Max-Planck-Institut
für Mathematik
in den Naturwissenschaften
Leipzig

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by

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Preprint no.: 146 2006

NEW MODULAR LIE SUPERALGEBRAS AS PROLONGS

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ABSTRACT. Over algebraically closed fields of characteristic $p > 2$, prolongations of the simple finite dimensional Lie algebras and Lie superalgebras with Cartan matrix are studied for certain gradings of these algebras. Three new series of Lie superalgebras are discovered.

1. Setting. We use standard notations of [FH, S]; for the precise definition of Cartan prolongation and its generalizations (Tanaka–Shchepochkina complete and partial prolongations), see [Shch]. Hereafter $\mathbb{K}$ is an algebraically closed field of characteristic p.

The works of S. Lie, Killing and Cartan, now classical, completed classification over $\mathbb{C}$ of

(1) **simple Lie algebras of finite dimension and of vector fields**.

Lie algebras and Lie superalgebras over fields in characteristic $p > 0$, a.k.a. *modular* Lie (super)algebras, were distinguished in topology in the 1930s. The **simple** Lie algebras drew attention (over finite fields $\mathbb{K}$) as a step towards classification of simple finite groups. Lie *super*algebras, even simple ones, did not draw much attention of mathematicians until their (outstanding) usefulness was observed by physicists in the 1970s, whereas new and new examples of simple modular Lie algebras were being discovered for decades until Kostrikin and Shafarevich ([KS]) formulated a conjecture embracing all previously found examples for $p > 7$. Its generalization reads: *select a $\mathbb{Z}$-form $\mathfrak{g}_{\mathbb{Z}}$ of every $\mathfrak{g}$ of type*[1)] (1), *take* $\mathfrak{g}_{\mathbb{K}} := \mathfrak{g}_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{K}$ *and its simple subquotient $\mathfrak{si}(\mathfrak{g}_{\mathbb{K}})$. Together with deformations*[2)] *of these examples we get in this way all simple finite dimensional Lie superalgebras over algebraically closed fields if $p > 5$. If $p = 5$, we should add Melikyan's examples to the above list.*

After 30 years of work, Block, Wilson, Premet and Strade proved the generalized KSh conjecture for $p > 3$, see [S].

For $p \leq 5$, the above KSh-procedure does not produce all simple finite dimensional Lie algebras; there are other examples. In [GL4], we returned to É. Cartan's description of $\mathbb{Z}$-graded Lie algebras as vectorial Lie algebras preserving certain distributions; we thus interpreted the "mysterious" exceptional examples of simple Lie algebras for $p = 3$ (Brown, Frank, Ermolaev and Skryabin algebras), further elucidated Kuznetsov's interpretation [Ku1] of Melikyan's algebras (as prolongs of the nonpositive part of the Lie algebra $\mathfrak{g}(2)$ in one of its $\mathbb{Z}$-gradings) and discovered three new series of simple Lie algebras. In [BjL], the same approach yielded $\mathfrak{bj}$, a super versions of $\mathfrak{g}(2)$, and $\mathfrak{Bj}(1; N|7)$, a super Melikyan algebra, both indigenous to $p = 3$. In [L], a super analog of the KSh conjecture is formulated in which the CTS prolongs play the main role. Here we support this conjecture.

1991 *Mathematics Subject Classification.* 17B50, 70F25.

Key words and phrases. Cartan prolongation, nonholonomic manifold, Lie superalgebra.

We are thankful to I. Shchepochkina for help; DL is thankful to MPIMiS, Leipzig, for financial support and most creative environment.

[1)] Observe that the algebra of divided powers (the analog of the polynomial algebra for $p > 0$) *and hence all prolongs* (Lie algebras of vector fields) acquire one more — shearing — parameter: $\underline{N}$, see [S].

[2)] It is not clear, actually, if the conventional notion of deformation can always be applied if $p > 0$ (for the arguments, see [LL]); to give the correct (better say, universal) notion is an open problem, but in some cases it is applicable, see [BGL4].

2. Yamaguchi's theorem. This theorem, reproduced in [GL4, BjL], states that *for almost all simple finite dimensional Lie algebras $\mathfrak{g}$ over $\mathbb{C}$ and their $\mathbb{Z}$-gradings $\mathfrak{g}=\mathop{\oplus}\limits_{-d\leq i}\mathfrak{g}_i$ of finite depth d, the generalized Cartan prolong of $\mathfrak{g}_{\leq}=\mathop{\oplus}\limits_{-d\leq i\leq 0}\mathfrak{g}_i$ is isomorphic to $\mathfrak{g}$, the rare exceptions being precisely the four series of simple vectorial algebras.*

Our general idea to construct a supply of new simple Lie algebras and superalgebras is as follows: For every simple finite dimensional $\mathbb{Z}$-graded Lie (super)algebra $\mathfrak{g}$, take its non-positive part and consider its complete and partial prolongs. (This is how Cartan got all simple $\mathbb{Z}$-graded Lie algebras of polynomila growth and finite depth — the Lie algebras of type (1) — at the time when root technique was not discovered yet.) We do NOT have to consider ALL $\mathbb{Z}$-gradings of every simple finite dimensional $\mathbb{Z}$-graded Lie (super)algebra $\mathfrak{g}$: only precious few suffice.

For a description of the exceptional Elduque superalgebras, see [CE, El1, CE2, El2]; for their description in terms of Cartan matrices and analogs of Chevalley relations and notations we use in what follows, see [BGL1, BGL2].

In what follows, we present the results of SuperLie-assisted ([Gr]) computations of CTS-prolongs of the non-positive parts of the simple finite dimensional Lie algebras and Lie superalgebras $\mathfrak{g}(A)$ with Cartan matrix A; we have only considered $\mathbb{Z}$-grading corresponding to each (or, for larger ranks, even certain chosen) of the **simplest** gradings $r=(r_1,\dots,r_{\mathrm{rk}\mathfrak{g}})$, where all but one coordinates of r are 0 and only one — *selected* — is equal to 1, and where we set $\deg X_i^{\pm}=\pm r_i$ for the Chevalley generators $\deg X_i^{\pm}$ of $\mathfrak{g}(A)$, see [BGL1]. **Conjecturally, other gradings (as well as algebras of higher ranks) do not yield new simple Lie (super)algebras as prolongs of the non-positive parts**.

3. Theorem. *Let $p=3$. For the Lie algebras $\mathfrak{g}=\mathfrak{f}(4)$, $\mathfrak{e}(6)$, $\mathfrak{e}(7)$ and $\mathfrak{e}(8)$ considered with the $\mathbb{Z}$-grading with one selected root corresponding to the endpoint of the Dynkin diagram, the CTS prolong returns $\mathfrak{g}$.*

Let $p=5$. For the Lie superalgebra $\mathfrak{g}=\mathfrak{el}(5)$ ([BGL2])*, and its $\mathbb{Z}$-grading with only one odd simple root and with one selected root corresponding to the endpoint of the Dynkin diagram, the CTS prolong returns $\mathfrak{g}$.*

4. Theorem. *Let $p=3$. For the known simple finite dimensional Lie superalgebras with Cartan matrix and rank ≤ 3, the CTS prolongs with the new simplest gradings r are given in the following table, where the $\mathfrak{Bj}$ series* (for details of their description, see an arXiv version of this paper) *are new simple Lie superalgebras.*

In the Melikyan algebra $\mathfrak{Me}(5;\underline{N})$ (defined for $p=5$), $\mathfrak{Me}_0=\mathfrak{cvect}(1;\underline{1})$ and $\mathfrak{Me}_{-1}=\mathcal{O}(1;\underline{1})/const$ (but can also be obtained as $\underline{N}$-prolongs of the non-negative part of $\mathfrak{g}(2)$, with $\mathfrak{g}(2)$ obtained now as a *partial prolong, see* [GL4]. *The same construction with $\mathfrak{g}_0=\mathfrak{ck}(1;\underline{1}|1)$ and $\mathfrak{g}_{-1}=\mathcal{O}(1;\underline{1}|1)/const$ yields another, non-isomorphic to $\mathfrak{Bj}(1;\underline{N}|7)$, super Melikyan algebra $\mathfrak{Me}(5;\underline{N}|5)$ if $p=3$, cf.* [BjL].

$\mathfrak{g}$	Cartan matrix	r	prolong
$\mathfrak{osp}(2\vert 4)$	$\begin{pmatrix}0&1&0\\-1&2&-2\\0&-1&2\end{pmatrix}$	(100) or (001) (010)	$\mathfrak{osp}(2\vert 4)$ $\begin{cases}\mathfrak{osp}(2\vert 4) & \text{if } p>3\\ \mathfrak{Bj}(3;N\vert 3) & \text{if } p=3\end{cases}$
$\mathfrak{g}(2\vert 3)$ see [BGL1]	$\begin{pmatrix}0&0&-1\\0&0&-2\\-1&-2&2\end{pmatrix}$	(100) (010) (001)	$\mathfrak{Bj}(2;N\vert 4)$ $\mathfrak{Bj}(3;N\vert 5)$ $\mathfrak{k}(1\vert 3;1)$

5. A description of $\mathfrak{Bj}(3;N|3)$. We have the following realization of the non-positive part:

(2)

$\mathfrak{g}_i$	the generators (even $\mid$ odd)
$\mathfrak{g}_{-2}$	$Y_6=\partial_1 \mid Y_8=\partial_4$
$\mathfrak{g}_{-1}$	$Y_2=\partial_2,\ Y_5=x_2\partial_1+x_5\partial_4+\partial_3, \mid Y_4=\partial_5,\ Y_7=2\,x_2\partial_4+\partial_6,$
$\mathfrak{g}_0$	$Y_3={x_2}^2\partial_1+x_2x_5\partial_4+x_2\partial_3+2x_5\partial_6,\ Z_3={x_3}^2\partial_1+2x_3x_6\partial_4+x_3\partial_2+2x_6\partial_5$
	$H_2=2\,x_1\partial_1+2\,x_2\partial_2+x_4\partial_4+x_5\partial_5+2\,x_6\partial_6,\ H_1=[Z_1,Y_1],H_3=[Z_3,Y_3] \mid$
	$Y_1=x_1\partial_4+2\,x_2\partial_5+x_3\partial_6,\ Z_1=2\,x_4\partial_1+2\,x_5\partial_2+x_6\partial_3$

We have

$$\mathfrak{g}_0\simeq\mathfrak{sl}(1|1)\oplus\mathfrak{sl}(2)\oplus\mathbb{K}. \tag{3}$$

The $\mathfrak{g}_0$-module $\mathfrak{g}_{-1}$ is irreducible, having one highest weight vector Y_2. In $N=(N_1,N_2,N_3)$, only N_1 can vary; $N_2=N_3=1$.

6. A description of $\mathfrak{Bj}(2;N|4)$. We have $\mathrm{sdim}(\mathfrak{g}(2,3)_-)=2|4$. Since the $\mathfrak{g}(2,3)_0$-module action is not faithful, we consider the quotient algebra $\mathfrak{g}_0=\mathfrak{g}(2,3)_0/\mathfrak{ann}(\mathfrak{g}_{-1})$ and embed $(\mathfrak{g}(2,3)_-,\mathfrak{g}_0)\subset\mathfrak{vect}(2|4)$. This realization is given by the following table:

$\mathfrak{g}_i$	the generators (even $\mid$ odd)
$\mathfrak{g}_{-1}$	$Y_6=\partial_2,\ Y_8=\partial_1 \mid Y_{11}=\partial_3,\ Y_{10}=\partial_4,\ Y_4=\partial_5,\ Y_1=\partial_6$
$\mathfrak{g}_0$	$Y_3=x_2\partial_1+2\,x_4\partial_3+x_6\partial_5,Y_9=[Y_2,[Y_3,Y_5]],Z_3=x_1\partial_2+2\,x_3\partial_4+x_5\partial_6,Z_9=[Z_2,[Z_3,Z_5]],$
	$H_2=[Z_2,Y_2],H_3=[Z_3,Y_3] \mid Y_2=x_1\partial_4+x_5\partial_2,Y_5=[Y_2,Y_3],Y_7=[Y_3,[Y_2,Y_3]],$
	$Z_2=x_2\partial_5+2x_4\partial_1,Z_5=[Z_2,Z_3],Z_7=[Z_3,[Z_2,Z_3]]$

We have $\mathfrak{g}_0\simeq\mathfrak{osp}(3|2)$ and $\mathrm{sdim}(\mathfrak{g}_0)=6|6$.

The $\mathfrak{g}_0$-module $\mathfrak{g}_{-1}$ is irreducible, with highest weight vector Y_1. The CTS prolong $(\mathfrak{g}_-,\mathfrak{g}_0)_*$ gives a Lie superalgebra of dimension $13|14$. This Lie superalgebra has THREE ideals $I_1\subset I_2\subset I_3$ with the same non-positive part BUT different positive parts: $\mathrm{sdim}(I_1)=10|14$, $\mathrm{sdim}(I_2)=11|14$, $\mathrm{sdim}(I_3)=12|14$. The ideal I_1 is just our $\mathfrak{bj}$, see [BjL, CE].

7. A description of $\mathfrak{Bj}(3;N|5)$. We have $\mathrm{sdim}(\mathfrak{g}(2,3)_-)=3|5$. Since the $\mathfrak{g}(2,3)_0$-module action is again not faithful, we consider the quotient module $\mathfrak{g}_0=\mathfrak{g}(2,3)_0/\mathfrak{ann}(\mathfrak{g}_{-1})$ and embed $(\mathfrak{g}(2,3)_-,\mathfrak{g}_0)\subset\mathfrak{vect}(3;\underline{N}|5)$. This realization is given by the following table:

$\mathfrak{g}_i$	the generators (even $\mid$ odd)
$\mathfrak{g}_{-2}$	$Y_9=\partial_1 \mid Y_{10}=\partial_3,Y_{11}=\partial_2$
$\mathfrak{g}_{-1}$	$Y_8=\partial_4,\ Y_6=\partial_5 \mid Y_5=2\,x_4\partial_2+2\,x_5\partial_3+2\,x_7\partial_1+\partial_7,Y_2=x_4\partial_3-2\,x_6\partial_1+\partial_8$
	$Y_7=x_5\partial_2+\partial_6$
$\mathfrak{g}_0$	$H_1=[Z_1,Y_1],H_3=[Z_3,Y_3],Y_3=2\,x_3\partial_2+2\,x_7x_8\partial_1+x_5\partial_4+2\,x_7\partial_6+x_8\partial_7,$
	$Z_3=2\,x_2\partial_3+2\,x_6x_7\partial_1+x_4\partial_5+x_6\partial_7+2\,x_7\partial_8 \mid Y_1=2\ (2\,x_1\partial_3+2\,x_6x_7\partial_2+x_6\partial_4+x_7\partial_5)$
	$Z_1=2\quad x_3\partial_1+2\,x_4x_5\partial_2+2\,{x_5}^2\partial_3+2\,x_5x_7\partial_1+2\,x_4\partial_6+x_5\partial_7\quad,Y_4=[Y_1,Y_3],Z_4=[Z_1,Z_3]$

We have $\mathfrak{g}_0\simeq\mathfrak{sl}(1|2)$ and therefore $\mathrm{sdim}(\mathfrak{g}_0)=4|4$.

The $\mathfrak{g}_0$-module $\mathfrak{g}_{-1}$ is irreducible, having one lowest weight vector Y_8 and one highest weight vector Y_2. We have $\mathrm{sdim}(\mathfrak{g}_1)=6|4$. The $\mathfrak{g}_0$-module $\mathfrak{g}_1$ has two lowest weight vectors given by

V_1'	$x_1x_5\partial_2+2\,x_5x_6x_8\partial_2+x_5x_7x_8\partial_3+2\,x_1\partial_6+2\,x_3\partial_4+x_5x_7\partial_4+x_5x_8\partial_5+2\,x_7x_8\partial_7$
V_1''	$x_6x_7x_8\partial_2+2\,x_1\partial_4+x_7x_8\partial_5$

Now, the $\mathfrak{g}_0$-module generated by the the vectors V_1' and V_1'' is not the whole $\mathfrak{g}_1$ but a $\mathfrak{g}_0$-module, denoted by $\mathfrak{g}_1''$, of sdim $=4|4$. The CTS prolong $(\mathfrak{g}_-,\mathfrak{g}_0,\mathfrak{g}_1)_*$ is not simple, so consider the Lie subsuperalgebra $\widetilde{\mathfrak{Bj}(3;\underline{N}|5)}:=(\mathfrak{g}_-,\mathfrak{g}_0,\mathfrak{g}_1'')_*$. Since none of the known simple finite dimensional Lie superalgebra over (algebraically closed) fields of characteristic 0 or > 3 has such a non-positive part in any $\mathbb{Z}$-grading, it follows that $\widetilde{\mathfrak{Bj}(3;\underline{N}|5)}$ is new.

Let us investigate if $\mathfrak{Bj}(3;\underline{N}|5)$ has subalgebras — partial prolongs.

(i) Denote by $\mathfrak{g}_1'$ the $\mathfrak{g}_0$-module generated by V_1'. We have $\mathrm{sdim}(\mathfrak{g}_1') = 2|3$. The CTS partial prolong $(\mathfrak{g}_{-1}, \mathfrak{g}_0, \mathfrak{g}_1')_*$ gives a graded Lie superalgebra with $\mathrm{sdim}(\mathfrak{g}_2') = 2|2$ and $\mathfrak{g}_i' = 0$ for $i > 3$. An easy computation shows that $[\mathfrak{g}_{-1}, \mathfrak{g}_1'] = \mathfrak{g}_0$ and $[\mathfrak{g}_1', \mathfrak{g}_1'] \subsetneq \mathfrak{g}_2'$. Since we are investigating simple Lie superalgebra, we take the simple part of $(\mathfrak{g}_{-1}, \mathfrak{g}_0, \mathfrak{g}_1')_*$. This simple Lie superalgebra is $\mathfrak{bj} = \mathfrak{g}(2,3)/\mathfrak{c}$.

(ii) Denote by $\mathfrak{g}_1''$ the $\mathfrak{g}_0$-module generated by $\widetilde{V_1''}$. We just got have $\mathrm{sdim}(\mathfrak{g}_1'') = 4|4$. The CTS partial prolong $(\mathfrak{g}_{-1}, \mathfrak{g}_0, \mathfrak{g}_1'')_*$ gives also $\widetilde{\mathfrak{Bj}(3;\underline{N}|5)}$.

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