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Construction

by

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Abstract

Given a smooth vector field Γ and assuming the knowledge of an infinitesimal symmetry X, Hojman [J. Phys. A 29 (1996), no. 3, 667–674] proposed a method for finding both a Poisson tensor and a function H such that Γ is the corresponding Hamiltonian system. In this paper we approach the problem from geometrical point of view. The geometrization leads to the clarification of several concepts and methods used in Hojman's paper. In particular the relationship between the nonstandard Hamiltonian structure proposed by Hojman and the degenerate quasi-Hamiltonian structures introduced by Crampin and Sarlet [J.Math.Phys 43 (2002) 2505-2517] is unveiled in this paper. We also provide some applications of our construction.

1 Introduction

About a decade ago Hojman [12] proposed a general technique to find a Hamiltonian structure for a given equation of motion using one infinitesimal symmetry transformation and one constant of motion. This method to construct Hamiltonian structures applies for system of both ordinary and partial differential equations.

Afterwards Hojman gave several examples [11, 14] of the construction of such Hamiltonian structures for dynamical systems in field theory without using any Lagrangian. This method includes a procedure for increasing the rank of the resulting (very singular) Poisson Bracket by using additional symmetries.

The standard way to construct Hamiltonian theories can be found in numerous textbooks. It starts from a regular Lagrangian and the Legendre transformation is used to define momenta and a Hamiltonian function. This is not the most general situation and actually there has been a tremendous interest in studying nonstandard approaches to produce Hamiltonian structures starting from the equations only, without using a Lagrangian, and for instance the minimal coupling can be obtained under some locality assumptions [2, 4, 10].

In the most general construction the knowledge of a Lagrangian is not at all necessary. Hojman's method works even when the Lagrangian description fails to exist because the method needs only the knowledge of a constant of motion and a solution of the infinitesimal symmetry equation. Note that the choice of the symmetry vector field needed to define the Poisson matrix is determined solely by the requirement of getting a nonvanishing deformation of the Hamiltonian for a given H.

The *main aim* of this article is to explore the geometry of Hojman's construction and its close relationship with the degenerate quasi-Hamiltonian theory proposed by Crampin and Sarlet [7]. They demonstrated how to represent a cofactor system as a Hamiltonian vector field with respect to a Poisson structure defined on an extended manifold. This method illustrated the generalization of the work on Euclidean spaces of Lundmark [17] and involves an application of quasi-Hamiltonian systems. A vector field Z on a manifold M is said to be quasi-Hamiltonian with respect to a Poisson bivector Π with a corresponding linear map $\widehat{\Pi} = \Pi^{\sharp} : \bigwedge^1(M) \to \mathfrak{X}(M)$ if $FZ = -\widehat{\Pi}(dH)$ for some functions F and H on M with F nonvanishing. Then

$$\tilde{\Pi} = \Pi + (Z + zF^{-1}\widehat{\Pi}(dF)) \wedge \frac{\partial}{\partial z}$$

is a Poisson bivector on $M \times \mathbb{R}$ [7]. We further discuss these structures with an illustration.

This paper is *organized* as follows: In Section 2 we review Hojman's construction of a Poisson structure out of a symmetry and a conservation law of a dynamical system. We give a geometrical description of Hojman's construction in Section 3. In other words we explore a more geometrical approach to Hojman's problem. The case of partial differential equations and classical field theory is analysed in Section 4 and the theory

is illustrated with some examples. Section 5 is devoted to quasi-Hamiltonian structures and to discussing its connection with degenerate quasi-Hamiltonian structure. The example of KdV equation is given in Section 6. We finish our paper with a modest conclusion in Section 7.

2   Hojman's Construction

In this section we present a rapid introduction to Hojman's construction. Consider an autonomous differential equation,

$$\frac{dx^a}{dt} = f^a(x^b), \qquad a, b = 1, \cdots, N. \tag{1}$$

We seek a Poisson tensor, J, and a smooth function, H, such that

$$J^{ab}\frac{\partial H}{\partial x^b} = f^a \tag{2}$$

(summation on repeated indexes is understood).

We recall that the Poisson tensor is a skew-symmetric tensor satisfying

$$J^{ab}{}_{,d}J^{dc} + J^{bc}{}_{,d}J^{da} + J^{ca}{}_{,d}J^{db} = 0 \tag{3}$$

so that the Poisson Bracket between any two dynamical variables, $P(x^a)$ and $Q(x^b)$, defined by

$$\{P, Q\} = \frac{\partial P}{\partial x^a}J^{ab}\frac{\partial Q}{\partial x^b} \tag{4}$$

satisfies the antisymmetry condition and Jacobi identity

$$\{P, \{Q, R\}\} + \{Q, \{R, P\}\} + \{R, \{P, Q\}\} = 0$$

for any three functions.

Note that as a consequence of the skew-symmetry of J the function H is a constant of the motion because

$$f^a\frac{\partial H}{\partial x^a} = J^{ab}\frac{\partial H}{\partial x^b}\frac{\partial H}{\partial x^a} = 0\,.$$

Definition 2.1 *An infinitesimal symmetry of the given differential equation (1) is an infinitesimal transformation*

$$\tilde{x}^a = x^a + \epsilon\,\eta^a(x^b, t), \tag{5}$$

which maps each solution into a solution of the given differential equation.

3

It was shown by Lie (see e.g. [13, 15, 22]) that the condition for such a transformation to be a symmetry is the existence of a function $\lambda(x,t)$ such that η satisfies

$$\partial_t \eta^a + \eta^a{}_{,b} f^b - f^a{}_{,b} \eta^b = \lambda(x,t)\, f^a(x) \quad \forall a = 1, \ldots, N\,. \tag{6}$$

It is to be remarked that given an infinitesimal symmetry we can define new infinitesimal symmetries just replacing $\eta^a(x,t)$ by $\bar{\eta}^a(x,t) = \eta^a(x,t) + \mu(x,t)\, f^a(x)$. In fact

$$\partial_t \bar{\eta}^a + \bar{\eta}^a{}_{,b} f^b - f^a{}_{,b} \bar{\eta}^b = \partial_t \eta^a + \eta^a{}_{,b} f^b - f^a{}_{,b} \eta^b + f^a \partial_t \mu + \mu\, f^a{}_{,b} f^b + \mu_{,b} f^a f^b - f^a{}_{,b} \mu f^b$$

and therefore

$$\partial_t \bar{\eta}^a + \bar{\eta}^a{}_{,b} f^b - f^a{}_{,b} \bar{\eta}^b = (\lambda + \partial_t \mu + \mu_b\, f^b) f^a \quad \forall a = 1, \ldots, N\,,$$

i.e., the infinitesimal transformation defined by $\bar{\eta}^a$ is also a symmetry of the given autonomous system. In particular, if $\lambda(x,t)$ in (6) reduces to a real number, we can choose $\mu = -\lambda\, t$ and then $\bar{\eta}^a = \eta^a - \lambda\, t\, f^a$ is an infinitesimal symmetry for which the right hand side of (6) vanishes.

In the framework of autonomous systems in which no time-reparametrization is allowed the function λ appearing in (6) must be zero.

Moreover, if the function $K(x^b, t)$ is a constant of the motion and the infinitesimal transformation (5) is a symmetry, then

$$\tilde{x}^a = x^a + \epsilon K(x^b, t)\, \eta^a(x^b, t), \quad a = 1, \ldots, N\,,$$

is also an infinitesimal symmetry. In fact, if $\tilde{\eta}^a = K\, \eta^a$, then

$$\partial_t \tilde{\eta}^a + \tilde{\eta}^a{}_{,b} f^b - f^a{}_{,b} \tilde{\eta}^b = K\, (\partial_t \eta^a + \eta^a{}_{,b} f^b - f^a{}_{,b} \eta^b) + \eta^a\, (\partial_t K + K_{,b}\, f^b) = 0\,.$$

Definition 2.2 *The deformation K of a function $H(x^a)$ corresponding to the infinitesimal transformation (5) is given by*

$$K \equiv \frac{\partial H}{\partial x^a} \eta^a\,. \tag{7}$$

Note that, if the function $H(x^a)$ is a constant of the motion given by (1), then the deformation K of H along an infinitesimal symmetry of the given system of differential equations is also a constant of the motion because

$$\frac{\partial K}{\partial t} + \frac{\partial K}{\partial x^a}\, f^a = \frac{\partial H}{\partial x^a}\, \frac{\partial \eta^a}{\partial t} + \frac{\partial^2 H}{\partial x^a \partial x^b} f^b\, \eta^a + \frac{\partial H}{\partial x^a}\, \frac{\partial \eta^a}{\partial x^b}\, f^b$$

and using (6) we find

$$\frac{\partial K}{\partial t} + \frac{\partial K}{\partial x^a}\, f^a = \frac{\partial H}{\partial x^a}\, (\lambda\, f^a + f^a{}_{,b} \eta^b) + \frac{\partial^2 H}{\partial x^a \partial x^b} f^b\, \eta^a = \lambda\, f^a\, \frac{\partial H}{\partial x^a} + \eta^b\, \frac{\partial}{\partial x^b}\left(f^a\, \frac{\partial H}{\partial x^a} \right) = 0\,.$$

4

Theorem 2.3 *Let (5) be an infinitesimal symmetry of the autonomous system (1) and $H(x^b)$ be a constant of the motion for evolution such that its deformation K does not vanish. Then*

$$J^{ab} = \frac{1}{K}(f^a \, \eta^b - f^b \, \eta^a) \tag{8}$$

is a Poisson structure for such a dynamics and the system is Hamiltonian with Hamiltonian H. Here K is the deformation of H along the infinitesimal transformation.

The matrix J^{ab} for a nonvanishing K is a Poisson matrix such that

$$J^{ab}\frac{\partial H}{\partial x^b} = \frac{1}{K}(f^a\eta^b - f^b\eta^a)\frac{\partial H}{\partial x^b} = f^a \, . \tag{9}$$

3 Geometrical description of Hojman's construction

In the geometric approach to ordinary differential equations the autonomous system (1) is replaced by a vector field Γ in a manifold M with a local coordinate expression

$$\Gamma = f^a(x)\,\frac{\partial}{\partial x^a}. \tag{10}$$

Its integral curves are solution of (1). An infinitesimal symmetry of the system is given by a vector field $X \in \mathfrak{X}(M)$ with a local coordinate expression

$$X = \eta^a \, \frac{\partial}{\partial x^a}$$

such that $[X, \Gamma] = 0$.

A Poisson structure on a manifold M is a bivector field Π satisfying $[\Pi, \Pi] = 0$, where $[\cdot, \cdot]$ is the Schouten Bracket. Such a bivector field provides us with a $C^\infty(M)$-linear map $\widehat{\Pi} : \bigwedge^1(M) \to \mathfrak{X}(M)$ of the set of 1-forms in M into that of vector fields by means of $\langle \beta, \widehat{\Pi}(\alpha) \rangle = \Pi(\alpha, \beta)$. The Poisson bracket of two functions $f_1, f_2 \in C^\infty(M)$ associated to Π is defined as $\{f_1, f_2\} = \Pi(df_1, df_2)$ and the vanishing of the Schouten Bracket demonstrates the Jacobi identity of the Poisson Bracket [16, 20, 23, 25].

We recall that the Schouten bracket $[\cdot, \cdot]$ is the unique extension of the Lie bracket of vector fields to the exterior algebra of multivector fields, making it into a graded Lie algebra (the grading in this algebra is given by the ordinary degree as multivectors minus one). Given a multivector field V on M, the linear operator $[V, \cdot]$ defined a derivation on the exterior algebra of multivector fields on M, the degree of which is the ordinary degree of V. In particular, if $V = X \wedge Y$ is a monomial bivector field, then

$$[V, V] = 2X \wedge Y \wedge [X, Y]. \tag{11}$$

Proposition 3.1 *i) If the vector fields X and Y generate a two-dimensional integrable distribution, then $V = X \wedge Y$ is a Poisson bivector.*

ii) If the vector field X together with the dynamical vector field Γ generate a two-dimensional integrable distribution, then $\Pi = \Gamma \wedge X$ is a Poisson bivector.

iii) If X and Y are two commuting vector fields in M, then $V = X \wedge Y$ is a Poisson bivector.

iv) In particular, if X is an infinitesimal symmetry of Γ, then $\Pi = \Gamma \wedge X$ is a Poisson bivector field.

Proof: *i)* is a consequence of the above-mentioned relation (11). Then *ii)* and *iii)* are particular cases of *i)* and *iv)* is a particular case of *iii)*.

$\square$

Note that the bivector field $\Pi = \Gamma \wedge X$ is of rank two and therefore degenerate when $\dim M$ is higher than two. Moreover in this case, if f is a constant of the motion for Γ, $f\Gamma \wedge X$ is a Poisson bivector too because then fX would also be a symmetry of Γ. Similarly, if the function f is such that $Xf = 0$, then $f\Gamma \wedge X$ is also a Poisson bivector.

If $H \in C^{\infty}(M)$ is a function in a Poisson manifold (M, Π), then $\Gamma = -\widehat{\Pi}(dH)$ is said to be the Hamiltonian vector field corresponding to the Hamiltonian function H. This vector field, Γ, satisfies $\mathcal{L}_\Gamma \Pi = 0$, which is equivalent to $\mathcal{L}_\Gamma \widehat{\Pi} = 0$.

When X is an infinitesimal symmetry of Γ, the bivector field $\Pi = \Gamma \wedge X$ is a Poisson bivector such that

$$\mathcal{L}_\Gamma \Pi = 0$$

because

$$\mathcal{L}_\Gamma \Pi = [\Gamma, \Gamma] \wedge X + \Gamma \wedge [\Gamma, X] = 0 \,.$$

Definition 3.2 *A vector field Γ on a Poisson manifold is called quasi-Hamiltonian if there exists a nowhere-vanishing function K such that $K\Gamma$, is a Hamiltonian vector field:*

$$K\,\Gamma = -\widehat{\Pi}(dH) \qquad H \in C^{\infty}(M)\,. \tag{12}$$

If M is connected, then K must be either everywhere positive or everywhere negative.

The geometric version of the result of Hojman given in [12] is the following:

Theorem 3.3 *Let $X \in \mathfrak{X}(M)$ be an infinitesimal symmetry of the dynamical vector field Γ and $H \in C^{\infty}(M)$ a constant of the motion for evolution. Then the vector field Γ is quasi-Hamiltonian with respect to the Poisson structure $\Pi = \Gamma \wedge X$. Moreover, if the deformation of H along X is a nowhere vanishing function, then $J = (1/K)\,\Pi$ is also a Poisson structure and the vector field Γ is the Hamiltonian vector field determined by H and J.*

Proof: If $\Pi = \Gamma \wedge X$, then

$$-\widehat{\Pi}(dH) = -(\Gamma H)\,X + (XH)\,\Gamma = K\,\Gamma\,,$$

while, as K is a constant of motion, the bivector, $J = (1/K)\,\Pi$, is also a Poisson vector and then

$$-\widehat{J}(dH) = \Gamma\,.$$

One of the most important advantages of this coordinate-free presentation is that proofs are also valid for infinite-dimensional manifolds and the results of this theorem are used below in the infinite-dimensional case. As far as the coordinate dependence is concerned note that the components of J are those of (8). It is also to be remarked that one can generate constants of motion by recursive applications of the vector field X on H, i.e., $(X^n H)$, because they are also constants of motion. These yield many quasi-Hamiltonian structures.

We can also construct Poisson bivectors for dynamics of rank higher than two when not only one infinitesimal symmetry, X_1, is known but three of them, X_1, X_2, X_3, independent and commuting among themselves. In fact, as $[X_2, X_3] = 0$, $X_2 \wedge X_3$ is a Poisson bivector which is compatible with $\Gamma \wedge X_1$ because

$$[\Gamma \wedge X_1, X_2 \wedge X_3] = [\Gamma \wedge X_1, X_2] \wedge X_3 + X_2 \wedge [\Gamma \wedge X_1, X_3]\,,$$

then

$$[\Gamma \wedge X_1, X_2 \wedge X_3] = [\Gamma, X_2] \wedge X_1 \wedge X_3 + \Gamma \wedge [X_1, X_2] \wedge X_3 + X_2 \wedge [\Gamma, X_3] \wedge X_1 + X_2 \wedge \Gamma \wedge [X_1, X_3]$$

and all terms on the right hand side vanish. The bivector field $\Gamma \wedge X_1 + X_2 \wedge X_3$ is therefore a Poisson bivector field and, as $\mathcal{L}_\Gamma([X_2, X_3]) = 0$, it is also admissible for Γ.

Consider the geometric theory from the t-dependent approach. The time t plays the same role as the usual coordinates and curves obtained by reparametrization are to be considered as equivalent curves. This amounts to considering a new system replacing (1),

$$\begin{cases} \dfrac{dx^a}{d\tau} &= f^a(x^b), \qquad a, b = 1, \cdots, N, \\ \dfrac{dt}{d\tau} &= 1 \end{cases}$$

or even better the family of systems

$$\begin{cases} \dfrac{dx^a}{d\tau} &= \lambda(x^b, t)\, f^a(x^b), \qquad a, b = 1, \cdots, N, \\ \dfrac{dt}{d\tau} &= \lambda(x^b, t) \end{cases}$$

for any nonvanishing arbitrary function λ. In other words the relevant object is not the vector field in the space M anymore but the one-dimensional distribution in $M \times \mathbb{R}$ generated by the vector field

$$\Gamma = \frac{\partial}{\partial t} + f^a(x) \frac{\partial}{\partial x^a} .$$
(13)

Infinitesimal symmetries of the system are given by vector fields Y preserving the distribution, i.e., there exists a function $\lambda(x^b, t)$ such that

$$[\Gamma, Y] = \lambda \, \Gamma .$$
(14)

If Y is an infinitesimal symmetry, then for each function $\mu(x^b, t)$ the vector field $Y + \mu \Gamma$ is also a symmetry because

$$[\Gamma, Y + \mu \, \Gamma] = [\Gamma, Y] + (\Gamma \, \mu)\Gamma = (\lambda + \Gamma \, \mu) \, \Gamma .$$

In particular, when λ reduces to a real number, it suffices to take $\mu = -\lambda \, t$ in order to obtain

$$[\Gamma, Y - \lambda \, t \, \Gamma] = 0 .$$

4 Partial differential equations and classical field theory

In the case of an infinite-dimensional system corresponding to a classical field ordinary differential equations are replaced by partial differential equations. The superscripts i are replaced by the continuous variable x and the manifold M is replaced by a Banach space $\mathbb{F}$ of functions $u(x)$ (for instance a $\mathcal{L}^2(\mathbb{R}, \mu)$). Functions on M become now functionals $F[u]$ on such subset of functions and their derivatives and in particular we are interested in functionals like

$$G[u] = \int_{\mathbb{R}} dx \, \mathcal{G}(u, u_x, \ldots, u_{xx \cdots x}) .$$

They constitute an associative and commutative algebra. Vector fields appear now as derivations of such an algebra.

The functional derivative of a functional G in the η direction is

$$dG[u](\eta) = \int_{\mathbb{R}} \frac{\delta \mathcal{G}}{\delta u(x)} \eta(x) dx,$$
(15)

where

$$\frac{\delta \mathcal{G}}{\delta u(x)} = \frac{\partial \mathcal{G}}{\partial u} - \frac{\partial}{\partial x} \left(\frac{\partial \mathcal{G}}{\partial u_x} \right) + \frac{\partial^2}{\partial x^2} \left(\frac{\partial \mathcal{G}}{\partial u_{xx}} \right) + \cdots .$$

8

All the structures appearing in finite-dimensional dynamical systems can be translated to this new framework and play a similar rôle. For instance vector fields are written as

$$Y[u] = \int_{\mathbb{R}} dx \ \mathcal{Y}(u, u_x, \ldots, u_{x \cdots x}) \frac{\delta}{\delta u(x)}. \tag{16}$$

Infinitesimal transformations are described by vector fields like (16) while differential equations describing time evolution as

$$u_t = \mathcal{F}(u, u_x, \ldots, u_{x \cdots x}) \tag{17}$$

are described by the vector field

$$\Gamma[u] = \int_{\mathbb{R}} dx \ \mathcal{F}(u, u_x, \ldots, u_{x \cdots x}) \frac{\delta}{\delta u(x)} . \tag{18}$$

As in the finite-dimensional case the infinitesimal transformation (16) is said to be a symmetry of the dynamics given by (18) if there exists a function λ such that $[\Gamma, Y] = \lambda \Gamma$ and in this case for any function μ the infinitesimal transformation determined by $Y + \mu \Gamma$ is a symmetry too. Moreover, if λ reduces to a real number, then $Y - \lambda \Gamma$ is such that $[\Gamma, Y] = 0$.

Similarly bivector fields are given by

$$\Pi[u] = \int_{\mathbb{R}^2} dx \ dy \ J(x, y, [u]) \ \frac{\delta}{\delta u(x)} \wedge \frac{\delta}{\delta u(y)} \tag{19}$$

which allow one to define a skew-symmetric bilinear bracket in the space of functionals by

$$\{P, Q\}[u] \equiv \int_{\mathbb{R}^2} \frac{\delta P}{\delta u(x)} J(x, y, [u]) \frac{\delta Q}{\delta u(y)} \ dx \ dy \tag{20}$$

and the bivector field is said to be a Poisson bivector if the Jacobi identity holds for any three functionals. Given such a Poisson bivector and choosing a Hamiltonian

$$H[u] = \int_{\mathbb{R}} \mathcal{H}(u, u_x, \ldots) \, dx$$

one obtains the time evolution which is given by

$$u_t(x) = \{u(x), H\} = \int_{\mathbb{R}} dy \ J(x, y, [u]) \frac{\delta H}{\delta u(y)}. \tag{21}$$

The theory developed for finite-dimensional systems generalizes easily to this more general context by the replacement of partial derivatives by variational derivatives. As a specific example of the procedure described so far we use the KdV equation (see e.g. [9]). The equation of motion is

$$u_t + u u_x + u_{xxx} = 0 \tag{22}$$

9

or written in a different way

$$u_t = -uu_x - u_{xxx}$$

which corresponds to (1) with the following choice for the function f (only one component) appearing in such equation:

$$f \equiv -u_{xxx} - uu_x. \tag{23}$$

In other words the dynamics is given by the vector field

$$\Gamma[u] = \int_{\mathbb{R}} dx \ (-u_{xxx} - uu_x) \frac{\delta}{\delta u(x)} \ . \tag{24}$$

It is easy to check that the time-independent infinitesimal transformation defined by the function

$$\eta_0 = -2u - xu_x \tag{25}$$

is an infinitesimal symmetry of (22). In fact, if X is the vector field

$$X[u] = \int_{\mathbb{R}} dx \ (-2u - xu_x(x)) \frac{\delta}{\delta u(x)}$$

and Γ is given by (24), then, as

$$\frac{\delta}{\delta u(x)}(-u_{xxx} - uu_x) = 0, \qquad \frac{\delta}{\delta u(x)}(-2u - xu_x) = -1 \, ,$$

we obtain that

$$[X, \Gamma][u] = \left[\int_{\mathbb{R}} dx \ (-2u - xu_x(x)) \frac{\delta}{\delta u(x)}, \int_{\mathbb{R}} dx \ (-u_{xxx} - uu_x) \frac{\delta}{\delta u(x)} \right] = \Gamma[u] \, .$$

Consequently the transformation given by

$$\eta = -2u - xu_x - t(u_{xxx} + uu_x), \tag{26}$$

which only differs in the addition of an appropriate term proportional to f, is a strict infinitesimal symmetry, i.e., $[\Gamma, Y] = 0$.

Note that, if $F[u]$ is the functional given by

$$F[u] = \int_{\mathbb{R}} x \, u_x \ dx,$$

then $\Gamma F[u] = -1$ and therefore all vector fields of the family

$$X_k[u] = \int_{\mathbb{R}} dx \ (-2u - kxu_x(x)) \frac{\delta}{\delta u(x)}$$

10

are also infinitesimal symmetries of the dynamics. This is so because $xu_x = d/dx(xu) - u$ and the functional

$$M[u] = \int_{\mathbb{R}} u \, dx$$

is a constant of the motion. The corresponding strict infinitesimal symmetry would be given by

$$\eta_k = -2u + xu_x - (2 - k)\, t(u_{xxx} + uu_x)\,.$$

It has been proved in [9] that the functionals H_1 and H_2 defined by

$$H_1[u] = \int_{\mathbb{R}} \left(-\frac{u_x^2}{2} + \frac{1}{3!}u^3 \right) dx\,, \qquad H_2[u] = \frac{1}{2}\int_{\mathbb{R}} u^2 \, dx$$

are constants of the motion. Moreover it has been proved that the KdV equation admits two different Poisson structures for which the Hamiltonians are H_1 and H_2.

The deformations K_1 and K_2 of the above mentioned functionals H_1 and and H_2 along η_0 given by (25) are

$$K_1[u] \equiv \int_{\mathbb{R}} \frac{\delta \mathcal{H}_1}{\delta u(x)} \eta_0 \, dx = -5H_1$$

$$K_2[u] \equiv \int_{\mathbb{R}} \frac{\delta \mathcal{H}_2}{\delta u(x)} \eta_0 \, dx = -3H_2 \qquad (27)$$

because

$$\frac{\delta \mathcal{H}_1}{\delta u(x)} = \frac{1}{2}u^2 + u_{xx}\,, \qquad \frac{\delta \mathcal{H}_2}{\delta u(x)} = u$$

and then

$$K_1[u] \equiv \int_{\mathbb{R}} \left(\frac{1}{2}u^2 + u_{xx} \right)(-2u - xu_x)\, dx = \int_{\mathbb{R}} \left(-u^3 - 2uu_{xx} - xu_x\left(\frac{1}{2}u^2 + u_{xx} \right) \right) dx$$

and having in mind that

$$u_x\left(\frac{1}{2}u^2 + u_{xx} \right) = \frac{d}{dx}\left(\frac{1}{3!}u^3 + \frac{1}{2}u_x^2 \right)\,, \qquad u\,u_{xx} = \frac{d}{dx}(u\,u_x) - u_x^2$$

an integration by parts shows us that

$$\int_{\mathbb{R}} x\,u_x\left(\frac{1}{2}u^2 + u_{xx} \right) dx = -\int_{\mathbb{R}} \left(\frac{1}{3!}u^3 + \frac{1}{2}u_x^2 \right) dx\,, \qquad -2\int_{\mathbb{R}} u\,u_{xx}dx = 2\int_{\mathbb{R}} u_x^2$$

and therefore

$$K_1[u] = \int_{\mathbb{R}} \left[-u^3 + 2u_x^2 + \left(\frac{1}{3!}u^3 + \frac{1}{2}u_x^2 \right) \right] dx = -5\,H_1[u]\,.$$

In a similar way

$$K_2[u] \equiv \int_{\mathbb{R}} \frac{\delta \mathcal{H}_2}{\delta u(x)} \eta \, dx = \int_{\mathbb{R}} u\,(-2u - xu_x)\, dx$$

11

and using integration by parts in the last term having in mind that $u\,u_x = (1/2)d/dx(u^2)$ we obtain

$$K_2[u] \equiv \int_{\mathbb{R}} \left(-2 + \frac{1}{2}\right) u^2 = -3H_2[u]\,.$$

The theory developed above for finite-dimensional systems also applies in this case. Therefore, given a differential equation like (17) for which an infinitesimal symmetry (16) and a constant of the motion H such that its deformation K does not vanish are known, then the bivector field (19), where

$$J(x,y)[u] = \mathcal{F}(u(x))\mathcal{Y}(u(y)) - \mathcal{F}(u(y))\mathcal{Y}(u(x))\,, \tag{28}$$

is a Poisson bivector and the dynamics is quasi-Hamiltonian. Moreover

$$\{P,Q\}[u](z) \equiv \frac{1}{K(z)} \int_{\mathbb{R}^2} \frac{\delta \mathcal{P}}{\delta u(x)} \left[\mathcal{F}(u(x))\mathcal{Y}(u(y)) - \mathcal{F}(u(y))\mathcal{Y}(u(x))\right] \frac{\delta \mathcal{Q}}{\delta u(y)}\, dx\, dy, \tag{29}$$

where K denotes the deformation of H, is also a Poisson Bracket and the dynamics is Hamiltonian with respect to this new Poisson structure with Hamiltonian function H.

Theorem 4.1 *Consider the infinitesimal transformation determined by $\eta_0 = -2u - xu_x$ which is a symmetry of the KdV equation*

$$u_t + uu_x + u_{xxx} = 0$$

and consider the functional $H_1[u] = \int_{\mathbb{R}} u^2\, dx$ which is a constant of the motion. Then the system is quasi-Hamiltonian with respect to the Poisson bivector field defined by

$$J(y,z) = (u(y)u_x(y) + u_{xxx}(y))(2u(z) - zu_x(z)) - (2u(y) - yu_x(y))(u(z)u_x(z) + u_{xxx}(z))$$

and, as $H_1[u] = \int_{\mathbb{R}} u^2\, dx$ is a constant of the motion,

$$\{P,Q\}[u(z)] \equiv \frac{1}{u(z)} \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{\delta \mathcal{P}}{\delta u(x)} \left[(u(y)u_x(y) + u_{xxx}(y))(2u(z) - zu_x(z))\right.$$
$$\left. - (2u(y) - yu_x(y))(u(z)u_x(z) + u_{xxx}(z))\right] \frac{\delta \mathcal{Q}}{\delta u(y)}\, dx\, dy$$

also defines a Poisson structure such that the Hamiltonian flow corresponding to H_1 is given by the KdV equation

$$u_t + uu_x + u_{xxx} = 0.$$

The theorem is a straightforward consequence of the theory we have developed and

$$\frac{\delta H}{\delta u} = \int_{\mathbb{R}} u\, dx\,.$$

5 Hamiltonization of quasi-Hamiltonian dynamics

Let Γ be a vector field on a Poisson manifold (M, Π) which is quasi-Hamiltonian with respect to a Poisson bivector Π, i.e., there exists a nowhere-vanishing function $K \in C^\infty(M)$ such that $K\,\Gamma$ is a Hamiltonian vector field: $K\,\Gamma = -\hat{\Pi}(dH)$. Then, as it has been shown by Crampin and Sarlet [6], there is a Poisson bivector $\widetilde{\Pi}$ on $M \times \mathbb{R}$ which projects onto Π and a vector field X, Hamiltonian with respect to $\widetilde{\Pi}$, the restriction of which to the zero section of the bundle $\pi : M \times \mathbb{R} \to \mathbb{R}$ is the given Γ. Such a bivector field is given by

$$\widetilde{\Pi} = \Pi + (\Gamma + tX) \wedge \frac{\partial}{\partial t}, \tag{30}$$

where t is the coordinate on $\mathbb{R}$ and $X = K^{-1}\hat{\Pi}(dK)$. In fact they proved that the conditions for $\hat{\Pi}$ to be Poisson again, i.e., the vanishing of the Schouten bracket, $[\widetilde{\Pi}, \widetilde{\Pi}] = 0$, yields

$$\mathcal{L}_\Gamma \Pi = \Gamma \wedge X, \qquad \mathcal{L}_X \Pi = 0, \tag{31}$$

and both requirements are satisfied if $X = K^{-1}\hat{\Pi}(dK)$.

Fortunately in the case we are analysing the fact that $\Pi = \Gamma \wedge X$ with K a constant of motion is enough to assure that $(1/K)\Pi$ is also a Poisson bivector and Γ is not only Hamiltonian with respect to Π but also Hamiltonian with respect to the new Poisson structure $(1/K)\Pi$. Therefore this Hamiltonizing process is unnecessary.

6 Example: KdV equation

We can reproduce the Poisson structure of the KdV equation from our approach. We define the quasi-Hamiltonian vector field

$$\Gamma = -\int (u_{xxx} + uu_x)\, \frac{\delta}{\delta u(x)} \tag{32}$$

and the symmetry vector field

$$X = \int (-2u(y) - yu_x(y)\, \frac{\delta}{\delta u(y)} \tag{33}$$

for the KdV equation.

We start with the variational formulation of the KdV. We introduce the Clebsch velocity potential $u = v_x$. The potential KdV is defined as

$$v_t + \frac{1}{2}v_x^2 + v_{xxx} = 0. \tag{34}$$

It can be directly checked that the equations of motion can be obtained from the variational principle

$$\delta I = 0, \qquad I = \int \mathcal{L}dtdx$$

13

with Lagrangian density

$$\mathcal{L} = \frac{1}{2}v_t v_x + \frac{1}{6}v_x^3 - \frac{1}{2}v_{xx}^2. \tag{35}$$

In fact corresponding to the Lagrangian density (35) we find

$$\mathcal{H} = -\frac{1}{6}v_x^3 + \frac{1}{2}v_{xx}^2 \tag{36}$$

for Dirac's total Hamiltonian density of the KdV equation.

The first symplectic form of the KdV equation is defined as

$$\omega = \delta v \wedge \delta v_x. \tag{37}$$

It can be easily checked that ω satisfies Hamilton's equation in the symplectic form

$$i_{\tilde{\Gamma}}\omega = \delta\mathcal{H}, \tag{38}$$

where the vector field

$$\tilde{\Gamma} = -(\frac{1}{2}v_x^2 + v_{xxx})\frac{\delta}{\delta v} \tag{39}$$

defines the flow for the potential KdV equation.

Proposition 6.1 *The quasi-Hamiltonian vector field Γ yields a Poisson bivector given by*

$$\Pi(x, y) = \int dx \, dy \, \phi(x)\psi(y) \, \frac{\delta}{\delta u(x)} \wedge \frac{\delta}{\delta u(y)}, \tag{40}$$

where $\phi(x) = -(u_{xxx} + uu_x)$ and $\psi(y) = (-2u + yu_y)$. This exactly coincides with Hojman's definition of the Poisson structure of the KdV equation.

Proof: By direct computation.

$\square$

7 Conclusion and Outlook

In this paper we have studied from the geometrical point of view Hojman's construction of Hamiltonian structures for dynamical systems in field theory without using Lagrangians. This geometrization has led to the clarification of several techniques of Hojman, which otherwise looked mysterious. We have established a close link between Hojman's construction and degenerate quasi-Hamiltonian structures studied by Crampin and Sarlet. A generalization of Hojman's construction for finding Poisson tensor to the construction of Nambu-Poisson tensor is given. What is required next is a careful study of bi-Hamiltonian structures associated to Hojman's construction and how they are connected to quasi-bi-Hamiltonian systems.

In future we will study the generalization of degenerate quasi-Hamiltonian structure. The obvious generalization will be towards the Nambu–Poisson direction [1]. Hopefully we will consider a quasi-Nambu-Hamiltonian structure in our forthcoming work.

14

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