Max-Planck-Institut
für Mathematik
in den Naturwissenschaften
Leipzig

Noncommutative Integrable Systems and Diffeomorphism on Quantum Spaces

by

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Preprint no.: 142 2006

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Abstract

Following Aschieri et al. [Classical Quantum Gravity 22 (2005), no. 17, 3511–3532] construction of deformed algebra of diffeomorphism group the $\star$-deformed $Vect(S^1)$ action on tensor densities of arbitrary degree λ on a circle S^1 is studied. We derive Noncommutative Korteweg-de Vries (KdV) and Noncommutative Burgers equations in this method.

PACS: 02.40.Gh, 02.20.Uw

Keywords: Vector field, deformed spaces, deformed diffeomorphisms.

1 Introduction

Noncommutative geometry [5] extends the notions of classical differential geometry from differential manifold to discrete spaces, like finite sets and fractals and noncommutative spaces which are given by noncommutative associative algebras. Noncommutative geometry has recently been involved in a noncommutaive gauge theory related to strings.

Noncommutative spaces are characterized by the noncommutative coordinates

$$[x_i, x_j] = i\theta^{ij}, \tag{1}$$

where θ^{ij} are real constants. Noncommutative gauge theories are naively realized from ordinary commutative theories just by replacing all products of the fields with $\star$ product. Thus, several classical integrable models have been generalized to noncommutative spaces [10,17]. Also, under the deformation, the self-dual Yang-Mills equation is considered to preserve the integrability in the same sense as in commutative cases. Noncommutative KdV and nonlinear Schrödinger equations are derived from the reduction of self dual Yang-Mills equation [12,13] and other methods [11]. There exist a method, namely the bicomplex method [6,7], to yield noncommutative integrable equations which have many conserved quantities. Certainly all these equations are derived formally from the Lax representation by replacing the product by $\star$ product.

It is known that the periodic Korteweg-de Vries (KdV) can be interpreted as geodesic flow of the right invariant metric on the Bott-Virasoro group [19], which at the identity is given by L^2-metric. This is nothing but the Euler-Poincaré flow [18] on the coadjoint orbit of Virasoro algebra.

Recently a deformation of the algebra of diffeomorphism is constructed by Aschieri et. al.[1,2,8,20] for canonically deformed spaces with constant deformation parameter θ. The algebra remains the same for deformed theory, only the coproduct rule is different from the undeformed one. In this paper we tacitly accept this formalism to construct the noncommuative version of periodic KdV and the Burgers equation. Let $\mathcal{F}_\lambda(S^1)$ be spaces of tensor densities of degree λ on the circle S^1. It is known that the spaces $\mathcal{D}^k_{\lambda,\mu}(S^1)$ of kth order linear differential operators from $\mathcal{F}_\lambda(S^1)$ to $\mathcal{F}_\mu(S^1)$ is a natural modules over $Diff(S^1)$. We study the algebraic deformation of $Vect(S^1)$ action on these spaces of differential operators which leads to $\star$-product formulation of integrable systems. This article shows another applications of [1,2].

Acknowledgement The author is profoundly grateful to Professor Christian Duval for several stimulating discussions during the conference on "Noncommutative Geometry and Quantum Physics" at Kolkata January 2006. He is also grateful to Professor Valentin Ovsienko for valuable remarks. He expresses grateful thanks to Jürgen Jost for gracious hospitality at the Max Planck Insitute for Mathematics in the Sciences.

2 Background: construction of KdV and Burgers from the action of $Vect(S^1)$ on Hill's operator

We quickly go through the background materials needed for this paper.

Let $Diff_+(S^1)$ be the group of orientation preserving diffeomorphism [14,15]. We represent an element of $Diff_+(S^1)$ as a diffeomorphism $f : \mathbf{R} \longrightarrow \mathbf{R}$ such that

- $f \in C^\infty(\mathbf{R})$
- $f'(x) > 0$
- $f(x + 2\pi) = f(x) + 2\pi$.

The group multiplication is defined by the composition maps, such that $(g \circ f)(x) = f(g(x))$. The Lie algebra of this group is the algebra $Vect(S^1)$ of smooth vector fields on S^1.

Locally an element of $\mathcal{F}_m(S^1)$ is given by $s = g(x)dx^m$, where $g(x) = g(x + 2\pi)$. There is a natural action of $Diff(S^1)$ on the sections of $\mathcal{F}_m(S^1)$. We express this action by

$$\mathcal{L}^m_{f\frac{d}{dx}} s \;=\; (fg' + mf'g)dx^{\otimes m}, \tag{2}$$

where $\mathcal{L}^m_{f\frac{d}{dx}}$ is known as Lie derivative with respect to the vector field $f\frac{d}{dx} \in Vect(S^1)$ and m stands for conformal weight.

Suppose $v \;=\; g\frac{d}{dx} \in \mathcal{F}_{-1}$, then we get

$$\mathcal{L}_u v \;=\; [u, v] \;=\; (gf' - g'f)\frac{d}{dx}. \tag{3}$$

Therefore $\mathcal{F}_{-1}$ is identified to $Vect(S^1)$.

2.1 Construction of the KdV in Lie derivative method

It is known that the space of the Hill's operator

$$\Delta \;=\; \frac{d^2}{dx^2} + u(x) \tag{4}$$

stands for the dual space of Virasoro algebra.

It can be shown easily that the Lie derivative $\mathcal{L}_{f\frac{d}{dx}}$ satisfies

$$[\mathcal{L}^m_{f\frac{d}{dx}}, \mathcal{L}^m_{g\frac{d}{dx}}] = \mathcal{L}^m_{[f,g]\frac{d}{dx}}. \tag{5}$$

Definition 2.1 *The action of $Vect(S^1)$ on the space of Hill's operator Δ is defined by the commutation with the Lie derivative*

$$[\mathcal{L}_{f(x)\frac{d}{dx}}, \Delta] := \mathcal{L}^{\frac{3}{2}}_{f(x)\frac{d}{dx}} \circ \Delta - \Delta \circ \mathcal{L}^{-\frac{1}{2}}_{f(x)\frac{d}{dx}}. \tag{6}$$

The result of this action is a scalar operator, i.e. the operator of multiplication by a function:

$$[\mathcal{L}_{f(x)\frac{d}{dx}}, \Delta] \;=\; \frac{1}{2}(f''' + 4uf' + 2u'f). \tag{7}$$

This can be viewed as a coadjoint action of $Vect(S^1)$ on its dual.

The coadjoint operator corresponding to KdV equation is given by

$$\mathcal{O} = \frac{1}{2}(\partial^3 + 2u\partial + 2\partial u).$$

Thus it is easy to see that the KdV equation

$$u_t = 6uu_x + u_{xxx} \tag{8}$$

follows from the Hamiltonian equation

$$u_t = \mathcal{O}\frac{\delta H}{\delta u} \tag{9}$$

for $H =< u, u >= \int_{S^1} u^2\, dx$.

Remark Let us consider the action of the Hamiltonian vector field $\mathcal{O} = ad^*_u$ on an operator $L = \partial_x^2 + u$. This yields the Lax flow

$$\frac{dL}{dt} = [L, P] \qquad P = \partial^3 + 2u\partial + 2\partial u. \tag{10}$$

In this case we obtain $u_t = u_{xxx} + 4uu_x$, and it is different from (8) by a scale factor.

2.2 Construction of the Burgers in Lie derivative method

Consider an operator

$$\mathbf{\Delta}_1 = \frac{d}{dx} + u(x), \tag{11}$$

acting on $\mathcal{F}_{-\frac{1}{2}} \in \Gamma(\Omega^{-\frac{1}{2}})$, square root of the tangent bundle on S^1, defined as

$$\Delta_1 = \frac{d}{dx} + u(x) \ : \ \mathcal{F}_{\frac{1}{2}} \longrightarrow \mathcal{F}_{-\frac{1}{2}}. \tag{12}$$

Definition 2.2 *The $Vect(S^1)$- action on Δ_1 is defined by the commutator with the Lie derivative*

$$[\mathcal{L}_{f(x)\frac{d}{dx}}, \Delta_1] \ := \ \mathcal{L}^{\frac{1}{2}}_{f(x)\frac{d}{dx}} \circ \Delta_1 - \Delta_1 \circ \mathcal{L}^{-\frac{1}{2}}_{f(x)\frac{d}{dx}}. \tag{13}$$

The result of this action is also a scalar operator, i.e., the operator of multiplication by a function.

Lemma 2.3

$$[\mathcal{L}_{f(x)\frac{d}{dx}}, \Delta_1] \ = \ \frac{1}{2}f''(x) + uf'(x) + u'f(x). \tag{14}$$

Proof: By direct computation.
□
Hence the implectic operator is

$$\mathcal{O}_{Burgers} = \frac{1}{2}\frac{d^2}{dx^2} + u\frac{d}{dx} + u'(x). \tag{15}$$

Therefore using the Hamiltonian equation (9) we obtain the Burgers equation

$$u_t = u_{xx} + 4uu_x \qquad \text{for} \qquad H = \int_{S^1} u^2\, dx. \tag{16}$$

3 Noncommutative framework

In this Section we discuss briefly some salient features of noncommutative quantization, concept of derivatives on quantum spaces and deformation of vector fields and its action on noncommutative spaces. We recapitulate this material from the papers of J. Wess and his coworkers [1,2, 8, 20].

The Moyal $\star$ product is defined by

$$\begin{aligned} F\star G(x) &= exp(\frac{i}{2}\theta^{ij}\partial_{x^i}\partial_{x^j}|_{x^i=x^j=x} \\ &= F(x)G(x)+\frac{i}{2}\theta\partial_i F(x)\partial_j G(x)+\mathcal{O}(\theta^2) \end{aligned} \tag{17}$$

This algebra is realized on the linear space of functions of commuting variables, and the space of this algebra is denoted by $\mathcal{A}_\theta$.

We give the definition of Moyal bracket:

$$\{F,G\}_{Moyal} := \frac{F\star G - G\star F}{2}. \tag{18}$$

The $\star$ product of two functions is again a function. The $\star$ product defines associativity: $f\star(g\star h)=(f\star g)\star h$, but noncommutativity algebra $\mathcal{A}_\theta$ and reduces to ordinary product in the limit $\theta^{ij}\longrightarrow 0$. The modification of the product makes the ordinary coordinate "noncommutative", i.e.

$$[x^i,x^j]_\star := x^i\star x^j - x^j\star x^i = i\theta^{ij}. \tag{19}$$

These are the defining relations for the generators of the algebra $\mathcal{A}_\theta$.

The pointwise product of two functions can be defined via trace map on the tensor product of vector spaces of commuting variables. Thus we define a bilinear map m which maps the tensor product to the space of functions

$$m:\mathcal{F}\otimes\mathcal{F}\to\mathcal{F} \qquad m(F(x)\otimes G(x)) := F(x)G(x). \tag{20}$$

This bilinear map can be extended to the space of noncommutating variables with the help of an abelian twist

$$\mu = exp(\frac{i}{2}\theta^{ij}\partial_i\otimes\partial_j). \tag{21}$$

This allows us to define $\star$-product in the form that makes use of the tensor product of vector spaces $\mathcal{F}$:

$$m_t(F(x)\otimes G(x)) := F\star G = m\circ exp(\frac{i}{2}\theta^{ij}\partial_i\otimes\partial_j)(F(x)\otimes G(x)), \tag{22}$$

where twist is explicitly given by

$$\mu = \sum \frac{1}{n!}(\frac{i}{2})^n\theta^{i_1j_1}\cdots\theta^{i_nj_n}\partial_{i_1}\cdots\partial_{i_n}\otimes\partial_{j_1}\cdots\partial_{j_n} \tag{23}$$

It is clear from the definition that for the noncommutative case we twist the bilinear map m with the Poisson bivector.

For comutational purposes it is customary to use following notation [1,2]

$$m_t = \sum_i \mu_1^i \otimes \mu_2^i \qquad m_t^{-1} = \sum_i \bar{\mu}_1^i \otimes \bar{\mu}_2^i, \tag{24}$$

where i is the multi-index notation.

One can compute the commutator of coordinates

$$m_t(x_\mu \otimes x_\nu) \equiv x_\mu \star x_\nu = x_\mu x_\nu + \frac{i}{2}\theta^{ij}$$

$$m_t(x_\nu \otimes x_\mu) = x_\nu x_\mu - \frac{i}{2}\theta^{ij},$$

and thus it is consistent with equation (19).

Remark The space of relativistic theory is four-dimensional Minkowski space-time, with coordinates x^μ. The Poincaré algebra acts on x_{mu}. To construct noncommutative field theory in a Lorentz invariant way one invokes the concept of Poincaré symmetry [3,4,16], and modified bilinear map is defined as

$$m_t = m \circ e^{-\frac{i}{2}\theta^{ij} P_i \otimes P_j},$$

where P_i are the generators of translation in standard Poincaré algebra. In fact, earlier Drinfeld [9] has shown that there exits a transformation of the structure maps of a Hopf algebra, which is an equivalence relation among Hopf algebras H, preserving the category of representations. This transformations $H \to H_t$ is generated by an intertible twist element

$$m_t = \sum_i \mu_1^i \otimes \mu_2^i \in H \otimes H.$$

Lax approach to Moyal-type quantization The most trivial way to express the noncommutative KdV or any other integrable system to replace the Lax equation by noncommutative Lax equation

$$\frac{dL}{dt} = [L, P]_\star = L \star P - P \star L, \tag{25}$$

where the Moyal $\star$–product is defined by (17).

Thus it is fairly straight forward to see that the Noncommutative KdV equation satisfies

$$u_t = u_{xxx} + 2(u \star u_x + u_x \star u). \tag{26}$$

This method is adopted by some string theorists [12,13,17]. Dimakis and Müller-Hoissen [6,7] of course started from a different point of view (bicomplex method) but finally they relied on the same procedure.

3.1 Deformation of the algebra of diffeomorphism

There is a natural way to introduce derivatives [21,22] on $\mathcal{A}_\theta$ based on $\star$-product formulation. It defines ∂_ν^* acting on $f \in \mathcal{A}_\theta$

$$\partial_i^* \triangleright f := (\partial_i f).$$

Therefore, the derivatives act as the usual derivatives. But the Leibniz rule will change in general. Wess et. al. have derived it in the $\star$-product formalism and use the fact that $f \star g$ is a function again

$$\partial_i(f \star g) = (\partial_i f) \star g + f \star (\partial_i g) + f(\partial_i \star)g, \tag{27}$$

and this exhibits Leibniz rule when $\star$–operation is x independent.

In an interesting paper Aschieri et.al. [1] studied a deformation of the algebra of diffeomorphisms for canonically deformed spaces with constant deformation parameter θ. They studied the algebra generated by vector fields which exhibit Hopf algebraic structure.

The star product of partial derivative and function is given by

$$\partial_\nu \star f = f\partial_\nu + (\partial_\nu f) + \frac{i}{2}(\partial_\nu \theta^{ij})(\partial_i f)\partial_j. \tag{28}$$

We can generalize the higher order differemtial operators $D = \sum_k d^{\nu_1,\nu_2\cdots\nu_k}\partial_{\nu_1}\cdots\partial_{\nu_k}$ to operators on noncommutative space acting on $\mathcal{A}_\theta$, the elements are

$$D^* = \sum_k d^{\nu_1,\nu_2\cdots\nu_k}\partial^*_{\nu_1}\cdots\partial^*_{\nu_k}, \tag{29}$$

where the coefficient function $d^{\nu_1,\nu_2\cdots\nu_k}$ is an element of $\mathcal{A}_\theta$.

3.2 Diffeomorphism and noncommutative space

Let us consider a formalism of the action of infinitesimal diffeomorphism on $\mathcal{A}_\theta$. Diffeomorphisms are generated by vector fields acting on a differential manifold. The infinite-dimensional Lie algebra of diffeomorphisms is isomorphic to the set of the first-order differential operators $Vect(M)$, defined as

$$\xi = \xi^i(x)\frac{\partial}{\partial x^i}.$$

The commutators of two operators ξ and η yields the Lie bracket formula for $Vect(M)$

$$[\xi, \eta] = (\xi^i\partial_i\eta^k - \eta^i\partial_i\xi^k)\frac{\partial}{\partial x^k}. \tag{30}$$

This new approach to noncommutative gravity has been proposed in [1]. This approach is based on the twist deformation of the diffeomorphism group in the real 4–dimensional space.

Let us study the realization of vector fields on $\mathcal{A}_\theta$. The action of vector fields $\xi = \xi^\nu \partial_\nu$ on functions $f \in \mathcal{A}_\theta$ is given as

$$\xi \triangleright f := \xi^\nu \star (\partial_\nu f).$$

The $\star$-product of two vector fields (for θ = constant) $\xi = \xi^\nu \partial_\nu$ and $\phi = \phi^\mu \partial_\mu$ is defined as

$$\xi \star \phi = (\xi^\nu \star (\partial_\nu \phi^\mu))\partial_\mu + (\xi^\nu \star \phi^\mu)\partial_\nu \partial_\mu.$$

It can be shown from the associativity condition

$$(\xi^* \star \eta^*) \star f = \xi^* \star (\eta^*) \star f) = \xi^* \star (\eta f) = \xi \eta f$$

that the deformed vector fields also satisfy

$$[\xi^*, \eta^*]_\star \equiv \xi^* \star \eta^* - \eta^* \star \xi^* = (\xi \times \eta)^*. \tag{31}$$

4 Moyal action of $Vect(S^1)$ on the space of tensor densities on S^1

In this Section we describe the construction of NCKdV equation. Since deformation by quantization does not incorporate the dynamics, so we have to take slightly different approach to construct the noncommutative version of KdV and Burgers equations. We follow the Lie derivative method. In this way we bypass all the difficulties of the construction of coadjoint orbits.

A function $f(x)$ is a primary field with conformal weight h if under the diffeomorphism $x \to \tau(x)$ it transforms as

$$f(x) \longmapsto \tilde{f}(\tau(x)) = \phi^{-h} \star f(x),$$

where $\phi(x) = \frac{d\tau}{dx}$.

We will compute the infinitesimal action of $Vect(S^1)$ on $\Omega^m_*(M)$. It is known that infinitesimal change of coordinates are realized by vector fields. Let $h_\epsilon(x)$ be a one-parameter family of diffeomorphisms of the circle, such that

$$h_\epsilon(x) = x + \epsilon \star f(x) + O(\epsilon^2).$$

Thus the infinitesimal action yields

$$\begin{aligned} g(h_\epsilon(x))(d(h_\epsilon(x)))^{\otimes m}_\star &= g(x + \epsilon \star f + O(\epsilon^2))d(x + \epsilon \star f + O(\epsilon^2))^{\otimes m}_\star \\ &= (g(x) + \epsilon \star g_x \star f + O(\epsilon^2))(dx + \epsilon \star df + O(\epsilon^2)^{\otimes m}_\star \\ &= g(x)(dx)^{\otimes m}_\star + \epsilon \star (g_x \star f + mg \star f_x)(dx)^{\otimes m}_\star + O(\epsilon^2). \end{aligned}$$

Hence,

$$\delta g = (g_x \star f + mg \star f_x)(dx)^{\otimes m}_\star.$$

One parameter family of $Vect(S^1)$ acts on the space of smooth functions $C^\infty(S^1)$ with respect to the $\star$ product by

$$\mathcal{L}_{\star}{}^{(m)}_{f(x)\frac{d}{dx}} g(x) = g_x \star f + mg \star f_x. \tag{32}$$

This equation implies a one parameter family of $Vect(S^1)$ action with respect to $\star$-product on the space of smooth functions $C^\infty(S^1)$. This definition exactly coincides with the definition of Aschieri et. al. [1]. This cane be checked easily.

Let us combine the ordinary Lie derivative $\mathcal{L}_{f(x)\frac{d}{dx}}$ with the twist. For simplicity we assume conformal weight $\lambda = 0$. Thus we obtain

$$\begin{aligned}\mathcal{L}_{\star f(x)\frac{d}{dx}}(g) = \bar{\mu}_1^i(f(x)\partial_x)(\bar{\mu}_2^i)(g)) = \bar{\mu}_1^i(f(x))\partial_x(\bar{\mu}_2^i)(g))\\ = f(x) \star \partial_x(g)).\end{aligned}$$

Thus we have lifted the algebra of diffeomorphisms to the $\star$–product realization. It is not hard to proof that the Lie derivative $\mathcal{L}_{\star}{}^{(m)}_{f(x)\frac{d}{dx}}$ on the noncommutative space satisfies

$$[\mathcal{L}_{\star}{}^{(m)}_{f(x)\frac{d}{dx}}, \mathcal{L}_{\star}{}^{(m)}_{g(x)\frac{d}{dx}}]_\star = \mathcal{L}_{\star}{}^{m}_{[f(x),g(x)]_\star \frac{d}{dx}}. \tag{33}$$

There is a natural action of deformed vector fields on $\star$-deformed tensor densities on S^1. We find that almost all the properties of the vector fields can be lifted to the noncommutative space.

4.1 Construction of Noncommutative integrable systems

The Hill's operator acting on a noncommmutating spaces that maps $\mathcal{F}_{-\frac{1}{2}}$ to $\mathcal{F}_{\frac{3}{2}}$ transforms according to

$$\tilde{\Delta} = \phi^{-3/2} \star \Delta \star \phi^{-1/2}.$$

Thus, the Hill's operator maps

$$\Delta_\star : \mathcal{F}^\star_{-\frac{1}{2}} \longrightarrow \mathcal{F}^\star_{\frac{3}{2}}, \tag{34}$$

where $\mathcal{F}^\star_{\frac{n+1}{2}}$ is the space of $\star$-deformed tensor densities of degree $n + 1/2$.

Definition 4.1 *The $\star$ -product action of $Vect(S^1)$ on the space of Hill's operator Δ is defined by the commutation with the Lie derivative*

$$[\mathcal{L}_{\star f(x)\frac{d}{dx}}, \Delta_\star]_\star := \mathcal{L}^{\frac{3}{2}}_{\star f(x)\frac{d}{dx}} \star \Delta_\star - \Delta_\star \star \mathcal{L}^{-\frac{1}{2}}_{\star f(x)\frac{d}{dx}}. \tag{35}$$

Lemma 4.2

$$[\mathcal{L}_{\star f(x)\frac{d}{dx}}, \Delta]_\star \quad = \quad \frac{1}{2} f_{xxx} + 2u \star f_x + u_x \star f \tag{36}$$

Proof: As we learned from the previous Section that the action of vector field on noncommutative spaces is similar to that commutative case. Thus by using Eqn. (35) we obtain this result by direct computation.

□

4.1.1 Action of vector fields and Burgers equation

We apply the $\star$–product deformed first order differential operator $\Delta_{\star 1}$ on $\mathcal{F}^{\star}_{\frac{n+1}{2}}$ to derive the $\star$-deformed Burgers operator.

The operator $\Delta_{\star 1}$ satisfies

$$\Delta_{\star 1} = \nu \frac{d}{dx} + u(x) \ : \ \mathcal{F}^{\star}_{-\frac{1}{2}} \longrightarrow \mathcal{F}^{\star}_{\frac{1}{2}}. \tag{37}$$

Definition 4.3 *The $\star$-product action of $Vect(S^1)$ on Δ_1 is defined by the commutator with the Lie derivative*

$$[\mathcal{L}_{\star f(x)\frac{d}{dx}}, \Delta_1]_\star \ := \ \mathcal{L}^{\frac{1}{2}}_{\star f(x)\frac{d}{dx}} \star \Delta_{\star 1} - \Delta_{\star 1} \star \mathcal{L}^{-\frac{1}{2}}_{\star f(x)\frac{d}{dx}}, \tag{38}$$

where LHS denotes the Lie derivative action of vector fields on Δ_1.

The result of this action is a scalar operator, i.e. the operator of multiplication by a function.

Lemma 4.4

$$[\mathcal{L}_{f(x)\frac{d}{dx}}, \Delta_1]_\star \ = \ \nu \frac{1}{2} f''(x) + u \star f_x(x) + u_x \star f(x). \tag{39}$$

Proof: We obtain this result by direct computation from equation (32).
□

4.1.2 Construction of Noncommutative KdV and Burgers equations

In this Section we construct the Noncommutative integrable systems from the knowledge of our previous sections.

At first we recall the time dependent noncommutative Schrödinger equation

$$\frac{\partial \psi}{\partial t} = \hat{H} \star \psi,$$

where ψ is a wave-function and $\hat{H}$ is the Hamiltonian operator. Now if we try to implement the above scheme by replacing the $\hat{H}$ by the noncommutative Hamiltonian operators of ncKdV and ncBurgers equations then we immediately run into difficulties, since infinite time derivatives hidden inside the $\star$–product. Some Physicists [12,13,17] avoided this difficulty by formulating the entire theory in terms of the $\star$ deformed Lax equation

$$L_t = [B, L]_\star \equiv B \star L - L \star B.$$

Certainly they sacrificed the geometry in this process.

We use some kind of "operator technique". Let us define the noncommutative flow by

$$u_t = [\mathcal{L}_{\star \frac{\delta H}{\delta u}}, \Delta_{i\star}]_\star. \tag{40}$$

Thus for $H = \int_{S^1} u^2\, dx$ equation (38) boils down to

$$u_t = [\mathcal{L}_{\star 2u(x)}, \Delta_{i\star}]_\star, \tag{41}$$

where $\Delta_{i\star}$ stands for $\Delta_\star$ or $\Delta_{1\star}$. In this way we overcome the problem of the absence of Hamiltonian theory on noncommutative space-time.

Equation (41) certainly yields ncKdV and ncBurgers.

Proposition 4.5 *The Noncommutative KdV equation and the Noncommutative Burgers equations are given as*

$$u_t = u_{xxx} + 4u \star u_x + 2u_x \star u. \tag{42}$$

$$u_t = u_{xx} + 2u \star u_x + 2u_x \star u \tag{43}$$

respectively.

5 Conclusion

In this article we explore the derivation of the Noncommutative KdV and the Noncommutative Burgers equation from a geometric point of view. We have studied the Moyal deformed action of $Vect(S^1)$ on the spaces of $\star$- deformed tensor product densities $\mathcal{F}_\lambda^\star$ of degree λ on the circle S^1 to derive the noncommutative KdV and Burgers equation. Using Lie derivative method we have tacitly avoided any connection to coadjoint orbit and quantization of Lie-Poisson structures. But certainly we have faced difficulties for not having a (noncommutative) Hamiltonian formalism. The lifting of vector fields and its action on tensor densities to noncommutative spaces is smooth but the main problem lies in the construction of appropriate noncommutative Hamiltonian formalism. Nevertheless, in this paper we have presented a far more geometrical method using diffeomorphism to study the $\star$-product deformed noncommutative KdV and the Noncommutative Burgers equations.

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