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für Mathematik
in den Naturwissenschaften
Leipzig

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Preprint no.: 135 2006

RELATING THE CURVATURE TENSOR AND THE COMPLEX JACOBI OPERATOR OF AN ALMOST HERMITIAN MANIFOLD

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Abstract. Let J be a unitary almost complex structure on a Riemannian manifold (M,g). If x is a unit tangent vector, let $\pi := \operatorname{Span}\{x, Jx\}$ be the associated complex line in the tangent bundle of M. The complex Jacobi operator and the complex curvature operators are defined, respectively, by $\mathcal{J}(\pi) := \mathcal{J}(x) + \mathcal{J}(Jx)$ and $\mathcal{R}(\pi) := \mathcal{R}(x, Jx)$. We show that if (M,g) is Hermitian or if (M,g) is nearly Kähler, then either the complex Jacobi operator or the complex curvature operator completely determine the full curvature operator; this generalizes a well known result in the real setting to the complex setting. We also show this result fails for general almost Hermitian manifolds.

1. Introduction

We shall let $\mathcal{M} := (M,g)$ be a Riemannian manifold of dimension m. Let

$$\mathcal{R}(x,y) := \nabla_x\nabla_y - \nabla_y\nabla_x - \nabla_{[x,y]},$$
$$R(x,y,z,w) := g(\mathcal{R}(x,y)z,w)$$

be the curvature operator and the curvature tensor, respectively; R has the symmetries:

$$\begin{aligned} &R(x,y,z,w) = R(z,w,x,y) = -R(y,x,z,w),\\ &R(x,y,z,w) + R(y,z,x,w) + R(z,x,y,w) = 0\,. \end{aligned} \tag{1.a}$$

It is convenient to work in the algebraic setting. Let V be a vector space of dimension m. We say that $A \in \otimes^4(V^*)$ is an *algebraic curvature tensor* if A has the symmetries given in Equation (1.a). We consider a *model* $\mathfrak{M} := (V, \langle\cdot,\cdot\rangle, A)$ where $\langle\cdot,\cdot\rangle$ is an auxiliary positive definite inner product on V. Every model is geometrically realizable; given a model $\mathfrak{M}$, one can construct a Riemannian manifold $\mathcal{M}$ so that $\mathfrak{M}$ is isomorphic to (T_PM, g_P, R_P) for some point $P \in M$.

Given a model $\mathfrak{M}$, one uses the inner product $\langle\cdot,\cdot\rangle$ to raise indices and define an associated curvature operator $\mathcal{A}$. The *Jacobi operator* $\mathcal{J} : y \to \mathcal{A}(y,x)x$ is characterized by the identity

$$\langle \mathcal{J}(x)y, z\rangle = A(y,x,x,z)\,.$$

The Jacobi operator determines the full curvature tensor. The following theorem is well known; Assertion (2) in the geometric setting is an immediate consequence of the corresponding Assertion (1) in the algebraic setting:

Theorem 1.1.

(1) *Let* $\mathfrak{M}_i = (V_i, \langle\cdot,\cdot\rangle_i, A_i)$ *be models for* $i = 1,2$. *Suppose there exists an isometry* $\theta : (V_1, \langle\cdot,\cdot\rangle_1) \to (V_2, \langle\cdot,\cdot\rangle_2)$ *so that* $\mathcal{J}_{\mathfrak{M}_2}(\theta x)\theta = \theta\mathcal{J}_{\mathfrak{M}_1}(x)$ *for all* $x \in V_1$. *Then* $\theta^*A_2 = A_1$.

(2) *Let* $\mathcal{M}_i = (M_i, g_i)$ *be Riemannian manifolds for* $i = 1,2$. *Suppose there is an isometry* $\theta : (T_PM_1, g_1) \to (T_QM_2, g_2)$ *so that* $\mathcal{J}_{\mathcal{M}_2,Q}(\theta x)\theta = \theta\mathcal{J}_{\mathcal{M}_1,P}(x)$ *for all* $x \in T_PM_1$. *Then* $\theta^*R_2(Q) = R_1(P)$.

Key words and phrases. complex curvature operator, complex Jacobi operator, almost Hermitian manifold, Hermitian manifold, nearly Kähler manifold.
2000 *Mathematics Subject Classification.* 53C15, 53C55.

If J is a unitary almost complex structure on a Riemannian manifold (M, g), then $\mathcal{C} := (M, g, J)$ is said to be an *almost Hermitian manifold*. In the algebraic setting, $\mathfrak{C} := (V, \langle\cdot,\cdot\rangle, J, A)$ is said to be a *complex model* if J is a unitary complex structure and if A is an algebraic curvature tensor. Any point P of an almost Hermitian manifold $\mathcal{C}$ determines a corresponding complex model $\mathfrak{C}(\mathcal{C}, P) := (T_PM, g_P, J_P, R_P)$ in a natural fashion.

Let $\mathfrak{C}$ be a complex model. The Ricci tensor ρ and the $\star$-Ricci tensor $\rho^\star$ are defined by contracting indices. If $\{e_1, ..., e_m\}$ is an orthonormal basis for V, then

$$\rho(x, y) = \sum_{i=1}^{m} R(e_i, x, y, e_i) \quad \text{and} \quad \rho^\star(x, y) := \sum_{i=1}^{m} R(x, Je_i, Jy, e_i)\,.$$

We note that $\rho^\star$ is not in general a symmetric 2-tensor; however one does have that $\rho^\star(x, y) = \rho^\star(y, x)$ if the compatibility condition given below in Lemma 1.4 is satisfied. The scalar curvature τ and the $\star$-scalar curvature $\tau^\star$ are defined by a final contraction:

$$\tau = \sum_{i=1}^{m} \rho(e_i, e_i) \quad \text{and} \quad \tau^\star = \sum_{i=1}^{m} \rho^\star(e_i, e_i)\,.$$

We say a 2-dimensional subspace π of V is a *complex line* if $J\pi = \pi$. Let $\mathbb{CP}(V, J)$ be the complex projective space of complex lines in V. If $\pi \in \mathbb{CP}(V, J)$, let $S(\pi)$ be the set of unit vectors in π. Let $x \in S(\pi)$ for $\pi \in \mathbb{CP}(V, J)$. Then one has $\pi = \pi_x := \operatorname{Span}\{x, Jx\}$. The *holomorphic sectional curvature* $Q(\pi)$, *complex Jacobi operator* $\mathcal{J}(\pi)$, and *complex skew-symmetric curvature operator* $\mathcal{R}(\pi)$ are then defined for $\pi \in \mathbb{CP}(V, J)$, respectively, by setting

$$\begin{array}{ll} Q(\pi) := A(x, Jx, Jx, x), & \mathcal{J}(\pi) := \mathcal{J}(x) + \mathcal{J}(Jx), \\ \mathcal{R}(\pi) := \mathcal{R}(x, Jx) & \text{for any } x \in S(\pi)\,. \end{array} \tag{1.b}$$

We shall be considering several important families of almost Hermitian manifolds. $\mathcal{C}$ is said to be *Hermitian* if the Nijenhuis tensor vanishes, i.e. if

$$[X, Y] + J[JX, Y] + J[X, JY] - [JX, JY] = 0 \quad \text{for all} \quad X, Y\,.$$

Equivalently, see [10], this means that we can find local holomorphic coordinates $z_\nu = x_\nu + \sqrt{-1}y_\nu$ for $1 \le \nu \le \frac{1}{2}m$ so that $J\partial_{x_\nu} = \partial_{y_\nu}$ and $J\partial_{y_\nu} = -\partial_{x_\nu}$; the transition functions relating two such coordinate systems are then complex analytic. There are other natural assumptions that may be imposed and which define other important families. For example, one says that $\mathcal{C}$ is *nearly Kähler* if $(\nabla_x J)x = 0$ for all tangent vectors x; we refer to [9] for further information concerning this class of manifolds. We say that $\mathcal{C}$ is *almost Kähler* if the two form $\Omega(x, y) := \langle Jx, y\rangle$ is closed; we refer to [3] for a survey and to [2], [8] and [11] for some recent results concerning this class of manifolds.

The following result, which generalizes Theorem 1.1 to the complex setting, is the central result of this paper. It shows that the full curvature tensor is determined either by the complex Jacobi operator or by the complex curvature operator in certain natural geometric contexts:

Theorem 1.2. *For $i = 1, 2$, let $\mathcal{C}_i = (M_i, g_i, J_i)$ be either Hermitian or nearly Kähler manifolds. Let $\theta : (T_PM_1, g_1, J_1) \longrightarrow (T_QM_2, g_2, J_2)$ be a complex isometry. The following assertions are equivalent:*

(1) $\theta\mathcal{J}_{\mathcal{C}_1,P}(\pi) = \mathcal{J}_{\mathcal{C}_2,Q}(\theta\pi)\theta$ *for all* $\pi \in \mathbb{CP}(T_PM_1, J_{1,P})$.
(2) $\theta R_{\mathcal{C}_1,P}(\pi) = R_{\mathcal{C}_2,Q}(\theta\pi)\theta$ *for all* $\pi \in \mathbb{CP}(T_PM_1, J_{1,P})$.
(3) $\theta^* R_{\mathcal{C}_2,Q} = R_{\mathcal{C}_1,P}$.

Our result for almost Kähler setting manifolds is a bit weaker:

Theorem 1.3. *Let $\mathcal{C} = (M, g, J)$ be an almost Kähler manifold. Assume either that $\mathcal{J}(\pi) = 0$ for all $\pi \in \mathbb{CP}(TM, J)$ or that $\mathcal{R}(\pi) = 0$ for all $\pi \in \mathbb{CP}(TM, J)$. Then M is flat.*

Let $\mathfrak{C}$ be a complex model. There is a basic compatibility condition we work with that relates the structures J and A:

Lemma 1.4. *Let $\mathfrak{C} = (V, \langle\cdot,\cdot\rangle, J, A)$ be a complex model. The following conditions are equivalent and if any is satisfied, we shall say that $\mathfrak{C}$ is a* compatible complex model.

1. $J^*A = A$, *i.e.* $A(x, y, z, t) = A(Jx, Jy, Jz, Jt)$ *for all* $x, y, z, t \in V$.
2. $\mathcal{J}(\pi)J = J\mathcal{J}(\pi)$ *for all* $\pi \in \mathbb{CP}(V, J)$.
3. $\mathcal{A}(\pi)J = J\mathcal{A}(\pi)$ *for all* $\pi \in \mathbb{CP}(V, J)$.

We note, see Lemma 3.1, that if $\mathcal{C} = (M, g, J)$ is a nearly Kähler manifold, then $\mathfrak{C}(\mathcal{C}, P) = (T_PM, g_P, J_P, R_P)$ is a compatible complex model for any point $P \in M$. In general, a manifold satisfying this condition at every point is known in the literature as a RK-manifold.

What is perhaps rather surprising is that Theorem 1.2 does not have a purely algebraic analogue even if we impose the compatibility condition of Lemma 1.4:

Theorem 1.5. *If $m \equiv 0$ mod 4, there exists a complex model $\mathfrak{C} = (V, \langle\cdot,\cdot\rangle, J, A)$ with $A \neq 0$ so that $\mathcal{J}(\pi) = 0$ and so that $\mathcal{A}(\pi) = 0$ for every $\pi \in \mathbb{CP}(V, J)$.*

Let $\mathcal{C} = (M, g, J)$ be an almost Hermitian manifold. We let $\mathcal{U}_\mathcal{C}$ be the bundle of complex isometries of TM; the fibers of this bundle are the associated unitary group of the fibers. If $\Theta \in C^\infty\{\mathcal{U}_\mathcal{C}\}$ and if $P \in M$, then $\theta_P := \Theta(P)$ is a complex isometry of (T_PM, g_P, J_P) for any $P \in M$. We show that Theorem 1.2 fails in the almost Hermitian context by establishing the following result:

Theorem 1.6. *Let $m \equiv 0$ mod 4. There exists an almost Hermitian manifold $\mathcal{C}$ and there exists $\Theta \in C^\infty\{\mathcal{U}_\mathcal{C}\}$ so that for any point P in M we have:*

1. $\theta_P\mathcal{J}_\mathcal{C}(\pi) = \mathcal{J}_\mathcal{C}(\theta_P\pi)\theta_P$ *for all* $\pi \in \mathbb{CP}(T_PM, J)$.
2. $\theta_P\mathcal{R}_\mathcal{C}(\pi) = \mathcal{R}_\mathcal{C}(\theta_P\pi)\theta_P$ *for all* $\pi \in \mathbb{CP}(T_PM, J)$.
3. $\theta_P^*R_P \neq R_P$.

Here is a brief outline to this paper. Section 2 is algebraic in nature. We begin by establishing Lemma 1.4. Next, in Lemma 2.1, we prove a result of Vanhecke [17] which expresses $R(x, y, y, x)$ for a compatible complex model in terms of the holomorphic sectional curvature Q defined in Equation (1.b) and in terms of an additional tensor

$$\lambda(x, y) = R(x, y, y, x) - R(x, y, Jy, Jx)\,. \tag{1.c}$$

The identity of Lemma 2.1 is polarized to establish a result of Sato [12] in Lemma 2.2. We then turn to a study of the complex Jacobi operator by studying the condition $\mathcal{J}(\pi) = 0$ for all $\pi \in \mathbb{CP}(V, J)$ in Lemma 2.3 and show this implies that ρ and ρ^* both vanish. We then prove Theorem 1.5. We conclude Section 2 by relating Lemma 2.3 to the curvature decompositions of Gray [7] in Lemma 2.4.

In Section 3 we apply the results of Section 2 to the geometric context. We begin by recalling certain results of Gray and Yano concerning Hermitian, nearly Kähler and almost Kähler manifolds. These results are then applied to prove Theorems 1.2 and 1.3. The construction used to establish Theorem 1.5 in the algebraic setting is then used to prove Theorem 1.6 in the geometric setting.

There are many example in the literature of almost Kähler manifolds which are not Kähler [1, 5, 18]. Also, a great interest has been shown in finding conditions for an almost Kähler manifold to be Kähler. For example, the Goldberg conjecture states: *A compact Einstein almost Kähler manifold is Kähler.* This conjecture has generated extensive literature, see for example [14, 15]. Many other conditions have been studied which might imply that an almost Kähler manifold is Kähler, see [2, 4] for example. We conclude the paper in Section 4 with two related results.

We shall adopt the following notational conventions. The curvature tensor and curvature operator of a Riemannian manifold will be denoted by R and $\mathcal{R}$, respectively; an algebraic curvature tensor and the corresponding algebraic curvature operator will be denoted by A and $\mathcal{A}$, respectively. A real Riemannian manifold and a real model will be denoted by $\mathcal{M}=(M,g)$ and $\mathfrak{M}=(V,\langle\cdot,\cdot\rangle,A)$, respectively. An almost Hermitian manifold and a complex model will be denoted by $\mathcal{C}=(M,g,J)$ and $\mathfrak{C}=(V,\langle\cdot,\cdot\rangle,J,A)$, respectively. The Jacobi operator will be denoted by $\mathcal{J}$; we will subscript as appropriate when more than one curvature operator is under consideration.

2. Algebraic Results

Proof. We first establish Lemma 1.4. Suppose first that Assertion (1) holds, i.e. that $A(x,y,z,t)=A(Jx,Jy,Jz,Jt)$ for all x, y, z, t. Then

$$A(y,x,x,Jz)+A(y,Jx,Jx,Jz)=-A(Jy,x,x,z)-A(Jy,Jx,Jx,z)\,, \tag{2.a}$$

which implies $\langle\mathcal{J}(\pi_x)y,Jz\rangle=-\langle\mathcal{J}(\pi_x)Jy,z\rangle$ and, hence, $\mathcal{J}(\pi_x)J=J\mathcal{J}(\pi_x)$. Assume conversely that $\mathcal{J}(\pi_x)J=J\mathcal{J}(\pi_x)$ or equivalently that Equation (2.a) holds for all x. Polarizing this identity and replacing z by $-Jz$ yields

$$\begin{aligned} & A(y,x,w,z)+A(y,w,x,z)+A(y,Jx,Jw,z)+A(y,Jw,Jx,z)\\ = & A(Jy,x,w,Jz)+A(Jy,w,x,Jz)+A(Jy,Jx,Jw,Jz)\\ + & A(Jy,Jw,Jx,Jz)\,. \end{aligned} \tag{2.b}$$

Interchanging arguments $1\leftrightarrow 2$ and $3\leftrightarrow 4$ in the curvature tensors then yields:

$$\begin{aligned} & A(x,y,z,w)+A(w,y,z,x)+A(Jx,y,z,Jw)+A(Jw,y,z,Jx)\\ = & A(x,Jy,Jz,w)+A(w,Jy,Jz,x)+A(Jx,Jy,Jz,Jw)\\ + & A(Jw,Jy,Jz,Jx)\,. \end{aligned}$$

If we interchange x and y and we interchange z and w in this identity, we get

$$\begin{aligned} & A(y,x,w,z)+A(z,x,w,y)+A(Jy,x,w,Jz)+A(Jz,x,w,Jy)\\ = & A(y,Jx,Jw,z)+A(z,Jx,Jw,y)+A(Jy,Jx,Jw,Jz)\\ + & A(Jz,Jx,Jw,Jy)\,. \end{aligned} \tag{2.c}$$

Adding (2.b) and (2.c) and simplifying yields:

$$A(y,x,w,z)+A(y,w,x,z)=A(Jy,Jx,Jw,Jz)+A(Jy,Jw,Jx,Jz) \tag{2.d}$$

We permute the indices in Equation (2.d) to change $y\to x\to w\to y$. This yields:

$$A(x,w,y,z)+A(x,y,w,z)=A(Jx,Jw,Jy,Jz)+A(Jx,Jy,Jw,Jz)\,. \tag{2.e}$$

We add 2(2.d) and (2.e) and use the Bianchi identity to see

$$\begin{aligned} & 3A(y,w,x,z)=A(y,x,w,z)+2A(y,w,x,z)+A(x,w,y,z)\\ = & A(Jy,Jx,Jw,Jz)+2A(Jy,Jw,Jx,Jz)+A(Jx,Jw,Jy,Jz)\\ = & 3A(Jy,Jw,Jx,Jz)\,. \end{aligned}$$

The desired identity now follows.

We now prove that Assertion (1) implies Assertion (3). We have:

$$\begin{aligned} & \langle J\mathcal{A}(\pi_x)y,z\rangle=-\langle\mathcal{A}(\pi_x)y,Jz\rangle=-A(x,Jx,y,Jz)\\ = & -A(Jx,JJx,Jy,JJz)=A(x,Jx,Jy,z)=\langle\mathcal{A}(\pi_x)Jy,z\rangle\,. \end{aligned}$$

Thus $J\mathcal{A}(\pi_x)=\mathcal{A}(\pi_x)J$ as desired.

We finally show that Assertion (3) implies Assertion (1). We have

$$\begin{aligned} & J\mathcal{A}(x,Jx)=\mathcal{A}(x,Jx)J,\\ \Rightarrow\quad & \langle JA(x,Jx)z,w\rangle-\langle A(x,Jx)Jz,w\rangle=0,\\ \Rightarrow\quad & A(x,Jx,z,Jw)+A(x,Jx,Jz,w)=0\,. \end{aligned}$$

Polarizing yields an identity for all x, y, z, w:

$$\begin{aligned} 0 &= A(y,Jx,z,Jw)+A(x,Jy,z,Jw)+A(y,Jx,Jz,w)\\ &+ A(x,Jy,Jz,w)\,. \end{aligned}$$

Interchange the first two arguments in the first and third term to see:

$$\begin{aligned} 0 &= -A(Jx,y,z,Jw)+A(x,Jy,z,Jw)-A(Jx,y,Jz,w)\\ &+ A(x,Jy,Jz,w)\,. \end{aligned}$$

Replace (x, w) by (Jx, Jw) to show:

$$\begin{aligned} 0 &= -A(x,y,z,w)-A(Jx,Jy,z,w)+A(x,y,Jz,Jw)\\ &+ A(Jx,Jy,Jz,Jw)\,. \end{aligned}\tag{2.f}$$

Interchange the first two arguments with the final two arguments:

$$\begin{aligned} 0 &= -A(z,w,x,y)-A(z,w,Jx,Jy)+A(Jz,Jw,x,y)\\ &+ A(Jz,Jw,Jx,Jy)\,. \end{aligned}$$

Change notation to interchange x and z, and y and w, to see:

$$\begin{aligned} 0 &= -A(x,y,z,w)-A(x,y,Jz,Jw)+A(Jx,Jy,z,w)\\ &+ A(Jx,Jy,Jz,Jw)\,. \end{aligned}\tag{2.g}$$

We add Equations (2.f) and (2.g) to conclude

$$-A(x,y,z,w)+A(Jx,Jy,Jz,Jw)=0$$

and complete the proof that Assertion (3) implies Assertion (1). □

The following result is due to Vanhecke [17]; it was originally stated in a purely geometrical setting. Let Q and λ be defined by Equations (1.b) and (1.c), respectively.

Lemma 2.1. *Let $\mathfrak{C}=(V,\langle\cdot,\cdot\rangle,A,J)$ be a compatible complex model. Then*

$$\begin{aligned} 32A(x,y,y,x) &= 3Q(x+Jy)+3Q(x-Jy)-Q(x+y)-Q(x-y)\\ &- 4Q(x)-4Q(y)+4\{5\lambda(x,y)+\lambda(x,Jy)\}\,. \end{aligned}$$

Proof. First note that

$$\begin{aligned} Q(x+y) &= A(x+y,Jx+Jy,Jx+Jy,x+y)\\ &= A(x,Jx,Jx,x)+A(x,Jy,Jy,x)\\ &\quad +A(y,Jx,Jx,y)+A(y,Jy,Jy,y)\\ &\quad +2A(x,Jx,Jx,y)+2A(x,Jy,Jy,y)+2A(x,Jx,Jy,x)\\ &\quad +2A(y,Jx,Jy,y)+2A(x,Jx,Jy,y)+2A(x,Jy,Jx,y). \end{aligned}$$

Hence

$$\begin{aligned} &3Q(x+Jy)+3Q(x-Jy)-Q(x+y)-Q(x-y)-4Q(x)-4Q(y)\\ &\quad =3\{A(x,Jx,Jx,x)+A(x,y,y,x)+A(Jy,Jx,Jx,Jy)+A(Jy,y,y,Jy)\\ &\quad +2A(x,Jx,Jx,Jy)+2A(x,y,y,Jy)-2A(x,Jx,y,x)\\ &\quad -2A(Jy,Jx,y,Jy)-2A(x,Jx,y,Jy)-2A(x,y,Jx,Jy)\}\\ &\quad +3\{A(x,Jx,Jx,x)+A(x,y,y,x)+A(Jy,Jx,Jx,Jy)+A(Jy,y,y,Jy)\\ &\quad -2A(x,Jx,Jx,Jy)-2A(x,y,y,Jy)+2A(x,Jx,y,x)\\ &\quad +2A(Jy,Jx,y,Jy)-2A(x,Jx,y,Jy)-2A(x,y,Jx,Jy)\}\\ &\quad -\{A(x,Jx,Jx,x)+A(x,Jy,Jy,x)+A(y,Jx,Jx,y)+A(y,Jy,Jy,y) \end{aligned}$$

$$\begin{aligned}
&+2A(x,Jx,Jx,y)+2A(x,Jy,Jy,y)+2A(x,Jx,Jy,x)\\
&+2A(y,Jx,Jy,y)+2A(x,Jx,Jy,y)+2A(x,Jy,Jx,y)\}\\
&-\{A(x,Jx,Jx,x)+A(x,Jy,Jy,x)+A(y,Jx,Jx,y)+A(y,Jy,Jy,y)\\
&-2A(x,Jx,Jx,y)-2A(x,Jy,Jy,y)-2A(x,Jx,Jy,x)\\
&-2A(y,Jx,Jy,y)+2A(x,Jx,Jy,y)+2A(x,Jy,Jx,y)\}\\
&-4\{A(x,Jx,Jx,x)+A(y,Jy,Jy,y)\}\\
&=6A(x,y,y,x)+6A(Jx,Jy,Jy,Jx)-2A(x,Jy,Jy,x)-2A(y,Jx,Jx,y)\\
&-12A(x,y,Jx,Jy)-12A(x,Jx,y,Jy)-4A(x,Jx,Jy,y)-4A(x,Jy,Jx,y)\\
&=12A(x,y,y,x)-4A(x,Jy,Jy,x)\\
&-12A(x,y,Jx,Jy)-8A(x,Jx,y,Jy)-4A(x,Jy,Jx,y).
\end{aligned}$$

We may now compute:

$$\begin{aligned}
&3Q(x+Jy)+3Q(x-Jy)-Q(x+y)-Q(x-y)\\
&\quad-4Q(x)-4Q(y)+4\{5\lambda(x,y)+\lambda(x,Jy)\}\\
&=12A(x,y,y,x)-4A(x,Jy,Jy,x)\\
&\quad-12A(x,y,Jx,Jy)-8A(x,Jx,y,Jy)-4A(x,Jy,Jx,y)\\
&\quad+20(A(x,y,y,x)-A(x,y,Jy,Jx))\\
&\quad+4(A(x,Jy,Jy,x)+A(x,Jy,y,Jx))\\
&=32A(x,y,y,x)-8A(x,y,Jy,Jx)-8A(x,Jx,y,Jy)-8A(x,Jy,Jx,y).
\end{aligned}$$

The desired identity now follows from the Bianchi identity. □

The following tensors arise naturally and play a fundamental role in studying complex models. The tensor A_0 has constant sectional curvature $+1$ and the curvature tensor of complex projective space with the Fubini-Study metric is given by $A_0 + A_J$ where we define:

$$\begin{aligned}
&A_0(x,y,z,w):=\langle x,w\rangle\langle y,z\rangle-\langle x,z\rangle\langle y,w\rangle\,,\\
&A_J(x,y,z,w):=\langle x,Jw\rangle\langle y,Jz\rangle-\langle x,Jz\rangle\langle y,Jw\rangle-2\langle x,Jy\rangle\langle z,Jw\rangle\,.
\end{aligned}\tag{2.h}$$

The following result is due to Sato [12]; again, it was stated in a geometrical context.

Lemma 2.2. *Let $\mathfrak{C}=(V,\langle\cdot,\cdot\rangle,A,J)$ be a compatible complex model.*

(1) *If $\mathfrak{C}$ has constant holomorphic sectional curvature c, then*

$$\begin{aligned}
&A(x,y,z,w)=\tfrac{c}{4}\{A_0(x,y,z,w)+A_J(x,y,z,w)\}\\
&\quad+\tfrac{1}{8}\{5A(x,y,z,w)-3A(x,y,Jz,Jw)+A(x,z,Jw,Jy)\\
&\quad-A(x,w,Jz,Jy)-A(x,Jz,w,Jy)+A(x,Jw,z,Jy)\}.
\end{aligned}$$

(2) *If $\mathfrak{C}$ has constant zero holomorphic sectional curvature, then*

$$\begin{aligned}
&3A(x,y,z,w)+3A(x,y,Jz,Jw)\\
&=A(x,z,Jw,Jy)-A(x,w,Jz,Jy)-A(x,Jz,w,Jy)+A(x,Jw,z,Jy)\,.
\end{aligned}$$

Proof. As the holomorphic sectional curvature is constant, $Q(x)=c\langle x,x\rangle^2$. We use the identity of Lemma 2.1 to see:

$$\begin{aligned}
&32A(x,y,y,x)\\
&=3c\{\langle x,x\rangle+\langle y,y\rangle+2\langle x,Jy\rangle\}^2+3c\{\langle x,x\rangle+\langle y,y\rangle-2\langle x,Jy\rangle\}^2\\
&-c\{\langle x,x\rangle+\langle y,y\rangle+2\langle x,y\rangle\}^2-c\{\langle x,x\rangle+\langle y,y\rangle-2\langle x,y\rangle\}^2\\
&-4c\langle x,x\rangle^2-4c\langle y,y\rangle^2+4\{5\lambda(x,y)+\lambda(x,Jy)\}\\
&=8c\{\langle x,x\rangle\langle y,y\rangle-\langle x,y\rangle^2+3\langle x,Jy\rangle^2\}+4\{5\lambda(x,y)+\lambda(x,Jy)\}.
\end{aligned}$$

We now polarize this identity to see:

$$\begin{array}{rl} & 8(A(x,y,z,w)+A(x,z,y,w)) \\ = & 2c\{2\langle x,w\rangle\langle y,z\rangle-\langle x,y\rangle\langle z,w\rangle-\langle x,z\rangle\langle w,y\rangle \\ + & 3\langle x,Jy\rangle\langle w,Jz\rangle+3\langle x,Jz\rangle\langle w,Jy\rangle\} \\ + & 5\{A(x,y,z,w)+A(x,z,y,w)-A(x,y,Jz,Jw)-A(x,z,Jy,Jw)\} \\ + & A(x,Jy,z,Jw)+A(x,Jz,y,Jw)+A(x,Jy,Jz,w)+A(x,Jz,Jy,w)\,. \end{array} \tag{2.i}$$

Interchanging x and y in Equation (2.i) yields:

$$\begin{array}{rl} & 8(A(y,x,z,w)+A(y,z,x,w)) \\ = & 2c\{2\langle y,w\rangle\langle x,z\rangle-\langle y,x\rangle\langle z,w\rangle-\langle y,z\rangle\langle w,x\rangle \\ + & 3\langle y,Jx\rangle\langle w,Jz\rangle+3\langle y,Jz\rangle\langle w,Jx\rangle\} \\ + & 5\{A(y,x,z,w)+A(y,z,x,w)-A(y,x,Jz,Jw)-A(y,z,Jx,Jw)\} \\ + & A(y,Jx,z,Jw)+A(y,Jz,x,Jw)+A(y,Jx,Jz,w)+A(y,Jz,Jx,w)\,. \end{array} \tag{2.j}$$

We now subtract Equation (2.j) from Equation (2.i) and simplify to obtain:

$$\begin{array}{l} 8(A(x,y,z,w)+A(x,z,y,w)-A(y,x,z,w)-A(y,z,x,w)) \\ =24A(x,y,z,w) \\ =2c\{3\langle x,w\rangle\langle y,z\rangle-3\langle x,z\rangle\langle w,y\rangle \\ +3\langle x,Jw\rangle\langle y,Jz\rangle-3\langle x,Jz\rangle\langle y,Jw\rangle-6\langle x,Jy\rangle\langle z,Jw\rangle\} \\ +10A(x,y,z,w)+5A(x,w,z,y)-5A(x,z,w,y) \\ -10A(x,y,Jz,Jw)+A(x,Jw,Jz,y)-A(x,Jz,Jw,y) \\ -5A(x,w,Jz,Jy)+A(x,Jy,Jz,w)-A(x,Jz,w,Jy)+A(x,Jy,Jz,w) \\ +5A(x,z,Jw,Jy)+A(x,Jy,z,Jw)+A(x,Jw,z,Jy)+A(x,Jy,z,Jw) \\ =2c\{3R_0(x,y,z,w)+3R_J(x,y,z,w)\}+15A(x,y,z,w)-9A(x,y,Jz,Jw) \\ -3A(x,w,Jz,Jy)-3A(x,Jz,w,Jy)+3A(x,Jw,z,Jy)+3A(x,z,Jw,Jy). \end{array}$$

The desired results now follow. □

We begin the proper study of the complex Jacobi operator by examining the condition $\mathcal{J}(\cdot)=0$.

Lemma 2.3. *Let $\mathfrak{C}=(V,\langle\cdot,\cdot\rangle,J,A)$ be a complex model.*

(1) *The following conditions are equivalent:*
   (a) $\mathcal{J}(\pi)=0$ *for all* $\pi\in\mathbb{CP}(V,J)$.
   (b) $\mathcal{A}(x,y)=-\mathcal{A}(Jx,Jy)$ *for all* x,y.
   (c) $\mathcal{A}(\pi)=0$ *for all* $\pi\in\mathbb{CP}(V,J)$.
   (d) $\mathcal{A}(Jx,y)z=\mathcal{A}(x,Jy)z=\mathcal{A}(x,y)Jz$ *for all* x,y,z.

(2) *If any of the conditions in (1) are satisfied, then:*
   (a) $\mathfrak{C}$ *is compatible.*
   (b) $\mathfrak{C}$ *is Ricci flat and* $\star$*-Ricci flat.*

Proof. Suppose Condition (1a) holds. Then $\mathfrak{C}$ is compatible. Furthermore,

$$Q(x)=R(x,Jx,Jx,x)=R(x,Jx,Jx,x)+R(x,x,x,x)=\langle\mathcal{J}(\pi_x)x,x\rangle=0\,,$$

so $\mathfrak{C}$ has constant holomorphic sectional curvature 0. Thus Lemma 2.2 (2) applies and we show that Condition (1b) holds by computing:

$$\begin{array}{rl} & 3A(x,y,z,w)+3A(x,y,Jz,Jw) \\ = & A(x,z,Jw,Jy)-A(x,w,Jz,Jy)-A(x,Jz,w,Jy)+A(x,Jw,z,Jy) \\ = & \langle\{\mathcal{J}(\pi_{z+Jw})-\mathcal{J}(\pi_z)-\mathcal{J}(\pi_{Jw})\}x,Jy\rangle=0\,. \end{array}$$

Suppose Condition (1b) holds. We establish Condition (1c) by computing:

$$\mathcal{A}(\pi_x)=\mathcal{A}(x,Jx)=-\mathcal{A}(Jx,JJx)=\mathcal{A}(Jx,x)=-\mathcal{A}(x,Jx)=-\mathcal{A}(\pi_x)\,.$$

Suppose Condition (1c) holds. Then $\mathfrak{C}$ is compatible. Again Lemma 2.2 is applicable. We set $w = Jz$ in Lemma 2.2 (2) to show Condition (1a) holds by computing:

$$\begin{aligned} 0 &= 6\langle\mathcal{A}(\pi_z)x, y\rangle = 6A(x, y, z, Jz) \\ &= -2A(x, z, z, Jy) - 2A(x, Jz, Jz, Jy) = -2\langle\mathcal{J}(\pi_z)x, Jy\rangle\,. \end{aligned}$$

We have shown that Conditions (1a), (1b), and (1c) are equivalent. Suppose that Condition (1d) holds. We show that Condition (1c) holds by computing:

$$\mathcal{A}(\pi_x) = \mathcal{A}(x, Jx) = \mathcal{A}(Jx, x) = -\mathcal{A}(x, Jx) = -\mathcal{A}(\pi_x)\,.$$

Finally suppose that Condition (1a) holds. We must show Condition (1d) holds. Since Condition (1a) implies Condition (1b), we have $\mathcal{A}(x, y) = -\mathcal{A}(Jx, Jy)$. Thus

$$A(Jx, y)z = -A(JJx, Jy)z = \mathcal{A}(x, Jy)z\,.$$

Since $\mathcal{J}(\pi_y) = 0$ and since $\mathfrak{C}$ is compatible,

$$0 = A(x, y, y, w) + A(x, Jy, Jy, w) = A(x, y, y, w) + A(Jx, y, y, Jw)\,.$$

Polarize this identity to see

$$0 = A(x, y, z, w) + A(x, z, y, w) + A(Jx, y, z, Jw) + A(Jx, z, y, Jw)\,. \tag{2.k}$$

Since $\mathcal{A}(x, w) = -\mathcal{A}(Jx, Jw)$, $A(Jx, Jw, y, z) = -A(x, w, y, z)$. We may therefore use the First Bianchi identity to see:

$$\begin{aligned} 0 &= A(Jx, y, z, Jw) + A(Jx, Jw, y, z) + A(Jx, z, Jw, y) \\ &+ A(x, y, z, w) + A(x, w, y, z) + A(x, z, w, y) \\ &= A(Jx, y, z, Jw) - A(Jx, z, y, Jw) + A(x, y, z, w) - A(x, z, y, w)\,. \end{aligned} \tag{2.l}$$

Adding (2.k) and (2.l) we get

$$0 = 2A(x, y, z, w) + 2A(Jx, y, z, Jw)\,.$$

Replacing w by Jw and changing the order then shows Condition (1a) implies Condition (1d) since:

$$A(x, y, Jw, z) = A(Jx, y, w, z)\,.$$

This completes the proof of Assertion (1). It is clear that (1a) implies $\mathfrak{C}$ is compatible, as we said before. Assume the conditions of Assertion (1) hold. We may then compute:

$$\begin{aligned} 2\rho(x, y) &= \sum_{i=1}^{m}\{A(e_i, x, y, e_i) + A(Je_i, x, y, Je_i)\} = \sum_{i=1}^{m}\langle\mathcal{J}(\pi_{e_i})x, y\rangle) = 0, \\ \rho^\star(x, y) &= \sum_{i=1}^{m} A(x, Je_i, Jy, e_i) = \sum_{i=1}^{m} A(x, e_i, JJy, e_i) = \rho(x, y) = 0\,. \end{aligned}$$

where $\{e_1, \ldots, e_m\}$ forms an orthonormal basis. This completes the proof of the Lemma. □

Proof. We now establish Theorem 1.5. Since the dimension of V is a multiple of 4, we may choose almost complex structures J, K such that $JK + KJ = 0$. Consider the algebraic curvature tensor $A := A_K - A_{JK}$. Note that the Jacobi operator is given by

$$\mathcal{J}(x)y = 3\langle y, Kx\rangle Kx - 3\langle y, JKx\rangle JKx\,,$$

and, since $\mathcal{J}(x)Kx = Kx$ for any unit vector x, $A \neq 0$. However, the complex Jacobi operator vanishes identically:

$$\begin{aligned} \mathcal{J}(\pi_x) &= 3\langle y, Kx\rangle Kx - 3\langle y, JKx\rangle JKx \\ &\quad +3\langle y, KJx\rangle KJx - 3\langle y, JKJx\rangle JKJx \\ &= 0\,. \end{aligned}$$

We may now apply Lemma 2.3 to see that $\mathcal{A}(\pi)$ also vanishes identically as well. $\square$

We conclude this section by putting things in a slightly different invariant framework. Let $\mathfrak{A}(V)$ be the vector space of all algebraic curvature tensors on V. We use $\langle\cdot,\cdot\rangle$ to define a natural inner product on $\mathfrak{A}(V)$ by setting:

$$\langle A_1, A_2\rangle := \sum_{i,j,k,l} A_1(e_i,e_j,e_k,e_l)A_2(e_i,e_j,e_k,e_l)\,;$$

this is independent of the particular orthonormal basis $\{e_i\}$ chosen. Consider the following subspaces [7]:

$$\begin{aligned}
\mathfrak{A}_1(V,J) &= \{A\in\mathfrak{A}(V): A(x,y,z,w)=A(Jx,Jy,z,w)\}\,,\\
\mathfrak{A}_2(V,J) &= \{A\in\mathfrak{A}(V): A(x,y,z,w)=A(Jx,Jy,z,w)\\
&\qquad +A(Jx,y,Jz,w)+A(Jx,y,z,Jw)\},\\
\mathfrak{A}_3(V,J) &= \{A\in\mathfrak{A}(V): A(x,y,z,w)=A(Jx,Jy,Jz,Jw)\}\,.
\end{aligned}$$

Note that $\mathfrak{A}_1(V,J)$ is the space of algebraic curvature tensors which verify the Kähler identity and $\mathfrak{A}_3(V,J)$ is the space of compatible curvature tensors. We have

$$\mathfrak{A}_1(V,J)\subset\mathfrak{A}_2(V,J)\subset\mathfrak{A}_3(V,J)\,.$$

Denote by $\mathfrak{A}_2^\perp(V,\langle\cdot,\cdot\rangle,J)$ the orthogonal complement of $\mathfrak{A}_2(V,J)$ in $\mathfrak{A}_3(V,J)$.

Lemma 2.4. *Let $\mathfrak{C}=(V,\langle\cdot,\cdot\rangle,A)$ be a complex model. The following assertions are equivalent:*

(1) *$\mathcal{J}(\pi)=0$ for all $\pi\in\mathbb{CP}(V,J)$.*
(2) *$A\in\mathfrak{A}_2^\perp(V,\langle\cdot,\cdot\rangle,J)$.*

Proof. Projection in $\mathfrak{A}_2(V,J)$ restricted to $\mathfrak{A}_3(V,J)$ is given by

$$\begin{aligned}
\mathcal{P}_2(A)(x,y,z,w) &= \tfrac12\{A(x,y,z,w)+A(Jx,Jy,z,w)\\
&\qquad +A(Jx,y,Jz,w)+A(Jx,y,z,Jw)\}\,.
\end{aligned}$$

$\mathcal{P}_2$ is an involutive isometry and $\mathfrak{A}_2^\perp(V,\langle\cdot,\cdot\rangle,J)$ is the (-1)-eigenspace of $\mathcal{P}_2$ by [16]. For $A\in\mathfrak{A}_3(V,J)$ compute

$$\mathcal{P}_2(A)(x,y,z,w)+\mathcal{P}_2(A)(Jx,Jy,z,w)=A(x,y,z,w)+A(Jx,Jy,z,w)\,.$$

From here, it is easy to verify that

$$\mathfrak{A}_2^\perp(V,\langle\cdot,\cdot\rangle,J)=\{A\in\mathfrak{A}(V): A(x,y,z,w)=-A(Jx,Jy,z,w)\}\,.$$

The desired result now follows from Lemma 2.3. $\square$

3. Geometrical Results

We begin our study of the geometrical context by recalling several well known results. We refer to Gray [7] for the proof of Assertions (1) and (2), see also [6], and to Yano [19] for the proof of Assertion (3) in the following Lemma:

Lemma 3.1. *Let $\mathcal{C}=(M,g,J)$ be an Hermitian manifold and let $\mathfrak{C}=\mathfrak{C}(\mathcal{C},P)$ be the almost complex model determined by $\mathcal{C}$ at a point $P\in M$. Then:*

(1) *If $\mathcal{C}$ is Hermitian, then*

$$\begin{aligned}
&R(x,y,z,w)+R(Jx,Jy,Jz,Jw)=R(Jx,Jy,z,w)+R(x,y,Jz,Jw)\\
&\qquad +R(Jx,y,Jz,w)+R(x,Jy,z,Jw)+R(Jx,y,z,Jw)+R(x,Jy,Jz,w)\,.
\end{aligned}$$

(2) *If $\mathcal{C}$ is nearly Kähler, then $\mathcal{C}$ is compatible and*

$$\begin{aligned}
&R(x,y,z,w)+R(Jx,Jy,Jz,Jw)=R(Jx,Jy,z,w)+R(x,y,Jz,Jw)\\
&\qquad +R(Jx,y,Jz,w)+R(x,Jy,z,Jw)+R(Jx,y,z,Jw)+R(x,Jy,Jz,w)\,.
\end{aligned}$$

(3) *If $\mathcal{C}=(M,g,J)$ is an almost Kähler manifold, then $\|\nabla J\|^2=2(\tau^\star-\tau)$.*

Let $\mathcal{C}_i$ be almost Hermitian manifolds. We suppose given a complex isometry $\theta : (T_PM_1, g_1, J_1) \to (T_QM_2, g_2, J_2)$. We let $V = T_PM_1$, $\langle\cdot,\cdot\rangle = g_1$, $J = J_1$, and $A := R_1 - \theta^* R_2$. Let

$$\mathfrak{C} = \mathfrak{C}(\mathcal{C}_1, P, \mathcal{C}_2, Q, \theta) := (V, \langle\cdot,\cdot\rangle, J, A)\,.$$

Theorem 1.2 will follow from the following Lemma:

Lemma 3.2. *Adopt the notation established above. Assume that $\mathcal{C}_i$ are Hermitian or nearly Kähler manifolds. The following assertions are equivalent:*

1. $\mathcal{J}(\pi) = 0$ *for all* $\pi \in \mathbb{CP}(V, J)$.
2. $\mathcal{R}(\pi) = 0$ *for all* $\pi \in \mathbb{CP}(V, J)$.
3. $A = 0$.

Proof. Assume that either Condition (1) or Condition (2) holds; these are equivalent by Lemma 2.3. Since the curvature tensors $A_{\mathcal{C}_i}$ satisfy the identity of Lemma 3.1, so does their difference. We use the relations provided by Lemma 2.3 to show that $A = 0$ by computing:

$$\begin{aligned}
0 &= R(x,y,z,w) + R(Jx,Jy,Jz,Jw) - R(Jx,Jy,z,w) - R(x,y,Jz,Jw)\\
&- R(Jx,y,Jz,w) - R(x,Jy,z,Jw) - R(Jx,y,z,Jw) - R(x,Jy,Jz,w)\\
&= R(x,y,z,w) + R(x,y,z,w) - R(JJx,y,z,w) - R(JJx,y,z,w)\\
&- R(JJx,y,z,w) - R(JJx,y,z,w) - R(JJx,y,z,w) - R(JJx,y,z,w)\\
&= 8R(x,y,z,w)\,.
\end{aligned}$$

Conversely, of course, if $A = 0$, then $\mathcal{J}(\pi) = \mathcal{R}(\pi) = 0$ for all $\pi \in \mathbb{CP}(V, J)$. □

Proof. We now prove Theorem 1.3. We use Lemma 2.3 to see that $\mathcal{M}$ is both Ricci flat and $\star$-Ricci flat. Hence $\tau = \tau^\star = 0$. Therefore, by Lemma 3.1 (3), $\nabla J = 0$ and the manifold is Kähler. This implies the almost complex structure is in fact integrable so $\mathcal{C}$ is Hermitian. The desired conclusion now follows from Theorem 1.2. □

Proof. We now prove Theorem 1.6. Our construction is motivated by the construction of Theorem 1.5 and is based on work of Sato [13]. Let $m = 4n$. Let $(\mathbb{CP}^{2n}, g, J)$ be complex projective space with the Fubini-Study metric g and usual complex structure J_0; this is a Kähler manifold. The canonical embedding of $\mathbb{C}^{2n} \subset \mathbb{C}^{2n+1}$ defines an isometric embedding of $\mathbb{CP}^{2n-1}$ in $\mathbb{CP}^{2n}$. Let $M := \mathbb{CP}^{2n} - \mathbb{CP}^{2n-1}$. Let $\mathcal{H}$ be the fiber bundle of all unitary quaternion structures $\{J_1, J_2, J_3\}$ on the tangent bundle of M which satisfy $J_1 = J$. Since M is contractable, $\mathcal{H}$ is a trivial fiber bundle so we can define a global quaternion structure $\{J_1, J_2, J_3\}$ on TM so that $J = J_1$. This is, of course, just the usual twistor construction.

Let $\mathcal{C} := (M, g, J_2)$. Let

$$\Theta : x \to (1 + J_2)/\sqrt{2}\,.$$

This defines an isometry of T_PM with $\Theta J_2 = J_2\Theta$. Furthermore

$$\Theta J_1 = -J_3\Theta \quad \text{and} \quad \Theta J_3 = J_1\Theta\,.$$

The curvature tensor of the Fubini-Study metric is given by $R_0 + R_{J_1}$. Let x be a unit tangent vector. We use the defining relations of Equation (2.h) to see that:

$$\mathcal{J}_R(x)y = \begin{cases} 0 & \text{if} \quad y \in \operatorname{Span}\{x\},\\ 4y & \text{if} \quad y \in \operatorname{Span}\{J_1x\},\\ y & \text{if} \quad y \perp \operatorname{Span}\{x, J_1x\}\,.\end{cases}$$

As $\Theta^*R = R_0 + R_{J_3}$ and as $J_1x \perp J_3x$, $\Theta^*R \neq R$. Since $\mathcal{J}_R(\pi_x) = \mathcal{J}_R(x) + \mathcal{J}_R(J_2x)$,

$$\mathcal{J}_R(\pi_x)y = \begin{cases} 4y & \text{if} \quad y \in \operatorname{Span}\{x, J_2x\},\\ 5y & \text{if} \quad y \in \operatorname{Span}\{J_1x, J_1J_2x = J_3x\},\\ 2y & \text{if} \quad y \perp \operatorname{Span}\{x, J_1x, J_2x, J_3x\}\,.\end{cases}$$

Since J_1 and J_3 play symmetric roles in this identity, $\mathcal{J}_{\Theta^* R}(\pi_x) = \mathcal{J}_R(\pi_x)$ as desired. Lemma 2.3 now shows $\mathcal{R}_{\Theta^* R}(\pi_x) = \mathcal{R}_R(\pi_x)$ as well. □

4. Conformal and almost Kähler geometry

Theorem 4.1. *Let $\mathcal{C} = (M, g, J)$ be an almost Kähler manifold such that there exists a Kähler metric $\tilde{g}$ on (M, J) such that $\mathcal{R}(\pi) = \tilde{\mathcal{R}}(\pi)$ for all $\pi \in \mathbb{CP}(TM, J)$ (equivalently, $\mathcal{J}_R(\pi) = \mathcal{J}_{\tilde{R}}(\pi)$ for all $\pi \in \mathbb{CP}(TM, J)$). Then $\mathcal{C}$ is Kähler and $R = \tilde{R}$.*

Proof. Suppose $\mathcal{R}(x, Jx) = \tilde{\mathcal{R}}(x, Jx)$. Set $\bar{R} := R - \tilde{R}$. Then $\bar{\mathcal{R}}(\pi) = 0$ for all π. Since $\tau^\star = \tau$ for any Kähler manifold and $\bar{R}$ is Ricci flat and $\star$-Ricci flat by Lemma 2.3 (2), one has $\|\nabla J\|^2 = 0$ by Lemma 3.1 (3); hence $\nabla J = 0$ and the original manifold $\mathcal{C}$ is indeed Kähler. That the curvature tensors are equal follows from Theorem 1.2. □

Theorem 4.2. *Let $\mathcal{C} := (M, g, J)$ and $\mathcal{C}^\alpha := (M, e^\alpha g, J)$ be conformally equivalent almost Hermitian manifolds. If $\mathcal{J}_{\mathcal{C}}(\pi_x) = \mathcal{J}_{\mathcal{C}^\alpha}(\pi)$ for all $\pi \in \mathbb{CP}(TM, J)$, then $R = R^\alpha$.*

Proof. Let $\sigma(g, J)$ be projection on $\mathfrak{A}_2^\perp(g, J)$. A priori this projection depends on the choice of g. It is, however conformally invariant, i.e. $\mathfrak{A}_2^\perp(g, J) = \mathfrak{A}_2^\perp(e^\alpha g, J)$ and $\sigma(g, J) = \sigma(e^\alpha g, J)$. Set $\sigma := \sigma(g, J)$. Furthermore, see [16], $\sigma(R) = \sigma(R^\alpha)$; thus $\sigma(R - R^\alpha) = 0$. Since by hypothesis $(\mathcal{J}_{\mathcal{C}} - \mathcal{J}_{\mathcal{C}^\alpha})(\pi_x) = 0$ for all x, we have as desired that $R = R^\alpha$, using Theorem 1.2. □

Acknowledgments

The research of M. Brozos-Vázquez was partially supported by project MTM2005-08757-C04-01 (Spain) and a FPU grant. The research of E. García-Río was partially supported by project BFM2003-02949 (Spain). The research of both M. Brozos-Vázquez and P. Gilkey was partially supported by the Max Planck Institute for Mathematics in the Sciences (Leipzig, Germany).

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