{\subseteq} & M \end{array} \tag{16}$$

where I is an iso due to Prop.1, one can see that $(I \circ f, g' = (\mathrm{Im}(g) \subset M), \eta, \varepsilon' = g(\varepsilon))$ is an adjunction isomorphic to $(f, g, \eta, \varepsilon)$ as an object of **AdMon**, where the isomorphism is given by $(1_M, I)$; that all this iso's together generate the natural equivalence of **AdMon** with its full subcategory consisting of just those adjunctions $(f, g, \eta, \varepsilon)\colon M \rightharpoonup N$ in which N is a submonoid of M and g is the inclusion map $N \subset M$.

Define the category $\mathbf{AdMon_L}$ as follows: objects of $\mathbf{AdMon_L}$ are all data of the type $(N \subset M, f\colon M \to M, \eta \in M, \varepsilon \in M)$, where M is a monoid, N its submonoid, f a monoid endomorphism such that $f(M) \subset N$ and, besides, the identities

$$f(m)\eta = \eta m \quad (m \in M) \tag{17a}$$
$$\varepsilon f(n) = n\varepsilon \quad (n \in N) \tag{17b}$$
$$\varepsilon\eta = 1 \tag{17c}$$
$$\varepsilon f(\eta) = 1 \tag{17d}$$

hold. In other words, objects of $\mathbf{AdMon_L}$ are just monoids equipped with some additional structure (a submonoid $N \subset M$, an endomorphism $f\colon M \longrightarrow M$ such that $f(M) \subset N$ and elements $\eta \in M$, $\varepsilon \in M$ satisfying eqs.(17a)-(17d)); then a morphism in $\mathbf{AdMon_L}$ is just a monoid morphism respecting this structure. So we have (see above):

Proposition 2 *The category* **AdMon** *is naturally equivalent to the category* $\mathbf{AdMon_L}$.

Note that though the transition from **AdMon** to $\mathbf{AdMon_L}$ breaks the "$\mathbb{Z}/2$-symmetry" ($f \leftrightarrow g$, $\varepsilon \leftrightarrow \eta$) we get simpler objects instead and simpler relations (17a)-(17d) instead of "symmetric" ones (10)-(11°).

Now we will give some conditions on an object $(M \supset N, f, \eta, \varepsilon)$ equivalent to the statement that f is an isomorphism.

Proposition 3 *Let* $(M \supset N, f, \eta, \varepsilon)$ *be an object of* $\mathbf{AdMon_L}$. *Then the following conditions* (a)-(g) *are equivalent:*
(a) f *is surjective;*
(b) f *is an isomorphism;*
(c) $N = M$*;*
(d) $f(\eta) = \eta$*;*
(e) $f(\varepsilon) = \varepsilon$*;*
(f) $\eta\varepsilon = 1$ (i.e., η is invertible in M due to eq.(17c))*;*
(g) *for every* $m \in M$ *one has* $f(m) = \eta m \varepsilon$ (i.e., f is an inner automorphism of M due to (f)).

Proof. (a)$\Longleftrightarrow$(b) due to Prop.1; (b)$\Longrightarrow$(c) is evident, because N contains $f(M) = M$;

(c)$\Longrightarrow$(d): Multiplying eq.(17a) by ε from the left (resp. multiplying eq.(17b) by $f(\eta)$) from the right) one gets:

$$m = \varepsilon f(m)\eta \quad (m \in M), \tag{18a}$$
$$n = \varepsilon f(n\eta) \quad (n \in N). \tag{18b}$$

If $N = M$, then due to eq.(18b) every $m \in M$ has a representation $m = \varepsilon f(m')$ for some $m' \in M$. In particular,

$$\eta = \varepsilon f(\eta^2) = (\varepsilon f(\eta)) f(\eta) = f(\eta); \tag{19}$$

(d)$\Longrightarrow$(e): $\varepsilon \in N$ implies

$$\varepsilon = \varepsilon 1 = \varepsilon f^2(1) \overset{(17c)}{=} \varepsilon f^2(\varepsilon\eta) = \varepsilon f^2(\varepsilon) f^2(\eta) \overset{(d)}{=} \varepsilon f^2(\varepsilon)\eta \overset{(17a)}{=} \varepsilon\eta f(\varepsilon) \overset{(17c)}{=} f(\varepsilon); \tag{20}$$

(e)$\Longrightarrow$(f): Indeed:

$$\eta\varepsilon \overset{(17a)}{=} f(\varepsilon)\eta \overset{(e)}{=} \varepsilon\eta = 1; \tag{21}$$

(f)$\Longrightarrow$(g): One has: $f(m) \overset{(f)}{=} f(m)\eta\varepsilon \overset{(17a)}{=} \eta m \varepsilon$;
(g)$\Longrightarrow$(b), because $1 = f(1) = \eta\varepsilon$, i.e., (g)$\Longrightarrow$(f) and defining

$$f^{-1}(m) := \varepsilon m \eta \tag{22}$$

one sees that $f^{-1}f(m) = ff^{-1}(m) = m$ ∎

Return now to the definition of $\mathbf{AdMon_L}$. One sees that $\mathbf{AdMon_L}$ is "almost" the variety of algebras in the Birkhoff's sense (see, e.g., [3]). In more detail, let

$$\Omega = \Omega_0 \cup \Omega_1 \cup \Omega_2, \text{ where } \Omega_0 = \{1, \eta, \varepsilon\}, \, \Omega_1 = \{f\} \text{ and } \Omega_2 = \{\centerdot\}$$

i.e, Ω_0, Ω_1 and Ω_2 are, respectively, the set of 0-ary, unary and binary operations. The set of "equations" consists of equations stating that $\centerdot$ and 1 determine a monoid structure, f is an endomorphism of the corresponding structure, with eqs.(17a)-(17d) added. One sees that eq.(17b) is not an equation in Birkhoff's sense, because n there is *restricted* to the subset $N \subset M$. This means that, generally speaking, $\mathbf{AdMon_L}$ is a variety of "sorted" algebras with two sorts of algebras: "N-like" and "M-like". But, noting that due to eq.(5°):

$$N = \varepsilon f(M) \tag{23}$$

one can derive from eq.(17b) that the equality

$$\varepsilon f(\varepsilon f(m)) = \varepsilon f(m)\varepsilon \quad (m \in M) \tag{24}$$

is valid in which m runs over the *whole* M.

We will prove now that vice versa, the data $(M, f\colon M \longrightarrow M, \eta, \varepsilon)$ together with equations (17a), (17c)-(17d), equations stating that f is an automorphism of monoids as well as eq.(24) (instead of eq.(17b)), reconstruct the remaining data, namely, the submonoid N and the equation (17b) valid on N.

Indeed, *define* N by eq.(23) as a set; we have to prove that this set is, in fact, a submonoid (eq.(17b) for $n \in N$ follows immediately from eq.(24)).

We see, first of all, that $1 \in N$ due to eq.(17d). Suppose now that $n_1, n_2 \in N$, i.e., for some $m_1, m_2 \in M$ one has $n_1 = \varepsilon f(m_1)$, $n_2 = \varepsilon f(m_2)$. Then:

$$n_1 n_2 = \varepsilon f(m_1)\varepsilon f(m_2) \overset{(24)}{=} \varepsilon f(\varepsilon f(m_1)) f(m_2) = \varepsilon f(\varepsilon f(m_1) m_2) \in N. \tag{25}$$

This proves that N is, actually, a submonoid of M.

Proposition 4 *The category* **AdMon** *is naturally equivalent to the Birkhoff variety of monoids M equipped with the structure $(M,\ f\colon M \longrightarrow M,\ \eta, \varepsilon \in M)$, where f is an endomorphism of monoids, satisfying the following conditions:*

$$\begin{aligned}
\varepsilon\eta &= 1, && \text{(26a)}\\
\varepsilon f(\eta) &= 1, && \text{(26b)}\\
\varepsilon f(\varepsilon) &= \varepsilon^2, && \text{(26c)}\\
\varepsilon f^2(m) &= f(m)\varepsilon \quad (m \in M), && \text{(26d)}\\
f(m)\eta &= \eta m \quad (m \in M). && \text{(26e)}
\end{aligned}$$

Proof. It remains to prove only that eqs.(26c)-(26d) together are equivalent to eq.(24) above. Indeed, eq.(26c) is a particular case of eq.(24) for $m = 1$, whereas eq.(26d) is obtained from eq.(24) if one substitutes $m = \eta m'$ and takes into account eq.(26a). On the other hand, for $m \in M$ one has:

$$\varepsilon f(\varepsilon f(m)) = \varepsilon f(\varepsilon) f^2(m) \overset{(26c)}{=} \varepsilon^2 f^2(m) \overset{(26d)}{=} \varepsilon f(m)\varepsilon \ \blacksquare \tag{27}$$

Denote **AdMonB** the Birkhoff variety described by Prop.4. We will prove next that the equality $\eta\varepsilon = 1$ is not satisfied in **AdMonB**. To this end, we will consider in details the "minimal model of the theory **AdMonB**", in other words, the free **AdMonB** algebra $\mathcal{F}(\emptyset)$ (which is an initial object in **AdMonB**).

Define elements $\eta_k, \varepsilon_k \in \mathcal{F}(\emptyset)$ $(k \in \mathbb{N})$ as follows:

$$\begin{aligned}
&\eta_0 := \eta, \quad \eta_{k+1} := f(\eta_k) && \text{(28a)}\\
&\varepsilon_0 := \varepsilon, \quad \varepsilon_{k+1} := f(\varepsilon_k) && \text{(28b)}
\end{aligned}$$

It is clear that, as monoid, $\mathcal{F}(\emptyset)$ is generated by elements $\eta_k,\ \varepsilon_k$. In other words, the monoid $\mathcal{F}(\emptyset)$ can be represented as

$$\mathcal{F}(\emptyset) = \mathcal{F}_{\mathbf{Mon}}(\eta_0, \eta_1, \ldots, \eta_k, \ldots \varepsilon_0, \varepsilon_1, \ldots, \varepsilon_k, \ldots)/\mathrm{R} \tag{29}$$

for some set of relations R, where $\mathcal{F}_{\mathbf{Mon}}(\eta_0, \eta_1, \ldots, \eta_k, \ldots \varepsilon_0, \varepsilon_1, \ldots, \varepsilon_k, \ldots)$ is the free monoid generated by $2 \times \mathbb{N}$ variables $\{\eta_k, \varepsilon_m\}_{k,m \in \mathbb{N}}$. The following proposition describes the corresponding set of relations R.

Proposition 5 *The set of relations* R *in* $\mathcal{F}(\emptyset)$ *is generated by the following set* R_0 *of relations:*

$$\begin{aligned}
\varepsilon_i\varepsilon_j &= \varepsilon_{j-1}\varepsilon_i \ (j > i), && \text{(30a)}\\
\eta_j\eta_i &= \eta_i\eta_{j-1} \ (j > i), && \text{(30b)}\\
\varepsilon_i\eta_j &= \begin{cases} \eta_{j-1}\varepsilon_i & (j > i+1)\\ \eta_j\varepsilon_{i-1} & (i > j)\\ \varepsilon_{i-1}\eta_i & (i = j > 0)\\ \varepsilon_i\eta_i & (j = i+1)\\ 1 & (i = j = 0) \end{cases} && \text{(30c)}
\end{aligned}$$

Proof. One easily checks that eqs.(30a)-(30c) are satisfied, being either particular cases of some of the relations (26a)-(26e), or can be obtained from the latter ones after applying f^k to both sides of (26a)-(26e) (for some $k > 0$).

Vice versa, define the elements η and ε and an automorphism f of the monoid $\mathcal{F}_{\mathbf{Mon}}(\eta_0, \eta_1, \ldots, \eta_k, \ldots \varepsilon_0, \varepsilon_1, \ldots, \varepsilon_k, \ldots)$ by "inverting" definitions (28) above:

$$\eta := \eta_0, \quad f(\eta_k) := \eta_{k+1}, \tag{31a}$$

$$\varepsilon := \varepsilon_0, \quad f(\varepsilon_k) := \varepsilon_{k+1}. \tag{31b}$$

One easily checks, that the automorphism f "survives" the factorization by relations (30) above and induction on the length of words proves that, in the monoid

$$\mathcal{F}_{\mathbf{Mon}}(\eta_0, \eta_1, \ldots, \eta_k, \ldots \varepsilon_0, \varepsilon_1, \ldots, \varepsilon_k, \ldots)/\mathrm{R}_0, \tag{32}$$

relations (26d)-(26e) are satisfied. ■

From now on we will identify $\mathcal{F}(\emptyset)$ with the monoid (32) equipped with η, ε and f defined by eqs.(31).

Returning now to our original problem: one sees that it is exactly equivalent to the question whether or not the identity

$$\eta\varepsilon = 1 \tag{33}$$

holds in the monoid $\mathcal{F}(\emptyset)$.

The following theorem provides us with the canonical form for elements of $\mathcal{F}(\emptyset)$ and simultaneously gives the negative answer to the last question.

Theorem 6 *For every element m of $\mathcal{F}(\emptyset)$, there exist the only pair $k, l \in \mathbb{N}$ and the only pair of sequences*

$$0 \leq i_1 \leq \ldots \leq i_k, \quad j_1 \geq \ldots \geq j_l \geq 0 \tag{34a}$$

such that

$$m = \eta_{i_1} \ldots \eta_{i_k} \varepsilon_{j_1} \ldots \varepsilon_{j_l}. \tag{34b}$$

(We asume that if $k = 0$ (resp. $l = 0$), then the corresponding sequence in (34a) *above is empty and the corresponding product in* (34b) *is replaced with the neutral element* 1 *of the monoid $\mathcal{F}(\emptyset)$.)*

Proof. On $\mathcal{F}_{\mathbf{Mon}}(\eta_0, \eta_1, \ldots, \eta_k, \ldots \varepsilon_0, \varepsilon_1, \ldots, \varepsilon_k, \ldots)$, define the binary relation $\models$ as follows. First of all, set:

$$\varepsilon_i \varepsilon_j \models \varepsilon_{j-1} \varepsilon_i \ (j > i) \tag{35a}$$

$$\eta_j \eta_i \models \eta_i \eta_{j-1} \ (j > i) \tag{35b}$$

$$\varepsilon_i \eta_j \models \begin{cases} \eta_{j-1}\varepsilon_i & (j > i+1) \\ \eta_j \varepsilon_{i-1} & (i > j) \\ \varepsilon_{i-1}\eta_i & (i = j > 0) \\ \varepsilon_i \eta_i & (j = i+1) \\ 1 & (i = j = 0) \end{cases} \tag{35c}$$

Let now $\geq$ be the smallest preorder on $\mathcal{F}_{\mathbf{Mon}}(\eta_0, \eta_1, \ldots, \eta_k, \ldots \varepsilon_0, \varepsilon_1, \ldots, \varepsilon_k, \ldots)$ containing $\models$ and turning $\mathcal{F}_{\mathbf{Mon}}(\eta_0, \eta_1, \ldots, \eta_k, \ldots \varepsilon_0, \varepsilon_1, \ldots, \varepsilon_k, \ldots)$ into a preordered monoid.

One sees that $m \geq m'$ if and only if there exists a sequence

$$m = m_0, m_1, \ldots, m_n = m' \tag{36}$$

such that, for any $0 \leq i < n$, there exist

$$L, R, \mu, \mu' \in \mathcal{F}_{\mathbf{Mon}}(\eta_0, \eta_1, \ldots, \eta_k, \ldots \varepsilon_0, \varepsilon_1, \ldots, \varepsilon_k, \ldots)$$

for which $\mu \models \mu'$ and both $m_i = L\mu R$ and $m_{i+1} = L\mu' R$.

Let now $\sim$ be the equivalence relation on $\mathcal{F}_{\mathbf{Mon}}(\eta_0, \eta_1, \ldots, \eta_k, \ldots \varepsilon_0, \varepsilon_1, \ldots, \varepsilon_k, \ldots)$ generated by the relation $\geq$ (i.e., $m \sim m'$ if and only if both m and m' belong to the same connected component of the preorder relation $\geq$).

It is rather clear that the relation $\sim$ coincides with the relation R from eq.(29) defining $\mathcal{F}(\emptyset)$.

It is also clear that the r.h.s. of the canonical representation (34b), considered as an element of $\mathcal{F}_{\mathbf{Mon}}(\eta_0, \eta_1, \ldots, \eta_k, \ldots \varepsilon_0, \varepsilon_1, \ldots, \varepsilon_k, \ldots)$ is a minimal element with respect to the preorder relation $\geq$. The only thing to prove is that *every equivalence class of the relation $\sim$ contains the only minimal element with respect to $\geq$.*

To prove this, it suffices to prove that the relation $\geq$ satisfies conditions of reduction theorem of M. Newman [4], or its weaker version given in [3]. The latter conditions on $\geq$ are the following conditions (A) and (B):

(A) *For any $m \in \mathcal{F}_{\mathbf{Mon}}(\eta_0, \eta_1, \ldots, \eta_k, \ldots \varepsilon_0, \varepsilon_1, \ldots, \varepsilon_k, \ldots)$, there exists $k \in \mathbb{N}$ such that for any decreasing sequence $m = m_0 > m_1 > \ldots > m_{k'}$ one has $k \leq k'$;*

(B) *Any pair $m, m' \in \mathcal{F}_{\mathbf{Mon}}(\eta_0, \eta_1, \ldots, \eta_k, \ldots \varepsilon_0, \varepsilon_1, \ldots, \varepsilon_k, \ldots)$ with a common parent is bounded from below, i.e., there exists $b \in \mathcal{F}_{\mathbf{Mon}}(\eta_0, \eta_1, \ldots, \eta_k, \ldots \varepsilon_0, \varepsilon_1, \ldots, \varepsilon_k, \ldots)$ such that both $m \geq b$ and $m' \geq b$.* Here p is said to be a *parent* of m if it is the smallest element such that $p \geq m,\ p \neq m$.

To prove (A), consider the morphism of monoids

$$d \colon \mathcal{F}_{\mathbf{Mon}}(\eta_0, \eta_1, \ldots, \eta_k, \ldots \varepsilon_0, \varepsilon_1, \ldots, \varepsilon_k, \ldots) \longrightarrow \mathbb{N} \tag{37a}$$

uniquelly determined by the images

$$d(\eta_i) = i + 1, \quad d(\varepsilon_i) = i + 1. \tag{37b}$$

One easily sees from relations (35) that d is a morphism of preordered monoids (i.e., respects preorders). Moreover, it is clear now that the relation $\geq$ is, in fact, an order relation (because $d(m) > d(m')$ for any pair $m, m' \in \mathcal{F}_{\mathbf{Mon}}(\eta_0, \eta_1, \ldots, \eta_k, \ldots \varepsilon_0, \varepsilon_1, \ldots, \varepsilon_k, \ldots)$ such that $m \models m'$). (A) is obvious now.

To prove (B), observe first that

(C) *p is a parent of m if and only if $p = L\mu R$, $m = L\mu' R$ and $\mu \models \mu'$.*

Indeed, the l.h.s. of any of the particular cases (35) of the relation $\models$, except, perhaps, the last one, is a parent of its r.h.s. just because $d(\text{l.h.s.}) - d(\text{r.h.s.}) = 1$; as to the last particular case, $\varepsilon_0\eta_0 = 1$, this is the only case such that the length of l.h.s. $\neq$ the length of r.h.s, which implies that in this case as well the l.h.s. is the parent of the r.h.s.

Let $m, m' \in \mathfrak{F}_{\mathbf{Mon}}(\eta_0, \eta_1, \ldots, \eta_k, \ldots \varepsilon_0, \varepsilon_1, \ldots, \varepsilon_k, \ldots)$ have a common parent, p.

Now follows the most boring part of all this mess.

There are five possible cases:

(I) $p = a\mu b\nu c$, $m = a\mu' b\nu c$, $m' = a\mu b\nu' c$, where $\mu \models \mu'$ and $\nu \models \nu'$.

Clearly, in this case $b := a\mu' b\nu' c$ is a common lower bound for both m and m'.

In all of the cases (II)-(V) below, p is of the form $L\mu_1\mu_2\mu_3R$, where every μ_i is of length 1 (i.e., is either η_j for some j, or ε'_j for some j'). In what follows, terms L and R will be omitted, because they take no explicit part in the process of finding of the lower bound b.

(II) $p = \varepsilon_i\varepsilon_j\varepsilon_k \ \ (i < j < k)$; Applying relation (35a) one gets:

$$\begin{aligned} p \models m &:= \varepsilon_{j-1}\varepsilon_i\varepsilon_k \geq \varepsilon_{j-1}\varepsilon_{k-1}\varepsilon_i \geq \varepsilon_{k-2}\varepsilon_{j-1}\varepsilon_i \\ p \models m' &:= \varepsilon_i\varepsilon_{k-1}\varepsilon_j \geq \varepsilon_{k-2}\varepsilon_i\varepsilon_j \geq \varepsilon_{k-2}\varepsilon_{j-1}\varepsilon_i \end{aligned}$$

So in this case $b = \varepsilon_{k-2}\varepsilon_{j-1}\varepsilon_i$ is the lower bound of both m and m'.

(III) $p = \eta_i\eta_j\eta_k \ \ (i > j > k)$;

This case is "dual" in an obvious sense to case (II).

(IV) $p = \varepsilon_i\varepsilon_j\eta_k \ \ (i < j)$;

This case is subdivided into following 7 subcases below:

a) $k > j + 1$

$$\begin{aligned} p \models m &:= \varepsilon_{j-1}\varepsilon_i\eta_k \geq \varepsilon_{j-1}\eta_{k-1}\varepsilon_i \geq \eta_{k-2}\varepsilon_{j-1}\varepsilon_i \\ p \models m' &:= \varepsilon_i\eta_{k-1}\varepsilon_j \geq \eta_{k-2}\varepsilon_i\varepsilon_j \geq \eta_{k-2}\varepsilon_{j-1}\varepsilon_i \end{aligned}$$

i.e., $b = \eta_{k-2}\varepsilon_{j-1}\varepsilon_i$ is a common lower bound of both m and m'.

b) $k = j + 1$

$$\begin{aligned} p = \varepsilon_i\varepsilon_j\varepsilon_{j+1} \models m &:= \varepsilon_{j-1}\varepsilon_i\eta_{j+1} \geq \varepsilon_{j-1}\eta_j\varepsilon_i \geq \ldots \geq \varepsilon_i \\ p \models m' &:= \varepsilon_i\varepsilon_j\eta_j \geq \ldots \geq \varepsilon_i \end{aligned}$$

i.e., $b = \varepsilon_i$ is a common lower bound of both m and m'.

c) $k = j$

$$\begin{aligned} p = \varepsilon_i\varepsilon_j\eta_j \models m &:= \varepsilon_{j-1}\varepsilon_i\eta_j \geq \begin{cases} \varepsilon_{j-1} = \varepsilon_i & (i = j-1) \\ \varepsilon_{j-1}\eta_{j-1}\varepsilon_i \geq \ldots \geq \varepsilon_i & (i < j-1) \end{cases} \\ p \models m' &:= \varepsilon_i\varepsilon_{j-1}\eta_j \geq \ldots \geq \varepsilon_i \end{aligned}$$

i.e., $b = \varepsilon_i$ is a common lower bound of both m and m'.

d) $i + 1 < k < j$

$$\begin{aligned} p \models m &:= \varepsilon_{j-1}\varepsilon_i\eta_k \geq \varepsilon_{j-1}\eta_{k-1}\varepsilon_i \geq \eta_{k-1}\varepsilon_{j-2}\varepsilon_i \\ p \models m' &:= \varepsilon_i\eta_k\varepsilon_{j-1} \geq \eta_{k-1}\varepsilon_{j-2}\varepsilon_i \end{aligned}$$

i.e., $b = \eta_{k-1}\varepsilon_{j-2}\varepsilon_i$ is a common lower bound of both m and m'.
e) $k = i + 1 < j$

$$\begin{aligned} p = \varepsilon_i\varepsilon_j\eta_{i+1} \models m &:= \varepsilon_{j-1}\varepsilon_i\eta_{i+1} \geq \ldots \geq \varepsilon_{j-1} \\ p \models m' &:= \varepsilon_i\eta_{i+1}\varepsilon_{j-1} \geq \ldots \geq \varepsilon_{j-1} \end{aligned}$$

i.e., $b = \varepsilon_{j-1}$ is a common lower bound of both m and m'.
f) $k = i$

$$\begin{aligned} p \models m &:= \varepsilon_{j-1}\varepsilon_i\eta_i \geq \ldots \geq \varepsilon_{j-1} \\ p \models m' &:= \varepsilon_i\eta_i\varepsilon_{j-1} \geq \ldots \geq \varepsilon_{j-1} \end{aligned}$$

i.e., $b = \varepsilon_{j-1}$ is a common lower bound of both m and m'.
g) $k < i$

$$\begin{aligned} p \models m &:= \varepsilon_{j-1}\varepsilon_i\eta_k \geq \varepsilon_{j-1}\eta_k\varepsilon_{i-1} \geq \eta_k\varepsilon_{j-2}\varepsilon_{i-1} \\ p \models m' &:= \varepsilon_i\eta_k\varepsilon_{j-1} \geq \eta_k\varepsilon_{j-2}\varepsilon_{i-1} \end{aligned}$$

i.e., $\eta_k\varepsilon_{j-2}\varepsilon_{i-1}$ is a common lower bound of both m and m'.

(V) $p = \varepsilon_i\eta_j\eta_k \quad (j > k)$;

This case is in a sense dual to case (IV) and is subdivided into following 7 subcases below:
a) $i > j$

$$\begin{aligned} p \models m &:= \varepsilon_i\eta_k\eta_{j-1} \geq \eta_k\varepsilon_{i-1}\eta_{j-1} \geq \eta_k\eta_{j-1}\varepsilon_{i-2} \\ p \models m' &:= \eta_j\varepsilon_{i-1}\eta_k \geq \eta_j\eta_k\varepsilon_{i-2} \geq \eta_k\eta_{j-1}\varepsilon_{i-2} \end{aligned}$$

i.e., $b = \eta_k\eta_{j-1}\varepsilon_{i-2}$ is a common lower bound of both m and m'.
b) $i = j$

$$\begin{aligned} p = \eta_j\eta_j\varepsilon_k \models m &:= \varepsilon_{j-1}\eta_j\eta_k \geq \ldots \geq \eta_k \\ p \models m' &:= \varepsilon_j\eta_k\eta_{j-1} \geq \eta_k\varepsilon_{j-1}\eta_{j-1} \geq \ldots \geq \eta_k \end{aligned}$$

i.e., $b = \eta_k$ is a common lower bound of both m and m'.
c) $i = j - 1$

$$\begin{aligned} p = \varepsilon_{j-1}\eta_j\eta_k \models m &:= \varepsilon_{j-1}\eta_{j-1}\eta_k \geq \ldots \geq \eta_k \\ p \models m' &:= \varepsilon_{j-1}\eta_k\eta_{j-1} \geq \begin{cases} \eta_{j-1} = \eta_k & (k = j-1) \\ \eta_k\varepsilon_{j-2}\eta_{j-1} \geq \ldots \geq \eta_k & (k < j-1) \end{cases} \end{aligned}$$

i.e., $b = \eta_k$ is a common lower bound of both m and m'.
d) $k < i < j - 1$

$$\begin{aligned} p \models m &:= \varepsilon_i\eta_k\eta_{j-1} \geq \eta_k\varepsilon_{i-1}\eta_{j-1} \geq \eta_k\eta_{j-2}\varepsilon_{i-1} \\ p \models m' &:= \eta_{j-1}\varepsilon_i\eta_k \geq \eta_{j-1}\eta_k\varepsilon_{i-1} \geq \eta_k\eta_{j-2}\varepsilon_{i-1} \end{aligned}$$

i.e., $b = \eta_k\eta_{j-2}\varepsilon_{i-1}$ is a common lower bound of both m and m'.
e) $k = i < j-1$

$$\begin{aligned} p = \varepsilon_i\eta_j\eta_i \models m &:= \varepsilon_i\eta_i\eta_{j-1} \geq \ldots \geq \eta_{j-1} \\ p \models m' &:= \eta_{j-1}\varepsilon_i\eta_i \geq \ldots \geq \eta_{j-1} \end{aligned}$$

i.e., $b = \eta_{j-1}$ is a common lower bound of both m and m'.
f) $k = i+1$

$$\begin{aligned} p = \varepsilon_i\eta_j\eta_{i+1} \models m &:= \varepsilon_i\eta_{i+1}\eta_{j-1} \geq \ldots \geq \eta_{j-1} \\ p \models m' &:= \eta_{j-1}\varepsilon_i\eta_{i+1} \geq \ldots \geq \eta_{j-1} \end{aligned}$$

i.e., $b = \eta_{j-1}$ is a common lower bound of both m and m'.
g) $k > i+1$

$$\begin{aligned} p \models m &:= \varepsilon_i\eta_k\eta_{j-1} \geq \eta_{k-1}\varepsilon_i\eta_{j-1} \geq \eta_{k-1}\eta_{j-2}\varepsilon_i \\ p \models m' &:= \eta_{j-1}\varepsilon_i\eta_k \geq \eta_{j-1}\eta_{k-1}\varepsilon_i \geq \eta_{k-1}\eta_{j-2}\varepsilon_i \end{aligned}$$

i.e., $\varepsilon_k\eta_{j-2}\eta_{i-1}$ is a common lower bound of both m and m'. ∎

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