Max-Planck-Institut
für Mathematik
in den Naturwissenschaften
Leipzig

Explicit realization of induced and coinduced modules over Lie superalgebras by differential operators

by

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Preprint no.: 129 2006

Explicit Realization of Induced and Coinduced modules over Lie Superalgebras by Differential Operators

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September 9, 2005

This text is an extended version of a part of my 1980 Trieste preprint [16]. In this preprint I gave, in particular, an expression for the action of a Lie superalgebra $\mathfrak{g}$ on the $\mathfrak{g}$-module coinduced from an $\mathfrak{h}$-module V for the case where both $\mathfrak{g}$ and V are finite-dimensional and $\mathfrak{g}$ decomposes (as a *linear space*, not Lie superalgebra) into a direct sum $\mathfrak{g}_- \oplus \mathfrak{h}$ with $\mathfrak{g}_-$ a Lie superalgebra.[1]

The importance of coinduced and induced $\mathfrak{g}$-modules is a consequence of the fact that the vast majority of representations of Lie superalgebras occurring both in mathematical and physical practice, belongs to one of these two classes (e.g., the "universal" $\mathfrak{g}$-module $U(\mathfrak{g})$ and its dual "couniversal" module $U(\mathfrak{g})^*$, Verma and Harish-Chandra modules, etc.). Nevertheless, even some mathematicians working with these representations do not realize that they are "speaking prose", to say nothing of physicists who study "representations by creation and annihilation operators" while speaking about Verma modules or their duals.

The standard method used by both mathematicians and physicists to calculate the action for basis elements of $\mathfrak{g}_-$ in coinduced modules (where they act as differential operators) in case where $\mathfrak{g}$ is a finite-dimensional Lie algebra, was to calculate the differential of the *induced* representation of the corresponding Lie group.

This method is inapplicable both if $\mathfrak{g}$ is infinite-dimensional and if $\mathfrak{g}$ is a superalgebra, even a finite-dimensional one.

In the infinite-dimensional case the reason is that there is no correspondence between abstract infinite-dimensional Lie algebras and infinite-dimensional Lie groups, and no correspon-

2000 *Mathematics Subject Classification.* 17B15 (Primary), 17B65 (Secondary).

Key words and phrases. Lie superalgebras, representations.

*This work was written with financial support of MPIMiS (Leipzig) and thanks to creative atmosphere of this Institution. I am especially thankful to A. Lebedev, D. Leites and Ch. Sachse for helpful discussion, triggering the idea of representing $T^{\mathfrak{g}}_{\mathfrak{h}}(g)$ and $I^{\mathfrak{g}}_{\mathfrak{h}}(g)$ (actions of an element $g \in \mathfrak{g}$ in coinduced and induced $\mathfrak{g}$-module) as "path integral" in the action graph, which inspired me to write a program making all calculations: PCs are very capable playing with graphs.

[1]As far as I know, the first paper in which there appeared an explicit expression for the action of a finite-dimensional Lie algebra $\mathfrak{g} = \mathfrak{g}_- \oplus \mathfrak{h}$ (represented as a direct sum of Lie subalgebras) by differential operators on the space $\mathfrak{g}_-^*$ was I. Kantor's paper [12] of whose existence I became aware only last year. In [13], I. Kantor gave an expression for this action in a more general situation, when (1) $\mathfrak{g}_-$ may be not a sub*algebra* but only a finite-dimensional sub*space* of $\mathfrak{g}$ and (2) the Lie algebra $\mathfrak{h}$ may be infinite-dimensional.

In both [12] and [13], I. Kantor only considered Lie algebras and their quasiregular representations, which are, in terms of coinduced modules, $\mathfrak{g}$-modules coinduced from a trivial 1-dimensional $\mathfrak{h}$-module.

dence between their representations.

For finite dimensional Lie superalgebras, the correspondences in question definitely do exist, but up to now there is no printed text[2] with a *definition* of induced representations of Lie supergroups. Not that it is difficult to define these creatures,[3] but the current trend in "supermath" is to develop the corresponding issues on "local" purely Lie superalgebraic level, ignoring the lifting to the Lie supergroup level, in cases where such a lifting exists.

In this situation, many people dealing with coinduced representations of Lie superalgebras, including myself, were forced to make the necessary calculations introducing "odd parameters θ" and formally imitating the corresponding calculations for Lie algebra representations.

In [16] I gave a solid mathematical background to these formal manipulations for the case most often occurring in practice (finite-dimensional $\mathfrak{g}$ and $\mathfrak{g}_-$ a Lie subsuperalgebra of $\mathfrak{g}$), replacing "quick and dirty"[4] trick with induced representations by correct manipulations with certain formal power series.

Here the result of [16] (as well as of I. Kantor's [13] main theorem) is extended for the most general case where both $\mathfrak{g}_-$ and $\mathfrak{h}$, as well as V, may be infinite-dimensional and, moreover, $\mathfrak{g}_-$ need not be a subsuperalgebra of $\mathfrak{g}$. I also give an expression for this action as a "path integral" over paths in a graph generated by the action of $\mathfrak{g}_-$ on $\mathfrak{g}$. This expression permits one to write a program, calculating differential operators representing the basis elements of coinduced representations. The program realizing the corresponding algorithm is also written. Since the algorithm for calculating the structure constants of the simple Lie algebras (described in [9]) is especially simple for simply laced Lie algebras A_n, D_n and E_n in their Chevalley bases and with respect to the standard decomposition $\mathfrak{g}_- = \oplus_{\alpha<0}\mathfrak{g}_\alpha$, $\mathfrak{h} = \oplus_{\alpha\geq 0}\mathfrak{g}_\alpha$, the current (beta) version of this program calculates the generators and other elements of the basis of the Lie superalgebra in these cases (in particular, for $\mathfrak{g}$-modules dual to Verma modules, the latter being "free" highest weight modules). An extension of the program to the non-simply laced simple Lie algebras, and to simple Lie superalgebras is being developed.

Notations and conventions.

Throughout the paper all Lie superalgebras and vector superspaces are over a ground field $\Bbbk$ of characteristic 0. This restriction is forced mainly by the fact that, in our explicit constructions, we use formal series in various vector spaces over $\Bbbk$, whose coefficients contain factorials in denominators.

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[2] "Induced representations of a supergroup" $\mathcal{G}$, defined in [6], are, in fact, representations of the super*algebra* of $\mathcal{G}$, corresponding to induced representations of $\mathcal{G}$.

[3] The topos of sets is not appropriate external universe for the category of supermanifolds. In particular, coinduced representations of supergroups can not be defined as sets with some structure (though all modern and future mathematics is nothing but conservative extension of set theory (any category is a set or class), not every category is a category of sets with *structure*). There is a universal way to embed any category into appropriate universe (of contravariant set-valued functors) – Yoneda's "point functor". For the category of supermanifolds there is an alternative solution: the category of set-valued functors from the category of finite-dimensional Grassmann superalgebras [18]. The latter approach permits one to extend the category of supermanifolds itself, including in it infinite-dimensional supermanifolds. Then induced representations of finite-dimensional supergroups can be defined as *internal* objects: Frechet supermanifolds of "supersections" of a vector superbundle with supersmooth action of a supergroup on them.

[4] In programmer's slang this means a quicky bug fix without localizing in source code the bug itself, i.e. without real understanding of the bug's reason.

Vector superspaces will be considered, if necessary, as *topological* vector superspaces (over $\Bbbk$ equipped with the discrete topology). We tacitly assume that the default topology of a vector superspace E is discrete again, whereas that of E^* has the set of all subspaces orthogonal to finite-dimensional subspaces of E as the base of neighborhoods of zero.

For any Lie superalgebra $\mathfrak{g}$, let $U(\mathfrak{g})$ be the universal enveloping superalgebra of $\mathfrak{g}$. Hereafter if A is an element of some superspace (see Ref. [15] or Appendix A), we denote the even (resp. odd) component of A by ${}_{\bar{0}}A$ (resp. ${}_{\bar{1}}A$); let $[\cdot,\cdot]$ denote the composition (bracket) in Lie superalgebras.

The parity of an homogeneous element X of a vector superspace will be denoted by $|X|$.

We will often define multilinear maps by their values on homogeneous elements (i.e., either even or odd) without mentioning this explicitly. The same practice will be applied while checking identities between multilinear maps, to avoid writing out numerous "sums over parities" widening the formulas to an abnormal extent. The presence of expressions like $(-1)^{|X|\cdot|Y|}$ in a formula will indicate that the formula is represented in its "short-hand" form.

$\mathfrak{S}_n$ denotes the group of permutations of the set $\{1,\ldots,n\}$.

$\widehat{E}$ denotes the completion of a topological vector (super)space E.

$\overline{X}$ denotes the closure of a subset X of a topological space.

$\Bbbk\langle\, P\,\rangle$ denotes the $\Bbbk$-linear span of a subset (or of an indexed family) P of a topological vector (super)space.

If E and W are topological vector superspaces, then $L(E,W)$ denotes the superspace af all *continuous* linear maps from E to W, whereas $\mathcal{L}(E,W)$ denotes the superspace $L(E,W)$ equipped with lpco topology, defined in Sect. 2.4.

$\mathrm{Bil}(E\times E',W)$ denotes the superspace of all continuous bilinear maps from $E\times E'$ to W.

Throughout the paper $\mathfrak{g}$ is a Lie superalgebra, $\mathfrak{h}$ a Lie subsuperalgebra of $\mathfrak{g}$, $\mathfrak{g}_-$ a subsuperspace of $\mathfrak{g}$ such that $\mathfrak{g}=\mathfrak{g}_-\oplus\mathfrak{h}$. Depending on the context, V will denote either an $\mathfrak{h}$-module, or a superspace.

1 The induced and coinduced modules over Lie superalgebras.

Here we reproduce, for the reader's convenience, general definitions of induced and coinduced modules as well as some of their properties, see [3] and [11].

1.1 The induced modules. Let $\mathfrak{h}$ be a Lie subsuperalgebra of a Lie superalgebra $\mathfrak{g}$ and V an $\mathfrak{h}$-module. The $\mathfrak{g}$-module

$$I^{\mathfrak{g}}_{\mathfrak{h}}(V)\overset{\mathrm{def}}{=}U(\mathfrak{g})\otimes_{U(\mathfrak{h})}V \tag{1.1}$$

is said to be **induced** by the $\mathfrak{h}$-module V. As a vector space, $I^{\mathfrak{g}}_{\mathfrak{h}}(V)$ is the quotient space of tensor product $U(\mathfrak{g})\otimes V$ modulo the relations

$$gh\otimes v=g\otimes\rho(h)v\quad(h\in\mathfrak{h}). \tag{1.2}$$

The action of the Lie superalgebra $\mathfrak{g}$ on $I^{\mathfrak{g}}_{\mathfrak{h}}(V)$ is defined as follows:

$$g(u\otimes v)=gu\otimes v\quad(g\in\mathfrak{g},\ u\otimes v\in I^{\mathfrak{g}}_{\mathfrak{h}}(V)), \tag{1.3}$$

where $u\otimes v$ denotes, by abuse of notation, the element $u\otimes_{U(\mathfrak{h})}v$ of $I^{\mathfrak{g}}_{\mathfrak{h}}(V)$.

There is a canonical morphism of $\mathfrak{h}$-modules

$$\varphi : V \longrightarrow I_{\mathfrak{h}}^{\mathfrak{g}}(V)|_{\mathfrak{h}} \quad (\varphi(v) \stackrel{\text{def}}{=} 1 \otimes v) \tag{1.4}$$

such that the pair (V, φ) is universal: for any other pair $(W, \sigma : V \longrightarrow W|_{\mathfrak{h}})$, there exists a unique morphism $\theta : I_{\mathfrak{h}}^{\mathfrak{g}}(V) \longrightarrow W$ of $\mathfrak{g}$-modules such that $\sigma = \theta\varphi$.

1.2 The coinduced modules. Let $\mathfrak{h}$, $\mathfrak{g}$ and V be as above. The $\mathfrak{g}$-module

$$T_{\mathfrak{h}}^{\mathfrak{g}}(V) \stackrel{\text{def}}{=} \mathrm{Hom}_{U(\mathfrak{h})}(U(\mathfrak{g}), V) \tag{1.5}$$

is said to be **coinduced** (or **produced** in the original Blattner's terminology [3]) by the $\mathfrak{h}$-module V. The module $T_{\mathfrak{h}}^{\mathfrak{g}}(V)$ as a vector space is the subspace of $\mathrm{Hom}(U(\mathfrak{g}), V)$ consisting of all $\mathfrak{h}$-invariant elements f:

$$hf(g) = \sum_{i,j \in \mathbb{Z}_2} (-1)^{ij} {}_i f({}_j hg) \quad (h \in \mathfrak{h}). \tag{1.6}$$

(i.e., intertwining maps).

The action of Lie superalgebra $\mathfrak{g}$ on $T_{\mathfrak{h}}^{\mathfrak{g}}(V)$ is defined as follows:

$$(gf)(g') = \sum_{i,j,k \in \mathbb{Z}_2} (-1)^{i(j+k)} {}_j f({}_k g' {}_i g) \quad (g, g' \in \mathfrak{g},\ f \in T_{\mathfrak{h}}^{\mathfrak{g}}(V)). \tag{1.7}$$

There is a canonical morphism of $\mathfrak{h}$-modules

$$\varphi : T_{\mathfrak{h}}^{\mathfrak{g}}(V)|_{\mathfrak{h}} \longrightarrow V \quad (\varphi(f) \stackrel{\text{def}}{=} f(1)) \tag{1.8}$$

such that the pair (V, φ) is universal: for any other pair $(W, \sigma : W|_{\mathfrak{h}} \longrightarrow V)$, there exists a unique morphism $\theta : W \longrightarrow T_{\mathfrak{h}}^{\mathfrak{g}}(V)$ of $\mathfrak{g}$-modules such that $\sigma = \varphi\theta$.

1.3 A fine structure of coinduced modules. Consider the space $\Bbbk$ as the trivial $\mathfrak{h}$-module. Given an $\mathfrak{h}$-module V, define the $\Bbbk$-bilinear map

$$T_{\mathfrak{h}}^{\mathfrak{g}}(\Bbbk) \times T_{\mathfrak{h}}^{\mathfrak{g}}(V) \stackrel{\cdot}{\longrightarrow} T_{\mathfrak{h}}^{\mathfrak{g}}(V) \tag{1.9}$$

as follows. Let

$$\Delta : U(\mathfrak{g}) \longrightarrow U(\mathfrak{g}) \otimes U(\mathfrak{g}) \tag{1.10}$$

be a comultiplication in $U(\mathfrak{g})$, i.e., the only morphism of associative superalgebras that extends the linear map

$$\Delta_0 : \mathfrak{g} \longrightarrow U(\mathfrak{g}) \otimes U(\mathfrak{g}) \quad (\Delta_0(g) = g \otimes 1 + 1 \otimes g) \tag{1.11}$$

from $\mathfrak{g} \subset U(\mathfrak{g})$ to the whole $U(\mathfrak{g})$.

For any $f \in T^{\mathfrak{g}}_{\mathfrak{h}}(\mathbb{k})$ and $s \in T^{\mathfrak{g}}_{\mathfrak{h}}(V)$, define $f \cdot s \in T^{\mathfrak{g}}_{\mathfrak{h}}(V)$ as follows. For any $g \in U(g)$, let $\Delta(g) = \sum_i a_i \otimes b_i$. Then:

$$(f \cdot s)(g) \stackrel{\text{def}}{=} \sum_{i,\varepsilon,\varepsilon'} (-1)^{\varepsilon\varepsilon'} f({}_{\varepsilon}a_i)\, {}_{\varepsilon'}s(b_i). \tag{1.12}$$

The definition (1.12) is just the straightforward extension (via "sign rule") to the case of Lie superalgebras of the particular case of the corresponding definition from [3].

The following proposition is the generalization of Proposition 10 from [3] to the case of Lie superalgebras.

Proposition 1 **a)** *The map* (1.9) *defines on* $T^{\mathfrak{g}}_{\mathfrak{h}}(\mathbb{k})$ *the structure of a supercommutative superalgebra and, for any* $\mathfrak{h}$*-module* V*, turns* $T^{\mathfrak{g}}_{\mathfrak{h}}(V)$ *into an* $T^{\mathfrak{g}}_{\mathfrak{h}}(\mathbb{k})$*-module.*
b) $\mathfrak{g}$ *acts on* $T^{\mathfrak{g}}_{\mathfrak{h}}(V)$ *via differentiations with respect to the just defined structure. In other words,*

$$g(f \cdot s) = (gf) \cdot s + \sum_{\varepsilon,\varepsilon' \in \mathbb{Z}_2} (-1)^{\varepsilon\varepsilon'}\, {}_{\varepsilon}f \cdot ({}_{\varepsilon'}gs) \tag{1.13}$$

for any $f \in T^{\mathfrak{g}}_{\mathfrak{h}}(\mathbb{k})$, $s \in T^{\mathfrak{g}}_{\mathfrak{h}}(V)$, *and* $g \in \mathfrak{g}$.

Proof Heading a) is straightforward. Let us prove b). Let g' be a homogeneous element in $U(\mathfrak{g})$ and $\Delta(g') = \sum_i a_i \otimes b_i$. Then, for any homogeneous $g \in \mathfrak{g}$, we have

$$\Delta(g'g) = \Delta(g')\Delta(g) = \sum_i ((-1)^{|g|\cdot|b_i|} a_i g \otimes b_i + a_i \otimes b_i g)$$

and

$$\begin{aligned}
&(-1)^{|g|(|f|+|s|+|g'|)} g(f \cdot s)(g') = (f \cdot s)(g'g) \\
&= \textstyle\sum_i ((-1)^{|g|\cdot|b_i|+|s|(|a_i|+|g|)} f(a_i g)s(b_i) + (-1)^{|a_i|\cdot|s|} f(a_i)s(b_i g)) \\
&= \textstyle\sum_i ((-1)^{|g|\cdot|b_i|+|s|(|a_i|+|g|)+|g|(|f|+|a_i|)} gf(a_i)s(b_i) + (-1)^{|a_i|\cdot|s|+|g|(|s|+|b_i|)} f(a_i)gs(b_i)) \\
&= \textstyle\sum_i ((-1)^{|g|\cdot|g'|+|s|(|a_i|+|g|)+|g|\cdot|f|} gf(a_i)s(b_i) + (-1)^{|a_i|\cdot|s|+|g|(|s|+|b_i|)} f(a_i)gs(b_i)) \\
&= \textstyle\sum_i ((-1)^{|g|(|f|+|s|+|g'|)+|s|\cdot|a_i|} gf(a_i)s(b_i) + (-1)^{|a_i|\cdot|s|+|g|(|s|+|b_i|)} f(a_i)gs(b_i)) \\
&= (-1)^{|g|(|f|+|s|+|g'|)}((gf) \cdot s)(g') + (-1)^{|g|(|f|+|s|+|g'|+|f|\cdot|g|)}(f \cdot (gs))(g') \\
&= (-1)^{|g|(|f|+|s|+|g'|)}(((gf) \cdot s) + (f \cdot (gs)))(g').
\end{aligned}$$

The above calculation uses definitions (1.7), (1.12) and the fact that $|a_i| + |b_i| = |g'|$. ■

Proposition 1 states, roughly, that the action of $\mathfrak{g}$ in coinduced modules is *always* realized by differential operators of the first order. But it gives no prescription as to how to find these differential operators. The main purpose of this paper is just to find *explicit* realizations for these differential operators in a "natural" basis in $T^{\mathfrak{g}}_{\mathfrak{h}}(V)$. The first step is to define a "natural" basis.

1.4 $I^{\mathfrak{g}}_{\mathfrak{h}}(V)$ and $T^{\mathfrak{g}}_{\mathfrak{h}}(V)$ as superspaces. Let $\mathfrak{g}_-$ be a subsuperspace in $\mathfrak{g}$ such that $\mathfrak{g} = \mathfrak{g}_- \oplus \mathfrak{h}$. We do not require here that $\mathfrak{g}_-$ be a sub*superalgebra* of $\mathfrak{g}$. Moreover, no condition of finite-dimensionality is imposed on $\mathfrak{g}$, $\mathfrak{h}$ or $\mathfrak{g}_-$.

If $P = \{P_i\}_{i\in I}$ is a well-ordered basis of $\mathfrak{g}_-$, then the space $I^{\mathfrak{g}}_{\mathfrak{h}}(V)$ is spanned by $1 \otimes V$ and by monomials

$$P_{i_1}^{\alpha_1} \dots P_{i_k}^{\alpha_k} \otimes v \quad (k \in \mathbb{N} \setminus \{0\},\ i_1 < \dots < i_k,\ v \in V), \tag{1.15}$$

where $\alpha_i = 1$ if P_i is an odd basis element and α_i is a positive integer if P_i is an even basis element (The Poincaré-Birkhoff-Witt theorem [7], [11]).

Hence, every such ordered basis defines a linear isomorphism

$$I^{\mathfrak{g}}_{\mathfrak{h}}(V) \approx \Bbbk[\{P_i\}_{i\in I}] \otimes V \approx S(\mathfrak{g}_-) \otimes V\ , \tag{1.16}$$

where $\Bbbk[\{P_i\}_{i\in I}]$ is the superspace of supercommuting polynomials, see Appendix A, in the P_i (whose number may be infinite).

Remark. The isomorphism (1.16) is an analogue of transition from non-commuting operators in quantum field theory to operators commuting under the sign of normal product or T-product.

The counterpart of (1.16) for coinduced modules is the isomorphism:

$$T^{\mathfrak{g}}_{\mathfrak{h}}(V) \approx \mathrm{Hom}(\Bbbk[\{P_i\}_{i\in I}], V) \approx \mathrm{Hom}(S(\mathfrak{g}_-), V)\ . \tag{1.17}$$

The isomorphisms (1.16) and (1.17) respect, moreover, natural $\mathbb{Z}_+$-filtrations of $I^{\mathfrak{g}}_{\mathfrak{h}}(V)$ and $T^{\mathfrak{g}}_{\mathfrak{h}}(V)$ (for details, see [3]).

For our purposes (to find explicit expressions for (co)induced modules) more appropriate are "supersymmetrized" bases in $I^{\mathfrak{g}}_{\mathfrak{h}}(V)$ or $T^{\mathfrak{g}}_{\mathfrak{h}}(V)$ defined below.

Define the supersymmetric product on $\mathfrak{g}_-^k$ with values in $U(\mathfrak{g})$ as follows:

$$g_1 \circ \dots \circ g_k \stackrel{\text{def}}{=} \frac{1}{k!} \sum_{\sigma\in\mathfrak{S}_k} (-1)^{\varepsilon(\sigma_{\text{odd}})} g_{\sigma 1} \dots g_{\sigma k}\ , \tag{1.18}$$

where $\varepsilon(\sigma_{\text{odd}})$ is the parity of permutation of odd elements among $g_1, \dots, g_k$.

In what follows, we will denote $U(\mathfrak{g}_-, \mathfrak{g})$ the subsuperspace of $U(\mathfrak{g})$ spanned by all supersymmetric products including the empty product 1. The subsuperspace $U(\mathfrak{g}_-, \mathfrak{g})$ clearly is spanned by 1 and by "monomials"

$$P_{i_1} \circ \dots \circ P_{i_k} \quad (k \in \mathbb{N} \setminus \{0\},\ i_1 \leq \dots \leq i_k). \tag{1.19}$$

If $\mathfrak{g}_-$ is a subsuperalgebra of $\mathfrak{g}$, then $U(\mathfrak{g}_-, \mathfrak{g})$ is a subsuperalgebra of $U(\mathfrak{g})$ coinciding with "canonical" subsuperalgebra spanned by non-supersymmetrizied monomials and isomorphic to $U(\mathfrak{g}_-)$. This justifies the shorthand notation $U(\mathfrak{g}_-)$ for $U(\mathfrak{g}_-, \mathfrak{g})$, which will be used in what follows..

The Poincaré-Birkhoff-Witt theorem implies that for any direct sum decomposition $\mathfrak{g} = \mathfrak{g}_- \oplus \mathfrak{h}$ of $\mathfrak{g}$ into pair of subsuperspaces there is an isomorphism

$$U(\mathfrak{g}_-) \otimes U(\mathfrak{h}) \approx U(\mathfrak{g}) \quad (\sum_i g_i \otimes h_i \mapsto \sum g_i h_i). \tag{1.20}$$

The latter isomorphism implies that the superspace $I_{\mathfrak{h}}^{\mathfrak{g}}(V)$ is spanned by $1 \otimes V$ and by monomials

$$P_{i_1}^{\circ\alpha_1} \circ \cdots \circ P_{i_k}^{\circ\alpha_k} \otimes v \quad (k \in \mathbb{N} \setminus \{0\},\ i_1 < \cdots < i_k,\ v \in V), \tag{1.21}$$

where $g^{\circ k}$ denotes k-fold supersymmetric product of g.

The isomorphisms (1.16) and (1.17) arising from the choice of "supersymmetrized" basis in $U(\mathfrak{g})$ also respect, as is easy to see, the natural $\mathbb{Z}_+$-filtrations of $I_{\mathfrak{h}}^{\mathfrak{g}}(V)$ and $T_{\mathfrak{h}}^{\mathfrak{g}}(V)$.

2 Linearly topologized superspaces and superalgebras of formal power series.

Before proceeding further, let us describe some useful topologies on superspaces arising, in particular, on superalgebras of formal power series (in possibly infinite number of variables). These topologies will turn our superspaces into complete topological superspaces over the field $\Bbbk$ with the discrete topology.

The pragmatical reader interested in finite-dimensional case only, i.e. the case where both the superalgebra $\mathfrak{g}$ and the $\mathfrak{h}$-module V are finite-dimensional, can skip the most part of this section. In this case the rudimentary knowledge of elementary properties of linearly compact spaces (especially of the superalgebra of formal power series in finite number of indeterminates) is enough to understand constructions of next sections.

In the general case the superspaces arising are not linearly compact, so that this section contains all necessary results from the theory of linearly topologized spaces. And, for the sake of completeness, a number of more general results not used in this text, but lacking in canonical sources [14] and [5], where this subject is treated.

2.1 Linearly topologized superspaces. A superspace E equipped with a topology τ is said to be a **topological superspace** over $\Bbbk$ if it is a topological vector space over $\Bbbk$ and, besides, the decomposition $E = {}_{\bar{0}}E \oplus {}_{\bar{1}}E$ is a decomposition into *topological* direct sum (in particular, both ${}_{\bar{0}}E$ and ${}_{\bar{1}}E$ are closed subspaces of E).

A Hausdorff topology τ of a topological superspace E is said to be **linear** (and E itself as **linearly topologized**) if it has a base of open neighborhoods of 0, consisting of subspaces. Clearly, this is equivalent to the condition that τ has a basis of open neighborhoods of 0, consisting of sub*super*spaces. This is equivalent as well to the condition that both ${}_{\bar{0}}E$ and ${}_{\bar{1}}E$ are linearly topologized vector spaces (the theory of which is developed in Ch. 2 § 10 of [14] and, in more general context of modules over commutative rings, in Excercises 14-28 in Ch.III, § 2 of [5]).

2.2 Discrete superspaces. A superspace E equipped with the discrete topology is, clearly, linearly topologized topological superspace briefly called a **discrete superspace**. In what follows we assume that the Lie algebra $\mathfrak{g}$, $U(\mathfrak{g})$, and the $\mathfrak{h}$-module V are discrete superspaces. The commutative superalgebra $S(E)$ for a discrete superspace E will also be considered as discrete superspace. The topological direct sum of any family of discrete superspaces is a discrete superspace. Any discrete superspace is isomorphic to a topological direct sum of superspaces $\Bbbk^{1|0}$ or $\Bbbk^{0|1}$.

2.3 Cofinite topologies and linearly compact superspaces. A linear topology τ on V is said to be **cofinite** if it has a base of open neighborhoods of 0 consisting of superspaces of finite codimension. A *complete* topological superspace with cofinite topology is called **linearly compact**.

Proposition 2 *The completion $\widehat{E}$ of a superspace E with cofinite topology is linearly compact.* ■

This justifies the name **linearly precompact** for superspaces equipped with cofinite topology.

Example 1: superspaces E^*. If E is a discrete topological superspace, then the superspace E^* equipped with weak topology, i.e., arising from duality between E^* and E, is linearly compact. Recall that a base of neighborhoods of 0 of E^* (both open and closed) in weak topology is formed by subsuperspaces $K^\perp \subset E^*$ of linear forms annihilating some finite-dimensional subsuperspace K of E.

In what follows superspaces dual to discrete ones will be always considered as linearly compact topological superspaces. Any topological product of linearly compact superspaces is linearly compact and any linearly compact superspace is isomorphic to a topological product of superspaces $\Bbbk^{1|0}$ or $\Bbbk^{0|1}$. In particular, any linearly compact superspace is isomorphic (as topological superspace) to a superspace E^* for some discrete superspace E.

Example 2: the cofinite topology generated by a basis. Let $P = \{P_i\}_{i\in I}$ be a basis of a superspace E. Define a cofinite topology τP on E having a base of open neighborhoods of 0 consisting of all subsuperspaces

$$E_F \stackrel{\text{def}}{=} \Bbbk\langle\, \{P_i\}_{i\in I\setminus F}\,\rangle, \tag{2.1}$$

where F is a finite subset of I. For infinite-dimensional E the topology τP is strictly *weaker* than the weak topology of E (i.e., coming from duality between E and E^*). Recall ([14]) that a basis of open neighborhoods of 0 in weak topology is formed by *all* subsuperspaces of E of finite codimension. For example, the hyperplane $\langle\{P_i - P_j\}_{i,j\in I}$ is open in the weak topology of E but is not open in the topology τP.

Let $\widehat{E}_w$, resp. $\widehat{E}_{\tau P}$ be the completion of E with respect to the weak topology, resp. with respect to the topology τP. Both $\widehat{E}_w$ and $\widehat{E}_{\tau P}$ are linearly compact due to Prop. 2 and there is continuous linear map $\widehat{\mathrm{Id}} : \widehat{E}_w \longrightarrow \widehat{E}_{\tau P}$. We will see below that $\widehat{E}_{\tau P}$ is isomorphic (non-canonically) to E^*, whereas $\widehat{E}_w$ is known (see [14]) to be isomorphic to $(E^*)^*_{\text{disc}}$, where $(E^*)_{\text{disc}}$ is the superspace E^* with discrete topology. If E is infinite-dimensional, then $\dim(E^*)^*_{\text{disc}} = 2^{\dim E^*}$, so that the superspace $(E^*)^*_{\text{disc}}$ is "bigger" than E^* and the map $\widehat{\mathrm{Id}}$ is to have non-zero kernel of dimension $(E^*)^*_{\text{disc}}$.

Proposition 3 **(a)** *The product of any family of linearly precompact superspaces is linearly precompact;*
(b) *The Image $f(C)$ of any linearly precompact subsuperspace C of a linearly topologized superspace E under any continuous linear map f is linearly precompact;*
(c) *Any linearly precompact subsuperspace C of a topological direct sum $\oplus_{i\in I} E_i$ of a family of linearly topologized superspaces is contained in the direct sum of some finite subfamily.*

■

2.4 Topological superspaces $\mathcal{L}(E,W)$ and $E \otimes W$. Other types of topological superspaces, which will occur, are $L(E,W)$ and $E \otimes W$ for discrete or linearly compact superspaces E and W, equipped with appropriate topologies. But we will give the definitions of these topologies for the general case of arbitrary linear topologies on E and W. After all, the class of linearly topologized superspaces consisting of all discrete and all linearly compact superspaces only, is not closed with respect to such categorical operations as $\oplus$, $\otimes$, $\mathcal{L}(\cdot,\cdot)$, etc.

From [14] one can conclude that the category **LVect** of linearly topologized spaces is both complete and cocomplete, because it has arbitrary direct sums, products, kernels and cokernels. What about categorical tensor product and internal Hom functor? Recall that a bifunctor $(E,W) \mapsto E \otimes W$, resp. $(E,W) \mapsto \mathcal{H}om(E,W)$ is called a **tensor product** functor, resp. **internal Hom** functor, if there exists a natural isomorphism

$$\mathrm{Bil}(E \times W, V) \approx L(E \otimes W, V), \tag{2.2}$$

resp.

$$\mathrm{Bil}(E \times W, V) \approx L(E, \mathcal{H}om(W,V)), \tag{2.3}$$

where $\mathrm{Bil}(E \times W, V)$ is the set of continuous bilinear maps from $E \times W$ to V. We will see that categorical tensor product functor $\otimes$ exists in **LVect**. As to $\mathcal{H}om$, it seems to be non-existent on the whole category **LVect**. But there are 2 full coreflective subcategories of **LVect**, where internal Hom does exist.

Linearly precompact-open topology on $\mathcal{L}(E,W)$. Define a linear topology on the superspace $L(E,W)$ of continuous linear maps from E to W as follows. For $C \subset E$ and $O \subset W$, let $L(C,O)$ be the subset of $L(E,W)$, consisting of all f such that $f(C) \subset O$. The superspace $L(E,W)$ will be equipped with a linear topology having as a base of open neighborhoods of 0 all subsuperspaces $L(C,O)$, where C is a linearly precompact (closed) subsuperspace of E and O is an open subsuperspace in W. This topology will be called **linearly precompact-open** or, briefly, **lpco** topology on $L(E,W)$. In what follows we will use lpco topology as default one and the superspace $L(E,W)$ equipped with this topology will be denoted $\mathcal{L}(E,W)$.

Remark. Replacing in the above definition linearly precompact subsuperspaces of E by linearly compact ones one gets a **linearly compact-open** (or, briefly, **lco**) topology on $L(E,W)$. This topology is, generally speaking, *weaker* than lpco topology, but if E is complete, then any closed linearly precompact subsuperspace $C \subset E$ is linearly compact, so that in this case both topologies coincide. The superspace $L(E,W)$ equipped with lco topology will be denoted $\mathcal{L}(E,W)_{lco}$.

If W is discrete, then, clearly, the family of subsuperspaces $L(C,0)$ form a base of open neighborhoods of 0 in $\mathcal{L}(E,W)$. In particular, if $W = \Bbbk$ and E is discrete, then the lco topology on E^* coincides with the weak topology of E^*.

It is clear as well that if $E = V^*$, then the family of subsuperspaces $L(E,O)$ form a base of open neighborhoods of 0 in $\mathcal{L}(E,W)$. In particular, if $W = \Bbbk$, then the lco topology on $V^{**} \approx V$ coincides with the discrete topology of V.

Proposition 4 *Let E and W be discrete superspaces. The map*

$$* : \mathfrak{L}(E,W) \longrightarrow \mathfrak{L}(W^*,E^*) \qquad (f \mapsto f^*) \tag{2.4}$$

is a linear homeomorphism.

Proof The map $*$ is known to be a linear isomorphism (see [14]), so it only needs to prove that lpco topologies on $\mathfrak{L}(E,W)$ and $\mathfrak{L}(W^*,E^*)$ are isomorphic. Linearly compact subsuperspaces of a discrete superspace E are exactly finite-dimensional subsuperspaces K (see [14]). Given such a superspace $K \subset E$, we have:

$$\begin{aligned} *(L(K,0)) &= \{\, f^* \mid f(K) = 0 \,\} \\ &= \{\, f^* \mid \varphi(f(K)) = 0 \text{ for all } \varphi \in W^* \,\} \\ &= \{\, f^* \mid (f^*\varphi)(K) = 0 \text{ for all } \varphi \in W^* \,\} \\ &= L(W^*, K^{\perp}) \end{aligned}$$

And, by the definition of weak topology, the family of spaces $K^{\perp}$ is a base of open neighborhoods of 0 in E^*, hence the family of superspaces $L(W^*,K^{\perp})$ is a base of neighborhoods in $\mathfrak{L}(W^*,E^*)$ in lpco topology. ■

The completeness of the superspace $\mathfrak{L}(E,W)$ for discrete E and W follows from general Prop. 6 below. To formulate this proposition we need the notion of *linerly precompactly generated superspaces*.

Linerly (pre)compactly generated superspaces. Call a linearly topologized superspace E **linearly (pre)compactly generated** or, briefly, **l(p)cg** if any subsuperspace U of E is open if its intersection with any linearly (pre)compact subsuperspace C is open in C.

Any discrete or linearly compact superspace is, clearly, both lcg and lpcg. Any linearly precompact superspace is lpcg; it is lcg iff it is complete.

The main property of l(p)cg superspaces is:

Proposition 5 *A linear map $f : E \longrightarrow W$ of a l(p)cg E into a linearly topologized superspace W is continuous iff for any linearly (pre)compact subsuperspace C of E the restriction $f|_C$ is continuous.*

■

Proposition 6 *If E is lpcg* (resp. *lcg*) *and W is complete, then $\mathfrak{L}(E,W)$* (resp. $\mathfrak{L}(E,W)_{lco}$) *is complete.*

Proof The statement follows immediately from 5 and Theorem 2 in Chapter 10 §3 of [4]. ■

Denote **LVect** the category of linearly topologized vector superspaces, **PCGVect** (resp. **CGVect**) the full subcategory of **LVect**, consisting of all lpcg superspaces (resp. of all lcg superspaces).

Proposition 7 *The inclusion functor* **PCGVect** $\subset$ **LVect** (resp. **CGVect** $\subset$ **LVect**) *has right adjoint*

$$K : \mathbf{LVect} \longrightarrow \mathbf{PCGVect} \qquad (\text{resp.} K : \mathbf{LVect} \longrightarrow \mathbf{CGVect}). \tag{2.5}$$

Corollary 8 *The categories* **PCGVect** *and* **CGVect** *are complete and cocomplete.*

Theorem 9 *The category* **PCGVect** (resp. **CGVect**) *have internal* Hom *functor. This functor is defined as* $\mathcal{H}om(E,W) := K\mathcal{L}(E,W)$ (resp. as $\mathcal{H}om(E,W) \stackrel{\text{def}}{=} K\mathcal{L}(E,W)_{lco}$).

Proposition 10

$$(\bigoplus_{i\in I} KE_i)^* \approx \prod_{i\in I} (KE_i)^*; \qquad (\prod_{i\in I} KE_i)^* \approx \bigoplus_{i\in I} (KE_i)^*; \tag{2.6}$$

Considering, as usual, a linearly topologized superspace E as a subsuperspace of its double topological dual E^{**}, we call E **reflexive** if $E^{**} \approx E$ (as topological superspaces).

We have seen that both discrete superspaces and dual to them (i.e., linearly compact) superspaces are reflexive. The next Corollary states that the class of reflexive l(p)cg is big enough:

Corollary 11 *Any topological direct sum or product of reflexive l(p)cg superspaces is reflexive.*

Props. 7–11 above (which proofs and more details will be published soon in [17]) show that both categories **PCGVect** and **CGVect** are counterparts of the category of Kelley vector spaces ([8],[18]).

In fact, the direct counterpart is the subcategory **CGVect** of linearly compactly generated superspaces, but bigger subcategory **PCGVect** is more flexible in some respects.

Topologies on $E \otimes W$. We will be interested here only in topologies on superspaces of the type $E^* \otimes W$ with both E and W being discrete superspaces. Our aim is to "approximate" somehow the superspace $\mathcal{L}(E,W)$ by the superspace $E^* \otimes W$. But definitions and theorems of this section are given in their natural generality — for arbitrary linearly topologized spaces.

From purely categorical point of view "the natural" topology on the superspace $E \otimes W$ should be chosen in such a way, that it permits one to represent *bi*linear continuous maps from $E \times W$ (which are *not* morphisms of the category of topological superspaces) by continuous *linear* maps from $E \otimes W$. In other words it permits one to internalize definitions of $\Bbbk$-algebras, R-modules, etc. entirely inside the category of linearly topologized superspaces, not using external maps such as bilinear or, more generally, multilinear ones. So we will define the **default**[5] topology on $E \otimes W$ as the strongest linear topology such that the canonical bilinear map

$$E \times W \longrightarrow E \otimes W \qquad (\,(e,w) \mapsto e \otimes w)$$

is continuous. If such linear topology exists, then $E \otimes W$ equipped with this topology is clearly the categorical tensor product $E \boldsymbol{\otimes} W$, i.e., there is a natural isomorphism (2.2).

Proposition 12 *For any linearly topologized superspaces E and W, the default topology on $E \otimes W$ exists. A subsuperspace $L \subset E \otimes W$ is open in the default topology iff for any $e \in E$ and $w \in W$ there exist an open subsuperspace U of E and an open subsuperspace U' of W such that*

$$e \otimes U' + U \otimes w + U \otimes U' \subset L. \blacksquare$$

[5] I would prefer the name "tensor product topology" but this name is already reserved by Bourbaki :((for another linear topology on $E \otimes W$ (the definition in Exc. 28 in Ch.III, § 2 of [5], reproduced below).

By the way, the counterpart of default topology for the tensor product of locally convex spaces is called, strangely enough, *projective* topology (see e.g. [20]), being in fact a kind of *inductive* one.

In what follows $E \circledast W$ denotes the tensor product of E and W equipped with default topology, whereas $E \widehat{\circledast} W$ denotes the completion of $E \circledast W$ with respect to the default topology.

Another good property of categorical tensor product is that it respects arbitrary direct sums:

Proposition 13 *For any family $\{W_i\}_{i\in I}$ of linearly topologized superspaces and any linearly topologized superspace E there is a natural isomorphism*

$$E \circledast \bigoplus_{i\in I} W_i \approx \bigoplus_{i\in I} (E \circledast W_i)\,. \tag{2.7}$$

But just this "good" property prevents us from using categorical tensor product to approximate the superspace of linear maps. Because Prop. 13 implies immediately that, for discrete superspaces E and W, the superspace $E^* \circledast W$ is complete (as direct sum of a number of copies of the complete superspace E^*). And the image of this superspace in $\mathfrak{L}(E, W)$ (see Prop. 14 below) consists of the maps of finite rank only, so there is no way to approximate maps of infinite rank via completing $E^* \circledast W$ – it is complete already.

So we are forced to consider another topology on $E \otimes W$ (defined in Exc. 28 in Ch.III, § 2 of [5]). It has a base of open neighborhoods of 0 consisting of all superspaces $U \otimes W + E \otimes U'$, where U is an open subsuperspace of E and U' is an open subsuperspace of W. Bourbaki calls this topology the **tensor product topology** but, in my opinion, the name **projective** topology is more appropriate, because the completion of $E \otimes W$ in this topology coincides with the projective limit $\lim(E/U_\alpha \otimes W/U'_\beta)$, where U_α, resp. U'_β run over open subspaces of E, resp. of W. We will denote $E \otimes_{\rm p} W$ or, by abuse of notations, $E \otimes W$ the tensor product of E and W equipped with the projective topology.

We will need this definition only when W is discrete and E is either discrete or linearly compact (or vice versa). In the first case, $E \otimes_{\rm p} W$ is discrete again, hence, complete. In the second case ($E = V^*$), subsuperspaces $U \otimes W$ form a base of open neighborhoods of 0 and $E \otimes_{\rm p} W$ is not, generally speaking, either discrete, or linearly compact, or complete.

The completion of $E \otimes_{\rm p} W$ will be denoted $E \widehat{\otimes}_{\rm p} W$. The importance of this topology is caused by the following

Proposition 14 *Let E and W be linearly topologized superspaces. Let α be the canonical linear inclusion map*

$$\alpha : W \otimes E^* \longrightarrow \mathfrak{L}(E, W) \quad (\alpha(w \otimes f)e \stackrel{\rm def}{=} wf(e) \textit{ for any } e \in E,\ w \in W,\ f \in E^*). \tag{2.8}$$

Then:
(a) *The set of continuous linear maps from $\mathfrak{L}(E, W)$ belonging to image of α coincides with the set of continuous linear maps of* **finite rank***, i.e. maps with finite-dimensional image.*
(b) *α is continuous in projective topology on $W \otimes E^*$;*
(c) *Moreover, the projective topology on $W \otimes E^*$ coincides with the topology induced from the lpco topology of $\mathfrak{L}(E, W)$ along α.*
(d) *The set $\alpha(W \otimes E^*)$ is dense in $\mathfrak{L}(E, W)$. If, moreover, E is lpcg and W is complete then α extends by continuity to an isomorphism:*

$$\widehat{\alpha} : W \widehat{\otimes} E^* \approx \mathfrak{L}(E, W). \tag{2.9}$$

In what follows, we will identify topological superspace $\mathcal{L}(E, W)$ with $W \widehat{\otimes} E^*$ (or with $E^* \widehat{\otimes} W$) via the isomorphism (2.9), which clearly exists if both E and W are discrete superspaces.

Proof Heading (a) is evident.

The inverse image under α of a neighborhood $L(C, U)$ of 0 in $\mathcal{L}(E, W)$ is, clearly, $W \otimes C^\perp + U \otimes E^*$, so that α is continuous and heading (b) is valid. But the set of these inverse images is a base of neighborhoods of 0 in $W \otimes E^*$ by definition of the tensor product topology, so (c) is also valid.

Prove now heading (d). Let $f \in \mathcal{L}(E, W)$. We are to prove that for any linearly precompact closed subsuperspace C of E and any open subsuperspace U of W there exists $f_0 \in \mathcal{L}(E, W)$, such that $f - f_0 \in L(C, U)$ and f_0 is of finite rank.

Consider the subsuperspace $C \bigcap f^{-1}(U) \subset E$. It is of finite codimension in C, because $f^{-1}(U)$ is open and C is linearly precompact. Let $e_1, \dots, e_n$ be a basis in an algebraic complement of $C \bigcap f^{-1}(U)$ in C. The $\Bbbk$-span $\Bbbk\langle e_1, \dots, e_n \rangle$ is closed in C and E and its intersection with $f^{-1}(U)$ is 0. The counterpart of Hahn–Banach theorem for linearly topologized superspaces (see "Erweiterungszatz" on p. 89 of [14]) implies, that there exist continuous linear functionals $\varphi_1, \dots \varphi_n \in E^*$ such that for any $1 \leq i, j \leq n$ one has: $\varphi_i(e_j) = \delta_{ij}$ and, moreover, $f^{-1}(U) \subset \operatorname{Ker} \varphi_i$ for any i. One easily checks then that the finite rank map

$$f_0 := \alpha(\sum_i f(e_i) \otimes \varphi_i)$$

satisfies the desired condition.

The second part of heading (d) follows directly from Prop. 6. ■

Remark. The theory of linearly topologized superspaces is a "toy" counterpart of the theory of locally convex spaces (at least of this part of the latter theory, which is based on Hahn-Banach theorem). But the first part of heading (d) of Prop. 14 is the only result I know of, which has a "toy" proof, whereas its locally convex counterpart (denseness of $E^* \otimes V$ in $\mathcal{L}(E, V)$ equipped with compact-open topology) is a very difficult problem to solve. In fact, it was yet the open problem in 1971 (see e.g. [20]). I do not know whether this conjecture was proved since then.

2.5 Topological bases. The next thing to do is, given some basis in a (discrete) superspace E, to describe some kind of "dual basis" in E^*. Unfortunately, if E is infinite-dimensional, there is no *algebraic* basis which may be called dual to the original basis in E. Just because the algebraic dimension of E^* is then strictly *bigger* than that of E: $\dim E^* = 2^{\dim E}$. Nevertheless, in this situation there is an adequate replacement of algebraic basis, namely the *topological* basis of E^*.

Let E be a topological superspace. A family $P = \{P_i\}_{i \in I}$ of elements of E is called a **topological basis** of E (or **continuous basis** of E in the terminology of [14]) if any $e \in E$ has the unique representation $e = \sum_{i \in I} c_i P_i$ as a *topological* sum and, besides, for any $i \in E$, the map $\pi_i : E \longrightarrow \Bbbk$ $(e \mapsto c_i)$ is continuous.

For a discrete superspace E, the notions of topological and algebraic bases clearly coincide.

On the contrary, in E^*, there does not exists an algebraic basis which is, simultaneously, a topological one. Nevertheless ([14]):

Proposition 15 *Let $P = \{P_i\}_{i\in I}$ be a basis in E.*

Let the family $X = \{X^i\}_{i\in I}$ in E^* be defined by the equations:

$$X^i(P_j) = \delta^i{}_j\ . \tag{2.10}$$

Then X is a topological basis in E^*. ■

The topological basis X will be called **dual** to the basis P; in what follows we will write simply "dual basis" instead of "dual topological basis".

S. Köthe claimed in [14] that it is not known, for a general linearly topologized space E, whether or not there exists a topological basis of E. We will only use topological bases for the superspaces dual to discrete superspaces, and, more generally, for the superspaces $\mathcal{L}(E, W) \approx W\widehat{\otimes}E^*$ for discrete E and W.

For E^* such bases always exist due to Prop. 15, and, moreover, ([14]):

Proposition 16 *Any topological basis of E^* is dual to some basis of E.* ■

And what about $\mathcal{L}(E, W)$ and $W\widehat{\otimes}E^*$? The following Proposition generalizes Prop. 15 to this case.

Proposition 17 *Let $\{P_i\}_{i\in I}$ be a basis in E, $\{X^i\}_{i\in I}$ be a dual basis in E^* and let $\{P'_k\}_{k\in K}$ be a basis in W.*
(a) *Let the family $X = \{X^i_k : E \longrightarrow W\}_{i\in I, k\in K}$ be defined by equations:*

$$X^i_k(P_j) = \delta^i{}_j P'_k\ . \tag{2.11}$$

Then X is a topological basis in $\mathcal{L}(E, W)$. The inverse image of this basis in $W\widehat{\otimes}E^*$ under the isomorphism $\widehat{\alpha} : W\widehat{\otimes}E^* \approx \mathcal{L}(E, W)$ (see (2.9) in Prop. 14 (d)) *is a topological basis*

$$\widehat{\alpha}^{-1}(X) = \{\widehat{\alpha}^{-1}(X^i_k) = P'_k \otimes X^i\}_{i\in I, k\in K}. \tag{2.12}$$

(b) *The formal sum*

$$\sum_{i,k} c^k_i X^i_k \qquad (\textit{resp.}\ \sum_{i,k} c^k_i P'_k \otimes X^i)$$

is a topological sum representing some element of $\mathcal{L}(E, W)$ (resp. of $W\widehat{\otimes}E^*$) *iff, for any $i \in I$, there exists only finite number of $k \in K$ such that $c^k_i \neq 0$.* ■

Corollary 18 **(a)** *For any element $x \in W\widehat{\otimes}E^*$ there exists a unique family $\{w_i\}_{i\in I}$ of elements of W (resp. a unique family $\{f^i\}_{i\in I}$ of elements of E^*) such that x represents as a topological sum*

$$x = \sum_{i\in I} w_i \otimes X^i = \sum_{i\in I} P'_i \otimes f^i. \tag{2.13}$$

(b) *For any family $\{w_i\}_{i\in I}$ of elements of W the topological sum $\sum_{i\in I} w_i \otimes X^i \in W\widehat{\otimes}E^*$ exists.*
■

Remark. Heading (b) of Cor. 18 permits one to consider informally a basis $\{X^i\}_{i\in I}$ in E^* as a "virtual" basis of $W\widehat{\otimes}E^* \approx \mathcal{L}(E,W)$ "with coefficients in W". Meditate a bit over this informal interpretation :).

2.6 The superalgebra $S(E)^*$. Consider a superspace E as a commutative Lie superalgebra (with the zero bracket), let $S(E)^*$ be an E-module coinduced from trivial 1-dimensional representation of the trivial subalgebra $\mathfrak{h} = 0$ of E. Then $S(E)^*$ has a natural structure of a commutative associative superalgebra with unit element and multiplication defined by eqs. (1.12). How this superalgebra looks like?

First of all, the topological superspace E^* may be identified with a closed subsuperspace of $S(E^*)$ via continuous injective map

$$\iota : E^* = S^1(E)^* \hookrightarrow \bigoplus_{n\in\mathbb{N}} S^n(E)^* \longrightarrow \prod_{n\in\mathbb{N}} S^n(E)^* \approx (\bigoplus_{n\in\mathbb{N}} S^n(E))^* = S(E)^*. \tag{2.14}$$

In particular, E^*, considered as a subsuperspace of $S(E)^*$, coincides with the annihilator of the subsuperspace $\bigoplus_{n\neq 1} S^n(E)$ of $S(E)$.

In what follows we identify E^* with a subsuperspace of $S(E^*)$, writing just f instead of $\iota(f)$ for $f \in E^*$.

Proposition 19 *For any homogeneous $f_1, \ldots, f_n \in E^* \subset S(E)^*$ and any homogeneous $e_1, \ldots, e_m \in E$, we have*

$$(f_1 \ldots f_m)(e_n \ldots e_1) = \delta_{mn} \sum_{\sigma\in\mathfrak{S}_n} (-1)^{\varepsilon(\sigma_{\mathrm{odd}})} f_{\sigma(n)}(e_n) \ldots f_{\sigma(1)}(e_1). \tag{2.15}$$

Proof For a finite well ordered set S and a natural number i such that $i < \mathrm{card}(S)$, the image of i under the only order preserving isomorphism $\mathrm{card}(S) \longrightarrow S$ will be denoted S_i. For any nontrivial partition of a well ordered set $n = \{0, \ldots, n-1\} = S \cup S'$, where $S' = n \setminus S$, denote by $\sigma(S)$ the permutation

$$(0, \ldots, n-1) \mapsto (S_0, \ldots, S_{\mathrm{card}(S)-1}, S'_0, \ldots, S'_{\mathrm{card}(S')-1}).$$

For trivial partitions ($S = \emptyset$ or $S = n$), let $\sigma(S)$ be the identity permutation $\mathbb{1}_n$.

In these notations, we can express the action of comultiplication Δ on any element $e_0 \ldots e_{n-1}$ of $S^n(E)$ as follows:

$$\Delta(e_0 \ldots e_{n-1}) = \sum_{S\subset n} (-1)^{\varepsilon\sigma(S)_{\mathrm{odd}}} e_{S_0} \ldots e_{S_{\mathrm{card}(S)-1}} \otimes e_{S'_0} \ldots e_{S'_{\mathrm{card}(S')-1}}. \tag{2.16}$$

The expression (2.16) clearly follows from the fact that Δ is morphism of superalgebras:

$$\Delta(e_0 \ldots e_{n-1}) = \Delta(e_0) \ldots \Delta(e_{n-1}) = (e_0 \otimes 1 + 1 \otimes e_0) \ldots (e_{n-1} \otimes 1 + 1 \otimes e_{n-1}).$$

Now eq. (2.15) can be proved by induction on n if one applies the definition (1.12) to $f_1 \ldots f_n = f_1(f_2 \ldots f_n)$. ■

Corollary 20 *The choice of a basis P in E establishes an isomorphism of topological superspaces*

$$S(E)^* \approx \Bbbk[[X]]. \tag{2.17}$$

Moreover, this isomorphism respects multiplication, i.e. is an isomorphism of superalgebras.

We will identify $S(E)^*$ and $\Bbbk[[X]]$ via this isomorphism. In fact the isomorphism (2.17) implies that $S(E)^*$ is a complete *topological* superalgebra, because $\Bbbk[[X]]$ is (see Exc. (22) to § 2 of Ch. 3 of [5]).

The following proposition allows one to explicitly calculate the expression for $T^{\mathfrak{g}}_{\mathfrak{h}}(g)$ for any $g \in \mathfrak{g}$ in any given pair of homogeneous bases: $\{P_i\}_{i\in I}$ in $\mathfrak{g}_-$ and the dual basis $\{X^i\}_{i\in I}$ in $\mathfrak{g}_-^*$:

Proposition 21 *The isomorphism* (2.17) *establishes the "Fourier transform" of the operators* $\frac{\partial}{\partial X^i}$ *and left multiplications by* P_i *in* $S(E)$*:*

$$P_i^* = (-1)^{|X^i|}\frac{\partial}{\partial X^i}\ , \quad \left(\frac{\partial}{\partial P_i}\right)^* = X^i, \tag{2.18}$$

where we have identified P_i^* *and* $\left(\frac{\partial}{\partial P_i}\right)^*$ *with their images under the isomorphism* (2.17)*.*

Proof The validity of relations (2.18) on the dense subsuperspace $\Bbbk[X] \approx S(\Bbbk\langle X\rangle) \subset S(E)^*$ is verified straightforwardly by applying Eq. (2.15). Clearly, the operators P_i^* and $\left(\frac{\partial}{\partial P_i}\right)^*$ are continuous, so that eqs. (2.18) can be extended to hold on the whole $S(E)^*$. ■

Define now the bilinear map

$$S(E)^*\widehat{\otimes}S(E) \xrightarrow{\mathcal{D}} \mathrm{Diff}(S(E)^*)\ , \tag{2.19}$$

where $\mathrm{Diff}(S(E)^*)$ is the space of differential operators of finite order in $S(E)^*$. This map participates in the expression for the action of $\mathfrak{g}$ on $T^{\mathfrak{g}}_{\mathfrak{h}}(V)$. The map $\mathcal{D}$ is defined as follows:

$$\mathcal{D}(f\otimes e)\varphi \overset{\mathrm{def}}{=} f\cdot(e*\varphi) \quad (\varphi, f \in S(E)^*, \quad e \in E). \tag{2.20}$$

Here e^* is the operator dual to the operator of left multiplication by e in $S(E)$. Proposition 21 above implies that if $e = e_1\dots e_n$, where $e_i \in E$, then

$$e^* = (e_1\dots e_n)^* = e_1^*\dots e_n^*$$

is a product of differentiations in $S(E)^*$.

In particular, $P_i^* = (-1)^{|X^i|}\frac{\partial}{\partial X^i}$, so that one has the following expression for the action of $\mathcal{D}$ in the basis $P = \{P_i\}_{i\in I}$ on E and the dual basis $X = \{X^i\}_{i\in I}$ on E^*:

$$\mathcal{D}(f(X)\otimes P_{i_1}\dots P_{i_n}) = (-1)^{|X^{i_1}|+\dots+|X^{i_n}|}f(X)\frac{\partial}{\partial X^{i_1}}\dots\frac{\partial}{\partial X^{i_n}} \tag{2.21}$$

or, symbolically:

$$\mathcal{D}(\varphi(X,P)) = \varphi(X, (-1)^{|X|}\frac{\partial}{\partial X}). \tag{2.22}$$

Another bilinear map

$$S(E)\widehat{\otimes}S(E)^* \xrightarrow{\mathcal{D}_\infty} \mathrm{Diff}_\infty(S(E))\ , \tag{2.23}$$

where $\mathrm{Diff}_\infty(S(E))$ is the space of differential operators of *infinite* order with *polinomial* coefficients on $\widehat{S}(E)$, participates in the expression for the action of $\mathfrak{g}$ on $I_{\mathfrak{h}}^{\mathfrak{g}}(V)$. The map $\mathcal{D}_\infty$ is defined as follows:

$$\mathcal{D}_\infty(e\otimes f)v \stackrel{\mathrm{def}}{=} e\cdot(f^*v) \quad (e, v\in S(E),\quad f\in S(E)^*). \tag{2.24}$$

The counterparts of (2.21) and (2.22) are:

$$\mathcal{D}_\infty(P_{i_1}\dots P_{i_n}\otimes f(X)) = P_{i_1}\dots P_{i_n} f(\frac{\partial}{\partial P}) \tag{2.25}$$

or, symbolically:

$$\mathcal{D}_\infty(\varphi(P,X)) = \varphi(P,\frac{\partial}{\partial P}). \tag{2.26}$$

The reader who does not believe in existence of differential operators of infinite order may interpret (2.24) just as *definition* of the space $\mathrm{Diff}_\infty(S(E))$:-).

As to the space $\mathrm{Diff}(S(E)^*)$ (the existence of which is of no doubt), one can prove that $\mathcal{D}$ is a $\Bbbk$-linear *isomorphism*. We need, in fact, only the following easy result:

Proposition 22 *The restriction of $\mathcal{D}$ on the subsuperspace $S(E)^*\widehat{\otimes}E$ establishes an isomorphism of this subsuperspace with the superspace of all differentiations of the superalgebra $S(E)^* \approx S(E)^*$.*

Proof For any differentiation d, we have $d = \sum_i d(X^i)\frac{\partial}{\partial X^i}$. ■

3 Main theorem.

Let

$$\mathfrak{g} = \mathfrak{g}_- \oplus \mathfrak{h} \tag{3.1}$$

be a decomposition of a Lie superalgebra $\mathfrak{g}$ into the direct sum of subsuperspaces.

In what follows we will identify via Props. 14 (d) the topological superspace

$$T_{\mathfrak{h}}^{\mathfrak{g}}(V) \approx \mathcal{L}(S(\mathfrak{g}_-), V) \tag{3.2}$$

with the topological superspace

$$\widehat{S}(\mathfrak{g}_-^*, V) \stackrel{\mathrm{def}}{=} S(\mathfrak{g}_-)^*\widehat{\otimes}V \tag{3.3}$$

of **V-valued formal polynomials of $\mathfrak{g}_-^*$**.

Let $\varepsilon_{\bar{0}}$, resp. $\varepsilon_{\bar{1}}$, be an even, resp. odd, indeterminate. Let the supercommutative superalgebra A be defined as $A = \Bbbk[\varepsilon_{\bar{0}}, \varepsilon_{\bar{1}}]/(\varepsilon_{\bar{0}}^2)\otimes S(\mathfrak{g}_-)^*$. Fix some basis $\{P_i\}_{i\in I}$ of $\mathfrak{g}_-$ and the dual basis $\{X^i\}_{i\in I}$ of $\mathfrak{g}_-^*$. Then, as follows from Corollary 20 in sect. 2.6, one can identify the superalgebra A with the superalgebra $\Bbbk[[\varepsilon_{\bar 0},\varepsilon_{\bar 1},X]]/(\varepsilon_{\bar 0}^2)$ of formal polynomials in ε and X.

Let g be a homogeneous element of $\mathfrak{g}_-$, and $\varepsilon=\varepsilon_i$ if $g\in{}_i\mathfrak{g}_-$. Let

$$XP\stackrel{\rm def}{=}\sum_i X^i\otimes P_i\in S(\mathfrak{g}_-)^*\widehat{\otimes}\mathfrak{g}_-\subset A\widehat{\otimes}U(\mathfrak{g}).$$

Note that XP is just the image of the identity map $\mathrm{Id}_{\mathfrak{g}_-}$ under the canonical morphism $\mathrm{Hom}(\mathfrak{g}_-,\mathfrak{g}_-)\approx\mathfrak{g}_-^*\widehat{\otimes}\mathfrak{g}_-\subset S(\mathfrak{g}_-)^*\widehat{\otimes}\mathfrak{g}_-$, so it does not depend of the choice of the basis $\{P_i\}_{i\in I}$ and the dual basis $\{X^i\}_{i\in I}$.

In what follows we will often write just X^iP_i instead of $\sum_i X^i\otimes P_i$, etc., omitting $\otimes$ and assuming, (as is common practice in physical literature) that summation is taken over repeated pairs of indices (subscript and superscrip), unless the contrary is stated explicitly.

Consider an element

$$\exp(\varepsilon g)\exp(XP)\in A\widehat{\otimes}U(\mathfrak{g}).$$

The exponents above exist, evidently, within the superalgebra $A\widehat{\otimes}U(\mathfrak{g})$.

Let G_- and H be morphisms of superspaces

$$G_-:\mathfrak{g}\longrightarrow S(\mathfrak{g}_-)^*\widehat{\otimes}\mathfrak{g}_-,\quad g\mapsto G_-(g), \tag{3.4}$$

$$H:\mathfrak{g}\longrightarrow S(\mathfrak{g}_-)^*\widehat{\otimes}\mathfrak{h}\quad g\mapsto H(g). \tag{3.5}$$

Theorem 23 *The following properties* (a)–(c) *of maps* (3.4), (3.5) *are equivalent:*

(a) *For any $g\in\mathfrak{g}$, one has the equality of formal power series:*

$$e^{\varepsilon g}e^{XP}=e^{XP+\varepsilon G_-(g)}e^{\varepsilon H(g)} \tag{3.6}$$

(b) *For any $\mathfrak{h}$-module $\rho:\mathfrak{h}\longrightarrow\mathrm{End}(V)$, the representation $T_{\mathfrak{h}}^{\mathfrak{g}}(g)$ takes the following form in the realization* (3.3)*:*

$$T_{\mathfrak{h}}^{\mathfrak{g}}(g)=\mathcal{D}(\check{G}_-(g))\otimes\mathbb{1}_V+\widehat{\rho}_{S(\mathfrak{g}_-)^*}(\check{H}(g))\ ,\quad g\in\mathfrak{g}, \tag{3.7}$$

where $\mathcal{D}$ is defined by Eqs. (2.19)–(2.20) *and $\widehat{\rho}_A$ for a linearly topologized commutative superalgebra A is defined in Appendix A* (see the definition after (A.18))*; $\check{G}_-$ and $\check{H}$ means G_-, resp. H twisted by main antiautomorphism of the superalgebra $S(\mathfrak{g}_-)^*$, i.e. with X replaced by $-X$ in the corresponding formal series.*

(b′) *The equalities* (3.7) *are valid for the adjoint representation of $\mathfrak{h}$;*

(c) *The representation $I_{\mathfrak{h}}^{\mathfrak{g}}(g)$ takes the following form in the realization (1.21):*

$$I_{\mathfrak{h}}^{\mathfrak{g}}(g)=\mathcal{D}_\infty(IG_-(g))\otimes\mathbb{1}_V+\rho_{S(\mathfrak{g}_-)}(H(g))\ ,\quad g\in\mathfrak{g}, \tag{3.8}$$

where $\mathcal{D}_\infty$ is defined by Eqs. (2.19)–(2.20) and $I:S(\mathfrak{g}_-)^*\widehat{\otimes}\mathfrak{g}_-\longrightarrow\mathfrak{g}_-\widehat{\otimes}S(\mathfrak{g}_-)^*$ is the canonical isomorphism of permutation;

(c′) *The equalities* (3.8) *are valid for the adjoint representation of $\mathfrak{h}$.*

Proof

(a) $\Longrightarrow$ (c') Let

$$G_{-}(g) = \varphi^i(X,g)P_i, \qquad H(g) = h(X,g); \tag{3.9}$$

so that

$$e^{\varepsilon g}e^{XP} = e^{(X+\varepsilon\varphi(X,g))P}e^{\varepsilon h(X,g)}\ . \tag{3.10}$$

Expanding in powers of X gives:

$$e^{\varepsilon g}e^{XP} = \sum_n \frac{1}{n!}(1+\varepsilon g)(XP)^n = \sum_n \frac{1}{n!}(1+\varepsilon g)X^{\mu_1}P_{\mu_1}\cdots X^{\mu_n}P_{\mu_n}$$

and

$$e^{(X+\varepsilon\varphi(X,g))P}e^{\varepsilon h(X,g)} = \sum_n \frac{1}{n!}((X+\varepsilon\varphi(X,g))P)^n(1+\varepsilon h(X,g));$$

i.e., linear on ε terms in eq. (3.6) will give the following equality

$$\sum_n \frac{1}{n!}g(XP)^n = \sum_n \frac{1}{n!}\left\{\sum_{k=1}^n (XP)^{k-1}\varphi P(XP)^{n-k} + (XP)^n h(X,g)\right\}. \tag{3.11}$$

Further, $(XP)^n = (XP)^{\circ n}$, $\sum_{k=1}^n (XP)^{k-1}\varphi P(XP)^{n-k} = n(\varphi P)\circ(XP)^{\circ(n-1)}$, where $\circ$ means symmetric composition in $U(A\otimes\mathfrak{g}_{-})$ defined by eqn.(1.18).

Let

$$\begin{aligned}\varphi^i(X,g) &= \textstyle\sum_{\nu_1\le\overset{k}{\cdots}\le\nu_k} a^i_{\nu_1\ldots\nu_k}X^{\nu_1}\cdots X^{\nu_k} \qquad (a^i_{\nu_1\ldots\nu_k}\in\Bbbk, |X^{\nu_1}|+\ldots|X^{\nu_k}|+|\varepsilon| = |\varphi^i|),\\ h(X,g) &= \textstyle\sum_{\nu_1\le\overset{k}{\cdots}\le\nu_k} h_{\nu_1\ldots\nu_k}X^{\nu_1}\cdots X^{\nu_k} \qquad (h_{\nu_1\ldots\nu_k}\in\mathfrak{h})\end{aligned}$$

be the expansions of φ^i and h into formal power series.

Since $|\varepsilon|+|\varphi^i|+|P_i| = 0$ it follows that

$$\varphi^i P_i = (-1)^{|\varphi^i||P_i|}P_i\varphi^i = (-1)^{(1+|g|)|P_i|}P_i\varphi^i,$$

and hence we obtain:

$$\sum_n \frac{1}{n!}g(XP)^n =$$
$$\sum_n \frac{1}{(n-1)!}\sum\left\{(-1)^{|\varphi^i|P_i|}P_i\circ a^i_{\nu_1\ldots\nu_k}X^{\nu_1}\cdots X^{\nu_k}(XP)^{\circ(n-1)} + h_{\nu_1\ldots\nu_k}\circ X^{\nu_1}\cdots X^{\nu_k}(XP)^{\circ n}\right\},$$

where $h_{\nu_1\ldots\nu_k}\circ\ldots$ means that $h_{\nu_1\ldots\nu_k}$ must be placed on the right after all operations are performed.

Further, we have:

$$\frac{\partial}{\partial P_{\nu_1}}\cdots\frac{\partial}{\partial P_{\nu_k}}(XP)^m=\frac{m!}{(m-k)!}X^{\nu_1}\cdots X^{\nu_k}(XP)^{m-k} \qquad (3.13)$$

or, equivalently:

$$X^{\nu_1}\cdots X^{\nu_k}(XP)^l=\frac{l!}{(k+l)!}\frac{\partial}{\partial P_{\nu_1}}\cdots\frac{\partial}{\partial P_{\nu_k}}(XP)^{k+l}. \qquad (3.14)$$

Hence,

$$\begin{aligned}
&\sum_n\frac{1}{n!}g(XP)^{\circ n}=\\
&\sum_n\frac{1}{(n-1)!}\sum(-1)^{|\varphi^i||P_i|}P_i\circ a^i_{\nu_1\dots\nu_k}\frac{(n-1)!}{(k+n-1)!}\frac{\partial}{\partial P_{\nu_1}}\cdots\frac{\partial}{\partial P_{\nu_k}}(XP)^{k+n-1}\\
&+\sum_n\frac{1}{n!}\sum h_{\nu_1\dots\nu_k}\circ\frac{n!}{(k+n)!}\frac{\partial}{\partial P_{\nu_1}}\cdots\frac{\partial}{\partial P_{\nu_k}}(XP)^{k+n}\\
&=\sum_n\frac{1}{n!}\Big\{(-1)^{|\varphi^i||P_i|}P_i\circ\sum a^i_{\nu_1\dots\nu_k}\frac{\partial}{\partial P_{\nu_1}}\cdots\frac{\partial}{\partial P_{\nu_k}}\\
&+\sum h_{\nu_1\dots\nu_k}\circ\frac{\partial}{\partial P_{\nu_1}}\cdots\frac{\partial}{\partial P_{\nu_k}}\Big\}\circ(XP)^{\circ n}\\
&=\sum_n\frac{1}{n!}\Big\{\sum_i(-1)^{|\varphi^i||P_i|}P_i\varphi^i(\frac{\partial}{\partial P},g)+h(\frac{\partial}{\partial P},g)\Big\}\circ(XP)^{\circ n}.
\end{aligned}$$

Taking into account that $(-1)^{|\varphi^i||P_i|}=(-1)^{(1+|g|)|P_i|}$, we immediately get from the last equality the following expression for the operator $L(g)$ of left shift by g in the space

$$U(\mathfrak{g})\otimes_{U(\mathfrak{h})}\mathfrak{h}\approx S(\mathfrak{g})\otimes\mathfrak{h}$$

as a differential operator (of infinite order, generally speaking):

$$L(g)=(-1)^{(1+|g|)|P_i|}P_i\varphi^i(\frac{\partial}{\partial P},g)+h(\frac{\partial}{\partial P},g). \qquad (3.15)$$

This proves (c′).

The latter expression for a "generic" induced representation (i.e., induced from the adjoint representation of $\mathfrak{h}$) immediately implies that for any representation ρ of $\mathfrak{h}$ in V the operator $I^{\mathfrak{g}}_{\mathfrak{h}}(g)$ of the action of g in the superspace of induced representation

$$I(V)\equiv U(\mathfrak{g})\otimes_{U(\mathfrak{h})}V \qquad (3.16)$$

can be represented as

$$\boxed{I^{\mathfrak{g}}_{\mathfrak{h}}(g)=(-1)^{(1+|g|)|P_i|}P_i\varphi^i(\frac{\partial}{\partial P},g)+\rho(h(\frac{\partial}{\partial P},g)),} \qquad (3.17)$$

i.e., (c′) $\implies$ (c).

It is clear, that starting from (3.15) one can go backwards to get the equality (3.11), i.e., the implication (c′) $\implies$ (a) is also valid.

Now the expression (3.7) for the operators of coinduced representation follows from that for the induced ones by duality, if only the superspace V is finite-dimensional, due to an easily established (see [3]) canonical isomorphism

$$I_{\mathfrak{h}}^{\mathfrak{g}}(V)^* \approx T_{\mathfrak{h}}^{\mathfrak{g}}(V^*). \tag{3.18}$$

To extend the implication (a) $\implies$ (b) to the case where V has infinite-dimension (and is considered together with the discrete topology on it), we have to find an expression for the right action of an arbitrary homogeneous element of $\mathfrak{g}$ on $U(\mathfrak{g})$ in the corresponding bases.

Let $\varphi\colon S(\mathfrak{g}_-) \longrightarrow V$ be a linear map. Let $g \in \mathfrak{g}$ and $g' \in U(\mathfrak{g}_-) \subset U(\mathfrak{g})$ be arbitrary. Then the isomorphism (1.20) implies that there exists a decomposition

$$g'g = \sum_i h_i g_i \quad (h_i \in U(\mathfrak{h}),\ g_i, \in U(\mathfrak{g}_-))$$

such that

$$(T_{\mathfrak{h}}^{\mathfrak{g}}(g)f)(g') = (-1)^{|g|(|f|+|g'|)} \sum_i (-1)^{|h_i||f|} \rho(h_i) f(g_i).$$

So we must find the expression for the right "action" of $\mathfrak{g}$ on $U(\mathfrak{g}_-) \approx S(\mathfrak{g}_-)$ first.

Inverting the equality (3.6) one gets

$$e^{-XP} e^{-\varepsilon g} = e^{-\varepsilon h(X,g)} e^{-(X+\varepsilon\varphi(X,g))P}\ ,$$

or, replacing $\varepsilon \mapsto -\varepsilon$, $X \mapsto -X$, we get

$$e^{XP} e^{\varepsilon g} = e^{\varepsilon h(-X,g)} e^{(X+\varepsilon\varphi(-X,g))P}\ . \tag{3.6′}$$

Now, the counterpart to eq. (3.11) is:

$$\begin{aligned}&\sum_n \frac{1}{n!}(XP)^n g = \\ &\sum_n \frac{1}{n!}\left\{\sum_{k=1}^{n}(XP)^{k-1}\varphi^i(-X,g)P_i(XP)^{n-k} + (XP)^n h(-X,g)\right\} \\ &= \sum_n \frac{1}{n!}\left\{n\varphi^i(-X,g)P_i(XP)^{\circ(n-1)} + h(-X,g)(XP)^{\circ n}\right\}.\end{aligned} \tag{3.11′}$$

Expanding both $\varphi^i(-X,g)$ and $h(-X,g)$ in powers of X gives the following result:

$$\begin{aligned}&\sum_n \frac{1}{n!}(XP)^{\circ n} g = \dots \\ &= \sum_n \frac{1}{n!}\left\{\sum_i (-1)^{(1+|g|)|P_i|} P_i \varphi^i(-\frac{\partial}{\partial P}, g) + h(-\frac{\partial}{\partial P}, g)\right\} \circ (XP)^{\circ n}.\end{aligned}$$

In other words, for the operator $R(g)$ of right multiplication of $U(\mathfrak{g}_-) \approx S(\mathfrak{g}_-)$ by g we get the following expression:

$$R(g) = (-1)^{(1+|g|)|P_i|} P_i \varphi^i(-\frac{\partial}{\partial P}, g) + h(-\frac{\partial}{\partial P}, g). \tag{3.15'}$$

Recall now that the action of an element $f \otimes v \in S(\mathfrak{g}_-)^* \widehat{\otimes} V$ considered as an element of $\mathfrak{L}(S(\mathfrak{g}_-), V)$ is defined by

$$(f \otimes v)g = (-1)^{|v|+|g|} f(g)v \tag{3.19}$$

(see Prop. (14)). So, we have:

$$\begin{aligned}
T^{\mathfrak{g}}_{\mathfrak{h}}(g)(f \otimes v)(g') =&(-1)^{|g|(|f|+|v|+|g'|)}(f \otimes v)(g'g)\\
=&(-1)^{|g|(|f|+|v|+|g'|)+|g||g'|}(f \otimes v)(R(g)g')\\
=&(-1)^{|g|(|f|+|v|)}(f \otimes v)((-1)^{(1+|g|)|P_i|} P_i \varphi^i(-\frac{\partial}{\partial P}, g)g' + h(-\frac{\partial}{\partial P}, g)g')\\
=&(-1)^{|g|(|f|+|v|)+|v|(|g'|+|g|)} f((-1)^{(1+|g|)|P_i|} P_i \varphi^i(-\frac{\partial}{\partial P}, g)g' + h(-\frac{\partial}{\partial P}, g)g')v\\
=&(-1)^{|g||f|+|v||g'|+|g||f|}((-1)^{(1+|g|)|P_i|} P_i \varphi^i(-\frac{\partial}{\partial P}, g) + h(-\frac{\partial}{\partial P}, g))^* f(g')v\\
=&(-1)^{|v||g'|}(\varphi^i(-X, g)\frac{\partial}{\partial X^i} + h(-X, g))f(g')v\\
=&((-1)^{|X^i|}\varphi^i(-X, g))\frac{\partial}{\partial X^i} + h(-X, g)))(f \otimes v)(g').
\end{aligned}$$

I.e., the action $T^{\mathfrak{g}}_{\mathfrak{h}}(g)$ on elements of finite rank in $\mathfrak{L}(S(\mathfrak{g}_-), V)$ is given by

$$T^{\mathfrak{g}}_{\mathfrak{h}}(g) = (-1)^{|X^i|}\varphi^i(-X, g))\frac{\partial}{\partial X^i} + h(-X, g). \tag{3.20}$$

But elements of finite rank are dense in $\mathfrak{L}(S(\mathfrak{g}_-), V)$ by heading (d) of Prop. 14, so that (3.20) is valid for any element of $\mathfrak{L}(\mathfrak{g}_-, V) = S(\mathfrak{g}_-)^* \widehat{\otimes} V$.

The equality (3.20) is equivalent to the particular case of (3.7) by (2.21) and (2.18), proving the implication (a) $\Longrightarrow$ (b′). Other implications are proved similarly to already proved. ■

The main question now is whether the maps G_- and H do exist. And how they look like if they exist.

4 The case where $\mathfrak{g}_-$ is a Lie superalgebra.

In this section, let $\mathfrak{g}_-$ be a Lie superalgebra. In this case one can easily prove the existence of maps (3.4) and (3.5) satisfying conditions of Theorem 23 as well as to find an explicit expression for G_- and H.

Let $\Pi_{\mathfrak{g}_-}$ and $\Pi_{\mathfrak{h}}$ be the projections of $\mathfrak{g}$ onto $\mathfrak{g}_-$ and $\mathfrak{h}$, respectively:

$$\Pi_{\mathfrak{g}_-} + \Pi_{\mathfrak{h}} = \mathbb{1}\ , \quad \Pi^2_{\mathfrak{g}_-} = \Pi_{\mathfrak{g}_-}\ , \quad \Pi^2_{\mathfrak{h}} = \Pi_{\mathfrak{h}}\ , \quad \Pi_{\mathfrak{g}_-}\Pi_{\mathfrak{h}} = \Pi_{\mathfrak{h}}\Pi_{\mathfrak{g}_-} = 0. \tag{4.1}$$

Since $\varepsilon^2 = 0$, we have for any $g \in \mathfrak{g}$:

$$\begin{aligned} e^{\varepsilon g} e^{XP} &= e^{XP} e^{-XP} e^{\varepsilon g} e^{XP} = e^{XP} \exp(e^{-\mathrm{ad} XP} \varepsilon g) = \\ & e^{XP} e^{\varepsilon \Pi_{\mathfrak{g}_-} e^{-\mathrm{ad} XP} g} e^{\varepsilon \Pi_{\mathfrak{h}} e^{-\mathrm{ad} XP} g} . \end{aligned}$$

The same identity $\varepsilon^2 = 0$ implies that (a well known fact)

$$e^A e^{\varepsilon B} = \exp\left(A - \varepsilon \frac{\mathrm{ad} A}{e^{-\mathrm{ad} A} - 1} B \right) \tag{4.2}$$

for any A and B in any associative algebra where the corresponding expressions make sense.

Hence, for any $g \in \mathfrak{g}$

$$e^{\varepsilon g} e^{XP} = e^{(X + \varepsilon \varphi(X,g))P} e^{\varepsilon h(X,g)} , \tag{4.3}$$

where $\varphi^i(X,g) \in \Bbbk[[X]]$ and $h(X,g) \in \Bbbk[[X]] \widehat{\otimes} \mathfrak{h}$ are defined by

$$\varphi^i(X,g) P_i = - \frac{\mathrm{ad} XP}{e^{-\mathrm{ad} XP} - 1} \mathbb{1} \otimes \Pi_{\mathfrak{g}_-} e^{-\mathrm{ad} XP} g ; \tag{4.4}$$

$$h(X,g) = \mathbb{1} \otimes \Pi_{\mathfrak{h}} e^{-\mathrm{ad} XP} g . \tag{4.5}$$

In other words, in this case, the conditions of heading a) of Theorem 23 are satisfied with

$$G_-(g) = \varphi^i(X,g) P_i, \quad H(g) = h(X,g)$$

so that representations (3.7), (3.8) for $T_{\mathfrak{h}}^{\mathfrak{g}}(g)$ and $I_{\mathfrak{h}}^{\mathfrak{g}}(g)$ are valid.

We note that if $\mathfrak{g}$ is a $\mathbb{Z}$-graded Lie superalgebra

$$\mathfrak{g} = \bigoplus_{-l}^{\infty} \mathfrak{g}_i \qquad [\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_{i+j} \quad i, j \in \mathbb{Z} \tag{4.6}$$

and

$$\mathfrak{h} = \sum_{n=0}^{\infty} \mathfrak{g}_n , \quad \mathfrak{g}_- = \sum_{n=-l}^{-1} \mathfrak{g}_n , \tag{4.7}$$

such that the superalgebra $\mathfrak{g}_-$ is *finite*-dimensional, then the infinite sums in the r.h.s. of Eqs. (4.4) and (4.5) become finite and the differential operators (3.7) for $g \in \mathfrak{g}_d$ have polynomial coefficients with degrees of the corresponding polynomials not exceeding $l + d$.

5 The general case.

It turns out that in general case, where the subsuperspace $\mathfrak{g}_-$ need not be a subsuperalgebra of $\mathfrak{g}$, the maps G_- and H satisfying eq. (3.6) exist as well. Here, in order to compactify the formulas we use the shorthand expressions $\Pi_{\mathfrak{g}_-}$ and $\Pi_{\mathfrak{h}}$ to denote $\mathbb{1} \otimes \Pi_{\mathfrak{g}_-}$ and $\mathbb{1} \otimes \Pi_{\mathfrak{h}}$,respectively.

Theorem 24 *For any homogeneous $g \in \mathfrak{g}$ the equality*

$$e^{\varepsilon g} e^{XP} = e^{(X+\varepsilon\varphi(X,g))P} e^{\varepsilon h(X,g)} \tag{5.1}$$

is valid, where

$$\varphi^i(X,g)P_i = -\left(\Pi_{\mathfrak{g}_-} \frac{e^{-\mathrm{ad}XP}-1}{\mathrm{ad}XP}\right)\Big|^{-1}_{A\widehat{\otimes}\mathfrak{g}_-} \Pi_{\mathfrak{g}_-} e^{-\mathrm{ad}XP} g \ ; \tag{5.2}$$

$$h(X,g) = \left(\Pi_{\mathfrak{h}} \frac{\mathrm{ad}XP}{e^{-\mathrm{ad}XP}-1}\right)\Big|^{-1}_{A\widehat{\otimes}\mathfrak{h}} \Pi_{\mathfrak{h}} \frac{\mathrm{ad}XP}{e^{-\mathrm{ad}XP}-1} e^{-\mathrm{ad}XP} g \ . \tag{5.3}$$

Proof For the l.h.s. of (5.1) we have due to the equality (4.2):

$$e^{\varepsilon g} e^{XP} = e^{XP} e^{e^{-\mathrm{ad}XP}\varepsilon g} = \exp\left(XP - \frac{\mathrm{ad}XP}{e^{-\mathrm{ad}XP}-1} e^{-\mathrm{ad}XP} \varepsilon g\right). \tag{5.4}$$

On the other hand, the application of eq. (4.2) to the r.h.s. of (5.1) and taking into account that $\varepsilon^2 = 0$, yields:

$$e^{(X+\varepsilon\varphi(X,g))P} e^{\varepsilon h(X,g)} = \exp\left(XP + \varepsilon\varphi(X,g)P - \frac{\mathrm{ad}XP}{e^{-\mathrm{ad}XP}-1} \varepsilon h(X,g)\right). \tag{5.5}$$

Taking logarithm of right hand sides of (5.4) and (5.5) we see that the equation (5.2) is equivalent to the equation:

$$\varphi(X,g)P - \frac{\mathrm{ad}XP}{e^{-\mathrm{ad}XP}-1} h(X,g) = -\frac{\mathrm{ad}XP}{e^{-\mathrm{ad}XP}-1} e^{-\mathrm{ad}XP} g. \tag{5.6}$$

Applying to both sides of the latter equation the operator $\Pi_{\mathfrak{g}_-} \frac{e^{-\mathrm{ad}XP}-1}{\mathrm{ad}XP}$ we obtain

$$\Pi_{\mathfrak{g}_-} \frac{e^{-\mathrm{ad}XP}-1}{\mathrm{ad}XP} \varphi(X,g)P = -\Pi_{\mathfrak{g}_-} e^{-\mathrm{ad}XP} g. \tag{5.7}$$

Clearly, the latter equation implies (5.2).

To obtain (5.3), apply $\Pi_{\mathfrak{h}}$ to both sides of equation (5.6) to get

$$\Pi_{\mathfrak{h}} \frac{\mathrm{ad}XP}{e^{-\mathrm{ad}XP}-1} h(X,g) = \Pi_{\mathfrak{h}} \frac{\mathrm{ad}XP}{e^{-\mathrm{ad}XP}-1} e^{-\mathrm{ad}XP} g. \tag{5.8}$$

The latter equation implies evidently (5.3). ∎

6 The action graph and "path integral".

Expressions (4.4) and (4.5) are good enough for manual calculations of $T^{\mathfrak{g}}_{\mathfrak{h}}(g)$ and $I^{\mathfrak{g}}_{\mathfrak{h}}(g)$ in the case of $\mathbb{Z}$-graded Lie superalgebra of finite depth, considered in section 4: one expands the formal series consisting of exponents (Bernoulli coefficients appear at this stage); then one calculates multiple brackets of the element XP with g. There is only a finite number of non-zero brackets for any fixed g, so this process terminates even in the case of infinite dimensional $\mathfrak{g}$ if only $\mathfrak{g}_-$ is finite-dimensional.

But to make all calculations manually is a tedious task even for $\mathfrak{g}$ of dimension a dozen or two.

So it is desirable to convert expressions (4.4) and (4.5) into something potentially more digestible by computers.

A useful way to do this is by means of *action graphs* defined below. Our definition will be a bit more general than we need in this paper, so that it may be applied to local Lie superalgebras and Cartan-Tanaka-Shchepochkina prolongs ([19]), ([1]).

Given two superspaces E and H, any bilinear map

$$E \times H \overset{\cdot}{\longrightarrow} H \tag{6.1}$$

is said to be an **action** of E on H. Stress that neither E nor H are supposed to be equipped with any other structure, except a superstructure.

Let $P = \{P_i\}_{i \in I}$ be a basis in E and $h = \{h_j\}_{j \in J}$ be a basis in H.

In the bases P and h, the action (6.1) can be expressed as follows:

$$P_i \cdot h_j = c_{ij}^k h_k. \tag{6.2}$$

The family $C = \{c_{ij}^k\}_{i,j \in I; k \in J}$ will be called the **structure constants** of the action (6.1) in bases P and h.

Define the directed graph $G(P, h)$ as follows. Its vertices are elements of the basis h; for any vertices h_s and h_t and an element P_i of the basis P of E there exists an edge from h_s to h_t labelled by P_i iff $c_{is}^t \neq 0$. Note, that there may be more than one edge going from h_s to h_t.

The graph $G(P, h)$ will be called the **action graph** of the action (6.1) with respect to the bases P and h.

Before proceeding further we are to give some (ad hoc) definitions.

For any directed graph G denote $\mathcal{P}(G)$ the set of all directed paths of G.

$\mathcal{P}(G)$, equipped with partial operation of composition of paths, clearly may be considered as a category with the set of objects coinciding with the set of vertices of G.

The set of all paths beginning at vertex v, resp. ending at vertex v', resp. beginning at v and anding at v' will be denoted $\mathcal{P}_v(G)$, resp. $\mathcal{P}^{v'}(G)$, resp. $\mathcal{P}_v^{v'}(G)$.

Let A be a supercommutative superalgebra with unit element. A map

$$\mu : \mathcal{P}(G) \longrightarrow A \tag{6.3}$$

such that $\mu(p \circ p') = \mu(p)\mu(p')$ and $\mu(1_v \circ p) = \mu(p)$ (i.e. μ is a functor into multiplicative monoid of A) will be called a **multiplicative A-valued measure** on $\mathcal{P}(G)$. If, besides, V is an A-supermodule, then the map

$$\mu_V : \mathcal{P}(G) \longrightarrow V \tag{6.4}$$

such that $\mu_V(p \circ p') = \mu(p)\mu_V(p')$ will be called a **μ-compatible multiplicative V-valued measure**.

Clearly, a multiplicative A-valued measure on $\mathcal{P}(G)$ is uniquely determined by its values on edges of the graph G (i.e., paths of length 1). Moreover, any map $\text{Edges}(G) \longrightarrow A$ can be extended to a multiplicative A-valued measure on $\mathcal{P}(G)$.

To define path integrals over a graph, suppose first that the graph G is such that, for any vertex v of G, there exists only *finite* number of directed paths starting at v. This is the case of the action graph of the action $\mathfrak{g}_- \times \mathfrak{g} \longrightarrow \mathfrak{g}$, where $\mathfrak{g}$ is a $\mathbb{Z}$-graded Lie superalgebra and $\mathfrak{g}_- = \oplus_{i<0}\mathfrak{g}_i$ is finite-dimensional, considered at the end of preceeding section.

Then, for any function $c : \mathcal{P}(G) \longrightarrow A$, there exists

$$\int_v c\mu_V \stackrel{\text{def}}{=} \sum_{p\in\mathcal{P}_v(G)} c(p)\mu_V(p). \tag{6.5}$$

The sum in the r.h.s. of definition (6.5) may have sense (for particular choice of both μ_V and c) even if it is infinite in the case where A is a *topological* superalgebra and V is a *topological* supermodule.

We will not give here the most general definitions, restricting ourselves to the case of action graph $G(P, h)$ with a special kind of multiplicative measure defined on it. Namely, let $A = S(E)^*$ and $V = S(E)^*\widehat{\otimes}H$. Let $\{X_i\}_{i\in I}$ be the (topological) basis in E^* dual to the basis P in E.

Let μ be the multiplicative A-valued measure on $\mathcal{P}(G(P, h))$ which extends the map

$$\mu_0 : \text{Edges}(G(P, h)) \longrightarrow A \quad ((h_s \xrightarrow{P_i} h_t) \mapsto -c^t_{is}X_i). \tag{6.6}$$

Let μ_V be the multiplicative V-valued measure compatible with μ and defined as follows:

$$\mu_V(p) \stackrel{\text{def}}{=} \mu(p)T(p), \tag{6.7}$$

where $T(p)$ denotes the end (i.e., target) vertex of the path p.

Proposition 25 *For any action $E \times H \longrightarrow H$ and any bases P in E and h in H, the sum in the r.h.s. of eq.* (6.5) *exists for the measure μ_V defined by eq.* (6.7) *and any function*

$$c : \mathcal{P}(G(P, h)) \longrightarrow \Bbbk \hookrightarrow S(E)^*\widehat{\otimes}H.$$

Proof This trivially follows from the definition of formal series module $S(E^*)\widehat{\otimes}H$ and the fact that any monomial of degree n may occur in the sum (6.5) only finite number of times (not more than $n!$). ■

The next theorem states that the formal series $\varphi^i(X, g)$ and $h(X, g)$ defined by eqs. (4.4)–(4.5) giving the expressions for (co)induced modules in case where $\mathfrak{g}_-$ is a subsuperalgebra of $\mathfrak{g}$ can be calculated as path integral on the action graph with an appropriate function $c(p)$.

For nonnegative integers k and n such that $k \leq n$, define the rational number $c(k, n)$ as follows:

$$c(k, n) = \sum_{k\leq i\leq n} \frac{b_{n-i}}{i!(n-i)!}, \tag{6.8}$$

where b_i is the i-th Bernoulli number.

For a path $p \in \mathcal{P}(G(P,h))$, let $k(p)$ be the length of the longest subpath, whose source vertex coincides with that of p, whereas the target vertex belongs to $\mathfrak{h}$ (recall that vertices of $\mathcal{P}(G(P,h))$ are basis elements of $\mathfrak{g}$). Define now the map $K : \mathcal{P}(G(P,h)) \longrightarrow \Bbbk$ as follows:

$$K(p) \stackrel{\text{def}}{=} -c(k(p), \text{length}(p)), \tag{6.9}$$

where length(p) denotes the number of edges in the path p.

Theorem 26 *If $\mathfrak{g}_-$ is a Lie subsuperalgebra of $\mathfrak{g}$, then, for any basis element $M \in P \cup h$, the formal series $G_-(M)$ and $H(M)$ satisfying condition* (3.6) *are determined as follows:*

$$G_-(M) = \Pi_{\mathfrak{g}_-} \int_M K\mu_V, \qquad H(M) = \Pi_{\mathfrak{h}} \int_M K\mu_V, \tag{6.10}$$

where μ_V is defined by eq. (6.7) *and K is defined by eq.* (6.9)*, and the projections $\Pi_{\mathfrak{g}_-}$ and $\Pi_{\mathfrak{h}}$ are defined by eq.* (4.1)*. In other words, $G_-(M)$ is the sum over all paths starting at M and ending in $\mathfrak{g}_-$, whereas $H(M)$ is the sum over all paths starting at M and ending in $\mathfrak{h}$.*

Proof Meditate over eqs. (4.4) and (4.5) until being enlightened :). ■

7 The program `coindrep`.

The formulae (6.10) were converted into algorithm written in ANSI C++. The present beta version of the program `coindrep` which uses this algorithm calculates generators of series $\mathfrak{gl}(n)$, $A_n = \mathfrak{sl}(n+1)$, $D_n = \mathfrak{so}(2n)$ and E_n only, for the case where $\mathfrak{h}$ in the expansion $\mathfrak{g} = \mathfrak{g}_- \oplus \mathfrak{h}$ is a maximal solvable subalgebra of $\mathfrak{g}$. The possibility to make calculations for other series of simple Lie algebras as well as Lie superalgebras will be added to the program as well as the possibility to make calculations for arbitrary "good" expansion $\mathfrak{g} = \mathfrak{g}_- \oplus \mathfrak{h}$ defined by user.

The program output (statistics and expressions for generators of coinduced module) is in plain TEX format.

For the latest version of the windows executable of the program together with its source code, see
theo.inrne.bas.bg/~vmolot/coindrep.zip.

7.1 Some statistics.

Calculations for the case $\mathfrak{g} = \mathfrak{gl}(15)$ take about half an hour on Sun Solaris 9 system. The program is not optimized yet, but even for optimized future version this time is expected to be proportional to the number of paths in the action graph, so that, roughly, will grow exponentially as the depths of canonical $\mathbb{Z}$-grading grows.

The longest number of paths in path integral for the action graph of $\mathfrak{g} = \mathfrak{gl}(15)$ is 3052080, whereas the number of monomials after reduction is exactly 8K=8192 for all 15 generators from $\mathfrak{g}_+$. The maximal degree of these monomials is 15.

Statistics for the Lie algebra E_6: the longest number of paths is 73179, the maximal number of monomials after reduction is 1906, the maximal degree of monomials of generators from $\mathfrak{g}_1$ is 12. All calculations take less than 10 seconds.

In contrast, calculations of all generators for Lie algebra E_7 are estimated to take about 10–20 hours. I have no guts to run the program to the end, restricting myself to the calculations of the first few "easy" generators of maximal degree 16, which took about half an hour. I hope that after optimization of the program (replacing, where appropriate, STL[6] vectors by data structures based on trees, like maps and multisets) the run time can be reduced essentially. But I suspect that, even after eventual optimization, the calculations for the Lie algebra E_8 will occupy many years of present supercomputer's time. This is due to the fact that this time is expected to be proportional to the exponent of the depth of the main $\mathbb{Z}$-grading for E_8 which is 29.

A Concise Dictionary on Lie superalgebras.

Here we have placed together some basic definitions and constructions. For a detailed presentation of "the linear algebra in superspace", see Chapter 1 of Ref. [15] and that of Lie superalgebras can be obtained from reading Ref. [10].

It is supposed here that all vector spaces are over some field $\Bbbk$ of characteristic 0. $\mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z} = \{\bar{0}, \bar{1}\}$, $(-1)^{\bar{0}} = 1$ and $(-1)^{\bar{1}} = -1$.

Let V be a vector space. A **superstructure** on V is defined by any of the three equivalent structures:

a) the linear operator $P \in \mathrm{End}_{\Bbbk}(V)$, called the **parity operator**, such that

$$P^2 = \mathbb{1}_V \; ; \tag{A.1}$$

b) two linear operators ${}_i\Pi \in \mathrm{End}_{\Bbbk}(V)$ $(i = \bar{0}, \bar{1})$ such that

$${}_i\Pi^2 = {}_i\Pi \; , \quad {}_{\bar{0}}\Pi {}_{\bar{1}}\Pi = {}_{\bar{1}}\Pi {}_{\bar{0}}\Pi = 0 \;\text{ and }\; {}_{\bar{0}}\Pi + {}_{\bar{1}}\Pi = \mathbb{1}_V \; ; \tag{A.2}$$

c) the decomposition

$$V = {}_{\bar{0}}V \oplus {}_{\bar{1}}V \tag{A.3}$$

related to each other as follows:

$${}_i\Pi = \frac{\mathbb{1} + (-1)^i P}{2} \; , \quad P = {}_{\bar{0}}\Pi - {}_{\bar{1}}\Pi \; , \quad {}_iV = {}_i\Pi V \quad (i = \bar{0}, \bar{1}). \tag{A.4}$$

A **superspace** is a linear space V together with some superstructure on it. For any $x \in V$, the element ${}_{\bar{0}}x = {}_{\bar{0}}\Pi x$ is said to be the **even** component of x and ${}_{\bar{1}}x = {}_{\bar{1}}\Pi x$ is the **odd** component. If two or more superspaces are considered simultaneously, the operators P and ${}_i\Pi$ are sometimes supplied with indices.

Here are standard constructions of new superspaces from given ones. Given superspaces V and W, the following superstructures are naturally defined on $V \otimes W$ and $\mathrm{L}_{\Bbbk}(V, W)$ (linear maps from V to W), respectively:

$$P(v \otimes w) \stackrel{\text{def}}{=} Pv \otimes Pw \quad (v \in V, w \in W), \tag{A.5a}$$

$$P(F) \stackrel{\text{def}}{=} P_W \circ F \circ P_V \quad (F \in \mathrm{L}_{\Bbbk}(V, W)). \tag{A.5b}$$

[6]Standard Template Library of C++

In particular, the space $V^* = \mathrm{L}_{\Bbbk}(V, k)$ is a superspace if V is a superspace.

A **superalgebra** is a superspace A together with a bilinear map (composition):

$$A \times A \longrightarrow A \quad (x, y) \mapsto x \centerdot y \quad (x, y \in A) \tag{A.6}$$

such that

$$P(x \centerdot y) = Px \centerdot Py \quad (x, y \in A). \tag{A.7}$$

Associativity and the identity elements in superalgebras are defined as in ordinary algebras. A superalgebra A is said to be **supercommutative** if

$$xy = \sum_{i,j \in \mathbb{Z}_2} (-1)^{ij} {}_i y \, {}_j x \quad (x, y \in A). \tag{A.8}$$

Examples. If V is a superspace, then the tensor algebra

$$T(V) = \bigoplus_0^\infty T^n(V)$$

is an associative superalgebra with unity and **symmetric superalgebra**

$$S(V) = \bigoplus_0^\infty S^n(V)$$

of V defined as the quotient algebra of $T(V)$ modulo relations

$$x \otimes y - \sum_{i,j \in \mathbb{Z}_2} (-1)^{ij} {}_j y \otimes {}_i x$$

is a supercommutative associative superalgebra with identity.

The superspace $S(V)^*$ dual to the superalgebra $S(V)$ is a supercommutative superalgebra as well (see section 2.6). Moreover, it is a *topological* superalgebra, if equipped with weak topology, turning $S(V)^*$ into linearly compact space.

A superalgebra $\mathfrak{g}$ is a **Lie superalgebra** if the composition in $\mathfrak{g}$, denoted usually as $[\cdot, \cdot]$, satisfies the following relations:

$$[x, y] = -\sum_{i,j \in \mathbb{Z}_2} (-1)^{ij} [{}_i y, {}_j x] \quad (x, y \in \mathfrak{g}) \tag{A.9}$$

and

$$(-1)^{ij} [{}_i x, [{}_j y, {}_k z]] + (-1)^{jk} [{}_j y, [{}_k z, {}_i x]] + (-1)^{ki} [{}_k z, [{}_i x, {}_j y]] = 0 \;\; (x, y, z \in \mathfrak{g} \; i, j, k \in \mathbb{Z}_2). \tag{A.10}$$

The **tensor product** of the superalgebras A and B is the superspace $A \otimes B$ with composition defined as follows:

$$(a \otimes b)(a' \otimes b') = \sum_{i,j \in \mathbb{Z}_2} (-1)^{ij} a \, {}_i a' \otimes {}_j b b' \;\; (a, a' \in A \; b, b' \in B). \tag{A.11}$$

The tensor product of two associative (supercommutative) superalgebras is an associative (supercommutative) superalgebra. The tensor product of a supercommutative superalgebra and of Lie superalgebra is Lie superalgebra.

Examples. 1) If V is a superspace, then we have the isomorphism of supercommutative superalgebras:

$$S(V) \approx S({}_{\bar{0}}V) \otimes \Lambda({}_{\bar{1}}V), \tag{A.12}$$

where $\Lambda({}_{\bar{1}}V)$ is the exterior algebra of the vector space ${}_{\bar{1}}V$ supplied with a natural superstructure—the extension of the superstructure $Px = -x$ of ${}_{\bar{1}}V$ by means of relations (A.7)

2) If $\mathfrak{g}$ is a Lie superalgebra and V is a superspace, then $S(V)^* \otimes \mathfrak{g}$ (as well as its completion $S(V)^* \widehat{\otimes} \mathfrak{g}$) is a topological Lie superalgebra (see sect. 2.4).

A **linear representation** ρ of a Lie superalgebra $\mathfrak{g}$ in a superspace V is a linear map

$$\rho\colon \mathfrak{g} \longrightarrow \mathrm{End}_{\Bbbk}(V) \overset{\mathrm{def}}{=} \mathrm{L}_{\Bbbk}(V, V)$$

such that

$$\rho([g_1, g_2]) = \rho(g_1)\rho(g_2) - \sum_{i,j \in \mathbb{Z}_2} (-1)^{ij} \rho({}_i g_2)\rho({}_j g_1) \quad (g_i \in \mathfrak{g}) \tag{A.13}$$

and

$$P(\rho(g)) = \rho(P(g)) \quad (g \in \mathfrak{g}). \tag{A.14}$$

The representatation ρ^* in the superspace V^* defined as

$$\rho^*(g) = -(\rho(g))^* \tag{A.15}$$

is said to be **contragradient** to ρ. Here the operator $A^*\colon W^* \longrightarrow V^*$ **dual** to the operator $A\colon V \longrightarrow W$ is defined as follows ($\langle \cdot, \cdot \rangle$ is the "scalar product" in $V^* \times V$ or in $W^* \times W$):

$$\langle A^* w^*, v \rangle = \sum_{i,j \in \mathbb{Z}_2} (-1)^{ij} \langle {}_i w^*, {}_j A v \rangle \quad (v \in V, w^* \in W^*). \tag{A.16}$$

If $B\colon W \longrightarrow Y$ is another operator, then

$$(B \circ A)^* = \sum_{i,j \in \mathbb{Z}_2} (-1)^{ij} {}_i B^* \circ {}_j A^*. \tag{A.17}$$

Given a representatation ρ of a Lie superalgebra $\mathfrak{g}$, we can define another representatation, $\rho'(g) \overset{\mathrm{def}}{=} \rho(Pg)$, where P is the parity operator of the Lie superalgebra $\mathfrak{g}$.

Given a representatation ρ of a Lie superalgebra $\mathfrak{g}$ in a superspace V and any supercommutative associative superalgebra A, define the representatation ρ_A of the Lie superalgebra $A \otimes \mathfrak{g}$ in the superspace $A \otimes V$ as follows:

$$\rho_A(a \otimes g)(a' \otimes v) = \sum_{i,j \in \mathbb{Z}_2} (-1)^{ij} a\, {}_i a' \otimes {}_j g v \quad (a, a' \in A,\ g \in G,\ v \in V). \tag{A.18}$$

If, moreover, A is a linearly topologized superalgebra, then the representation (A.18)extends by continuity to the representation $\widehat{\rho}_A$ of the linearly topologized Lie superalgebra $A\widehat{\otimes}\mathfrak{g}$ in the superspace $A\widehat{\otimes}V$.

Given two representatations ρ and ρ' of a Lie superalgebra $\mathfrak{g}$ in superspaces V and V', respectively, we say that an operator $A\colon V \longrightarrow V'$ is an **intertwining operator** between representations ρ and ρ' if

$$\rho'(g) \circ A = \sum_{i,j \in \mathbb{Z}_2} (-1)^{ij} {}_iA \circ \rho({}_jg). \qquad \text{(A.19)}$$

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