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by

Christian Klein, Vladimir Matveev, and Alexander Smirnov

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Christian Klein
e-mail: klein@mis.mpg.de, Max Planck Institute for Mathematics in Sciences,
Inselstr. 22, 04103 Leipzig, Germany

Vladimir B. Matveev*
e-mail: matveev@u-bourgogne.fr, Institut de Mathématique de Bourgogne,
BP 47870 - F-21078 Dijon France,

Alexander O. Smirnov
e-mail: Alsmir@mail.ru, St. Petersburg University of Aerospace Instrumentation,
St. Petersburg, Bolshaya, Morskaya 67, St. Petersburg, 190000 Russia,

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Abstract

The aim of this article is to summarize results on the cylindrical KP equation also known as Johnson equation. In particular, we explain the gauge equivalence of the related Lax pairs. We also describe some important classes of explicit solutions obtained by the use of the Darboux transformation approach. Plots of explicit solutions including finite gap solutions to the CKP equation are presented.

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1 Introduction

The cylindrical KP equation (CKP) is a 2D generalization of the cylindrical KdV equation. It was introduced in 1978 in an article by Johnson [1] in the context of the description of surface waves in a shallow incompressible fluid. Later the same equation was derived by Lipowskij [2] for internal waves in a stratified medium. The Johnson equation has the following form:

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$$\frac{\partial}{\partial x}\left(u_t + 6uu_x + u_{xxx} + \frac{u}{2t}\right) = -3\alpha^2\frac{u_{yy}}{t^2}. \quad (1)$$

The same equation was recently derived in [15] for nonlinear acoustic waves. As for KP we can distinguish two cases depending on the sign of the real parameter α^2: the CKP I equation, corresponding to a positive value of α^2, and the CKP II case where α^2 is supposed to be negative. When $\alpha = 0$ the CKP equation obviously reduces to the CKdV equation. The Cauchy initial problem for the CKP equation can be posed correctly for any $t = t_0 > 0$.

Contrary to the widely known KP equation, the CKP equation has dissipative character: there is no soliton-like solution with a linear front localized along straight lines in the xy-plane. Also, the multi-lump-like solutions preserving their form during propagation that exist for the CKP II equation [11] are missing in the CKP I case. It was discovered in [4][1] that nevertheless in the CKP case, we have as much explicit solutions as in the KP case. The CKP equation allows to explain the existence of the horseshoe-like solitons and their multi-versions in quite a natural way. The horseshoe multi-soliton solutions correspond very well to real waves which can be observed in thin films of shallow water being cooled along the inclined plane. One can compare for instance the photo 19 on page 38 of the book [3] with some plots of this work. In both cases, the waves with crossed parabolic profiles are very well visible.

Following [4] each solution of the KP equation generates a solution of the CKP equation and vice versa. The related mapping was first constructed in [4] on the base of study of algebro-geometric solutions. The latter point somehow was never explained in the literature. The use of the aforementioned isomorphism between the whole varieties of solutions to the CKP and KP equations represents one of the most powerful approaches for solving the CKP equation by fully using the information on the solutions of the KP equation obtained by various methods. It is necessary to mention that Johnson [1] has gotten a weaker result allowing to construct from any solution of the KdV equation some solution of the CKP equation. His result follows from [5] as a natural reduction. The one to one correspondence between the solutions of the KP and CKP equations [4] preserves the class of solutions decreasing in all spatial directions, and the class of rational solutions. The same correspondence somehow transforms the line-front solitons of the KP equation to the parabolic front solitons of the CKP equation. It transforms the rational lump solutions of the KP equation to the dissipative rational solutions of the CKP model. It maps the bounded almost periodic solutions of the KP model to the class of bounded solutions periodic only with respect to the x variable. The plots of the related theta functional solutions of the CKP model also seem to correspond well to some real waves on shallow water. A different approach to the integration of the CKP equation which directly uses the Darboux covariance of the related Lax pair briefly mentioned in [5] will be also explained here. In our opinion the results of [4, 5] concerning the CKP equation and its link with the KP model

[1] See also [5] and the explanation below in section 5.

are underexplored. The appearance of powerful software allowing to visualize even theta-functional solutions of finite genera opens up new opportunities in the comparison of the related solutions with experimental data.

Therefore we decided to present an extended version of the results of [4, 5] including for the first time graphical material that is important to understand the qualitative features of the solutions having a complicated analytical structure[2].

2 CKP Lax pair

The Lax pair and its adjoint for the CKP equation are given by the formula:

$$\begin{aligned} \pm\alpha f_y^{\pm} &= t f_{xx}^{\pm} + (tu - x/12) f^{\pm}, \\ f_t^{\pm} &= -4 f_{xxx} - 6u f_x^{\pm} - (3u_x \pm 3\alpha v/t) f^{\pm}. \end{aligned} \tag{2}$$

The compatibility conditions of the $(+)$system (2) and of the $(-)$system (2) are both equivalent to the system

$$v_x = u_y, \qquad u_t + 6uu_x + u_{xxx} + \frac{u}{2t} = -3\alpha^2 \frac{v_y}{t^2},$$

which obviously produces the CKP equation after differentiation of the second equation by x and the use of the formula $v_{yx} = v_{xy} = u_{yy}$. The $(+)$ Lax pair was first found by Druma [9] and Salle [6] in 1983, and was rediscovered one year later by Ovel and Steeb [10].

3 Darboux transformation for the CKP equation

According to the general covariance theorem (Matveev 1979) [7, 4, 5], the equations (2) are both covariant with respect to a classical Darboux transformation (DT)

$$f \to \psi_1^{\pm}, u \to u_1^{\pm}, \qquad \psi_1^{\pm} := f_x^{\pm} - \sigma^{\pm} f \equiv \frac{W(f_1^{\pm}, f^{\pm})}{f_1^{\pm}},$$

$$\sigma^{\pm} := \frac{\partial_x f_1^{\pm}}{f_1^{\pm}}, \quad u_1^{\pm} := u + 2\sigma_x^{\pm},$$

where $f_1^{\pm}$ is some fixed solution of (2). The iterations of only $(+)$ or of only $(-)$ DT leads to formulas absolutely similar to the KP case:

$$u_n^{\pm} = u + 2\partial_x^2 \ln W(f_1^{\pm}, \ldots, f_n^{\pm}),$$

$$\psi_n^{\pm} = \frac{W(f_1^{\pm}, \ldots, f_n^{\pm}, f^{\pm})}{W(f_1^{\pm}, \ldots, f_n^{\pm})},$$

[2]In the recent work [14] the results of [4, 5] were used in order to prove the existence of nondecaying solutions of the CKP equation, vanishing as $x \to \infty$. Some additional results concerning physical applications of the CKP equation can be found in [13].

where the $f_j^{\pm}$ are arbitrary linearly independent solutions of the (+)- or (−)-Lax system respectively.

For any complex value of p the (+)-system (2) with $u = 0$ has a solution of exponential type:

$$f^+(k) := \exp\left[-\frac{x}{12\alpha}(y-p) + \frac{4t}{(12\alpha)^3}(y-p)^3\right]$$

$$= \exp\left[k\left(x - \frac{y^2 t}{12\alpha^2}\right) + k^2\alpha^{-1}yt - 4k^3 t - \frac{xy}{12\alpha} + \frac{4ty^3}{(12\alpha)^3)}\right], \qquad p \equiv 12k\alpha. \quad (3)$$

Similarly, for the (−)-system (2) with $u = 0$, a one parametric family of solutions is given by the formula

$$f^-(p) := \exp\left[\frac{x}{12\alpha}(y+p) - \frac{4t}{(12\alpha)^3}(y+p)^3\right].$$

Since the system (2) is linear, linear combinations of solutions corresponding to different values of p or k lead to new solutions. Also differentiation of $f^{\pm}(x, y, t, k)$ with respect to k to arbitrary order produces solutions to the same system. Next, taking $k = 0$ in $\partial_k^m f^{\pm}(x, y, t, k)$ we obtain an infinite number of solutions having the form of polynomials of x, y, t times a k independent exponential. Plugging them into a Wronskian we get an infinite family of rational solutions of the CKP equation. This construction obviously represents the natural analogue of the rational solutions of the KP equation expressed in terms of Faa de Bruno polynomials also frequently called Bell polynomials. In the KP case a similar construction of the rational solutions was first proposed in [4]. In the latter case, it contains as a very simple special reduction of the formulas of [4] the rational solutions of "general position" for the KP equation constructed by Krichever in 1978. In particular, we can construct a large family of solutions $f_j^{\pm}$ of the "starting" system (2) with $u = 0$, as follows:

$$f_j^+ = \int_{s_{j1}} f^+(k)d\,\mu_{j1}(k),$$

$$f_j^- = \int_{s_{j2}} f^-(k)d\,\mu_{j2}(k),$$

where $\mu_{j1}(k), \mu_{j2}(k)$ are some measures. It is clear that the related Wronskian can be written as

$$W(f_1^+, \ldots, f_n^+) = \exp\left[-\frac{nxy}{12\alpha} + \frac{4nty^3}{(12\alpha)^3)}\right] W(f_{01}^+, \ldots, f_{0n}^+),$$

$$f_{0j}^+ := \int_{s_{j1}} f_0^+(k)d\,\mu_{j1}(k),$$

where $f_0^+(k)$ denotes the k-dependent part of the exponential (3). The related Wronskians can also be written as multiple integrals of the following structure:

$$W = \exp\left[-\frac{nxy}{12\alpha} + \frac{4nty^3}{(12\alpha)^3)}\right] \int_{s_1} \dots \int_{s_n} \prod_1^n f_0^+(k_j) \prod_{j>i} (k_j - k_i) \prod_1^n d\,\mu_{j1}(k_j)$$

in the $(+)$ case or

$$W = \exp\left[-\frac{nxy}{12\alpha} + \frac{4nty^3}{(12\alpha)^3)}\right] \int_{s_1} \dots \int_{s_n} \prod_1^n f_0^-(k_j) \prod_{j>i} (k_j - k_i) \prod_1^n d\,\mu_{j2}(k_j),$$

in the $(-)$ case. It is clear that the external k independent factors do not contribute to the potential.

So, similarly to the KP I case [7, 5] we have a large number of solutions depending on an arbitrary number of functional parameters. It is quite obvious that the obtained solutions to the CKP equation are nonsingular and real valued provided that the following condition is satisfied:

$$s_j \bigcap s_m = \emptyset, \quad \forall j, \quad d\,\mu_j(k) = \rho_j(k) d\,k, \rho_j(k) \geq 0, \inf s_j > 0$$

where $s_j =: \mathrm{supp}\rho_j(k)$.

Therefore, one family of nonsingular N-solitons horseshoe line front solutions provided by the first nonsingularity condition formulated above can be written as

$$u = 2\partial_x^2 \ln W(f_1, \dots, f_n)$$

with

$$f_j = A_j e^{\vartheta_{2j-1}} + e^{\vartheta_{2j}}, \quad A_j > 0 \quad \forall j,$$

with

$$\vartheta_m =: k_m \left(x - \frac{y^2 t}{12\alpha^2} + \right) + k_m^2 \alpha^{-1} y t - 4k_m^3 t.$$

This solution is non singular if $0 < k_m < k_{m+1}, \quad \forall m$, which corresponds to the above nonsingularity condition with $\rho_j(k) = \delta(k - k_{2j-1}) + \delta(k - k_{2j})$.

An alternative sufficient non singularity condition, (see also [5]), is

$$\mathrm{supp}\rho_j = [a_j, b_j] \cup [c_j, d_j], \quad j = 1, \dots, n;$$

$$a_n \leq b_n \ldots < a_j \leq b_j \ldots < a_1 \leq b_1 < c_1 \leq d_1 < \ldots < c_j \leq d_j < \ldots < c_n \leq d_n$$

and

$$\rho_j(k_j) > 0 \quad \text{if} \quad k_j \in [c_j, d_j], \quad \forall j, \quad \text{or} \quad k_j \in [a_j, b_j], \quad j = 2k - 1,$$

$$\rho_j(k_j) < 0 \quad \text{if} \quad k_j \in [a_j, b_j], \quad j = 2k.$$

This condition guarantees that the product $\prod_{j=1}^n \rho_j(k_j) \det(k_j^{m-1})$ is positive, and hence the Wronskian $W(f_{01}^+, \dots, f_{0n}^+)$ is positive, too. In particular, taking

$$\rho_j(k_j) = A_j \delta(k_j - a_j) + \delta(k_j - c_j), A_{2J} < 0, \quad A_{2j-1} > 0$$

we we obtain the following nonsingular solutions of the CKP-I equation.

This second nonsingularity condition suggests another family of nonsingular horse shoe like N-solitons line front solutions which can be obtained by the following choice of f_j:

$$f_j = A_j e^{\theta_{2j-1}} + e^{\theta_{2j}}, \quad A_{2j-1} > 0, A_{2j} < 0, \forall j,$$

$$\theta_{2m-1} =: a_m \left(x - \frac{y^2 t}{12\alpha^2} + \right) + a_m^2 \alpha^{-1} y t - 4a_m^3 t,$$

$$\theta_{2m} =: c_m \left(x - \frac{y^2 t}{12\alpha^2} + \right) + c_m^2 \alpha^{-1} y t - 4c_m^3 t,$$

$$a_n < \ldots < a_1 < c_1 < \ldots c_n$$

In particular, taking $k_1 + k_2 = k_3 + k_4, \quad k_3 > k_2$ we can easily check that

$$u = 2\partial_x^2 \ln W \left(\cosh \frac{\vartheta_1 - \vartheta_2}{2}, \sinh \frac{\vartheta_3 - \vartheta_4}{2} \right)$$

represents a particular kind of non singular 2-solitons solution.

In the CKP II case, i.e., for $\Re\alpha = 0$ we introduce the following 1-form:

$$\Omega(f,g) = fgdx - \alpha^{-1} t(fg_x - f_x g)dy + (-4f_{xx}g + 4f_x g_x - 4fg_{xx} - 6ufg)dt.$$

The 1-form Ω is closed if f is an arbitrary solution of the $(+)$-system (2), and g is an arbitrary solution of the $(-)$-system (2) corresponding to the same starting solution $u(x,y,t)$ of the CKP equation. In terms of this form the following **binary Darboux dressing formula** for the solutions of the CKP equation holds:

$$_n u_m = u + 2\partial_x^2 \ln \det M, \quad n \geq m, \tag{4}$$

$$M := \begin{pmatrix} Q_{1,n+1} & Q_{2,n+1} & \cdots & Q_{n,n+1} \\ Q_{1,n+2} & Q_{2,n+2} & \cdots & Q_{n,n+2} \\ \vdots & \vdots & \vdots & \vdots \\ Q_{1,n+m} & Q_{2,n+m} & \cdots & Q_{n,n+m} \\ f_1^+ & f_2^+ & \cdots & f_n^+ \\ \partial_x f_1^+ & \partial_x f_2^+ & \cdots & \partial_x f_n^+ \\ \vdots & \vdots & \vdots & \vdots \\ \partial_x^{(n-m-1)} f_1^+ & \partial_x^{(n-m-1)} f_2^+ & \cdots & \partial_x^{(n-m-1)} f_n^+ \end{pmatrix}.$$

Taking $m = 0$ in (4) we recover the Wronskian formula written above.

In the case $n = m$ the matrix elements of M are given by the formula:

$$M_{jl} := Q_{j,n+l}, \quad Q_{jl} := A_{jl} + B_{jl} \int_{(x_0,y_0,t_0)}^{(x,y,t)} \Omega(f_j^+, f_l^-), \quad 1 \leq j, l \leq n.$$

All integrals in the last formulas are independent of the choice of the path of integration.

A particular smooth rational solution of CKP II equation with $\alpha = i\gamma/12$ following from (4) with $n = m = 1$ is given by the formula:

$$u = 2\partial_x^2 \ln \left[(12t^2) \left| \frac{x}{12t} + y^2 - \gamma^2 - 2i\gamma y \right|^2 + \frac{1}{4\gamma^2} \right]. \tag{5}$$

4 From CKdV to CKP

Assuming that the solution of the CKP equation satisfies the CKdV equation

$$u_t + 6uu_x + u_{xxx} + \frac{u}{2t} = 0, \tag{6}$$

i.e., it does not depend on y, we can construct solutions of the system (2) of the form

$$f(x, y, t) = e^{\lambda y} p(x, t, \lambda),$$

with p solving the Lax system for the CKdV equation first found by Druma [9]:

$$tp_{xx} + (tu - \frac{x}{12})p = \lambda p,$$

$$p_t = -4p_{xxx} - 6up_x - 3u_x p.$$

Particular solutions of this system for $u = 0$, can be constructed in the form

$$p(x, t, \lambda) = \frac{1}{\sqrt[3]{t}} w \left(\frac{x - \lambda}{\sqrt[3]{t}} \right),$$

where $w(z)$ is an arbitrary solution of the Airy equation

$$w'' = zw.$$

Therefore in the particular case $\alpha = \frac{i}{12}$, a class of solutions T-periodic in y of (2) with $u = 0$ is given by the formula

$$f^+ = \frac{1}{\sqrt[3]{t}} \sum_{m \in \mathbb{Z}} \left[a_m Ai \left(\frac{x - \frac{2\pi m}{T}}{\sqrt[3]{t}} \right) + b_m Bi \left(\frac{x - \frac{2\pi m}{T}}{\sqrt[3]{t}} \right) \right] e^{\frac{2iy\pi m}{T}}. \tag{7}$$

In this formula a_m, b_m are arbitrary coefficients providing the necessary smoothness of its RHS and $Ai(z), Bi(z)$ are the standard solutions of the Airy equation defined by the integrals:

$$Ai(z) = \frac{1}{\pi} \int_0^\infty \cos \left(\frac{t^3}{3} + tz \right) dt,$$

$$Bi(z) = \frac{1}{\pi} \int_0^\infty \left[\exp \left(-\frac{t^3}{3} - zt \right) + \sin \left(\frac{t^3}{3} + zt \right) \right] dt.$$

5 Gauge equivalence of the CKP equation and of the KP equation

There exists a remarkable link between the solutions of the widely known KP equation

$$\frac{\partial}{\partial \xi}\left(v_t + 6vv_\xi + v_{\xi\xi\xi}\right) = -3\alpha^2 v_{\eta\eta}, \tag{8}$$

and the solutions of the CKP equation discovered in 1986 [5]. Namely the map

$$v(\xi,\eta,\tau) \to u(x,y,t) := v(x - y^2t/12\alpha^2, yt, t), \tag{9}$$

transforms any solution of (8) to a solution of the CKP equation. Reciprocally for any solution $u(x,y,t)$ of the CKP equation, the function $v(\xi,\eta,\tau)$ defined by the formula

$$v(\xi,\eta,\tau) = u(\xi + \eta^2/12\alpha^2\tau, \eta/\tau, \tau),$$

satisfies the KP equation. In fact this correspondence can be detected already by observing the solutions obtained in section 3[3].

Moreover, the related Lax linear systems are connected by the following gauge equivalence also suggested by the formulas of section 3 or by the structure of the Baker-Akhiezer functions corresponding to the CKP Lax pair:

$$\begin{aligned} \psi_{CKP}(x,y,t) &= \psi_{KP}(x - y^2t/12\alpha^2, yt, t)\exp\left[-\frac{xy}{12\alpha} + \frac{4ty^3}{(12\alpha)^3}\right], \qquad (10)\\ \psi_{KP}(\xi,\eta,\tau) &= \psi_{CKP}(\xi + \eta^2/12\alpha^2\tau, \eta/\tau, \tau)\exp\left[\frac{\xi\eta}{12\alpha\tau} + \frac{4\eta^3}{(12\alpha\tau)^2}\left(1 - \frac{1}{3\alpha}\right)\right]. \end{aligned}$$

This gauge equivalence can also be checked by direct computation.

The related change of variables $\xi = x - y^2t/12\alpha^2, \eta = yt, \tau = t$ is invertible and its Jacobian is equal t for $t > 0$. In particular, it is clear that globally bounded solutions of the KP equation are mapped to globally bounded solutions of the CKP equation and vice versa. In order to solve the Cauchy initial problem posed at $t = t_0$ with initial data $u(x,y,t_0) = u_0(x,y)$, we have to find the solution $v(\xi,\eta,\tau)$ of the Cauchy initial problem for the KP equation with initial data

$$v(\xi,\eta) = u(\xi + \eta^2\tau/12\alpha^2\tau^0, \eta/\tau^0).$$

The solution of the related Cauchy problem for the CKP equation is given by the formula

$$u(x,y,t) = v(x - y^2t/12\alpha^2, yt, t).$$

The well known finite gap solutions of the KP equation (see for instance [16]) lead with the above relation to solutions of the CKP equation of the following form:

$$u = 2\partial_x^2 \ln\Theta\left[\left(x - \frac{y^2}{12\alpha t}\right)\vec{p} + yt\vec{v} + t\vec{q} + \vec{l}\right] + C.$$

[3] In fact it was detected by the 3rd author in the process of constructing the finite gap solutions of the CKP equation and was then confirmed by a direct computation. Somehow, at the time this remark was not included in the text [4].

Here Θ is the n-dimensional Riemann theta function. The related solutions are in general almost periodic in the x and t variables (as in the KP case) but, contrary to the KP case, they are not periodic or almost periodic with respect to y. (See Fig. 6 below).

It is also clear that

$$\begin{aligned} \int_{R^2}\int v^2(\xi,\eta,\tau)d\,\xi d\,\eta &= t\int_{R^2}\int u^2(x,y,t)d\,xd\,y, \\ u(x,y,t) &= v(x-y^2t/12\alpha^2, yt, t), \end{aligned} \tag{11}$$

where v is some solution of the KP equation rapidly decreasing at infinity. The LHS of (11) is an integral of motion in the KP case. It is quite clear that the related solution of the CKP equation has a dissipative nature: the integral of its square decreases like t^{-1}. Thus the CKP equation has no lump-like localized solutions propagating without change of shape.

Previously, some changes of variables were discovered by a number of authors [1, 9] not allowing to solve the Cauchy initial problem for the CKP equation on the base of the results obtained for the KP case. It is worthwhile to mention that before the discovery of the aforementioned change of variables, V. D. Lipovskii [12] developed a direct IST approach for solving the Cauchy initial problem with rapidly decreasing initial data for the CKP I and CKP II equation using the same technical tools (i.e., a nonlocal Riemann-Hilbert problem approach or $\bar{\partial}$-approach as in the KP case).

6 Hamiltonian framework for the CKP II equation

The auxiliary spectral problem has the form

$$\varphi_y = t\varphi_{xx} + 2t(ik - y/12)\varphi_x + tu\varphi. \tag{12}$$

By a Jost solution of (12) we shall denote a complex-valued function $\varphi(\mathbf{r},k,\overline{k})$, (where $\mathbf{r} := (x,y)$), bounded for all $k \in \mathbb{C}$, fixed by the asymptotic condition $\varphi(\mathbf{r},k,\overline{k}) \to 1$ when $|\mathbf{r}| \to \infty$. It is possible to prove that the following asymptotics introducing the "spectral data" always holds when u belongs to the Schwartz class as a function of $\mathbf{r}$:

$$\begin{gathered} \varphi_(\mathbf{r},k,\overline{k}) = 1 + \\ +i\cdot\mathrm{sgn}(k+\overline{k})\left[\frac{a(k,\overline{k})}{x-y^2t/12+2ikyt} - \frac{b(k,\overline{k})e^{s(\mathbf{r},k,\overline{k})}}{x-y^2t/12+2i\overline{k}yt}\right](1+o(1)), \end{gathered} \tag{13}$$

$$a(k,\overline{k}) = \frac{t}{2\pi}\int u(r)\varphi_(\mathbf{r},k,\overline{k})d\,xd\,y, \tag{14}$$

$$b(k,\overline{k}) = \frac{t}{2\pi}\int u(\mathbf{r})\varphi_(\mathbf{r},k,\overline{k})e^{-s(\mathbf{r},k,\overline{k})}d\,xd\,y, \tag{15}$$

$$s(\mathbf{r},k,\overline{k}) := -i(k+\overline{k})(x-y^2t/12)+(k^2-\overline{k}^2)yt. \quad (16)$$

It is assumed that the initial data $u(\mathbf{r},0)$ represent some real valued function belonging to the Schwartz class $\mathbb{S}(\mathbb{R}^2)$. This reality restriction implies that

$$a(k,\overline{k}) = \overline{a}(-\overline{k},-k), \quad b(k,\overline{k}) = \overline{b}(-\overline{k},-k).$$

For the solution $\tilde{\varphi}(\mathbf{r},k,\overline{k})$ to the "formal adjoint problem" fixed by the unit asymptotics at infinity, similar formulas hold modulo a change of the sign of the square bracket in (13) and a change $s \to -s$. The functions a and b are connected by the dispersion relation

$$a(k,\overline{k}) = a_0 - \int \frac{|b(p,\overline{p})|^2 \mathrm{sgn}(p+\overline{p})}{p-k} \frac{d\,p \bigwedge d\,\overline{p}}{2i\pi}. \quad (17)$$

The spectral data of the direct and the adjoint problem are connected by the relations

$$\overline{b}(k,\overline{k}) = \tilde{b}(k,\overline{k}), \qquad a(k,\overline{k}) = \tilde{a}(k,\overline{k}). \quad (18)$$

The functions φ and $\tilde{\varphi}$ admit, when $|k| \to \infty$, the following asymptotic expansions:

$$\varphi = 1 + \sum_{n=1}^{\infty} \varphi_n(\mathbf{r})k^{-n}, \qquad 1 + \sum_{n=1}^{\infty} \tilde{\varphi}_n(\mathbf{r})k^{-n},$$

where

$$2i\varphi_{1,x} + u = 0, \quad -2i\tilde{\varphi}_{1,x} + u = 0,$$

and the remaining coefficients can be computed from the recursive relations following from (12) and its formal adjoint. From the later relations we get after plugging them into (13) the asymptotic expansions for a and $\tilde{a}$:

$$a(k,\overline{k}) = \sum_{n=1}^{\infty} a_n k^{-n}, \quad \tilde{a}(k,\overline{k}) \sum_{n=1}^{\infty} \tilde{a}_n k^{-n}.$$

In addition we can conclude from the "reality condition" (18) that

$$a_n = \tilde{a}_n. \quad (19)$$

Comparison of this asymptotic expansion with the dispersion relation (17) leads to the infinite series of "trace identities". The first three of them are listed below.

$$P^x := -4\pi a_2 = \frac{t}{2} \int u^2 d\,x \bigwedge d\,y$$

$$= 4 \int_0^{\infty} d\,k_R \int_{\infty}^{\infty} d\,k_I |b(k,\overline{k})|^2 (k+\overline{k}),$$

$$P^y := -4\pi a_3 = \frac{1}{2} \int \left(u^2 \frac{yt}{6} + uw \right) d\,x \bigwedge d\,y$$

$$= 4i \int_0^{\infty} d\,k_R \int_{\infty}^{\infty} d\,k_I |b(k,\overline{k})|^2 (k^2 - \overline{k}^2),$$

$$H := -16\pi a_4 = \frac{t}{2} \int \left[u_x^2 - 2u^3 - 3\left(\frac{yu}{6} + \frac{w}{t}\right)^2 \right] d\,x \bigwedge d\,y$$

$$= 16 \int_0^{\infty} d\,k_R \int_{-\infty}^{\infty} d\,k_I |b(k,\overline{k})|^2 (k^3 + \overline{k}^3).$$

In the derivation of these formulas the second of the relations (18) should be taken into account. In the formulas above $w(x,y)$ is defined by the formula

$$w(x,y) := \int_{-\infty}^{x} u_y(x',y) d\,x'.$$

The relations $u \in \mathbf{S}(\mathbb{R}^2)$ and (19) define the phase space of the CKP II equation. The relations (19) impose a couple of constraints on the initial data:

$$\int u d\,x = 0, \text{and} \int w d\,x = 0.$$

The Hamiltonian form of the CKP II equation is described now by the following formulas:

$$\left(\frac{\partial}{\partial t} - \frac{y^2}{12}\frac{\partial}{\partial x} - \frac{y}{t}\frac{\partial}{\partial y}\right) u = \frac{1}{t}\frac{\partial}{\partial x}\frac{\delta H}{\delta u} = \{u, H\},$$

$$\{u(\mathbf{r}), u(\mathbf{r}')\} = \frac{1}{t}\delta(\mathbf{r} - \mathbf{r}').$$

6.1 Poisson brackets of the Spectral data and Action-Angle variables for the CKP II equation

$$\{a(k,\overline{k}), a(p,\overline{p})\} = 0,$$

$$\{a(k,\overline{k}), b(p,\overline{p})\} = \frac{ib(p,\overline{p})}{4\pi}\left(\frac{1}{k+\overline{p}} - \frac{1}{k-p}\right),$$

$$\{b(k,\overline{k}), b(p,\overline{p})\} = \frac{i}{4}\mathrm{sgn}(p_R)\delta(p_R + k_R)\delta(p_I - k_I).$$

The remaining brackets can be obtained from the reality conditions (18). It follows from the last three relations that $a(k,\overline{k})$ is a generating functional for the integrals of motion a_n, and $b(k,\overline{k},t)$ evolves according to the formula:

$$b(k,\overline{k},t) = b(k,\overline{k},t_0)\exp\left[-4i(k^3 + \overline{k}^3(t - t_0)\right].$$

The derivation of this formula is based on the first 3 formulas of this section and the equation of motion:

$$b_t = \{b, H\}.$$

The action-angle variables are:

$$J(k,\overline{k}) = -4|b(k,\overline{k})|^2, \Phi(k,\overline{k}) = \arg b(k,\overline{k})$$

There the time dependence is trivial:

$$J(k,\overline{k},t) = J(k,\overline{k},t_0), \quad \Phi(k,\overline{k},t) = \Phi(k,\overline{k},t_0) + \frac{\delta H}{\delta J(k,\overline{k})}(t-t_0).$$

It turns out that $b(k,\overline{k}) \in \mathbb{S}(\mathbb{R}^2)$. The condensed formulation of the result of this section is given by the following

Theorem 1. *On the class of Schwartz initial data with constraints $a_n = \tilde{a}_n$, the CKP II equation is a completely integrable Hamiltonian system, while the map $u \to b$ is the canonical transformation to action-angle variables J, Φ. Here*

$$\{\Phi(k,\overline{k}), \Phi(p,\overline{p})\} = \{J(k,\overline{k}), J(p,\overline{p})\} = 0,$$

$$\{\Phi(k,\overline{k}), J(p,\overline{p})\} = \delta(k_R - p_R)\delta(k_I - p_I),$$

$$H = -4\int_0^\infty d\,k_R \int_{-\infty}^\infty J(k,\overline{k})(k^3 + \overline{k}^3)d\,k_I.$$

In this formulation due to the "reality conditions", the spectral data are parameterized by the right half plane of the complex variable k.

7 Special solutions

In this section we present a collection of plots for solutions to the CKP equation. Fig. 1 corresponds to the image of the KP II lump solution,

$$u(x,y,t) = \frac{4\nu(1-\nu(x-3\nu t)^2 + \nu^2 y^2)}{(1+\nu(x-3\nu t)^2 + \nu^2 y^2)^2},$$

under the mapping (9). It is visible that when $t \to 0$, the front of the solution becomes more and more narrow, and it is concentrated in a small strip with respect to the variable y.

In Fig. 2 we show the evolution of the rational solution described by the formula (5). Here the pulse is for small t compressed in the x-direction and extended in the y-direction. For larger t, the y-extension reduces and the pulse gets stretched in the x-direction, the direction of propagation.

In Fig. 3 one can see the evolution of the rational solution of the KP equation obtained as an image of the inverse map applied to (5).

The single line soliton of the CKP equation is given by

$$u = 2\partial_x^2 \ln f_1 = 2\partial_x^2 \ln\cosh\frac{\vartheta_1 - \vartheta_2}{2},$$

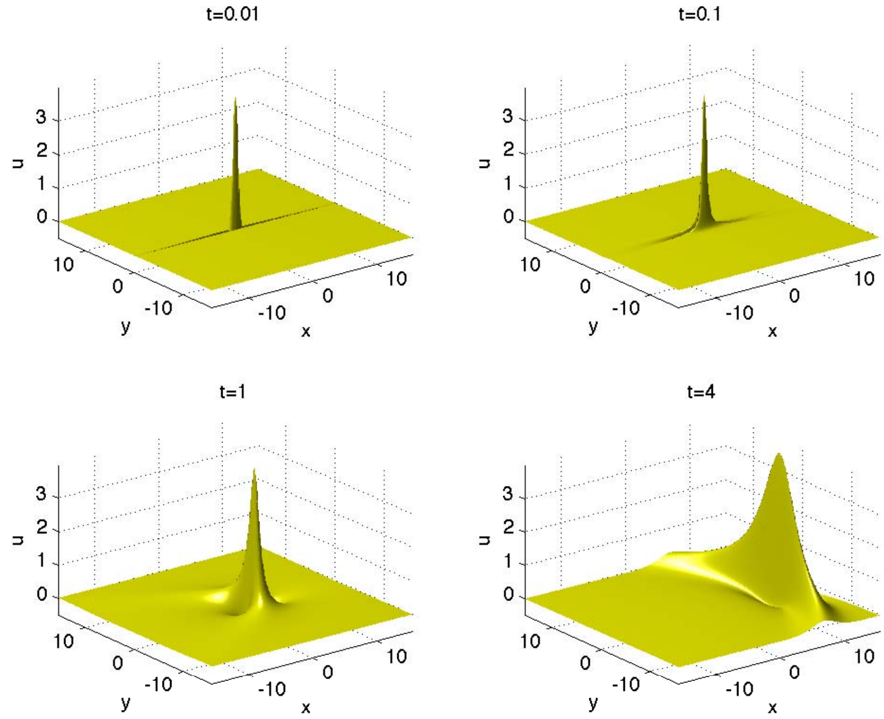

Figure 1: Solution to the CKP equation obtained as the image of the KP lump under the action of the map KP $\rightarrow$ CKP for several values of the time.

where

$$f_1 = e^{\vartheta_1} + e^{\vartheta_2}, \quad \vartheta_j := k_j \left(x - \frac{y^2 t}{12\alpha^2} \right) + k_j^2 \alpha^{-1} y t - 4k_j^3 t.$$

Its evolution from a KP-type line soliton to a horseshoe-type wave can be seen in Fig. 4.

We study the 2-soliton solution in the form

$$u = 2\partial_x^2 \ln W \left(\cosh \frac{\vartheta_1 - \vartheta_2}{2}, \sinh \frac{\vartheta_3 - \vartheta_4}{2} \right).$$

It is shown in Fig. 5 where the formation of horseshoe waves can be clearly recognized.

The last plot corresponds to the genus two hyperelliptic curve with branch points $-1, -2, -3, 0, 1, 2$. These solutions are numerically evaluated with the spectral code by Frauendiener and Klein [17, 18, 19]. The related solutions to the CKP equation are generated from the corresponding solutions of the KP

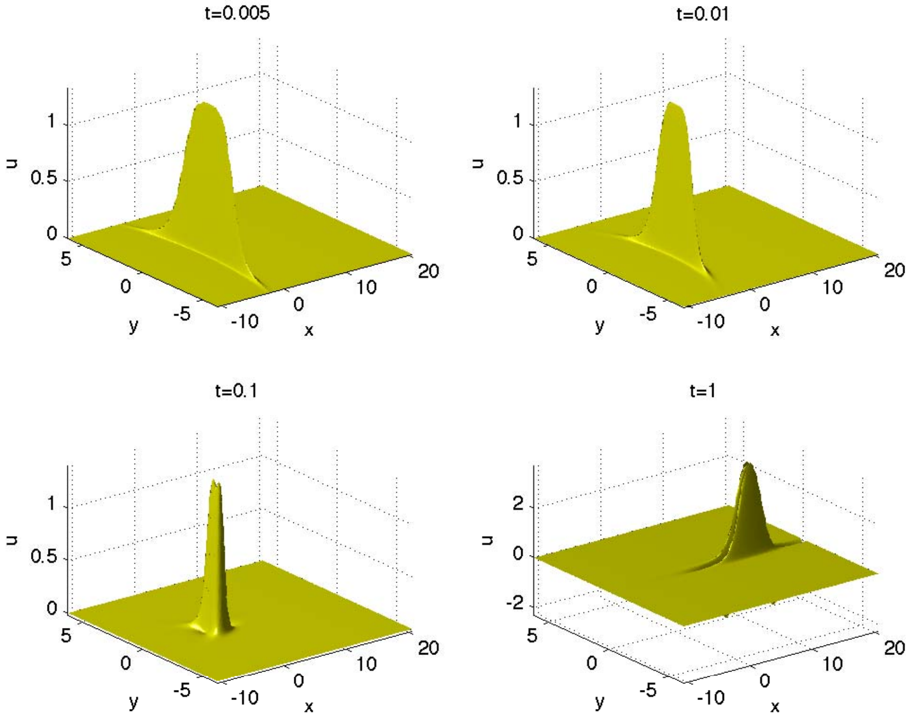

Figure 2: The rational solution of CKP obtained by the direct application of the binary Darboux dressing for several values of the time.

equation via the mapping (9). We clearly see the formation of intersecting families of parabolic fronts.

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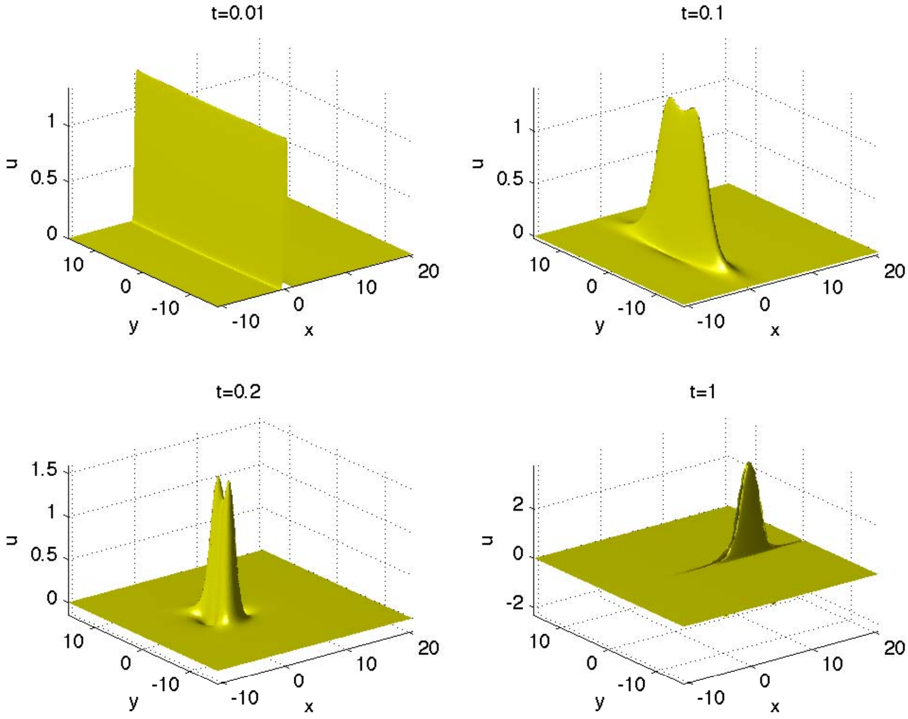

Figure 3: Rational solution to the KP equation obtained as an image of the inverse map CKP $\rightarrow$ KP applied to the rational solution shown before for several values of the time.

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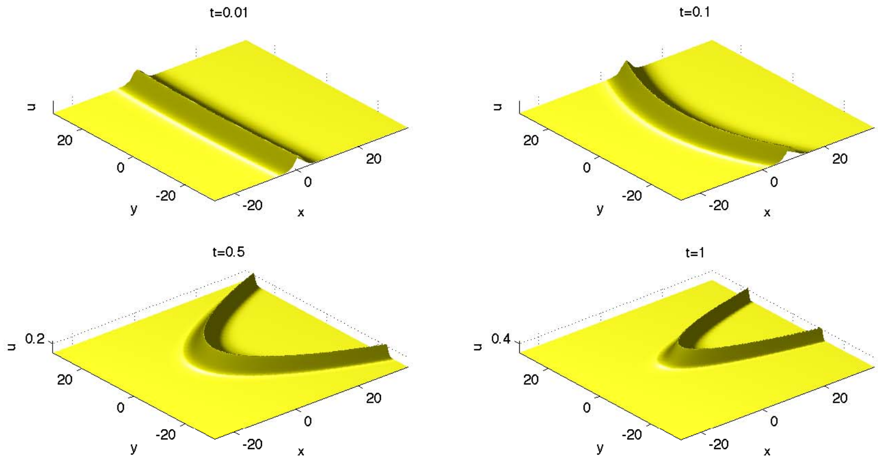

Figure 4: One soliton solution of the CKP equation with $k_1 = 1$, $k_2 = 0$ for several values of the time.

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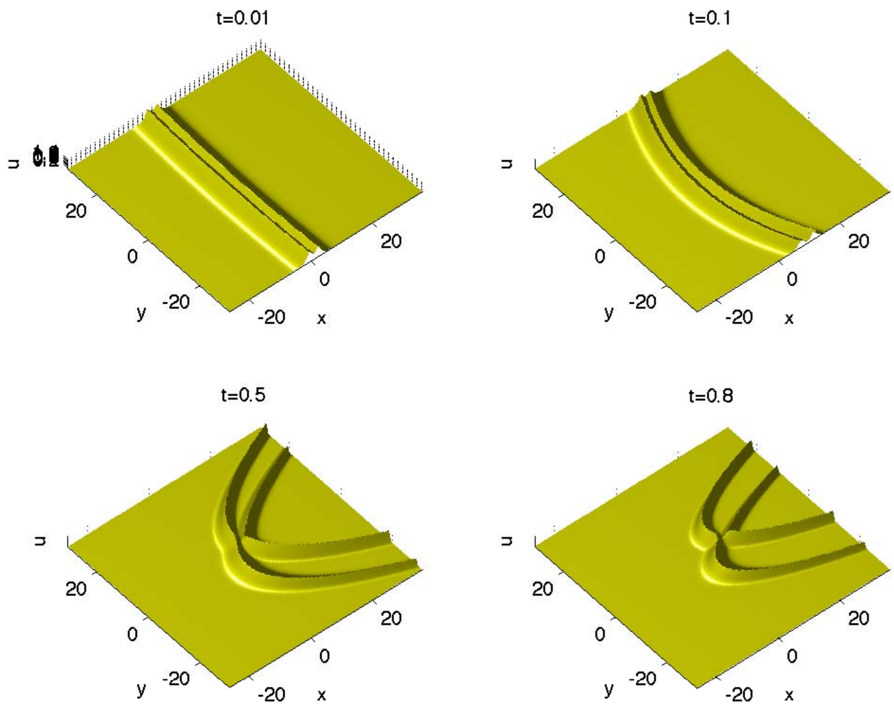

Figure 5: 2-soliton solution of the CKP equation with $k_1 = -1.5$, $k_2 = 0$, $k_3 = 2$, $k_4 = 0$ for several values of the time.

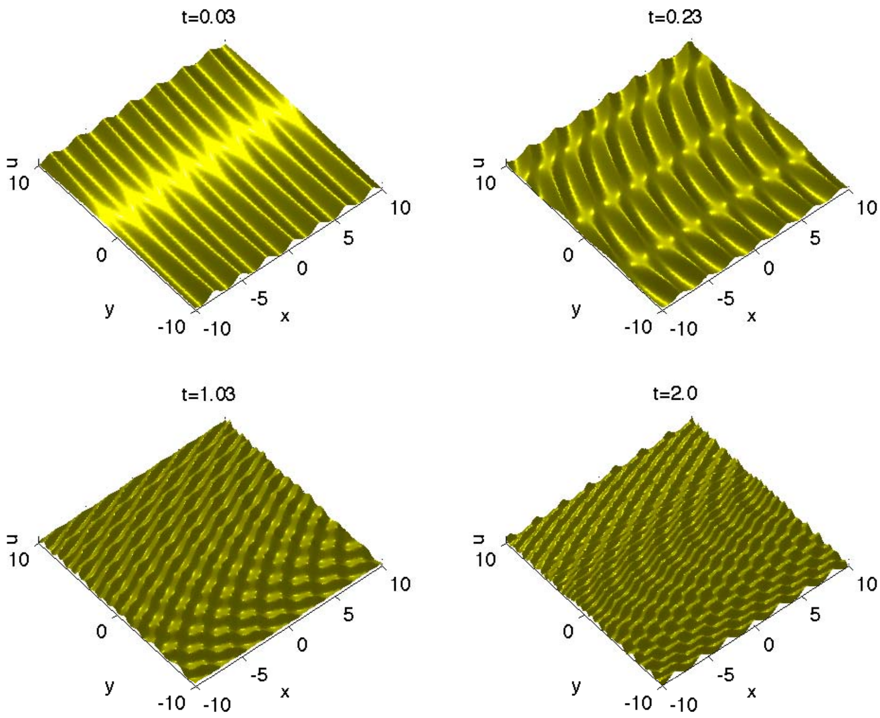

Figure 6: Genus 2 solution to the CKP equation generated by the curve $w^2 = \prod_{i=1}^{6}(z-e_i)$, $e_1 = -3, e_2 = -2, e_3 = -1, e_4 = 0, e_5 = 1, e_6 = 2$ for several values of t. The formation of intersecting families of parabolic fronts can be clearly seen.