Max-Planck-Institut
für Mathematik
in den Naturwissenschaften
Leipzig

Euler-Poincaré flows on sl_n Opers and Integrability

by

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Preprint no.: 144 2006

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Abstract

We consider the action of vector field $Vect(S^1)$ on the space of an sl_n - opers on S^1, i.e., a space of nth order differential operator $\Delta^{(n)} = \frac{d^n}{dx^n} + u_{n-2}\frac{d^{n-2}}{dx^{n-2}} + \cdots + u_1\frac{d}{dx} + u_0$. This action takes the sections of $\Omega^{-(n-1)/2}$ to those of $\Omega^{(n+1)/2}$, where Ω is the cotangent bundle on S^1. In this paper we study Euler-Poincaré (EP) flows on the space of sl_n opers, In particular, we demonstrate explicitly EP flows on the space of third and fourth order diffrential operators (or sl_3 and sl_4 opers) and its relation to Drienfeld-Sokolov, Hirota-Satsuma and other coupled KdV type systems. We also discuss the Boussinesq equation associated with the third order operator. The solutions of the sl_n oper defines an immersion $\mathbf{R} \longrightarrow \mathbb{R}P^{n-1}$ in homogeneous coordinates. We derive the Schwarzian KdV equation as an evolution of the solution curve associated to $\Delta^{(n)}$, We study the factorization of higher order operators and its compatibility with the action of $Vect(S^1)$. We obtain the generalized Miura transformation and its connection to the modified Boussinesq equation for sl_3 oper. We also study the eigenvalue problem associated to sl_4 oper. We discuss flows on the special higher order differential operators for all $u_i = f(u, u_x, u_{xx} \cdots)$ and its connection to KdV equation. Finally we explore a relation between projective vector field equation and generalized Riccati equations.

Mathematics Subject Classifications (2000): 53A07, 53B50, 35Q53, 14G32.

Keywords and Keyphrases: opers, Virasoro action, projective structure, Drienfeld-Sokolov equation, Hirota-Satsuma equation, coupled KdV equation Boussinesq equation and Riccati.

Contents

1 Introduction

The space of linear differential operators on a manifold M considered as a module over the group of diffeomorphisms is a well known classical text. This space has various algebraic structures, e.g. the structure of an associative algebra and of a Lie algebra [11]. In one dimensional case this was studied about a hundred years ago by Wilczynski [42] and more recently by E. Cartan [6].

It is well known [43] that the Korteweg-de Vries (KdV) equation is the canonical example of a scalar Lax equation, which is an equation defined by a Lax pair of scalar differential operators

$$\frac{d\Delta^{(n)}}{dt} = [P, \Delta^{(n)}]$$

where

$$\Delta^{(n)} = \frac{d^n}{dx^n} + u_{n-2}\frac{d^{n-2}}{dx^{n-2}} + \cdots + u_0. \tag{1}$$

Here P is a differential operator whose coefficients are differential polynomial in the variables, essentially determined by the requirement that $[P, \Delta^{(n)}]$ be an operator of order less than n.

The space of differential operators on S^1 has an interesting geometric and algebraic structure. This has been studied in various directions in various methods. We have studied [15-17] this space mainly from the point view of projective connections on the circle [20,28]. In the late 80th decades, many physicists studied the AGD operators and its connection to extended classical conformal algebras [4, 35].

Recently [5,9,10], the nth order differential operator is identified with the sl_n oper on a smooth curve (here S^1) is an equivalence class of operators of the form

$$\begin{pmatrix} * & * & * & \cdots & * \\ -1 & * & * & \cdots & * \\ 0 & -1 & * & \cdots & * \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & o & -1 & \cdots & * \end{pmatrix}$$

with respect to the gauge action of the group N of the upper triangular matrices with 1's on the diagonal. It is not difficult to show that each gauge class contains a unique operator of the form

$$\partial_x + \begin{pmatrix} 0 & u_{n-2} & u_{n-3} & \cdots & u_0 \\ -1 & 0 & 0 & \cdots & 0 \\ 0 & -1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & o & -1 & \cdots & 0 \end{pmatrix}$$

But the above operator is same as our scalar nth order differential operator. This is also known as projective connections on a circle. Thus *sl_n opers and projective connections are equivalent objects* . Moreover, the space of sl_n opers or space of projective connections is the phase space of the n-th KdV hierarchy introduced by Adler and Gelfand-Dickey. Hence the space of nth order differential operators on S^1 is also known as AGD space. It is worth to mention that for the classical Lie algebras sp_{2n} and o_{2n+1} may also be realized by differential operators. as shown by Drinfeld and Sokolov [7]

Let us focus on the integrable Hamiltonian systems. These systems carry additional structure, namely, they are bi-Hamiltonian systems, that is, they are Hamiltonian with respect to two different compatible Hamiltonian operators. Adler [2] proposed a scheme for deriving such Hamiltonian operators starting from a given Lax operator, and later Gelfand and Dickey gave a rigorous proof of Adler's construction. The space of nth order linear differential operators on S^1, is also known as Adler-Gelfand-Dickey (AGD) space [2,12], is connected to the $gl(n,R)$ current algebra, i.e. an algebra of loops $C^\infty(S^1, gl(n,R))$. This automatically reduces to the $sl(n,R)$ current algebra when $u_{n-1} = 0$. Gelfand and Dickey [12] established the relation between dual spaces of Kac-Moody algebras on the circle and the AGD space. The later is a Poisson subspace of the former [32]. They coincide only for $n = 2$.

Projective connections on the circle [20] were classified from a geometrical point of view by Kuiper [28]. Lazutkin and Pankratova [30] were the first to formulate this analytically. A projective connection on the circle is a linear second order differential operator, $\frac{d^2}{dx^2} + u(x)$, which acts on a periodic functions, known as Hill's operator. This is a dual space of the Virasoro algebra. It has a sequence of eigenvalues tending to infinity. If the eigenvalues are fixed, then the possible functions u form an infinite dimensional torus. One can imagine this to be a product of one circle for each pair of consecutive eigenvalues. The Korteweg-de Vries equation evolves in a straight line on one of the above tori.

The connection between the geodesic flow on the Virasoro-Bott group and the periodic KdV equation follows from the work of Kirillov [25-27], Segal [39,40] and Witten [46]. More direct proof was, of course, was given by Ovsienko and Khesin [36]. They showed that the KdV equation is the geodesic flow on the Bott-Virasoro group with respect to the right invariant L^2 metric.

It is known [15,18,37] that the solution of $\Delta^{(n)}$ defines an immersion in homogeneous coordinates. This immersion plays an important role to connect the KdV equation to the Schwarzian KdV equation. In our earlier paper [15,18], we have explored the connection between the Schwarzian (generalized) KdV equation and (generalized) KdV via projective geometry. It is known that the KdV and Schwarzian KdV formed and Antiplectic pair [44,45]. These are Euler-Poincaré type flows [3, 33, 34], and one of the flow takes place on an infinite-dimensional Poisson manifold and the other on a slightly degenerate infinite-dimensional Symplectic manifold.

1.1 Motivation

It is readily observable that a sl_2 oper is nothing but a Hill's operator of the form $\frac{d^2}{dx^2} + u(x)$ acting from $\Omega^{-1/2}$ to $\Omega^{-1/2}$, where $\Omega = TS^1$. Under the transformation of coordinates $x = z(x)$ we obtain the following transformation

$$u \mapsto \tilde{u}, \qquad \tilde{u} = u(z(x))(z'(x))^2 + \frac{1}{2}\mathcal{S}(z,x)$$

where $\mathcal{S}(z,x)$ is called the Schwarzian derivative, defined by

$$\mathcal{S}(z,x) = \frac{z'''}{z'} - \frac{3}{2}(\frac{z''}{z'})^2.$$

The infinitesimal transformation is turned out to be equivalent with the coadjoint action of vector fields on its dual. It is given by the action of Lie derivative $\mathcal{L}_{f(x)\frac{d}{dx}}$ on the space of the Hill's operator.

The space of sl_n opers has a $Vect(S^1)$ module structure. In this paper we study several evolution equations associated to the flow induced by the action of $Vect(S^1)$ on the space of higher order differential operators. This is a different than the Gelfand-Dickey method. We apply different geometrical technique to study such evolution equatiosn. This paper is not to point out any limitation of the Gelfand-Dickey formalism by any mean but it is an attempt to examine the problem from a different point of view, just to investigate if an alternative formalism is possible. Certainly the Gelfand-Dickey approach is very novel. But approach yields lots of new coupled KdV type equations, complex or quaternionic KdV, Schwarzian KdV and Camassa-Holm equations. It also yields Riccati and generalized Riccati equations geometrically.

In [15-17] we have discussed the prelimineries of the relation between projective connection and vector fields with the integrable systems. We have studied the Euler-Poincaré flows on the space of third order differential operators in [14]. In this paper, we extend our previous results. We will study the evolution defined by the action of a vector field $Vect(S^1)$ on a third order and fourth order differential operators.

We obtain the celebrated Drienfeld-Sokolov equation and other coupled KdV type systems as an Euler-Poincaré flows on the space of third order differential operators. It was realized [48] that the classical W_3 algebra can be constructed from the second Hamiltonian structure of the Boussinesq equation. We give a partial realization of classical W_3 algebra. We obtain the modified Boussinesq equation from the factorization of the third order differential operators. We obtain the Hirota-Satsuma [21] and other coupled KdV equations [1,22] from the flows on the space of fourth order operators. We study the Zakharov-Shabat type eigenvalue problem studied by Wadati [41] from the factorization of fourth order operators.

We also consider the space of differential operators for special values of u_is, where u_1 and u_0 are expressed in terms of single variable u and its derivatives. These are generalized projective connections. In this case, the evolution equation defined by action of $Vect(S^1)$ leads to KdV equation. We also study the motion of immersion curves and yields Schwarzian KdV and Camassa-Holm equations. We construct higher Riccati equations from the stabilizer set of the action of $Vect(S^1)$ on projective connections.

1.2 Organization

This paper is **organized** as follows:
In Section 2, we present a brief introduction to sl_n oper, the Adler and Gelfnad-Dickey structure and smooth immersion in homogeneous coordinates. We give a formal

introduction of sl_n oper (or projective connection) in Section 3. We describe the action of $Diff(S^1)$ on the space of differential operators in Section 4. In Section 5 we study Euler-Poincaré flows on the space of third order differential operators and these yield the Drienfeld-Sokolov type systems. We also study the Boussinesq flow. We study factorization of third order operators and generalized Miura transformation in Section 6. We study Euler-Poincaré flows on the space of fourth order differential operators or Sl_4 opers in Section 7. These yield the Hirota-Satsuma, the various coupled KdV type systems. We discuss Wadati type eigenvalue problem associated to the factorization of fourth order operator in Section 8. In Section 9 we also consider a special classes of higher order differential operators. We derive the KdV equation as the Euler-Poincaré flow on the space of these operators. We also briefly mention about the construction of dispersionless integrable systems. In Section 10 we discuss flows on the space of curve associated to immersion of operators considered in Section 9. We explore the connection between Riccati equation, higher order Riccati equation and modified Riccati equation with the projective vector field equation.

Acknowledgement : It is my pleasant duty to acknowledge gratefully for several stimulating discussions with Professors Sasha Kirillov, Jerry Marsden, Tudor Ratiu, Jose Carinena, George Wilson and Valentin Ovsienko. I am particularly grateful to Professor Valentin Ovsienko for introducing me the subject and to Professor Jerry Marsden for urging me to study EP flows. I am thankful to Professor David Ben-Zvi for explanation of projective connections and opers on S^1. Finally I would like to thank Professor Dieter Mayer at TU Clausthal and Professor Jürgen Jost at the Max Planck Insitute for Mathematics in the Sciences for their gracious hospitality where the paper was finished for kind hospitality and stimulating atmosphere. This work has been partially supported by the DFG Research Group " Zeta functions and locally symmetric spaces" which is gratefully acknowledged.

2 Background: Opers, AGD structure and immersion

In this section we consider a very special and simple class of opers, known as sl_n opers.

Let X be a smooth algebraic curve and Ω the line bundle of holomorphic differentials on X. Let us fix the square root $\Omega^{1/2}$ of the line bundle Ω. A sl_n-oper on X is an nth order differential operator acting from the holomorphic sections of $\Omega^{-(n-1)/2}$ to those of $\Omega^{(n+1)/2}$ whose (i) principle symbol equal to 1 and (ii) subprinciple symbol is equal to 0. Suppose x be a local coordinate, then we express this operator as

$$\Delta^{(n)} = \frac{d^n}{dx^n} + u_{n-2}\frac{d^{n-2}}{dx^{n-2}} + \cdots + u_1\frac{d}{dx} + u_0,$$

This operator are also known as projective connections. We will introduce formally in the next section.

Remark If we relax the second condition we obtain a gl_n oper on X, locally we can write this as

$$\bar{\Delta}^{(n)} = \frac{d^n}{dx^n} + u_{n-1}\frac{d^{n-1}}{dx^{n-1}} + u_{n-2}\frac{d^{n-2}}{dx^{n-2}} + \cdots + u_1\frac{d}{dx} + u_0.$$

Drinfeld and Sokolov introduced a space of matrix differential operators. Their idea was to replace the operator (1) by the first order order matrix differential operator

$$\frac{d}{dx} + \begin{pmatrix} 0 & u_{n-2} & u_{n-3} & \cdots & u_0 \\ -1 & 0 & 0 & \cdots & 0 \\ 0 & -1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & o & -1 & \cdots & 0 \end{pmatrix}$$

The group of upper triangular matrices with $1's$ on the diagonal acts on this space by gauge transformations

$$\partial_x + U(x) \longmapsto \partial_x + gU(x)g^{-1} - g^{-1}\partial_x g.$$

It is not hard to see that this action is free and each orbit contains a unique operator of the above form. The Poisson structure associated to this orbit is known as Adler-Gelfand.Dickey Poisson structure.

2.1 Immersion and Solution curve

At first we consider the $n = 2$ case. In this case, $\mathcal{S}$ is the space of Hill's operators of the form

$$\Delta^{(2)} = \frac{d^2}{dx^2} + u.$$

Lemma 2.1 *There is a one to one correspondence between (1) the Hill's equation on S^1*

$$\Delta\psi = \psi'' + u\psi = 0,$$

where $u \in C^\infty(S^1)$ and ψ is the unknown function.
(2) smooth orientation preserving immersions $g : S^1 \longrightarrow \mathbb{R}P^1$, modulo the equivalence upto $PSL(2, \mathbf{R})$.

Proof: This proof is very easy, it says that if we choose two independent solutions ψ_1 and ψ_2, then

$$x \longmapsto (\psi_1(x), \psi_2(x)) \tag{2}$$

defines an immersion $\mathbf{R} \longrightarrow \mathbb{R}P^1$ in homogeneous coordinates. This defines a curve in the projective line $\mathbb{R}P^1$. Since the Wronskian of the solution curve is constant upto multiplication by a matrix in $SL(2, \mathbf{R})$, then the Wronskian $\psi_1'\psi_2 - \psi_1\psi_2'$ of any immersion can be written in a form (2) equals one.

□

This picture can be easily extended to the case of n-th order scalar differential operator. Associating to the equation $\Delta\psi = 0$ we define n independent solutions $(\psi_1, \psi_2, \cdots, \psi_n)$. The map

$$x \longmapsto (\psi_1(x), \psi_2(x), \cdots, \psi_n(x)) \tag{3}$$

defines an immersion

$$g \ : \ \mathbf{R} \longrightarrow \mathbb{R}P^{n-1}$$

in homogeneous cooordinates. Thus we obtain a solution curve associated to L, once again the Wronskian of the components equals one. Since coefficients are periodic, hence, if $\psi(x)$ is a solution, then $\psi(x+2\pi)$ is also a solution. This implies

$$\psi(x+2\pi) = M_\psi \psi(x),$$

where

$$M_\psi = \psi(2\pi)\psi(0)^{-1}$$

is a monodromy matrix. This matrix preserves the skew form given by the Wronskian, so $det(M_\psi) = 1$, i.e. $M_\psi \in SL(n, \mathbf{R})$. If one chooses a different solution curve then the new monodromy matrix will appear, this will be the conjugate of M_ψ by an element of $SL(n, \mathbf{R})$. This means that for each Lax operator we can associate a projective curve whose monodromy will be an element of the conjugacy class $[M_\psi]$. This curve is unique up to the projective action of $SL(n, \mathbf{R})$.

3 Formal introduction to sl_n opers or projective Connection and $Vect(S^1)$ module

In this section we give a proper defintion of sl_n oper or projective connection. Let Ω denote the cotangent bundle of the circle. This is a (trivial) real line bundle on S^1. Its n-fold tensor product Ω^n is the line bundle of differentials of degree n.

Definition 3.1 (Projective Connection) *An extended projective connection on the circle is a class of differential (conformal) operators*

$$\Delta^{(n)} : \Gamma(\Omega^{-\frac{n-1}{2}}) \longrightarrow \Gamma(\Omega^{\frac{n+1}{2}})$$

such that

1. *The symbol of $\Delta^{(n)}$ is the identity.*
2. $\int_{S^1} (\Delta^{(n)} s_1) s_2 \ = \ \int_{S^1} s_1 (\Delta^{(n)} s_2)$ *for all* $s_i \in \Gamma(\Omega^{-\frac{n-1}{2}})$.

It is known that the symbol of a n-th order operator from a vector bundle U to V is a section of Hom $(U, V \otimes Sym^n T)$, where

$$U = \Omega^{-\frac{(n-1)}{2}} \ V = \Omega^{\frac{n+1}{2}}.$$

Since $T = \Omega^{-1}$, hence we get

$$V \otimes Sym^n T \cong U,$$

giving an invariant meaning to the first condition.

If $s_2 \in \Gamma(\Omega^{-\frac{n-1}{2}})$, then $s_1 \Delta^{(n)} s_2 \in \Gamma(\Omega)$ is a one form to integrate.

The consequence of the *first condition* is that all the differential operators are monic, that is, the coefficient of the highest derivative is always one. The *second condition* says that the term $u_{n-1} = 0$.

The weights $\frac{-(n-1)}{2}$ and $\frac{(n+1)}{2}$ related to the space of operator $\Delta^{(n)}$ is known to physicists and mathematicians [19,37], but not from the point view of projective connections.

Consider a one parameter family of $Vect(S^1)$ action on the space of smooth function $a(x) \in C^\infty(S^1)$ [31]

$$\mathcal{L}_v^\lambda a(x) := f(x)a'(x) + \lambda f'(x)a(x), \tag{4}$$

where $\mathcal{L}_v^\lambda$ is the Lie derivative with respect to $v = f(x)\frac{d}{dx} \in Vect(S^1)$, given by

$$\mathcal{L}_v^\lambda := f(x)\frac{d}{dx} + \lambda f'(x). \tag{5}$$

It is easy to verify that two Lie derivatives $\mathcal{L}^\lambda_{f(x)\frac{d}{dx}}$ and $\mathcal{L}^\lambda_{g(x)\frac{d}{dx}}$ satisfy

$$[\mathcal{L}^\lambda_{g(x)\frac{d}{dx}}, \mathcal{L}^\lambda_{f(x)\frac{d}{dx}}] = \mathcal{L}^\lambda_{(fg'-f'g)\frac{d}{dx}}.$$

Let us denote $\mathcal{F}_\mu(S^1)$ the space of tensor-densities of degree μ

$$\mathcal{F}_\lambda = \{a(x)dx^\lambda \mid a(x) \in C^\infty(S^1).$$

Thus, we say

$$\mathcal{F}_\lambda \in \Gamma(\Omega^{\otimes\lambda}) \quad \Omega^{\otimes\lambda} = (T^*S^1)^{\otimes\lambda},$$

where $\mathcal{F}_0(M) = C^\infty(M)$, the space $\mathcal{F}_1(M)$ coincides with the space differential forms.

Definition 3.2 *The action of $Vect(S^1)$ on the space of Hill's operator $\Delta \equiv \Delta^{(2)}$ is defined by the commutator with the Lie derivative*

$$[\mathcal{L}_v, \Delta] := \mathcal{L}_v^{3/2} \circ \Delta - \Delta \circ \mathcal{L}^{-1/2}. \tag{6}$$

Therefore the action of $f(x)\frac{d}{dx}$ on the space of Hill's operators satisfies

$$[\mathcal{L}_\mathcal{V}, \Delta] = f''' + 4f'u + 2fu'.$$

This action can be identified with the coadjoint action of Virasoro algebra on its dual. Similarly we can generalizes this action on $\Delta^{(n)}$

Definition 3.3 *The $Vect(S^1)$ action on Δ^n is defined by*

$$[\mathcal{L}_v, \Delta^{(n)}] := \mathcal{L}_v^{(n+1)/2} \circ \Delta^{(n)} - \Delta^{(n)} \circ \mathcal{L}_v^{-(n-1)/2}. \tag{7}$$

4 Action of $Diff(S^1)$ on the space of higher order differential operators

In this section, we describe the transformations of the higher order differential (or AGD) operators under the action of $Diff(S^1)$. This transformation has been known since last century. The action of $Diff(S^1)$ induces a change of variable in the independent parameter x.

Let $\Delta^{(n)}$ be a scalar differential operator. There exists a natural $Diff(S^1)$ action on this space. In the case of Hill's operator Δ (or $\Delta^{(2)}$) this action coincides with the coadjoint action.

Let us consider a one-parameter family of actions of $Diff(S^1)$ on the space of functions on S^1, given by

$$\sigma_\lambda^* f = f \circ \sigma^{-1}((\sigma^{-1})')^\lambda. \tag{8}$$

Definition 4.1 *The action of group $Diff(S^1)$ on the space of differential operators $\Delta^{(n)}$ is defined by*

$$\sigma^* \Delta^{(n)} := \sigma^*_{\frac{n+1}{2}} \circ \Delta^{(n)} \circ (\sigma^*_{-\frac{n-1}{2}})^{-1}. \tag{9}$$

The result of the $Diff(S^1)$ action on the Hill's operator is given by

$$\sigma^*(\Delta) = c\frac{d^2}{dx^2} + u^\sigma,$$

where

$$u^\sigma = u \circ \sigma^{-1}((\sigma^{-1})')^2 + \frac{c}{2}\mathcal{S}(\sigma^{-1}).$$

The action of $Diff(S^1)$ transform the solutions of $\Delta^{(n)}\psi = 0$ as densities of degree $\frac{n-1}{2}$

It should be noted that the operators $\Delta^{(n)}$ do not preserve their form under the action of $Diff(S^1)$, $x \to \sigma(x)$, due to the appearance of the $(n-1)$-th term $-\frac{1}{2}n(n-1)(\sigma''/\sigma'^{n+1}$. Hence we should think the operators are acting on *densities of weight* $-1/2(n-1)$ rather than on scalar functions, in this case we can always find $u_{n-1} = 0$ as a reparametrization invariant. Therefore, the action of $Diff(S^1)$ on Δ^n is given by

$$\partial_x^n + u_{n-2}(x)\partial_x^{n-2} + \cdots + u_0(x) \longrightarrow \sigma'^{-(n+1)/2}(\partial_x^n + \tilde{u}_{n-2}\partial_x^{n-2} + \cdots + \tilde{u}_0)\sigma'^{-(n-1)/2}, \tag{10}$$

where

$$\tilde{u}_{n-2} = \sigma'^2 u_{n-2}(\sigma(x)) + \frac{1}{12}n(n-1)(n+1)\mathcal{S}'(x).$$

In particular for $\mathbf{n} = \mathbf{3}$ we find

$$\tilde{u}_1(x) = \sigma'^2 u_1(\sigma(x)) + 2\mathcal{S}(x)$$

$$\tilde{u}_0(x) = {\sigma'}^3 u_0(\sigma(x)) + \sigma'\sigma'' u_1(\sigma(x)) + \mathcal{S}'(x).$$

For $\mathbf{n = 4}$ we obtain:

$$\tilde{u}_2(x) = {\sigma'}^2 u_2(\sigma(x)) + 5\mathcal{S}(x)$$
$$\tilde{u}_1(x) = {\sigma'}^3 u_1(\sigma(x)) + 2\sigma'\sigma'' u_2(\sigma(x)) + 5\mathcal{S}'(x).$$
$$\tilde{u}_0(x) = {\sigma'}^4(\sigma(x)) u_0(\sigma(x)) + \frac{3}{2}{\sigma'}^2\sigma'' u_1(\sigma(x)) + \frac{3}{2}{\sigma''}^2 u_2(\sigma(x))$$
$$+\frac{3}{2}{\sigma'}^2 u_2(\sigma(x))\mathcal{S}(x) + \frac{3}{2}\mathcal{S}''(x) + \frac{3}{2}\mathcal{S}^2(x).$$

This means, u_2 transforms as a potential of the Hill's operator [cf. 23], u_1 transforms as a cubic differential and u_0 has the sense of quartic differential.

Let us concentrate on third order operator, this has been poped up in various places in literatures [14 ,38].

Proposition 4.2 *A diffeomorphism σ transform a third order operator into the operators of the form (14) with coefficients:*

$$u_1^\sigma = u_1 \circ \sigma(\sigma')^2 + 2\mathcal{S}(\sigma) \qquad u_0^\sigma = u_0 \circ \sigma(\sigma')^3,$$

where $\mathcal{S}$ is the Schwarzian derivative.

Corollary 4.3 *The projection from the space of nth order operators to the space of Sturm -Liouville operators:*

$$\frac{d^n}{dx^n} + u_{n-2}\frac{d^{n-2}}{dx^{n-2}} + \cdots + u_0 \qquad \stackrel{\Pi_2}{\longmapsto} \qquad \frac{d^n}{dx^n} + u_{n-2}\frac{d^{n-2}}{dx^{n-2}}$$

is $Diff(S^1)$-equivariant.

Proposition 4.4 *The action of vector field $f(x)\frac{d}{dx} \in Vect(S^1)$ associates to $\Delta^{(3)}$ a first order operator*

$$[\mathcal{L}_{f(x)\frac{d}{dx}}, \Delta^{(3)}] = u_{1f}\frac{d}{dx} + (\frac{u'_{1f}}{2} + u_{0f}) \tag{11}$$

where

$$u_{1f} = (fu_1' + 2f'u_1 + 2f'''$$

$$u_{0f} = fu_0' + 3f'u_0.$$

Proof: By direct computation.
□

Thus, equation (11) is an element of the Lie algebra of first order differential operators on S^1. This Lie algebra is in fact the semidirect product of $Vect(S^1)$ by the module of functions $C^\infty(S^1)$.

Corollary 4.5 *The coadjoint action on u_1 and u_0 is given by*

$$ad^* u_1 = \partial u_1 + u_1\partial + 2\partial^3 \tag{12}$$
$$ad^* u_0 = \partial u_0 + 2u_0\partial. \tag{13}$$

5 Euler-Poincaré flows and coupled KdV equations

Let G be a Lie group and $\mathfrak{g}$ be its corresponding Lie algebra and its dual is denoted by $\mathfrak{g}^*$.

The dual space $\mathfrak{g}^*$ to any Lie algebra $\mathfrak{g}$ carries a natural Lie-Poisson structure:

$$\{f, g\}_{LP}(\mu) := < [df, dg], \mu >$$

for any $\mu \in \mathfrak{g}^*$ and $f, g \in C^\infty(S^1)$.

Lemma 5.1 *The Hamiltonian vector field on $\mathfrak{g}^*$ corresponding to a Hamiltonian function f, computed with respect to the Lie-Poisson structure is given by*

$$\frac{d\mu}{dt} = ad^*_{df}\mu \tag{14}$$

Proof: It follows from the following identities

$$\begin{aligned} &i_{X_f} dg|_\mu = L_{X_f} g|_\mu = \{f, g\}_{LP}(\mu) \\ &= < [dg, df], \mu > = < dg, ad^*_{df}\mu > . \end{aligned}$$

This implies that $X_f = ad^*_{df}\mu$. Thus the Hamiltonian equation $\frac{d\mu}{dt} = X_f$ yields our result.

$\square$

We write $E(\mu) = \frac{1}{2} < \mu, I\mu >$ for the quadratic energy form on $\mathfrak{g}$. $E(\mu)$ is used to define the Riemannian metric. We identify the Lie algebra and its dual with this quadratic form. This identification is done via an *inertia operator*.

Let I be an inertia operator

$$I : \mathfrak{g} \longrightarrow \mathfrak{g}^*$$

and then $\mu \in \mathfrak{g}^*$ evolve by

$$\frac{d\mu}{dt} = (I^{-1}\mu).\mu, \tag{15}$$

where right hand side denote the coadjoint action of $\mathfrak{g}$ on $\mathfrak{g}^*$. This equation is called the Euler-Poincaré equation.

Definition 5.2 *The Euler-Poincaré equation on $\mathfrak{g}^*$ corresponding to the Hamiltonian $H(\mu) = \frac{1}{2} < I^{-1}\mu, \mu >$ is given by*

$$\frac{d\mu}{dt} = -ad^*_{I^{-1}\mu}\mu.$$

It characterizes an evolution of a point $\mu \in \mathfrak{g}^*$.

Proposition 5.3 *Let ΩG be infnite dimensional Lie group equipped with a right invariant metric. A curve $t \longrightarrow c(t)$ in ΩG is a geodesic of this metric iff $u(t) = d_{c_t} R_{c_t^{-1}} \dot{c}(t)$ satisfes*

$$\frac{d}{dt} u(t) = -ad^*_{u(t)} u(t). \tag{16}$$

5.1 Coupled KdV type equations

Using the Euler-Poincaré framework we study the EP flow connected to third order differential operators. The coupled KdV equation is a generic example of multi-component systems. The classical Boussinesq system is connected to the cKdV system through nonsingular transformation.

Proposition 5.4 *The Euler-Poincaré flow on the space of third order differential operators yields following the Hamiltonian structure*

$$\begin{pmatrix} 2\partial^3 + 2u\partial + u_x \\ v_x + 3v\partial \end{pmatrix}.$$

This gives rise to (A) the Drienfeld-Sokolov equation

$$\begin{aligned} u_t + 2v_{xxx} + 2uv_x + u_x v &= 0 \ , \\ v_t + 4vv_x &= 0 \ . \end{aligned} \tag{17}$$

for $H = \frac{1}{2}\int v^2 \, dx$
(B) another coupled KdV equation

$$\begin{aligned} u_t + 2u_{xxx} + 3uu_x &= 0 \ , \\ v_t + (uv)_x + 2vu_x &= 0 \ . \end{aligned} \tag{18}$$

for Hamiltonian $H = \frac{1}{2}\int u^2 \, dx$.

Proof: By direct computation.
□

5.2 The Boussinesq equation

There is an interesting connection between the Hamiltonian structure of the Boussinesq equation

$$u_t = v_x, \qquad v_t = \frac{1}{3}u_{xxx} + \frac{8}{3}uu_x,$$

named after the French mathematician Joseph Boussinesq described in the 1870 model equation for the propagation of long waves on the surface of water with small amplitude. The Hamiltonian structure of the Boussinesq equation is given by

$$\mathcal{O} = (\mathbf{c}_1 \mid \mathbf{c}_2) = \left(\begin{array}{c|c} 2\,\partial^3 + u_1\partial + \partial u_1 & 2\partial u_0 + u_0\partial \\ \hline \partial u_0 + 2u_0\partial & h \end{array}\right),$$

where

$$h = \frac{1}{3}\partial^5 + \frac{5}{3}(u_1\partial^3 + \partial^3 u_1) - (u_{1xx}\partial + \partial u_{1xx}) + \frac{16}{3}u_1\partial u_1.$$

We obtain only first column matrix $\mathbf{c}_1$ of the Hamiltonian operator, and these only lead to the partial construction of W_3 algebra [45]:

$$\{u_1(x), u_1(y)\} = [2\partial^3 + 2u_1(x)\partial + u_1'(x)]\delta(x-y),$$

$$\{u_0(x), u_1(y)\} = [3u_0(x)\partial + u_0'(x)]\delta(x-y).$$

Unfortunately, the other part can not be obtained immediately in this way. The second column matrix $\mathbf{c}_2$ can be obtained from slightly different route. The element a_{12} is an adjoint of a_{21}. The a_{22} element is the total derivative of Boussinesq equation and some other higher order derivates. We must admit that this is the limitation of our method. In fact, Gelfand-Dickey method will be more appropriate than our method.

Similarly, one can compute $n = 4$ case. Here one must obtain a third order time derivative equation.

6 Factorization of $\Delta^{(3)}$ operators and generalized Miura transformation

In this section we will show that the action of vector field on S^1 is compatible with the factorization of the higher order linear operators.

If we assume that $\Delta^{(3)}$ is factorizable:

$$\Delta^{(3)} = (\partial^2 + u)(\partial + w).$$

It is compatible if and only if $w = 0$.

Let $K[u]$ be the algebra of differential polynomials in u, that is, polynomial with infinite set of variables u^i, $i \geq 0$, such that we write $u = u^0$,$u_x = u^{(1)}$ etc.

Let us extend our base field $K[u]$. Let us factorize

$$\Delta^{(3)} = (\partial^2 + u)\partial = (\partial + v)(\partial - v)\partial \tag{19}$$

over a differential field $K[v]$. This extension is given by

$$u = -(v_x + v^2), \tag{20}$$

and this extension is not Galois extension. The equation (20) is called Miura map.

This factorization was considered by Kupershmidt and Wilson [29]. In a remarkable paper, they gave simpler definition of Adler-Gelfand-Dickey bracket on the space of differential operators when the operator $\Delta^{(n)}$ is factorizable.

$$\Delta^{(n)} = (\partial + p_{n-1})(\partial + p_{n-2}) \cdots (\partial + p_1)(\partial + p_0),$$

where

$$p_k = \omega^k v_1 + \omega^{2k} v_2 + \cdots + \omega^{(n-1)k}, \qquad 0 \geq k \geq n-2; \qquad \omega = e^{2\pi i/n},$$

and $p_{n-1} = -\sum_{i=0}^{n-2} p_i$.

We consider a special case of above scheme.

Proposition 6.1

$$\mathcal{L} \circ (\partial + v)(\partial - v)\partial = (\partial^3 + u\partial + \partial u)\partial.$$

Outline of Proof: It is not difficult to show that the action of vector field $Vect(S^1)$ on $(\partial + v)$ is

$$[\mathcal{L}_{f(x)\frac{d}{dx}}, \partial + v] = (\partial^2 + \partial v) f.$$

Thus,

$$\begin{aligned}[\mathcal{L}, (\partial + v)](\partial - v)\partial &= (\partial^2 + \partial v)(\partial - v)\partial \\ &= \partial^4 - (v_x + v^2)\partial^2 - (v_x + v^2)'\partial \\ &= (\partial^3 + u\partial + \partial u)\partial.\end{aligned}$$

□

Modified Boussinesq equation Before finishing this section we briefly discuss the quadratic Miura transformation associated to a third order differential operator.

Let us consider the factorization of a third order differential operator

$$\begin{aligned}\Delta &= (\partial + p + q)(\partial - p)(\partial - q) \\ &= \partial^3 + (2q_x + p_x + p^2 + pq + q^2)\partial + pq(p + q)\end{aligned}$$

Thus the Miura type transformation is given by

$$u = 2q_x + p_x + p^2 + pq + q^2.$$

Therefore, the modified Boussinesq equation is given as

$$\begin{aligned} p_t &= \partial(-2q^2 - 2pq + p^2 + 2q_x + p_x) \;, \\ q_t &= \partial(q^2 - 2pq - 2p^2 - q_x - 2p_x) \;. \end{aligned} \tag{21}$$

7 Fourth order operators and integrable systems

In this section we study several integrable systems associated to the fourth order differential operator. Let us assume

$$\Delta^{(4)} = \partial^4 + u\partial^2 + u_x\partial + (\frac{9}{100}u^2 + \frac{3}{10}u'' + v). \tag{22}$$

It should be worth to say that for all conformal class of fourth order operators the coefficients of ∂^2 and ∂ must be related. In this section we study flows generated by various forms of such operators.

The $Vect(S^1)$ action on L is defined by the commutation with the Lie derivative

$$[\mathcal{L}_{f(x)\frac{d}{dx}}, \Delta^{(4)}] := \mathcal{L}^{5/2}_{f(x)\frac{d}{dx}} \circ L - L \circ \mathcal{L}^{-3/2}_{f(x)\frac{d}{dx}}.$$

Proposition 7.1 *The action of vector field* $f(x)\frac{d}{dx} \in Vect(S^1)$ *associates to* $\Delta^{(3)}$ *a second order operator*

$$[\mathcal{L}_{f(x)\frac{d}{dx}}, \Delta^{(4)}] = u_f \frac{d^2}{dx^2} + u'_f \frac{d}{dx} + (\frac{3}{10}u''_f + v_f) \tag{23}$$

where

$$u_f = (fu' + 2f'u + 5f'''$$
$$v_f = fv' + 4f'v.$$

Proof: By direct computation.
□

Corollary 7.2 *The Hamiltonian operator associated to the action of* $Vect(S^1)$ *on* u *and* u *are given as*

$$ad^*u = \partial u + u\partial + 5\partial^3 \tag{24}$$

$$ad^*v = \partial v + 3v\partial. \tag{25}$$

This Hamiltonian operator is almost similar to the Hamiltonian in Section 5.1. Therefore Euler-Poincaré flow yields here also Drinfeld-Sokolov equation.

7.1 Fifth order coupled integrable system

In this section we construct fifth order system. Let us consider following fourth order differential operator

$$\Delta^{(4)} = \partial^4 + u\partial^2 + u_x\partial + v. \tag{26}$$

Proposition 7.3 *The infinitesimal transformations of* u *and* v *under the* $Vect(S^1)$ *action are given by*

$$u_f \longmapsto 5f''' + 2f'u + fu' \tag{27}$$

$$v_f \longmapsto \frac{3}{2}f''''' + \frac{3}{2}uf''' + \frac{3}{2}u'f'' + 4vf' + v'f \tag{28}$$

Proof: By direct computation.
□

Corollary 7.4 *The Hamiltonian operator associated to the action of* $Vect(S^1)$ *on* u *and* u *are given as*

$$ad^*u = \partial u + u\partial + 5\partial^3 \tag{29}$$

$$ad^*v = \frac{3}{2}(\partial^5 + u\partial^3 + u'\partial^2) + 4v\partial + v' \tag{30}$$

Proposition 7.5 *The Euler-Poincaré flow with respect to the Hamiltonian structure*

$$\begin{pmatrix} 5\partial^3 + 2u\partial + u_x \\ \frac{3}{2}(\partial^5 + u\partial^3 + u'\partial^2) + v_x + 4v\partial \end{pmatrix}$$

(A) yields the Drienfeld-Sokolov equation

$$\begin{aligned} u_t + 5v_{xxx} + 2uv_x + u_x v &= 0 \ , \\ v_t + \frac{3}{2}(v_{xxxxx} + uv_{xxx} + u_x v_{xx}) + 5vv_x &= 0 \ . \end{aligned} \tag{31}$$

for $H = \frac{1}{2}\int v^2 \, dx$

(B) yields a fifth order coupled equation equation

$$\begin{aligned} u_t + 5u_{xxx} + 3uu_x &= 0 \\ v_t + 3u_{xxxxx} + 3uu_{xxx} + 3u_x u_{xx} + 8u_x v + 2uv_x &= 0 \ , \end{aligned} \tag{32}$$

for $H = \int u^2 \, dx$.

Proof: By direct computation
□

Remark It may be worth to say that the Equations (31) and (32) can be generalized simply by considering

$$\Delta^{(4)} = \partial^4 + (u+\lambda)\partial^2 + u_x\partial + (v+\mu)$$

instead of (26). This would induce slightly different Hamiltonian structure

$$\mathcal{O} = \begin{pmatrix} 5\partial^3 + 2(u+\lambda)\partial + u_x \\ \frac{3}{2}(\partial^5 + u\partial^3 + \lambda\partial^3 + u'\partial^2) + v_x + 4(v+\mu)\partial \end{pmatrix}$$

7.2 Hirota-Satsuma type equation

In this section we consider a factorizable fourth order differential operator of the following form

$$\Delta^{(4)} = (\partial^2 + u)(\partial^2 + v). \tag{33}$$

Lemma 7.6 *The Hamiltonian operators associated to the action of* $Vect(S^1)$ *on* $\Delta^{(4)} = (\partial^2 + u)(\partial^2 + v)$ *yields*

$$ad^* u = \partial u + u\partial + 5\partial^3 \tag{34}$$

$$ad^* v = \partial v + v\partial + 5\partial^3 \tag{35}$$

respectively.

Let us introduce a complex variable

$$p = u + iv. \tag{36}$$

Therefore, we combine Eqns. (34) and (35) to obtain new Hamiltonian operator

$$ad^* p = \partial p + p\partial + 5\partial^3 \tag{37}$$

Proposition 7.7 *The Euler-Poincaré flow with respect to the Hamiltonian structure*

$$ad^* p = \partial p + p\partial + 5\partial^3 \qquad p = u + iv$$

yields complex KdV or Hirota-Satsuma equation

$$p_t + p_{xxx} + 3pp_x = 0 \tag{38}$$

for

$$H = \frac{1}{2}\int p^2\, dx \equiv \frac{1}{2}\int (u^2 + v^2)\, dx.$$

We can also construct another coupled KdV equation from this pair of Hamiltonian operators.

Proposition 7.8 *The Euler-Poincaré flow with respect to the Hamiltonian structure*

$$\begin{pmatrix} 5\partial^3 + 2u\partial + u_x \\ 5\partial^3 + 2v\partial + v_x \end{pmatrix}$$

yields the following coupled equation

$$\begin{aligned} u_t + 5u_{xxx} + 3uu_x &= 0 \\ v_t + 5u_{xxx} + 2u_x v + uv_x &= 0\ , \end{aligned} \tag{39}$$

for $H = \frac{1}{2}\int u^2\, dx$.

8 Wadati type eigenvalue problem and generalized Miura map

In this section we study Zakharov-Shabat type eigenvalue [47] problem associated to the factorization of fourth order operators. We follow Wadati's technique [8,41] We will discuss two different cases.

Case I The operator (33) gives rise to the following eigenvalue problem

$$(\partial - p)(\partial + p)(\partial - q)(\partial + q)\psi = \lambda^4 \psi, \tag{40}$$

where

$$u = p_x - p^2, \qquad \text{and} \qquad v = q_x - q^2.$$

This may be deconposed again into two pairs of first-order equations

$$\begin{cases} (\partial + q)\psi = \lambda\phi \\ (\partial - q)\phi = \lambda\psi \end{cases} \tag{41}$$

$$\begin{cases} (\partial + p)\chi = \lambda\xi \\ (\partial - p)\xi = \lambda\chi \end{cases} \tag{42}$$

By introducing two sets of functions ($\psi_+ = \psi + \phi$, $\psi_- = \psi - \phi$) and ($\xi_+ = \xi + \chi$, $\xi_- = \xi - \chi$), we obtain

$$\partial \begin{pmatrix} \psi_+ \\ \psi_- \\ \xi_+ \\ \xi_- \end{pmatrix} = \begin{pmatrix} \lambda & q & & \\ q & -\lambda & & 0 \\ & 0 & \lambda & p \\ & & p & -\lambda \end{pmatrix} \begin{pmatrix} \psi_+ \\ \psi_- \\ \xi_+ \\ \xi_- \end{pmatrix}$$

which is the Zakharov-Shabat type eigenvalue problem. This was first introduced by Wadati to construct the Zakharov-Shabat eigenvalue problem from the factorization of the Strum-Liouville operator.

The time evolution of ψ, ϕ, ξ and χ is given by

$$\begin{aligned} \psi_t &= 5\partial^3 + 2u\partial + u_x \\ &= 5\partial^3 + 2(q_x - q^2)\partial + (q_{xx} - 2qq_x)\ , \end{aligned} \tag{43}$$

similarly we obtain the same for ϕ, ξ and χ

$$\phi_t = 5\partial^3 - 2(q_x + q^2)\partial - (q_{xx} + 2qq_x) \tag{44}$$

$$\xi_t = 5\partial^3 + 2(p_x - p^2)\partial + (p_{xx} - 2pp_x) \tag{45}$$

$$\chi_t = 5\partial^3 - 2(p_x + p^2)\partial - (p_{xx} + 2pp_x). \tag{46}$$

In general, for application purposes all these equations (43 - 46) are expressed in matrix of polynomials in λ, q and v and its derivatives.

<u>**Case II**</u> Let us factorize $\Delta^{(4)}$ of the following type:

$$\partial^4 + u\partial^2 + u_x\partial + v = (\partial^2 + \partial l - m)(\partial^2 - l\partial - m),$$

where l and m satisfy a pair of generalzed Miura system or coupled Riccti equation like system, given as

$$u = -(l_x + l^2 + 3m), m^2 - (lm + m')' = v. \tag{47}$$

$$v = (m^2 - (lm + m')'. \tag{48}$$

Thus, the above operator gives rise to the eigenvalue problem

$$(\partial^2 + \partial l - m)(\partial^2 - l\partial - m)\psi = \lambda^2\psi. \tag{49}$$

This can be decomposed into the pair of second-order equations

$$(\partial^2 + \partial l - m)\psi = \lambda\phi \tag{50}$$

$$(\partial^2 - l\partial - m)\phi = \lambda\psi. \tag{51}$$

Let us define two new wave functions

$$\begin{aligned} \psi_+ &= \psi + \phi \\ \psi_- &= \psi - \phi\ , \end{aligned} \tag{52}$$

Therefore, the eigenvalue problem becomes

$$\partial^2 \begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix} = \begin{pmatrix} \lambda + m - \frac{1}{2}l' & l\partial - \frac{1}{2}l' \\ -l\partial - \frac{1}{2}l' & m - \lambda - \frac{1}{2}l' \end{pmatrix} \begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix}. \tag{53}$$

Once again one can resolve the second order by invoking additional variables.

8.1 Construction of complex coupled system

Let us consider a new sixth order operator as the product of two different Lax operators of the Boussinesq type (or third order)

$$\Delta^{(6)} = (\partial^3 + u_1\partial + u_{1x} + v_1)(\partial^3 + u_2\partial + u_{2x} + v_2). \tag{54}$$

Lemma 8.1 *The Hamiltonian operators associated to the action of $Vect(S^1)$ on $\Delta^{(6)}$*

$$ad^*u_1 = 2\partial^3 + 2u_1\partial + u_{1x} \qquad ad^*v_1 = v_{1x} + 3v_1\partial \tag{55}$$

$$ad^*u_2 = 2\partial^3 + 2u_2\partial + u_{2x} \qquad ad^*v_2 = v_{2x} + 3v_2\partial \tag{56}$$

respectively.

By introducing

$$p = u_1 + iu_2, \qquad q = v_1 + iv_2,$$

we can combine these two pairs of operators. Thus, we would get

$$\mathcal{O} = \begin{pmatrix} 2\partial^3 + 2p\partial + p_x \\ q_x + 3q\partial \end{pmatrix} \tag{57}$$

Proposition 8.2 *The Euler-Poincaré flow with respect to the Hamiltonian structure (57) yields complex coupled KdV type equation*

$$p_t = p_{xxx} + 3pp_x \qquad p = u_1 + iu_2 \tag{58}$$

$$q_t = q_x p + 3qp_x \qquad p = v_1 + iv_2 \tag{59}$$

for

$$H = \frac{1}{2}\int p^2\, dx \equiv \frac{1}{2}\int (u_1^2 + u_2^2)\, dx.$$

Remark Similarly one can derive quaternionic KdV equation from the Euler-Poincaré flows associated to the eighth order differential operator

$$\Delta^{(8)} = (\partial^2 + u_1)(\partial^2 + u_2)(\partial^2 + u_3)(\partial^2 + u_4).$$

By introducing $P = u_1 + iu_2 + ju_3 + ku_3$, one would obtain quaternionic KdV in terms of variables P.

9 Special $\Delta^{(n)}$ operators and integrable flows

In this section we will study $\Delta^{(n)}$ operators with specific values of u_2, u_1, u_0 etc.

Let us start with a third order equation of the following type:

$$f''' + u_1 f' + u_0 f = 0. \tag{60}$$

To go further, consider solving (14) for u_1. We obtain

$$(ff'' - \frac{1}{2}(f')^2 + \frac{1}{2}(u_1 f^2)')_x + (u_0 - \frac{1}{2}u_1')f^2 = 0.$$

If we assume $u_0 = \frac{1}{2}u_1'$, then the above equation is solvable for u_0. After rescalling we assume $u_1 = 4u$ and $u_0 = 2u'$.

Thus, the special or factorizable differential operators of order $n = 3, 4$ are given as

$$\tilde{\Delta}^{(3)} = \partial_x^3 + 4u\partial_x + 2u', \tag{61}$$

$$\tilde{\Delta}^{(4)} = \partial_x^4 + 9u^2 + 3u'' + 10u'\partial_x + 10u\partial_x^2. \tag{62}$$

Proposition 9.1 *The action of a vector field $f(x)\frac{d}{dx} \in Vect(S^1)$ on $\tilde{\Delta}^{(3)}$ and $\tilde{\Delta}^{(4)}$ yield first order and second order operators*

$$[\mathcal{L}_{f(x)\frac{d}{dx}}, \tilde{\Delta}^{(3)}] = (f''' + 4f'u + 2fu')\partial + 2(f''' + 4f'u + 2fu')'$$

$$[\mathcal{L}_{f(x)\frac{d}{dx}}, \tilde{\Delta}^{(4)}] = (f''' + 4f'u + 2fu')\partial^2 + 5(f''' + 4f'u + 2fu')\partial$$
$$+(\partial^2 + 6u)(f''' + 4f'u + 2fu')$$

respectively.

Proof: By direct computation.

Definition 9.2 *A vector field is called projective vector field which keeps fixed a given projective connection $\tilde{\Delta}^{(n)}$*

$$[\mathcal{L}_{f(x)\frac{d}{dx}}, \tilde{\Delta}^{(n)}] = 0. \tag{63}$$

Corollary 9.3 *A projective vector field $v = f\frac{d}{dx} \in \Gamma(\Omega^{-1})$ satisfies*

$$f''' + 4f'u + 2fu' \;=\; 0. \tag{64}$$

Proposition 9.4 *The equation $\frac{\partial \tilde{\Delta}^{(n)}}{\partial t} = [\mathcal{L}_v, \tilde{\Delta}^{(n)}]$ generates the evolution as*

$$\frac{\partial u}{\partial t} = \frac{1}{2}f''' + 2f'u + fu'.$$

Proof: We prove it by direct computation.
□

We have argued that the action $[\mathcal{L}_v, \Delta^{(n)}]$, can be considered as a coadjoint action of $Vect(S^1)$ on $\Delta^{(n)}$. Hence, we say

$$\tilde{u} = \frac{1}{2}f''' + 2f'u + fu' = (\frac{1}{2}\partial_x^3 + 2u\partial_x + u_x)f.$$

The operator $(\frac{1}{2}\partial_x^3 + 2u\partial_x + u_x)$ is called the Poisson operator.

Proposition 9.5 *The Hamiltonian flow generated by action of $Vect(S^1)$ on the space of AGD or conformal operators for the Hamiltonian $H(u) = \frac{1}{2}\int_{S^1} u^2(x)\,dx$, yields KdV equation.*

9.1 Action of $Vect(S^1)$ and dispersionless class of systems

In this section we consider a special class of integrable systems, the dispersionless KdV or Riemann equation.

Dispersionless equations can be obtained as a quasi-classical limit of integrable ones. In this case we introduce scaling $\frac{\partial}{\partial t} \to \epsilon\frac{\partial}{\partial t}$, $\frac{\partial}{\partial x} \to \epsilon\frac{\partial}{\partial x}$ and take the limit $\epsilon \to 0$.

In the quasi-classical limit KdV equation

$$u_t = u_{xxx} + 3uu_x$$

becomes dispersionless KdV, defined as

$$u_t = 3uu_x. \tag{65}$$

This is the prototype for the hyperbolic systems and it has a Lax representation.

This equation can be derived easily from our method. We consider the special nth order operators.

Proposition 9.6 *The action of a constant vector field $\mu\frac{d}{dx} \in Vect(S^1)$ on $\tilde{\Delta}^{(n)}$ yields a unique expression*

$$[\mathcal{L}_{\mu\frac{d}{dx}}, \tilde{\Delta}^{(n)}] = \mu u'$$

for all values of n.

Proof: It is not difficult to check that the only expression which survives when the vector field $\mu \in Vect(S^1)$ (for μ = constant) acts on $\Delta^{(n)}$ is $\mu u'$.
$\square$

Proposition 9.7 *The Euler-Poincaré flow with respect to the Hamiltonian structure*

$$\mathcal{O} = u_x$$

yields dispersionless KdV equation

$$u_t = uu_x,$$

for $H = \frac{1}{2}\int_{S^1} u^2\, dx$.

10 Flows on the space of curve

In the earlier Section we have seen how an immersion associated to $\Delta^{(n)}$ yields a curve

$$\psi : \mathbf{R} \longrightarrow \mathbb{R}P^{n-1}$$

in the projective space.

Let us write ψ in terms of an inhomogeneous coordinates. We lift ψ to a curve on $\mathbf{R}^n$. This we may denote by $\tilde{\psi} = \eta(x)(1, \psi)$. We choose the factor $\eta(x)$ so that the Wronskian of the components of the new curve equals 1.

It turns out that there is a unique choice of $\eta(x)$ with such a property, and this is given by

$$\eta(x) = Wr(\psi_1', \cdots, \psi_{n-1}')^{-\frac{1}{n}}. \tag{66}$$

In particular, for $n = 2$ case,

$$\psi \equiv (\psi_1, \psi_2) = (\psi'^{-\frac{1}{2}}, \psi'^{-\frac{1}{2}}\psi) \tag{67}$$

is the solution curve [13]. It retains the unitarity of the Wronskian.

Corollary 10.1 *The three dimesional spaces of solutions to projective vector field equation is spanned by* $\psi_1^2, \psi_1\psi_2, \psi_2^2$, *and they form a* $sl(2, \mathbf{R})$ *algebra.*

Proposition 10.2 *(Kirillov) If f is a projective vector field, satisfies (62), then the square root of f is an element of scalar density of weight -$\frac{1}{2}$, denoted by ψ. Then (f, ψ) satisfies super algebra.*

The stabilizer of the action of this superalgebra on pairs $\tilde{\Delta}^{(n)}$ is given by (16) and

$$\Delta\psi \;=\; (\frac{d^2}{dx^2}+u)\psi \;=\; 0. \tag{68}$$

This gives us a geometrical explanation that if ψ_1, ψ_2 satisfy (68) then $\psi_1\psi_2$, ψ_1^2, and ψ_1^2 satisfy (64).

Let us study now the fourth order equation

$$f'''' + 10uf'' + 10u'f' + (9u^2 + 3u'')f = 0. \tag{69}$$

By direct computation one can show:

Proposition 10.3 *The equation $f'''' + 10uf'' + 10u'f' + (9u^2 + 3u'')f = 0$ traces out a four dimensional spaces of solutions spanned by*

$$\{\psi_1^3,\ \psi_1^2\psi_2,\ \psi_1\psi_2^2,\ \psi_2^3\}.$$

Lemma 10.4 *The equation $f''' + 2u'f + 4uf' = 0$ traces out a three dimesional spaces of solutions.*

Proof: If ψ_1 and ψ_2 are the solutions of

$$\Delta\psi \;=\; (\frac{d^2}{dx^2}+u)\psi \;=\; 0,$$

then it is easy to see that $\psi_i\psi_j \in \Gamma(\Omega^{-1})$ satisfies the above equation. Hence the solutions space is spanned by ψ_1^2, ψ_2^2 and $\psi_1\psi_2$.

□

Substituting (67) in equation $(\frac{d^2}{dx^2}+u)\psi_i \;=\; 0$ (for $i = 1, 2$), we obtain

$$u = \frac{1}{2}(\frac{\psi'''}{\psi'} - \frac{3}{2}\frac{\psi''^2}{\psi'^2}). \tag{70}$$

The right hand side is invariant under $PSL(2,\mathbf{R})(= SL(2,\mathbf{R}/\pm 1)$ group. If we substitute this expression in the Euler-Poincaré equation, we obtain the evolution equation of the solution curve on the projective space

$$\psi_t = \psi_{xxx} - \frac{3}{2}{\psi_{xx}}^2{\psi_x}^{-1}.$$

This equation is called the Schwarzian KdV (or Ur KdV by G. Wilson [44-45]). One can check directly that this equation is $SL(2,\mathbf{R})$ invariant.

Thus we obtain the following theorem

Theorem 10.5 *The solutions of an operator $\Delta^{(n)}$ define an immersion*

$$\mathbf{R} \longrightarrow \mathbb{R}P^{n-1}$$

in homogeneous coordinates. The evolution equation of the solution curve is a Hamiltonian flow, and it is given by the Schwarzian KdV equation.

The Schwarzian KdV has a bihamiltonian structure [44]

$$H_1 = 4u, \quad D_1 = -2{\psi_x}^{-1}\partial{\psi_x}^{-1},$$

$$H_2 = u^2, \quad D_2 = -\frac{1}{2}{\psi_x}^{-2}\partial^3 - 3\psi_{xx}{\psi_x}^{-3}\partial^2 + (3{\psi_{xx}}^2{\psi_x}^{-4} - \psi_{xxx}{\psi_x}^{-3})\partial.$$

10.1 Flows on the space of immersion and Schwarzian Camassa-Holm equation

Let ψ_1 and ψ_2 are the solutions to the second order, homogeneous, linear Schrödinger equation

$$\Delta\psi \;=\; (\frac{d^2}{dx^2} + m(u))\psi \;=\; 0 \qquad m = u - u_{xx}, \tag{71}$$

with potential $m(u)$.

Substituting (67) in equation $(\frac{d^2}{dx^2} + m)\psi_i \;=\; 0$ (for $i = 1, 2$), we obtain

$$m = \frac{1}{2}(\frac{\psi'''}{\psi'} - \frac{3}{2}\frac{\psi''^2}{\psi'^2}). \tag{72}$$

Therefore, we obtain

$$u = (1-\partial^2)^{-1}[\frac{1}{2}(\frac{\psi'''}{\psi'} - \frac{3}{2}\frac{\psi''^2}{\psi'^2})] \tag{73}$$

If we substitute this expression in the Hamiltonian equation,

$$u_t = \mathcal{O}\frac{\delta H}{\delta u}$$

we obtain the evolution equation of the solution curve on the projective space

$$(1-\partial^2)\psi_t = \psi_{xxx} - \frac{3}{2}{\psi_{xx}}^2{\psi_x}^{-1}.$$

This equation is called the Schwarzian CH. Once again, one can check directly that this equation is $SL(2,\mathbf{R})$ invariant.

Remark: The CH and Schwarzian CH equations are Euler-Poincaré type flows, and one of the flow takes place on an infinite-dimensional Poisson manifold and the other on a slightly degenerate infinite-dimensional Symplectic manifold. They form an Antiplectic pair.

We note that, if we interchange the roles of independent and dependent variables, then the Schwarzian derivative becomes

$$\mathcal{S} = -\frac{x_\phi x_{\phi\phi\phi} - \frac{3}{2}x_{\phi\phi}^2}{x_\phi^4}. \tag{74}$$

Remark Whenever a Lie group G acting on a space $M \simeq \Phi \times X$, where Φ represents the independent variables and X the dependent variables. There exist an induced action on the associated jet bundles $J^n M$, which is called the nth prolongation of G, denoted by $pr^n G$. Let $I(\phi, x^{(n)})$ be a scalar valued function depending on the independent and dependent variables nd their derivatives, which is invariant under $pr^n G$. Then $I(\phi, x^{(n)})$ is known as differential invariant of the group G. For the $SL(2)$ group action, Eqn. (74) is one of the two fundamental differential invariants, other one is x.

Let us substitute

$$w = \frac{1}{x_\phi}$$

in equation (74). We obtain the Harry Dym type equation

$$w_t = w^2[w(w\partial_\phi^{-1}(w^{-1})_t)_\phi]_\phi. \tag{75}$$

10.2 Riccati chain and stabilizer orbit

Let us consider the stabilizer equation of the odd part of Kirillov's superalgebra which coincides with the Hill's equation

$$a\psi_{xx} + u\psi = 0,$$

where a is a constant.

Let us make a change of variables

$$p(x) = \frac{\psi_x}{a\psi}, \qquad \text{then } p_x = \frac{\psi_{xx}}{a\psi} - \frac{\psi_x^2}{a\psi^2}.$$

Thus after substituting this into Hill's equation, we obtain the celebrated Riccati equation

$$p_x + a\,p^2 + u = 0. \tag{76}$$

Thus it is readily clear that the Riccati equation under Cole-Hopf transformation is connected to the stabilizer orbit of the "Fermionic" part of the Kirillov's superalgebra.

There are some interesting featues of the Riccati equation. If *one solution* of a some Riccati equation is known then we can get immediately general solutions of the whole family of Riccati equations obtained from the original one under the change of variables

$$\hat{p} = \frac{a(x)p + b(x)}{c(x)p + d(x)}. \tag{77}$$

It is also interesting to notice that for Riccati equation knowing any three solutions p_1, p_2, p_3 we can construct all other solutions p using a simple formula known as *cross ratio*:

$$\frac{p - p_1}{p - p_2} = \alpha\frac{p_3 - p_1}{p_3 - p_2} \tag{78}$$

with an arbitrary constant α.

10.2.1 Higher order Riccati and projective vector field equation

We wish to explore the connection between the projective vector field equation and second order Riccati equation.

Let us assume

$$v = \frac{f_x}{f} \tag{79}$$

where f satisfies projective vector field equation. Equation (79) implies

$$\frac{f_{xxx}}{f} = v_{xx} + 3vv_x + v^3,$$

and after substituting the results above in the projective vector field equation, it takes the following form

$$v_{xx} + 3vv_x + v^3 + 4uv + 2u_x = 0, \tag{80}$$

which is particular case of second order Riccati equation. The coefficients are fixed by the projective vector field equation.

Proposition 10.6 *1. The projective vector field equation is equivalent to a particular form of second order Riccati equation* $v_{xx} + 3vv_x + v^3 + 4uv + 2v_x = 0$*, where* $v = \frac{f_x}{f}$*.*

2. Suppose $p(x) = p_1$ *be the solution of the Riccati equation. Then the second order Riccati satisfies* $v(x) = 2p_1$*.*

Proof: By direct computation one can check this result.
□

Above proposition is the Riccati analogue of the relation between projective vector field equation and its partner equation $\psi_{xx} + u(x)\psi = 0$.

Therefore, above result yields the correspondences between solutions of second order Riccati and ordinary Riccati equation. At this stage we must give the definition of Riccati chains. In fact, all the higher order Riccati equations satisfy most of the properties of the Riccati equation .

Definition 10.7 *Let L be the following differential operator*

$$L = \frac{d}{dx} + v(x).$$

The n–order equation of the Riccati chain is given by the following formula

$$L^n v(x) + \sum_{j=1}^{n-1} \alpha_j(x)(L^{j-1}v(x)) + \alpha_0(x) = 0, \tag{81}$$

where n is an integer characterzing the order of the Riccati equation in the chain and $\alpha_j(x)$*,* $j = 0, 1, \cdots N$ *are arbitrary functions.*

The lowest-order equations in the chain after the ordinary Riccati equation are:

$$n = 2, \qquad v_{xx} + 3v(x)v_x + v^3(x) + \alpha_1(x)v(x) + \alpha_0(x) = 0 \tag{82}$$

$$n = 3, \qquad v_{xxx}+4vv_{xx}+3v_x^2+6v^2v_x+\alpha_2(x)v_x+v^4(x)+\alpha_2 v^2(x)+\alpha_1(x)v(x)+\alpha_0(x) = 0 \tag{83}$$

Thus, our second Riccati coincides with $n = 2$ member of the Riccati chain when $\alpha_1 = 4u$ and $\alpha_0(x) = 2u_x$.

10.3 Finite-gap potential and generalized Schwarzian equation

Let us consider a polynomial generalized potential

$$u(x, \lambda) = \lambda^n + u_1\lambda^{n-1} + \cdots + u_n. \tag{84}$$

Hence, the projective vector field equation becomes

$$f_{xxx} + 4u(x, \lambda)f_x + 2u_x(x, \lambda)f = 0. \tag{85}$$

A generalized potential $u(x, \lambda)$ is called N-phase potential if (84) has a solution which is a polynomial in λ of degree N, i.e.,

$$f(x, \lambda) = \lambda^N + f_1\lambda^{N-1} + \cdots + f_N.$$

It is easy to transform the projective vector field equation to

$$\frac{f''}{f} + 4u - \frac{f'^2}{f^2} = \frac{Wr^2}{f^2} \tag{86}$$

where the constant part can be fixed by the Wronskian of its partner equation

$$\psi'' + u(x)\psi = 0.$$

Proposition 10.8 *Let ψ_1 and ψ_2 be the solutions of $\psi'' + u(x)\psi = 0$. Let us define*

$$v_i = \frac{\psi_1\psi_2 \mp Wr}{2\psi_1\psi_2}, \tag{87}$$

where $Wr = \psi_1\psi_2' - \psi_2\psi_1'$ is the Wronskian. Then q_i maps the Riccati equation

$$v_{ix} + v_i^2 + u(x) = 0$$

to

$$ff'' + 2uf^2 - \frac{1}{2}(f')^2 = Wr^2.$$

Proof: By substituting v_i into the Riccati equation one obtains the proof.
$\square$

Suppose we assume $Wr(\psi_1, \psi_2) = \lambda$. Thus corresponding to (86) we obtain the following modified Schwarzian derivative equation

$$u(x, \lambda) = \frac{1}{2}\frac{g_{xx}}{g} - \frac{3}{4}\frac{g_x^2}{g^2} + \lambda^2 g^2. \tag{88}$$

This equation is called the modified Schwarzian equation by Kartashova and Shabat [24]. This equation has a profound application in integrable systems.

Then Eqn. (88) has unique asymptotic solution represented by formal Laurent series, such that

$$g(x, \lambda) = 1 + \sum_{l=1}^{\infty} \lambda^{-l} g_l(x),$$

where coefficients g_l are different polynomials in all $u_1, \cdots, u_n$.

11 Conclusion and Outlook

The space of differential operators has a nice geometrical structure. It is known that Lie derivative action of a vector field on the space of second order differential operators on a circle exactly coincides with the coadjoint action, and this action yields KdV equation. Thus, we became curious to study the flows associated to the space of higher order differential operators or opers.

This paper has focused on to study Euler-Poincaré flows associated to the action of $Vect(S^1)$ on the space of higher order differential operators. At first we have studied the action of $Vect(S^1)$ on the space of third order differential operators and EP flows yield Drinfeld-Sokolov and coupled KdV type system. This method is very geometrical based on projective connections on S^1. But one of the shortcomings of this construction is that the Boussinesq equation can not be derived explicitly.

We have explicitly demonstrated this action on the space of fourth order differential operators. We have constructed various coupled KdV type system, and the notable one is the Hirota-Satsuma or complex KdV equation. In this paper we also studied the eigenvalue problem of the fourth order differential operator. In one particular case we boiled down to the Zakharov-Shabat type eigenvalue problem.

We have also considered a conformal class of operators such that all the coefficients can be expressed in terms of a polynomial and derivatives of a single dependent variable u. This is a generalized projective connection. We have shown that the EP flow is associated to the space of such conformal operators is always the KdV equation. A nth order differential operator induces an immersion in projective space. In this paper we briefly study EP flow on the space of projective curve, and this yields Schwarzian KdV equation. Finally we have studied the famous Riccati equations and its deep connection to projective vector field equation.

There are several problems popped up in this article. One of them is to study EP flows on other kind of opers, in this paper we have considered only sl_n opers. Another way to generalize is to study such Euler-Poincaré flows on super spaces. In this case we have to consider the Lie derivative action of supervector field $Vect(S^{1|1})$ on the space of differential operators on a supercircle $S^{1|1}$. But the computation must be much more tedious than here.

12 References

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