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The classification problem for
pseudo-Riemannian symmetric spaces

by

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Abstract. Riemannian and pseudo-Riemannian symmetric spaces with semisimple transvection group are known and classified for a long time. Contrary to that the description of pseudo-Riemannian symmetric spaces with non-semisimple transvection group is an open problem. In the last years some progress on this problem was achieved. In this article we want to explain these results and some of their applications.

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1. Introduction

There are many basic problems in differential geometry that are completely solved for Riemannian manifolds, but that become really complicated in the pseudo-Riemannian situation. One of these problems is the determination of all possible holonomy groups of pseudo-Riemannian manifolds. While holonomy groups of Riemannian manifolds are classified the problem is open for general pseudo-Riemannian manifolds, only the Lorentzian case is solved. The difficulty is that in general the holonomy representation of a pseudo-Riemannian manifold does not decompose into irreducible summands. Of course, we can decompose the representation into indecomposable ones, i.e., into subrepresentations that do not have proper non-degenerate invariant subspaces. By de Rham's theorem the indecomposable summands are again holonomy representations. This reduces the problem to the classification of indecomposable holonomy representations. Indecomposable holonomy representations that are not irreducible have isotropic invariant subspaces. Such representations are especially difficult to handle if these invariant subspaces do not have an invariant complement. Manifolds that have an indecomposable holonomy representation are called indecomposable. Manifolds that are not indecomposable are at least locally a product of pseudo-Riemannian manifolds. Hence, we can speak of local factors of such a manifold.

Many open questions in pseudo-Riemannian differential geometry are directly related to the unsolved holonomy problem. One of these open questions is the classification problem for symmetric spaces. Pseudo-Riemannian symmetric spaces are in some sense the most simple pseudo-Riemannian manifolds. Locally they are characterised by parallelity of the curvature tensor. As global manifolds they are defined as follows. A connected pseudo-Riemannian manifold M is called a pseudo-Riemannian symmetric space if for any $x \in M$ there is an involutive isometry θ_x of M that has x as an isolated fixed point. In other words, for any $x \in M$ the geodesic reflection at x extends to a globally defined isometry of M .

The theory of Riemannian symmetric spaces was developed simultaneously with the theory of semisimple Lie groups and algebras by E. Cartan during the first decades of the twentieth century. It results in a complete classification of these spaces, see Helgason's beautiful book [30] on the subject. The theory remains similar in spirit as long as one is interested in pseudo-Riemannian symmetric spaces whose holonomy representation is completely reducible as an algebraic representation, i.e., if any invariant subspace of the holonomy representation has an invariant complement. These so called reductive symmetric spaces were classified by Berger [9] in 1957. This classification is essentially the classification of involutions on real semisimple Lie algebras.

In order to understand non-reductive pseudo-Riemannian symmetric spaces one has to consider more general Lie algebras, which are, moreover, equipped with an invariant inner product. Note that in contrast to the semisimple case this inner product is really an additional datum since it is not just a multiple of the Killing form. Such a pair $(\mathfrak{g}, \langle \cdot, \cdot \rangle)$ consisting of a finite-dimensional real Lie algebra and an ad-invariant non-degenerate symmetric bilinear form on it is called a *metric Lie algebra*. In the literature metric Lie algebras appear under various different names, e.g. as quadratic or orthogonal Lie algebras.

The transition from symmetric spaces to metric Lie algebras proceeds as follows. Let (M, g) be a pseudo-Riemannian symmetric space. The group G generated by compositions of geodesic reflections $\theta_x \circ \theta_y$, $x, y \in M$, is called the transvection group of (M, g) . It acts transitively on M . We fix a base point $x_0 \in M$. The reflection θ_{x_0} induces an involutive automorphism θ of the Lie algebra \mathfrak{g} of G , and thus a decomposition $\mathfrak{g} = \mathfrak{g}_+ \oplus \mathfrak{g}_-$. The natural identification $T_{x_0}M \cong \mathfrak{g}_-$ induces an $\text{ad}(\mathfrak{g}_+)$ -invariant bilinear form $\langle \cdot, \cdot \rangle_-$ on \mathfrak{g}_- . It is an important observation (see [16]) that $\langle \cdot, \cdot \rangle_-$ extends uniquely to an $\text{ad}(\mathfrak{g})$ - and θ -invariant inner product $\langle \cdot, \cdot \rangle$ on \mathfrak{g} . Thus, starting with a pseudo-Riemannian symmetric space (M, g) , we obtain a metric Lie algebra $(\mathfrak{g}, \langle \cdot, \cdot \rangle)$ together with an isometric involution θ on it. The resulting triple $(\mathfrak{g}, \theta, \langle \cdot, \cdot \rangle)$ will be called a *symmetric triple* later on. Up to local isometry, (M, g) can be recovered from this structure. We will call a symmetric space as well as its associated symmetric triple semisimple (reductive, solvable etc.) if its transvection group is semisimple (reductive, solvable etc.). The reader will find more details on the correspondence between symmetric spaces and symmetric triples in Section 3.1. For the general theory of symmetric spaces he may consult [10, 16, 30, 39, 40, 42]. The moral we want to stress at this point is that the understanding of metric Lie algebras is crucial for the understanding of symmetric spaces.

The present paper focuses on the classification problem for pseudo-Riemannian symmetric spaces. The above discussion reduces this problem to the classification of symmetric triples. It is easy to see that we can decompose every symmetric triple into a direct sum of a semisimple one and one whose underlying Lie algebra does not have simple ideals. Of course, pseudo-Riemannian symmetric spaces that are associated with semisimple symmetric triples are reductive, and thus, as explained above, already classified. This is the reason for our decision to concentrate here on metric Lie algebras and symmetric triples without simple ideals. Thus the investigation of the geometry of semisimple symmetric spaces, which is still an active and interesting field, will be left almost untouched in this paper.

The classification of metric Lie algebras appears to be very difficult. Most likely, one has to accept that one won't get a classification in the sense of a list that includes all metric Lie algebras for arbitrary index of the inner product. The same is true for symmetric triples, the existence of an involution neither decreases nor really increases the difficulties. Therefore the aim is to develop a structure theory for metric Lie algebras (and symmetric triples) that allows a systematic construction and that gives a "recipe" how to get an explicit classification under suitable additional conditions, e.g., for small index of the inner product. In

[34, 35, 36] we developed a new strategy to reach this aim. The initial idea of this strategy is due to Bérard-Bergery who observed that every symmetric triple without simple ideals arises in a canonical way by an extension procedure from “simpler” Lie algebras with involution. We used this idea to give a cohomological description of isomorphism classes of metric Lie algebras (and symmetric triples), which gives a suitable classification scheme. Here we will present this method, called quadratic extension, and some of its applications. Moreover, we will survey earlier and related results due to Cahen, Parker, Medina, Revoy, Bordemann, Alekseevsky, Cortés, and others concerning metric Lie algebras and symmetric triples from this new point of view.

We do not aim at a complete overview on the work on metric Lie algebras and symmetric spaces, for instance the basic results of Astrahancev (see e.g. [4]) will not be discussed. However, we try to present a quite complete up-to-date account for classification results for metric Lie algebras (Section 2.5), symmetric spaces (Section 3.3), and symmetric spaces with certain complex or quaternionic structures (Sections 4.2 and 4.3).

Note that metric Lie algebras are of interest in their own right, not only in the context of symmetric spaces. They naturally appear in various contexts, e.g., in Mathematical physics or in Poisson geometry. As an illustration, we shortly discuss the notions of Manin triples and pairs and present a new construction method for Manin pairs based on the theory of quadratic extensions of metric Lie algebras (Section 5.2). As a further application we study pseudo-Riemannian extrinsic symmetric spaces by our method (Section 5.1).

Though being a survey article the paper also contains some new results. A first group of new results appears in Section 2 and is due to the fact that we develop here a unified theory which works for metric Lie algebras, symmetric triples, and symmetric triples with additional structures at once. Most of these results are straightforward generalisations of the corresponding special results given in the original papers [35, 36, 37]. Proofs that really require new ideas will be given in the appendix. This generality makes Sections 2.2 and 2.3 a little bit more technical than usual for survey articles. However, having mastered these moderate technical difficulties the reader will see in the subsequent sections how quite different results follow easily from one general principle. The results in Section 3.2 concerning the geometric meaning of the quadratic extension procedure and the above mentioned construction method for Manin pairs appear here for the first time. In addition, we announce a new result on the structure of hyper-Kähler symmetric spaces (Theorem 4.4).

The theory of metric Lie algebras and pseudo-Riemannian symmetric spaces is far from being complete. In fact, there is a huge amount of open problems. The difficulty is to find those questions, which really lead to new theoretical insight and not just to messy calculations. We hope that the questions raised at several places in this paper belong to the first category.

Some conventions. We denote by \mathbb{N} the set of positive integers and we put $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$.

Let $(\mathfrak{a}, \langle \cdot, \cdot \rangle_{\mathfrak{a}})$ be a pseudo-Euclidean vector space. A subspace $\mathfrak{a}' \subset \mathfrak{a}$ is called isotropic if $\langle \cdot, \cdot \rangle_{\mathfrak{a}}|_{\mathfrak{a}'} = 0$. A basis $A_1, \dots, A_p, A_{p+1}, \dots, A_{p+q}$ of \mathfrak{a} is called orthonormal if $A_i \perp A_j$ for $i \neq j$, $\langle A_i, A_i \rangle_{\mathfrak{a}} = -1$ for $i = 1, \dots, p$ and $\langle A_j, A_j \rangle_{\mathfrak{a}} = 1$ for $j = p+1, \dots, p+q$. In this case (p, q) is called signature and p index of $\langle \cdot, \cdot \rangle_{\mathfrak{a}}$ (or of \mathfrak{a}). Let $\langle \cdot, \cdot \rangle_{p,q}$ be the inner product of signature (p, q) on \mathbb{R}^{p+q} for which the standard basis of \mathbb{R}^{p+q} is an orthonormal basis. Then we call $\mathbb{R}^{p,q} := (\mathbb{R}^{p+q}, \langle \cdot, \cdot \rangle_{p,q})$ standard pseudo-Euclidean space.

We will often describe a Lie algebra by giving a basis and some of the Lie bracket relations, e.g. we will write the three-dimensional Heisenberg algebra as $\mathfrak{h}(1) = \{[X, Y] = Z\}$. In this case we always assume that all other brackets of basis vectors vanish. If we do not mention the basis explicitly, then we suppose that all basis vectors appear in one of the bracket relations (on the left or the right hand side).

Let \mathfrak{l} be a Lie algebra and let \mathfrak{a} be an \mathfrak{l} -module. Then $\mathfrak{a}^{\mathfrak{l}}$ denotes the space of invariants in \mathfrak{a} , i.e., $\mathfrak{a}^{\mathfrak{l}} = \{A \in \mathfrak{a} \mid L(A) = 0\}$ for all $L \in \mathfrak{l}$.

2. Metric Lie algebras

2.1. Examples of metric Lie algebras. The easiest example of a metric Lie algebra is an abelian Lie algebra together with an arbitrary (non-degenerate) inner product. A further well-known example is a semisimple Lie algebra equipped with a non-zero multiple of its Killing form.

Let H be a Lie group and \mathfrak{h} its Lie algebra. The cotangent bundle T^*H can be given a group structure such that the associated Lie algebra equals $\mathfrak{h}_{\text{ad}^*} \ltimes \mathfrak{h}^*$. Now let $\langle \cdot, \cdot \rangle_{\mathfrak{h}}$ be any invariant symmetric bilinear form on \mathfrak{h} (which can degenerate). We can define on $\mathfrak{h}_{\text{ad}^*} \ltimes \mathfrak{h}^*$ a symmetric bilinear form $\langle \cdot, \cdot \rangle$ by adding $\langle \cdot, \cdot \rangle_{\mathfrak{h}}$ to the dual pairing of \mathfrak{h} and \mathfrak{h}^* , that is by

$$\langle H_1 + Z_1, H_2 + Z_2 \rangle = Z_1(H_2) + Z_2(H_1) + \langle H_1, H_2 \rangle_{\mathfrak{h}}$$

for $H_1, H_2 \in \mathfrak{h}$, $Z_1, Z_2 \in \mathfrak{h}^*$. It is not hard to prove that $\langle \cdot, \cdot \rangle$ is invariant and non-degenerate, its signature equals $(\dim \mathfrak{h}, \dim \mathfrak{h})$. Hence $(\mathfrak{h}_{\text{ad}^*} \ltimes \mathfrak{h}^*, \langle \cdot, \cdot \rangle)$ is a metric Lie algebra. In particular, $\langle \cdot, \cdot \rangle$ induces a biinvariant metric on T^*H .

The following construction is a generalisation of the previous example. It is due to Medina and Revoy [44]. Starting with an n -dimensional metric Lie algebra and an arbitrary m -dimensional Lie algebra it produces a metric Lie algebra of dimension $n + 2m$. Let $(\mathfrak{g}, \langle \cdot, \cdot \rangle_{\mathfrak{g}})$ be a metric Lie algebra and let $(\mathfrak{h}, \langle \cdot, \cdot \rangle_{\mathfrak{h}})$ be a Lie algebra with an invariant symmetric bilinear form (which can degenerate). Furthermore, let $\pi : \mathfrak{h} \rightarrow \text{Der}_{\mathfrak{a}}(\mathfrak{g}, \langle \cdot, \cdot \rangle_{\mathfrak{g}})$ be a Lie algebra homomorphism from \mathfrak{h} into the Lie algebra of all antisymmetric derivations of \mathfrak{g} . We denote by $\beta \in C^2(\mathfrak{g}, \mathfrak{h}^*) := \text{Hom}(\wedge^2 \mathfrak{g}, \mathfrak{h}^*)$ the 2-cocycle (see 2.3 for this notion)

$$\beta(X, Y)(H) := \langle \pi(H)X, Y \rangle_{\mathfrak{g}}, \quad X, Y \in \mathfrak{g}, H \in \mathfrak{h}.$$

On the vector space $\mathfrak{d} := \mathfrak{h}^* \oplus \mathfrak{g} \oplus \mathfrak{h}$ we define a Lie bracket $[\cdot, \cdot]$ by

$$[(Z, X, H), (\tilde{Z}, \tilde{X}, \tilde{H})] = (\beta(X, \tilde{X}) + \text{ad}_{\mathfrak{h}}^*(H)\tilde{Z} - \text{ad}_{\mathfrak{h}}^*(\tilde{H})Z, [X, \tilde{X}]_{\mathfrak{g}} + \pi(H)\tilde{X} - \pi(\tilde{H})X, [H, \tilde{H}]_{\mathfrak{h}})$$

and an inner product $\langle \cdot, \cdot \rangle$ by

$$\langle (Z, X, H), (\tilde{Z}, \tilde{X}, \tilde{H}) \rangle = \langle X, \tilde{X} \rangle_{\mathfrak{g}} + \langle H, \tilde{H} \rangle_{\mathfrak{h}} + Z(\tilde{H}) + \tilde{Z}(H)$$

for all $Z, \tilde{Z} \in \mathfrak{h}^*$, $X, \tilde{X} \in \mathfrak{g}$ and $H, \tilde{H} \in \mathfrak{h}$. Then $\mathfrak{d}_{\pi}(\mathfrak{g}, \mathfrak{h}) := (\mathfrak{d}, \langle \cdot, \cdot \rangle)$ is a metric Lie algebra. It is called double extension of \mathfrak{g} by \mathfrak{h} since it can be regarded as an extension of the semi-direct product $\mathfrak{g} \rtimes_{\pi} \mathfrak{h}$ by the abelian Lie algebra \mathfrak{h}^* . If the signature of \mathfrak{g} equals (p, q) and if $\dim \mathfrak{h} = m$, then the signature of $\mathfrak{d}_{\pi}(\mathfrak{g}, \mathfrak{h})$ equals $(p + m, q + m)$.

The importance of this construction becomes clear from the following structure theorem by Medina and Revoy. It says that we can inductively produce all metric Lie algebras from simple and one-dimensional ones by taking direct sums and applying the double extension procedure.

Theorem 2.1 (Medina/Revoy [44]). *If $(\mathfrak{g}, \langle \cdot, \cdot \rangle)$ is an indecomposable metric Lie algebra, then either \mathfrak{g} is simple or \mathfrak{g} is one-dimensional or \mathfrak{g} is a double extension $\mathfrak{d}_{\pi}(\tilde{\mathfrak{g}}, \mathfrak{h})$ of a metric Lie algebra $\tilde{\mathfrak{g}}$ by a one-dimensional or a simple Lie algebra \mathfrak{h} .*

We remark that for the special case of solvable metric Lie algebras this result can already be found in the form of exercises in Kac's book [31], Exercises 2.10, 11. For a complete proof in this case see also [23].

Using Theorem 2.1 it is not hard to see that any indecomposable non-simple Lorentzian metric Lie algebra is the double extension of an abelian Euclidean metric Lie algebra by a one-dimensional Lie algebra. This allows the classification of isomorphism classes of Lorentzian metric Lie algebras [43], compare Example 2.2 and Theorem 2.4.

In principle one can try to use this method to classify also metric Lie algebras of higher index. This was done in [5] for index two. However, now the following difficulty arises. In general a metric Lie algebra of index greater than one can be obtained in many different ways by double extension from a lower-dimensional one. Thus in order to solve the classification problem we would have to decide under which conditions two metric Lie algebras arising in different ways by repeated application of the double extension construction (and direct sums) are isomorphic. This seems to be very complicated. Therefore we are now looking for a way that avoids this difficulty. In the following we will develop a structure theory for metric Lie algebras which is more adapted to classification problems. The basic idea of this theory goes back to Bérard-Bergery [7] who suggested to consider indecomposable non-semisimple pseudo-Riemannian symmetric spaces as the result of two subsequent extensions. Our starting point is the following construction.

Let \mathfrak{l} be a Lie algebra and let $\rho : \mathfrak{l} \rightarrow \mathfrak{so}(\mathfrak{a})$ be an orthogonal representation of \mathfrak{l} on a pseudo-Euclidean vector space $(\mathfrak{a}, \langle \cdot, \cdot \rangle_{\mathfrak{a}})$. Take $\alpha \in C^2(\mathfrak{l}, \mathfrak{a}) := \text{Hom}(\wedge^2 \mathfrak{l}, \mathfrak{a})$

and $\gamma \in C^3(\mathfrak{l}) := \text{Hom}(\wedge^3 \mathfrak{l}, \mathbb{R})$. We consider the vector space $\mathfrak{d} := \mathfrak{l}^* \oplus \mathfrak{a} \oplus \mathfrak{l}$ and define an inner product $\langle \cdot, \cdot \rangle$ by

$$\langle Z_1 + A_1 + L_1, Z_2 + A_2 + L_2 \rangle := \langle A_1, A_2 \rangle_{\mathfrak{a}} + Z_1(L_2) + Z_2(L_1)$$

for $Z_1, Z_2 \in \mathfrak{l}^*$, $A_1, A_2 \in \mathfrak{a}$ and $L_1, L_2 \in \mathfrak{l}$. Moreover, we define an antisymmetric bilinear map $[\cdot, \cdot] : \mathfrak{d} \times \mathfrak{d} \rightarrow \mathfrak{d}$ by $[\mathfrak{l}^*, \mathfrak{l}^* \oplus \mathfrak{a}] = 0$ and

$$\begin{aligned} [L_1, L_2] &= \gamma(L_1, L_2, \cdot) + \alpha(L_1, L_2) + [L_1, L_2]_{\mathfrak{l}} \\ [L, A] &= -\langle A, \alpha(L, \cdot) \rangle + L(A) \\ [A_1, A_2] &= \langle \rho(\cdot) A_1, A_2 \rangle \\ [L, Z] &= \text{ad}^*(L)(Z) \end{aligned}$$

for $L, L_1, L_2 \in \mathfrak{l}$, $A, A_1, A_2 \in \mathfrak{a}$ and $Z \in \mathfrak{l}^*$. Then the Jacobi identity for $[\cdot, \cdot]$ is equivalent to a certain cocycle condition for α and γ . We will denote this condition by $(\alpha, \gamma) \in \mathcal{Z}_Q^2(\mathfrak{l}, \mathfrak{a})$ and postpone its exact formulation to Section 2.3. Thus, if $(\alpha, \gamma) \in \mathcal{Z}_Q^2(\mathfrak{l}, \mathfrak{a})$, then $[\cdot, \cdot]$ is a Lie bracket and it is easy to check that $\langle \cdot, \cdot \rangle$ is invariant with respect to this bracket. This gives the following result.

Proposition 2.1 ([35], Prop. 2.4). *If $(\alpha, \gamma) \in \mathcal{Z}_Q^2(\mathfrak{l}, \mathfrak{a})$, then $\mathfrak{d}_{\alpha, \gamma}(\mathfrak{l}, \mathfrak{a}) := (\mathfrak{d}, \langle \cdot, \cdot \rangle)$ is a metric Lie algebra.*

Two special cases of this construction were known previously. For the case $\alpha = \gamma = 0$ our $\mathfrak{d}_{\alpha, \gamma}(\mathfrak{l}, \mathfrak{a})$ is the double extension $\mathfrak{d}_{\rho}(\mathfrak{a}, \mathfrak{l})$ of the abelian metric Lie algebra \mathfrak{a} by \mathfrak{l} (in the sense of Medina and Revoy as explained above) and for $\mathfrak{a} = 0$ it coincides with the T^* -extension introduced by Bordemann [12].

Example 2.2. Take $\mathfrak{l} = \mathbb{R}$. Let \mathfrak{a} be the standard Euclidean vector space \mathbb{R}^{2m} with (orthonormal) standard basis e_1, \dots, e_{2m} . Take $\lambda = (\lambda^1, \dots, \lambda^m) \in (\mathfrak{l}^*)^m \cong \mathbb{R}^m$. We define an orthogonal representation of \mathfrak{l} on \mathfrak{a} by

$$\rho_{\lambda}(L)(e_{2i-1}) = \lambda_i(L) \cdot e_{2i}, \quad \rho_{\lambda}(L)(e_{2i}) = -\lambda_i(L) \cdot e_{2i-1}$$

for $L \in \mathfrak{l}$ and $i = 1, \dots, m$. We set $\mathfrak{a}_{\lambda} := (\rho_{\lambda}, \mathfrak{a})$. Then $\mathfrak{osc}(\lambda) := \mathfrak{d}_{0,0}(\mathbb{R}, \mathfrak{a}_{\lambda})$ is a metric Lie algebra of signature $(1, 2m + 1)$. This Lie algebra is often called oscillator algebra. As explained above, $\mathfrak{osc}(\lambda)$ can also be considered as double extension of \mathfrak{a} by \mathbb{R} .

In the following we will see that any metric Lie algebra without simple ideals is isomorphic to some $\mathfrak{d}_{\alpha, \gamma}(\mathfrak{l}, \mathfrak{a})$ for suitable data $\mathfrak{l}, \mathfrak{a}, (\alpha, \gamma) \in \mathcal{Z}_Q^2(\mathfrak{l}, \mathfrak{a})$ and how this fact can be used to describe isomorphism classes of metric Lie algebras.

2.2. Metric Lie algebras and quadratic extensions. As already mentioned in the introduction we are especially interested in metric Lie algebras without simple ideals. In this section we will learn more about the structure of such metric Lie algebras. Later on we wish to equip metric Lie algebras with additional structures, e.g. with an involution when we want to study symmetric

triples or with even more structure when we will be studying geometric structures on symmetric spaces. For this reason we develop a theory that is equivariant under a Lie algebra \mathfrak{h} and a Lie group K acting semisimply on \mathfrak{h} by automorphisms. We assume throughout the paper that K has only finitely many connected components. We suggest to take the trivial case $(\mathfrak{h}, K) = (0, \{e\})$ on first reading. In particular, this means that you may omit all maps called Φ in the following.

An (\mathfrak{h}, K) -module (V, Φ_V) consists of a finite-dimensional vector space V and a map $\Phi_V : \mathfrak{h} \cup K \rightarrow \text{Hom}(V)$ such that $\Phi_V|_{\mathfrak{h}} : \mathfrak{h} \rightarrow \text{Hom}(V)$ and $\Phi_V|_K : K \rightarrow GL(V) \subset \text{Hom}(V)$ are representations of \mathfrak{h} and K satisfying

$$\Phi_V(k \cdot X) = \Phi_V(k)\Phi_V(X)\Phi_V(k)^{-1}$$

for all $k \in K$ and $X \in \mathfrak{h}$. There is a natural notion of a (\mathfrak{h}, K) -submodule. An (\mathfrak{h}, K) -module is called semisimple if for any (\mathfrak{h}, K) -submodule there is a complementary (\mathfrak{h}, K) -submodule.

Definition 2.3. 1. An (\mathfrak{h}, K) -equivariant Lie algebra $(\mathfrak{l}, \Phi_{\mathfrak{l}})$ is a Lie algebra \mathfrak{l} that is equipped with the structure of a semisimple (\mathfrak{h}, K) -module such that $\text{im}(\Phi_{\mathfrak{l}}|_{\mathfrak{h}}) \subset \text{Der}(\mathfrak{l})$ and $\text{im}(\Phi_{\mathfrak{l}}|_K) \subset \text{Aut}(\mathfrak{l})$, where $\text{Der}(\mathfrak{l})$ and $\text{Aut}(\mathfrak{l})$ denote the Lie algebra of derivations and the group of automorphisms of \mathfrak{l} , respectively.

2. An (\mathfrak{h}, K) -equivariant metric Lie algebra is a triple $(\mathfrak{g}, \Phi, \langle \cdot, \cdot \rangle)$ such that

- (i) $(\mathfrak{g}, \langle \cdot, \cdot \rangle)$ is a metric Lie algebra,
- (ii) (\mathfrak{g}, Φ) is an (\mathfrak{h}, K) -equivariant Lie algebra,
- (iii) $\Phi(\mathfrak{h}) \subset \text{Der}_a(\mathfrak{g})$ and $\Phi(K) \subset \text{Aut}(\mathfrak{g}, \langle \cdot, \cdot \rangle)$, where $\text{Der}_a(\mathfrak{g})$ denotes the Lie algebra of antisymmetric derivations on $(\mathfrak{g}, \langle \cdot, \cdot \rangle)$.

The index (signature) of an (\mathfrak{h}, K) -equivariant metric Lie algebra $(\mathfrak{g}, \Phi, \langle \cdot, \cdot \rangle)$ is the index (signature) of $\langle \cdot, \cdot \rangle$. Sometimes we abbreviate $(\mathfrak{g}, \Phi, \langle \cdot, \cdot \rangle)$ as \mathfrak{g} .

A homomorphism (resp., isomorphism) of (\mathfrak{h}, K) -equivariant Lie algebras $F : (\mathfrak{l}_1, \Phi_1) \rightarrow (\mathfrak{l}_2, \Phi_2)$ is a homomorphism (resp., isomorphism) of Lie algebras $F : \mathfrak{l}_1 \rightarrow \mathfrak{l}_2$ that satisfies $F \circ \Phi_1(h) = \Phi_2(h) \circ F$ for all $h \in \mathfrak{h} \cup K$. Isomorphisms of (\mathfrak{h}, K) -equivariant metric Lie algebras are in addition compatible with the given inner products. We have a natural notion of direct sums of (\mathfrak{h}, K) -equivariant (metric) Lie algebras. An (\mathfrak{h}, K) -equivariant (metric) Lie algebra is called decomposable if it is isomorphic to the direct sum of two non-trivial (\mathfrak{h}, K) -equivariant (metric) Lie algebras. Otherwise it is called indecomposable.

In the following let $(\mathfrak{l}, \Phi_{\mathfrak{l}})$ always be an (\mathfrak{h}, K) -equivariant Lie algebra.

Definition 2.4. 1. An $(\mathfrak{l}, \Phi_{\mathfrak{l}})$ -module $(\rho, \mathfrak{a}, \Phi_{\mathfrak{a}})$ consists of

- (i) a semisimple (\mathfrak{h}, K) -module $(\mathfrak{a}, \Phi_{\mathfrak{a}})$,
- (ii) a representation $\rho : \mathfrak{l} \rightarrow \text{Hom}(\mathfrak{a}, \langle \cdot, \cdot \rangle)$ that satisfies

$$\rho(\Phi_{\mathfrak{l}}(k)L) = \Phi_{\mathfrak{a}}(k) \circ \rho(L) \circ \Phi_{\mathfrak{a}}(k)^{-1}, \quad \rho(\Phi_{\mathfrak{l}}(X)L) = [\Phi_{\mathfrak{a}}(X), \rho(L)]$$

for all $k \in K$, $X \in \mathfrak{h}$ and $L \in \mathfrak{l}$.

2. An orthogonal $(\mathfrak{l}, \Phi_{\mathfrak{l}})$ -module $(\rho, \mathfrak{a}, \langle \cdot, \cdot \rangle_{\mathfrak{a}}, \Phi_{\mathfrak{a}})$ consists of an $(\mathfrak{l}, \Phi_{\mathfrak{l}})$ -module $(\rho, \mathfrak{a}, \Phi_{\mathfrak{a}})$ and an inner product $\langle \cdot, \cdot \rangle_{\mathfrak{a}}$ on \mathfrak{a} such that

- (i) $(\mathfrak{a}, \langle \cdot, \cdot \rangle_{\mathfrak{a}}, \Phi_{\mathfrak{a}})$ is an abelian (\mathfrak{h}, K) -equivariant metric Lie algebra,
- (ii) ρ is an orthogonal representation, i.e. $\rho : \mathfrak{l} \rightarrow \mathfrak{so}(\mathfrak{a}, \langle \cdot, \cdot \rangle_{\mathfrak{a}})$.

We often abbreviate $(\rho, \mathfrak{a}, \Phi_{\mathfrak{a}})$ and $(\rho, \mathfrak{a}, \langle \cdot, \cdot \rangle_{\mathfrak{a}}, \Phi_{\mathfrak{a}})$ as (ρ, \mathfrak{a}) or \mathfrak{a} .

Let $(\mathfrak{l}_i, \Phi_{\mathfrak{l}_i})$, $i = 1, 2$, be two (\mathfrak{h}, K) -equivariant Lie algebras and let (ρ_i, \mathfrak{a}_i) , $i = 1, 2$, be orthogonal $(\mathfrak{l}_i, \Phi_{\mathfrak{l}_i})$ -modules. Let $S : \mathfrak{l}_1 \rightarrow \mathfrak{l}_2$ be a homomorphism of (\mathfrak{h}, K) -equivariant Lie algebras and let $U : \mathfrak{a}_2 \rightarrow \mathfrak{a}_1$ be an (\mathfrak{h}, K) -equivariant isometric embedding. Suppose that

$$U \circ \rho_2(S(L)) = \rho_1(L) \circ U$$

holds for all $L \in \mathfrak{l}$. Then we call (S, U) morphism of pairs. We will write this as $(S, U) : ((\mathfrak{l}_1, \Phi_{\mathfrak{l}_1}), \mathfrak{a}_1) \rightarrow ((\mathfrak{l}_2, \Phi_{\mathfrak{l}_2}), \mathfrak{a}_2)$, but remember that S and U map in different directions.

We will say that an ideal of an (\mathfrak{h}, K) -equivariant Lie algebra (\mathfrak{g}, Φ) is Φ -invariant if it is invariant under all maps belonging to $\text{im } \Phi$.

Definition 2.5. Let $(\rho, \mathfrak{a}, \langle \cdot, \cdot \rangle_{\mathfrak{a}}, \Phi_{\mathfrak{a}})$ be an orthogonal $(\mathfrak{l}, \Phi_{\mathfrak{l}})$ -module. A quadratic extension of $(\mathfrak{l}, \Phi_{\mathfrak{l}})$ by \mathfrak{a} is given by a quadruple $(\mathfrak{g}, \mathfrak{i}, i, p)$, where

- (i) \mathfrak{g} is an (\mathfrak{h}, K) -equivariant metric Lie algebra,
- (ii) \mathfrak{i} is an isotropic Φ -invariant ideal of \mathfrak{g} ,
- (iii) i and p are homomorphisms of (\mathfrak{h}, K) -equivariant Lie algebras constituting an exact sequence

$$0 \longrightarrow \mathfrak{a} \xrightarrow{i} \mathfrak{g}/\mathfrak{i} \xrightarrow{p} \mathfrak{l} \longrightarrow 0$$

that is consistent with the representation ρ of \mathfrak{l} on \mathfrak{a} and has the property that i is an isometry from \mathfrak{a} to $\mathfrak{i}^{\perp}/\mathfrak{i}$.

Example 2.6 (The standard model). First we consider a Lie algebra \mathfrak{l} without further structure, i.e. $\Phi_{\mathfrak{l}} = 0$. Let \mathfrak{a} be an orthogonal \mathfrak{l} -module and take $(\alpha, \gamma) \in \mathcal{Z}_{\mathbb{Q}}^2(\mathfrak{l}, \mathfrak{a})$. Let $\mathfrak{d}_{\alpha, \gamma}(\mathfrak{l}, \mathfrak{a}) = (\mathfrak{d}, \langle \cdot, \cdot \rangle)$ be the metric Lie algebra constructed in Section 2.1. We identify $\mathfrak{d}/\mathfrak{l}^*$ with $\mathfrak{a} \oplus \mathfrak{l}$ and denote by $i : \mathfrak{a} \rightarrow \mathfrak{a} \oplus \mathfrak{l}$ the injection and by $p : \mathfrak{a} \oplus \mathfrak{l} \rightarrow \mathfrak{l}$ the projection. Then $(\mathfrak{d}_{\alpha, \gamma}(\mathfrak{l}, \mathfrak{a}), \mathfrak{l}^*, i, p)$ is a quadratic extension of \mathfrak{l} by \mathfrak{a} .

Now suppose that we have in addition a (\mathfrak{h}, K) -structure on \mathfrak{l} and \mathfrak{a} , i.e. let $(\mathfrak{l}, \Phi_{\mathfrak{l}})$ be an (\mathfrak{h}, K) -equivariant Lie algebra and let $(\rho, \mathfrak{a}, \langle \cdot, \cdot \rangle_{\mathfrak{a}}, \Phi_{\mathfrak{a}})$ be an orthogonal $(\mathfrak{l}, \Phi_{\mathfrak{l}})$ -module. Then we can define a map $\Phi : \mathfrak{h} \cup K \rightarrow \text{Der}(\mathfrak{d}) \cup \text{Aut}(\mathfrak{d})$ by

$$\begin{aligned} \Phi(X)(Z + A + L) &= -\Phi_{\mathfrak{l}}(X)^*(Z) + \Phi_{\mathfrak{a}}(X)(A) + \Phi_{\mathfrak{l}}(X)(L) \\ \Phi(k)(Z + A + L) &= (\Phi_{\mathfrak{l}}(k)^*)^{-1}(Z) + \Phi_{\mathfrak{a}}(k)(A) + \Phi_{\mathfrak{l}}(k)(L). \end{aligned}$$

Then $\mathfrak{d}_{\alpha, \gamma}(\mathfrak{l}, \Phi_{\mathfrak{l}}, \mathfrak{a}) := (\mathfrak{d}, \Phi, \langle \cdot, \cdot \rangle)$ is an (\mathfrak{h}, K) -equivariant metric Lie algebra if (α, γ) satisfies a certain natural invariance condition with respect to $\Phi_{\mathfrak{l}}$ and $\Phi_{\mathfrak{a}}$. We

write $(\alpha, \gamma) \in \mathcal{Z}_Q^2(\mathfrak{l}, \Phi_{\mathfrak{l}}, \mathfrak{a})$ for this condition whose exact formulation we will give in Section 2.3. Hence, if $(\alpha, \gamma) \in \mathcal{Z}_Q^2(\mathfrak{l}, \Phi_{\mathfrak{l}}, \mathfrak{a})$, then $(\mathfrak{d}_{\alpha, \gamma}, \mathfrak{l}^*, i, p)$ is a quadratic extension of $(\mathfrak{l}, \Phi_{\mathfrak{l}})$ by \mathfrak{a} . It is called standard model since, as we will see, any quadratic extension of $(\mathfrak{l}, \Phi_{\mathfrak{l}})$ by \mathfrak{a} is in a certain sense equivalent to some $(\mathfrak{d}_{\alpha, \gamma}, \mathfrak{l}^*, i, p)$ for a suitable cocycle $(\alpha, \gamma) \in \mathcal{Z}_Q^2(\mathfrak{l}, \Phi_{\mathfrak{l}}, \mathfrak{a})$.

What makes the theory of quadratic extensions so useful is the fact that any (\mathfrak{h}, K) -equivariant metric Lie algebra without simple ideals admits such a structure. Essentially, this follows from Bérard-Bergery's investigations of pseudo-Riemannian holonomy representations and symmetric spaces in [6, 7, 8]. He proved that for any metric Lie algebra $(\mathfrak{g}, \langle \cdot, \cdot \rangle)$ there exists an isotropic ideal $\mathfrak{i}(\mathfrak{g}) \subset \mathfrak{g}$ such that $\mathfrak{i}(\mathfrak{g})^\perp/\mathfrak{i}(\mathfrak{g})$ is abelian. We want to describe the construction of this ideal now. However, instead of following [6, 7] we will give a description that is more adapted to the structure theory that we wish to develop here. In particular, we will give an (\mathfrak{h}, K) -equivariant formulation.

Let $(\mathfrak{g}, \Phi, \langle \cdot, \cdot \rangle)$ be an (\mathfrak{h}, K) -equivariant metric Lie algebra. There is a chain of Φ -invariant ideals

$$\mathfrak{g} = R_0(\mathfrak{g}) \supset R_1(\mathfrak{g}) \supset R_2(\mathfrak{g}) \supset \dots \supset R_{l_-}(\mathfrak{g}) = 0$$

which is defined by the condition that $R_k(\mathfrak{g})$ is the smallest ideal of \mathfrak{g} contained in $R_{k-1}(\mathfrak{g})$ such that the \mathfrak{g} -module $R_{k-1}/R_k(\mathfrak{g})$ is semisimple. The ideal $R(\mathfrak{g}) := R_1(\mathfrak{g})$ is called nilpotent radical of \mathfrak{g} . It has to be distinguished from the nilradical (i.e. the maximal nilpotent ideal) \mathfrak{n} and the (solvable) radical \mathfrak{r} . By Lie's Theorem

$$R(\mathfrak{g}) = \mathfrak{r} \cap \mathfrak{g}' = [\mathfrak{r}, \mathfrak{g}] \subset \mathfrak{n} \quad (1)$$

and $R(\mathfrak{g})$ acts trivially on any semisimple \mathfrak{g} -module [13]. We define an ideal $\mathfrak{i}(\mathfrak{g}) \subset \mathfrak{g}$ by

$$\mathfrak{i}(\mathfrak{g}) := \sum_{k=1}^l R_k(\mathfrak{g}) \cap R_k(\mathfrak{g})^\perp$$

and call it the canonical isotropic ideal of \mathfrak{g} .

Proposition 2.7 ([6, 7]; [35], Lemma 3.4). *If $(\mathfrak{g}, \Phi, \langle \cdot, \cdot \rangle)$ is an (\mathfrak{h}, K) -equivariant metric Lie algebra, then $\mathfrak{i}(\mathfrak{g})$ is a Φ -invariant isotropic ideal and the \mathfrak{g} -module $\mathfrak{i}(\mathfrak{g})^\perp/\mathfrak{i}(\mathfrak{g})$ is semisimple. If \mathfrak{g} does not contain simple ideals, then the Lie algebra $\mathfrak{i}(\mathfrak{g})^\perp/\mathfrak{i}(\mathfrak{g})$ is abelian.*

In particular, $\mathfrak{g}/\mathfrak{i}(\mathfrak{g})^\perp$ becomes an (\mathfrak{h}, K) -equivariant Lie algebra and $\mathfrak{i}(\mathfrak{g})^\perp/\mathfrak{i}(\mathfrak{g})$ a semisimple orthogonal $\mathfrak{g}/\mathfrak{i}(\mathfrak{g})^\perp$ -module. Moreover,

$$0 \longrightarrow \mathfrak{i}(\mathfrak{g})^\perp/\mathfrak{i}(\mathfrak{g}) \xrightarrow{i} \mathfrak{g}/\mathfrak{i}(\mathfrak{g}) \xrightarrow{p} \mathfrak{g}/\mathfrak{i}(\mathfrak{g})^\perp \longrightarrow 0$$

is an exact sequence of (\mathfrak{h}, K) -equivariant Lie algebras.

Corollary 2.8. *For any (\mathfrak{h}, K) -equivariant metric Lie algebra $(\mathfrak{g}, \Phi, \langle \cdot, \cdot \rangle)$ without simple ideals the quadruple $(\mathfrak{g}, \mathfrak{i}(\mathfrak{g}), i, p)$ is a quadratic extension of $\mathfrak{g}/\mathfrak{i}(\mathfrak{g})^\perp$ by $\mathfrak{i}(\mathfrak{g})^\perp/\mathfrak{i}(\mathfrak{g})$.*

This extension will be called the canonical quadratic extension associated with $(\mathfrak{g}, \Phi, \langle \cdot, \cdot \rangle)$.

Example 2.9. The following example shows that for a given metric Lie algebra $(\mathfrak{g}, \langle \cdot, \cdot \rangle)$ there may exist other quadratic extensions $(\mathfrak{g}, \mathfrak{i}, i, p)$ than the canonical one. Let $\mathfrak{h}(1) = \{[X_1, X_2] = X_3\}$ be the three-dimensional Heisenberg algebra and let $\sigma^1, \sigma^2, \sigma^3$ be the basis of $\mathfrak{h}(1)^*$ that is dual to X_1, X_2, X_3 . Let us consider the metric Lie algebra $\mathfrak{g} := \mathfrak{d}_{0,0}(\mathfrak{h}(1), 0) = \mathfrak{h}(1)^* \rtimes \mathfrak{h}(1)$. Example 2.6 says that $(\mathfrak{g}, \mathfrak{h}(1)^*, i, p)$ is a quadratic extension of $\mathfrak{h}(1)$ by 0, where (i, p) is defined by

$$0 \xrightarrow{i=0} \mathfrak{g}/\mathfrak{h}(1)^* \xrightarrow{p} \mathfrak{h}(1) \longrightarrow 0.$$

However, this quadratic extension is not the canonical one. Indeed, we have $R(\mathfrak{g}) = \mathfrak{g}' = \text{span}\{X_3, \sigma^1, \sigma^2\}$. In particular, $R(\mathfrak{g}) \subset \mathfrak{z}(\mathfrak{g})$, hence $R_2(\mathfrak{g}) = 0$. This implies

$$\mathfrak{i}(\mathfrak{g}) = R(\mathfrak{g})^\perp \cap R(\mathfrak{g}) = R(\mathfrak{g}) = \text{span}\{X_3, \sigma^1, \sigma^2\}.$$

In particular, the canonical quadratic extension associated with \mathfrak{g} is a quadratic extension of $\mathfrak{g}/\mathfrak{i}(\mathfrak{g})^\perp \cong \mathbb{R}^3 \not\cong \mathfrak{h}(1)$ by $\mathfrak{i}(\mathfrak{g})^\perp/\mathfrak{i}(\mathfrak{g}) = 0$.

Hence, at first glance we have the same difficulty as for double extensions, namely, in general an (\mathfrak{h}, K) -equivariant metric Lie algebra can be obtained in different ways by quadratic extensions. However, now we can always distinguish one of these extensions, namely the canonical one. As a quadratic extension this extension is characterised by the property to be balanced in the following sense.

Definition 2.10. A quadratic extension $(\mathfrak{g}, \mathfrak{i}, i, p)$ of an (\mathfrak{h}, K) -equivariant Lie algebra (\mathfrak{l}, Φ) by an orthogonal $(\mathfrak{l}, \Phi_\mathfrak{l})$ -module \mathfrak{a} is called balanced if $\mathfrak{i} = \mathfrak{i}(\mathfrak{g})$.

Since our aim is to determine isomorphism classes of (\mathfrak{h}, K) -equivariant metric Lie algebras Corollary 2.8 leads us to the problem to decide for which balanced quadratic extensions $(\mathfrak{g}_1, \mathfrak{i}_1, i_1, p_1)$ and $(\mathfrak{g}_2, \mathfrak{i}_2, i_2, p_2)$ the (\mathfrak{h}, K) -equivariant metric Lie algebras \mathfrak{g}_1 and \mathfrak{g}_2 are isomorphic. We will divide this problem into two steps. First we will introduce an equivalence relation for quadratic extensions that is stronger than isomorphism of the underlying (\mathfrak{h}, K) -equivariant metric Lie algebras. We will describe the corresponding equivalence classes. In the second step we have to decide which equivalence classes of quadratic extensions have isomorphic underlying (\mathfrak{h}, K) -equivariant metric Lie algebras.

Definition 2.11. Two quadratic extensions $(\mathfrak{g}_j, \mathfrak{i}_j, i_j, p_j)$, $j = 1, 2$, of $(\mathfrak{l}, \Phi_\mathfrak{l})$ by \mathfrak{a} are called equivalent if there exists an isomorphism $F : \mathfrak{g}_1 \rightarrow \mathfrak{g}_2$ of (\mathfrak{h}, K) -equivariant metric Lie algebras that maps \mathfrak{i}_1 to \mathfrak{i}_2 and satisfies $\overline{F} \circ i_1 = i_2$ and $p_2 \circ \overline{F} = p_1$, where $\overline{F} : \mathfrak{g}_1/\mathfrak{i}_1 \rightarrow \mathfrak{g}_2/\mathfrak{i}_2$ is the map induced by F .

Similar to the case of ordinary extensions of Lie algebras one can describe equivalence classes of quadratic extensions by cohomology classes. We will introduce a suitable cohomology theory in the next section. Actually, for a given (\mathfrak{h}, K) -equivariant Lie algebra $(\mathfrak{l}, \Phi_\mathfrak{l})$ and an orthogonal $(\mathfrak{l}, \Phi_\mathfrak{l})$ -module \mathfrak{a} we will define a cohomology set $\mathcal{H}_Q^2(\mathfrak{l}, \Phi_\mathfrak{l}, \mathfrak{a})$ and a subset $\mathcal{H}_Q^2(\mathfrak{l}, \Phi_\mathfrak{l}, \mathfrak{a})_b \subset \mathcal{H}_Q^2(\mathfrak{l}, \Phi_\mathfrak{l}, \mathfrak{a})$ such that the following holds.

Theorem 2.2. *There is a bijective map Ψ from the set of equivalence classes of quadratic extensions of $(\mathfrak{l}, \Phi_{\mathfrak{l}})$ by \mathfrak{a} to $\mathcal{H}_{\mathbb{Q}}^2(\mathfrak{l}, \Phi_{\mathfrak{l}}, \mathfrak{a})$. The image under Ψ of the subset of all equivalence classes of balanced extensions equals $\mathcal{H}_{\mathbb{Q}}^2(\mathfrak{l}, \Phi_{\mathfrak{l}}, \mathfrak{a})_{\mathfrak{b}} \subset \mathcal{H}_{\mathbb{Q}}^2(\mathfrak{l}, \Phi_{\mathfrak{l}}, \mathfrak{a})$.*

The set $\mathcal{H}_{\mathbb{Q}}^2(\mathfrak{l}, \Phi_{\mathfrak{l}}, \mathfrak{a})$ consists of equivalence classes $[\alpha, \gamma]$ of cocycles $(\alpha, \gamma) \in \mathcal{Z}_{\mathbb{Q}}^2(\mathfrak{l}, \Phi_{\mathfrak{l}}, \mathfrak{a})$ with respect to a certain equivalence relation. The inverse of Ψ then maps $[\alpha, \gamma]$ to the equivalence class of the standard model $\mathfrak{d}_{\alpha, \gamma}(\mathfrak{l}, \Phi_{\mathfrak{l}}, \mathfrak{a})$ (Example 2.6). For an explicit description of the map Ψ see Section 2.4. For a proof of this theorem in the non-equivariant case see [35], Thm. 2.7 and Thm. 3.12.

2.3. Quadratic cohomology. The aim of this section is the exact definition of the cohomology sets that appear in Theorem 2.2. Since quadratic extensions are not ordinary Lie algebra extensions we cannot expect to describe them by usual Lie algebra cohomology. We need a kind of non-linear cohomology. Such cohomology sets were first introduced by Grishkov [29] in a rather general setting. For the special case of cohomology needed for quadratic extensions we gave a self-contained presentation in [35]. Neither [29] nor [35] deals with the equivariant situation. As we will see here, the (\mathfrak{h}, K) -action can be easily incorporated.

Let us first recall the construction of the usual Lie algebra cohomology. Let $\rho : \mathfrak{l} \rightarrow \mathfrak{gl}(\mathfrak{a})$ be a representation of a Lie algebra \mathfrak{l} on a vector space \mathfrak{a} . Then we have the standard Lie algebra cochain complex $(C^*(\mathfrak{l}, \mathfrak{a}), d)$, where $C^p(\mathfrak{l}, \mathfrak{a}) = \text{Hom}(\bigwedge^p \mathfrak{l}, \mathfrak{a})$ and $d : C^p(\mathfrak{l}, \mathfrak{a}) \rightarrow C^{p+1}(\mathfrak{l}, \mathfrak{a})$ is defined by

$$\begin{aligned} d\tau(L_1, \dots, L_{p+1}) &= \sum_{i=1}^{p+1} (-1)^{i-1} \rho(L_i) \tau(L_1, \dots, \hat{L}_i, \dots, L_{p+1}) \\ &\quad + \sum_{i < j} (-1)^{i+j} \tau([L_i, L_j], L_1, \dots, \hat{L}_i, \dots, \hat{L}_j, \dots, L_{p+1}). \end{aligned}$$

The corresponding cohomology groups are denoted by $H^p(\mathfrak{l}, \mathfrak{a})$. In the special case where \mathfrak{a} is the one-dimensional trivial representation of \mathfrak{l} we denote the standard cochain complex also by $(C^*(\mathfrak{l}), d)$ and the cohomology groups by $H^p(\mathfrak{l})$.

Suppose that we have two Lie algebras \mathfrak{l}_i , $i = 1, 2$ and orthogonal \mathfrak{l}_i -modules \mathfrak{a}_i and that $(S, U) : (\mathfrak{l}_1, \mathfrak{a}_1) \rightarrow (\mathfrak{l}_2, \mathfrak{a}_2)$ is a morphism of pairs. Then we have pull back maps

$$\begin{aligned} (S, U)^* : \quad C^p(\mathfrak{l}_2, \mathfrak{a}_2) &\longrightarrow C^p(\mathfrak{l}_1, \mathfrak{a}_1) \\ (S, U)^* \alpha(L_1, \dots, L_p) &:= U \circ \alpha(S(L_1), \dots, S(L_p)) \\ (S, U)^* : \quad C^p(\mathfrak{l}_2) &\longrightarrow C^p(\mathfrak{l}_1), \\ (S, U)^* \gamma(L_1, \dots, L_p) &:= \gamma(S(L_1), \dots, S(L_p)). \end{aligned}$$

Now let $(\mathfrak{l}, \Phi_{\mathfrak{l}})$ be an (\mathfrak{h}, K) -equivariant Lie algebra and let \mathfrak{a} be an orthogonal (\mathfrak{h}, K) -module. Then we can consider

$$(e^{\Phi_{\mathfrak{l}}(X)}, e^{-\Phi_{\mathfrak{a}}(X)}) : (\mathfrak{l}, \mathfrak{a}) \longrightarrow (\mathfrak{l}, \mathfrak{a}), \quad X \in \mathfrak{h}, \quad (2)$$

and

$$(\Phi_l(k), (\Phi_a(k))^{-1}) : (l, \mathfrak{a}) \longrightarrow (l, \mathfrak{a}), \quad k \in K, \quad (3)$$

as morphism of pairs (without (\mathfrak{h}, K) -structure). Let $C^p(l, \mathfrak{a})^{(\mathfrak{h}, K)} \subset C^p(l, \mathfrak{a})$ denote the subspace of cochains that are invariant under these morphisms of pairs for all $X \in \mathfrak{h}$ and $k \in K$.

We define a product $C^p(l, \mathfrak{a})^{(\mathfrak{h}, K)} \times C^q(l, \mathfrak{a})^{(\mathfrak{h}, K)} \rightarrow C^{p+q}(l)^{(\mathfrak{h}, K)}$ by

$$\langle \cdot \wedge \cdot \rangle : C^p(l, \mathfrak{a})^{(\mathfrak{h}, K)} \times C^q(l, \mathfrak{a})^{(\mathfrak{h}, K)} \xrightarrow{\wedge} C^{p+q}(l, \mathfrak{a} \otimes \mathfrak{a})^{(\mathfrak{h}, K)} \xrightarrow{\langle \cdot, \cdot \rangle_{\mathfrak{a}}} C^{p+q}(l)^{(\mathfrak{h}, K)}.$$

Now we define the set of quadratic 1-cochains to be

$$C_Q^1(l, \Phi_l, \mathfrak{a}) = C^1(l, \mathfrak{a})^{(\mathfrak{h}, K)} \oplus C^2(l)^{(\mathfrak{h}, K)}.$$

This set is a group with group operation defined by

$$(\tau_1, \sigma_1) * (\tau_2, \sigma_2) = (\tau_1 + \tau_2, \sigma_1 + \sigma_2 + \frac{1}{2} \langle \tau_1 \wedge \tau_2 \rangle).$$

Let us consider the set

$$\mathcal{Z}_Q^2(l, \Phi_l, \mathfrak{a}) = \{(\alpha, \gamma) \in C^2(l, \mathfrak{a})^{(\mathfrak{h}, K)} \oplus C^3(l)^{(\mathfrak{h}, K)} \mid d\alpha = 0, d\gamma = \frac{1}{2} \langle \alpha \wedge \alpha \rangle\}$$

whose elements are called quadratic 2-cocycles. Then the group $C_Q^1(l, \Phi_l, \mathfrak{a})$ acts from the right on $\mathcal{Z}_Q^2(l, \Phi_l, \mathfrak{a})$ by

$$(\alpha, \gamma)(\tau, \sigma) = (\alpha + d\tau, \gamma + d\sigma + \langle (\alpha + \frac{1}{2}d\tau) \wedge \tau \rangle).$$

We define the quadratic cohomology set as the orbit space

$$\mathcal{H}_Q^2(l, \Phi_l, \mathfrak{a}) := \mathcal{Z}_Q^2(l, \Phi_l, \mathfrak{a}) / C_Q^1(l, \Phi_l, \mathfrak{a}).$$

The equivalence class of $(\alpha, \gamma) \in \mathcal{Z}_Q^2(l, \Phi_l, \mathfrak{a})$ in $\mathcal{H}_Q^2(l, \Phi_l, \mathfrak{a})$ is denoted by $[\alpha, \gamma]$. As usual, if (\mathfrak{h}, K) is trivial, then we omit Φ_l in all the notation above.

Now let us consider a morphism of pairs $(S, U) : ((l_1, \Phi_1), \mathfrak{a}_1) \rightarrow ((l_2, \Phi_2), \mathfrak{a}_2)$. As discussed above (S, U) acts on $C^p(l, \mathfrak{a})$. By (\mathfrak{h}, K) -equivariance (S, U) maps the subspace $C^p(l_2, \mathfrak{a}_2)^{(\mathfrak{h}, K)} \subset C^p(l_2, \mathfrak{a}_2)$ to $C^p(l_1, \mathfrak{a}_1)^{(\mathfrak{h}, K)} \subset C^p(l_1, \mathfrak{a}_1)$. It is not hard to prove that this map induces a map

$$(S, U)^* : \mathcal{H}_Q^2(l_2, \Phi_{l_2}, \mathfrak{a}_2) \longrightarrow \mathcal{H}_Q^2(l_1, \Phi_{l_1}, \mathfrak{a}_1)$$

(cf. [35] for a proof in the non-equivariant case).

In particular, for a given (\mathfrak{h}, K) -equivariant Lie algebra (l, Φ_l) and an orthogonal (l, Φ_l) -module \mathfrak{a} the morphisms of pairs in (2) and (3) induce maps on $\mathcal{H}_Q^2(l, \mathfrak{a})$. Let $\mathcal{H}_Q^2(l, \mathfrak{a})^{(\mathfrak{h}, K)} \subset \mathcal{H}_Q^2(l, \mathfrak{a})$ denote the subset of all cohomology classes that are invariant under these maps for all $X \in \mathfrak{h}$ and $k \in K$.

Proposition 2.12. *Under our assumptions on (\mathfrak{h}, K) the inclusion*

$$\mathcal{Z}_Q^2(l, \Phi_l, \mathfrak{a}) \hookrightarrow \mathcal{Z}_Q^2(l, \mathfrak{a})$$

induces a bijection

$$\mathcal{H}_Q^2(l, \Phi_l, \mathfrak{a}) \longrightarrow \mathcal{H}_Q^2(l, \mathfrak{a})^{(\mathfrak{h}, K)}.$$

You can find a proof of this proposition in the appendix.

Next we will define the subset $\mathcal{H}_Q^2(\mathfrak{l}, \Phi_{\mathfrak{l}}, \mathfrak{a})_b \subset \mathcal{H}_Q^2(\mathfrak{l}, \Phi_{\mathfrak{l}}, \mathfrak{a})$, which plays an important role in Theorem 2.2. It was first introduced in [35], where you can also find a proof of the fact that its elements correspond exactly to those extensions that are balanced.

As usual, an $(\mathfrak{l}, \Phi_{\mathfrak{l}})$ -module (ρ, \mathfrak{a}) will be called semisimple if every $(\mathfrak{l}, \Phi_{\mathfrak{l}})$ -submodule has an $(\mathfrak{l}, \Phi_{\mathfrak{l}})$ -invariant complement. This is the case if and only if ρ is semisimple. In the following definition we need the notion of the socle $S(\mathfrak{l})$ of a Lie algebra \mathfrak{l} , which is the maximal ideal of \mathfrak{l} on which \mathfrak{l} acts semisimply.

Definition 2.13. *Let $(\mathfrak{l}, \Phi_{\mathfrak{l}})$ be an (\mathfrak{h}, K) -equivariant Lie algebra, let (ρ, \mathfrak{a}) be a semisimple orthogonal $(\mathfrak{l}, \Phi_{\mathfrak{l}})$ -module and take $(\alpha, \gamma) \in \mathcal{Z}_Q^2(\mathfrak{l}, \Phi_{\mathfrak{l}}, \mathfrak{a})$. Since ρ is semisimple we have a decomposition $\mathfrak{a} = \mathfrak{a}^{\mathfrak{l}} \oplus \rho(\mathfrak{l})\mathfrak{a}$ and a corresponding decomposition $\alpha = \alpha_0 + \alpha_1$. Let m be such that $R_{m+1}(\mathfrak{l}) = 0$. Then $(\alpha, \gamma) \in \mathcal{Z}_Q^2(\mathfrak{l}, \Phi_{\mathfrak{l}}, \mathfrak{a})$ is called balanced if it satisfies the following conditions (A_k) and (B_k) for $0 \leq k \leq m$.*

(A_0) *Let $L_0 \in \mathfrak{z}(\mathfrak{l}) \cap \ker \rho$ be such that there exist elements $A_0 \in \mathfrak{a}$ and $Z_0 \in \mathfrak{l}^*$ satisfying for all $L \in \mathfrak{l}$*

$$(i) \quad \alpha(L, L_0) = \rho(L)A_0,$$

$$(ii) \quad \gamma(L, L_0, \cdot) = -\langle A_0, \alpha(L, \cdot) \rangle_{\mathfrak{a}} + \langle Z_0, [L, \cdot]_{\mathfrak{l}} \rangle \text{ as an element of } \mathfrak{l}^*,$$

then $L_0 = 0$.

(B_0) *The subspace $\alpha_0(\ker[\cdot, \cdot]_{\mathfrak{l}}) \subset \mathfrak{a}^{\mathfrak{l}}$ is non-degenerate.*

(A_k) $(k \geq 1)$

Let $\mathfrak{k} \subset S(\mathfrak{l}) \cap R_k(\mathfrak{l})$ be an \mathfrak{l} -ideal such that there exist elements $\Phi_1 \in \text{Hom}(\mathfrak{k}, \mathfrak{a})$ and $\Phi_2 \in \text{Hom}(\mathfrak{k}, R_k(\mathfrak{l})^)$ satisfying for all $L \in \mathfrak{l}$ and $K \in \mathfrak{k}$*

$$(i) \quad \alpha(L, K) = \rho(L)\Phi_1(K) - \Phi_1([L, K]_{\mathfrak{l}}),$$

$$(ii) \quad \gamma(L, K, \cdot) = -\langle \Phi_1(K), \alpha(L, \cdot) \rangle_{\mathfrak{a}} + \langle \Phi_2(K), [L, \cdot]_{\mathfrak{l}} \rangle + \langle \Phi_2([L, K]_{\mathfrak{l}}), \cdot \rangle \text{ as an element of } R_k(\mathfrak{l})^*,$$

then $\mathfrak{k} = 0$.

(B_k) $(k \geq 1)$

Let $\mathfrak{b}_k \subset \mathfrak{a}$ be the maximal submodule such that the system of equations

$$\langle \alpha(L, K), B \rangle_{\mathfrak{a}} = \langle \rho(L)\Phi(K) - \Phi([L, K]_{\mathfrak{l}}), B \rangle_{\mathfrak{a}}, \quad L \in \mathfrak{l}, K \in R_k(\mathfrak{l}), B \in \mathfrak{b}_k,$$

has a solution $\Phi \in \text{Hom}(R_k(\mathfrak{l}), \mathfrak{a})$. Then \mathfrak{b}_k is non-degenerate.

One can prove that for a cocycle the property to be balanced depends only on its cohomology class. Hence we may call a cohomology class $[\alpha, \gamma] \in \mathcal{H}_Q^2(\mathfrak{l}, \Phi_{\mathfrak{l}}, \mathfrak{a})$ balanced if $(\alpha, \gamma) \in \mathcal{Z}_Q^2(\mathfrak{l}, \Phi_{\mathfrak{l}}, \mathfrak{a})$ is balanced.

For an (\mathfrak{h}, K) -equivariant Lie algebra $(\mathfrak{l}, \Phi_{\mathfrak{l}})$ and an orthogonal $(\mathfrak{l}, \Phi_{\mathfrak{l}})$ -module (ρ, \mathfrak{a}) let $\mathcal{H}_Q^2(\mathfrak{l}, \Phi_{\mathfrak{l}}, \mathfrak{a})_b \subset \mathcal{H}_Q^2(\mathfrak{l}, \Phi_{\mathfrak{l}}, \mathfrak{a})$ be the set of all balanced cohomology classes if ρ is semisimple and put $\mathcal{H}_Q^2(\mathfrak{l}, \Phi_{\mathfrak{l}}, \mathfrak{a})_b := \emptyset$ if ρ is not semisimple. This finishes the definition of the cohomology sets used in Theorem 2.2.

Example 2.14. Take $\mathfrak{l} = \mathfrak{h}(1) = \{[X, Y] = Z\}$ and let (ρ, \mathfrak{a}) be a semisimple orthogonal \mathfrak{l} -module. Then the following two maps are bijective:

$$Z_{\mathfrak{l}} := \{\alpha \in C^2(\mathfrak{l}, \mathfrak{a}) \mid \alpha(X, Y) = 0, \alpha(Z, \mathfrak{l}) \subset \mathfrak{a}^{\perp}\} \longrightarrow H^2(\mathfrak{l}, \mathfrak{a}), \quad \alpha \mapsto [\alpha],$$

$$Z_{\mathfrak{l}, b} := \{\alpha \in Z_{\mathfrak{l}} \mid \alpha \neq 0, \alpha(Z, \mathfrak{l}) \subset \mathfrak{a}^{\perp} \text{ is non-degenerate}\} \longrightarrow \mathcal{H}_Q^2(\mathfrak{l}, \mathfrak{a})_b, \quad \alpha \mapsto [\alpha, 0].$$

Proof. Let us consider the first map. It is well-defined and injective. We will prove that it is surjective. Take $a \in H^2(\mathfrak{l}, \mathfrak{a})$. Since \mathfrak{l} is nilpotent and \mathfrak{a} is semisimple we have $H^*(\mathfrak{l}, \mathfrak{a}) = H^*(\mathfrak{l}, \mathfrak{a}^{\perp})$ [21]. Hence we can represent a by a cocycle α that satisfies $\alpha(\mathfrak{l}, \mathfrak{l}) \subset \mathfrak{a}^{\perp}$. We define $\tau \in C^1(\mathfrak{l}, \mathfrak{a})$ by $\tau(X) = \tau(Y) = 0$, $\tau(Z) = \alpha(X, Y)$. Then $\tilde{\alpha} := \alpha + d\tau$ satisfies $\tilde{\alpha}(X, Y) = \alpha(X, Y) - \tau([X, Y]) = 0$ and $\tilde{\alpha}(Z, \mathfrak{l}) \subset \mathfrak{a}^{\perp}$. Since $[\alpha] = [\tilde{\alpha}]$ the assertion follows.

Now let us turn to the second map. First we have to check that it is also well-defined. If $\alpha \in Z_{\mathfrak{l}}$, then $\langle \alpha \wedge \alpha \rangle = 0$. Hence $(\alpha, 0) \in \mathcal{Z}_Q^2(\mathfrak{l}, \mathfrak{a})$. We have to show that the cocycle $(\alpha, 0)$ is balanced if $\alpha(\mathfrak{l}, \mathfrak{l})$ is non-degenerate and $\alpha \neq 0$. We have to check the admissibility conditions (A_k) , (B_k) for $k = 0, 1$ (note that $R_2(\mathfrak{l}) = 0$). Since ρ is semisimple and $Z \in R(\mathfrak{l})$ we have $\rho(Z) = 0$, hence $\mathfrak{z}(\mathfrak{l}) \cap \ker \rho = \mathbb{R} \cdot Z$. Because of $\mathfrak{a}^{\perp} \supset \alpha(Z, \mathfrak{l}) \neq 0$ Conditions (A_0) and (A_1) are satisfied. Conditions (B_0) and (B_1) hold since $\alpha(Z, \mathfrak{l}) \subset \mathfrak{a}^{\perp}$ is non-degenerate by assumption. Hence the second map is well-defined. Obviously it is injective. Let us prove surjectivity. Suppose $a \in \mathcal{H}_Q^2(\mathfrak{l}, \mathfrak{a})_b$. By surjectivity of our first map we can represent a by a balanced cocycle (α, γ) with $\alpha \in Z_{\mathfrak{l}}$. Clearly, $\alpha = 0$ would contradict Condition (A_0) (choose $L_0 = Z$, $A_0 = 0$, $Z_0 = 0$). Hence $0 \neq \alpha \in Z_{\mathfrak{l}}$. Now it is easy to see that there exists a cochain $\tau \in C^1(\mathfrak{l}, \mathfrak{a})$ such that $d\tau = 0$ and $\langle \alpha \wedge \tau \rangle = \gamma$. Consequently, $a = [\alpha, \gamma] = [\alpha, 0]$, which proves the assertion. \square

Since we want to construct indecomposable metric Lie algebras by quadratic extensions we also need a notion of indecomposability for quadratic cohomology classes.

Definition 2.15. A non-trivial decomposition of a pair $((\mathfrak{l}, \Phi_{\mathfrak{l}}), \mathfrak{a})$ consists of two non-zero morphisms of pairs $(q_i, j_i) : ((\mathfrak{l}, \Phi_{\mathfrak{l}}), \mathfrak{a}) \longrightarrow ((\mathfrak{l}_i, \Phi_{\mathfrak{l}_i}), \mathfrak{a}_i)$, $i = 1, 2$, such that $(q_1, j_1) \oplus (q_2, j_2) : ((\mathfrak{l}, \Phi_{\mathfrak{l}}), \mathfrak{a}) \rightarrow ((\mathfrak{l}_1, \Phi_{\mathfrak{l}_1}), \mathfrak{a}_1) \oplus ((\mathfrak{l}_2, \Phi_{\mathfrak{l}_2}), \mathfrak{a}_2)$ is an isomorphism.

A cohomology class $\varphi \in \mathcal{H}_Q^2(\mathfrak{l}, \Phi_{\mathfrak{l}}, \mathfrak{a})$ is called decomposable if it can be written as a sum

$$\varphi = (q_1, j_1)^* \varphi_1 + (q_2, j_2)^* \varphi_2$$

for a non-trivial decomposition (q_i, j_i) of $((\mathfrak{l}, \Phi_{\mathfrak{l}}), \mathfrak{a})$ and certain $\varphi_i \in \mathcal{H}_Q^2(\mathfrak{l}_i, \Phi_{\mathfrak{l}_i}, \mathfrak{a}_i)$, $i = 1, 2$. Here addition is induced by addition in the vector space $C^2(\mathfrak{l}, \mathfrak{a}) \oplus C^3(\mathfrak{l})$. A cohomology class which is not decomposable is called indecomposable.

Then we have the following relation between indecomposability of cohomology classes and indecomposability of metric Lie algebras.

Proposition 2.16 ([35], Prop. 4.5). An (\mathfrak{h}, K) -equivariant metric Lie algebra $(\mathfrak{g}, \Phi, \langle \cdot, \cdot \rangle)$ is indecomposable if and only if the image under Ψ of the canonical quadratic extension associated with $(\mathfrak{g}, \Phi, \langle \cdot, \cdot \rangle)$ is an indecomposable cohomology class.

2.4. A classification scheme. According to Corollary 2.8, each (\mathfrak{h}, K) -equivariant metric Lie algebra without simple ideals comes with a distinguished structure of a quadratic extension which is balanced. By Thm. 2.2 balanced quadratic extensions are characterised by balanced quadratic cohomology classes. We obtain a (functorial) assignment

$$\begin{aligned} & \{(\mathfrak{h}, K)\text{-equivariant metric Lie algebras } (\mathfrak{g}, \Phi, \langle \cdot, \cdot \rangle) \text{ without simple ideals}\} \\ & \implies \left\{ \text{quadruples } \left(\mathfrak{l}, \Phi_{\mathfrak{l}}, \mathfrak{a}, [\alpha, \gamma] \in \mathcal{H}_Q^2(\mathfrak{l}, \Phi_{\mathfrak{l}}, \mathfrak{a})_b \right) \right\} \end{aligned} \quad (4)$$

In order to make (4) concrete let us write down the map Ψ appearing in Thm. 2.2 explicitly. Let $(\mathfrak{g}, \mathfrak{i}, i, p)$ be a quadratic extension of $(\mathfrak{l}, \Phi_{\mathfrak{l}})$ by \mathfrak{a} . Let $\tilde{p} : \mathfrak{g} \rightarrow \mathfrak{l}$ be the map induced by p . We choose an (\mathfrak{h}, K) -equivariant section $s : \mathfrak{l} \rightarrow \mathfrak{g}$ of \tilde{p} with isotropic image (which exists by semisimplicity of Φ) and define $\alpha \in C^2(\mathfrak{l}, \mathfrak{a})^{(\mathfrak{h}, K)}$ and $\gamma \in C^3(\mathfrak{l})^{(\mathfrak{h}, K)}$ by

$$\begin{aligned} i(\alpha(L_1, L_2)) & := [s(L_1), s(L_2)] - s([L_1, L_2]_{\mathfrak{l}}) + \mathfrak{i} \in \mathfrak{g}/\mathfrak{i} \\ \gamma(L_1, L_2, L_3) & := \langle [s(L_1), s(L_2)], s(L_3) \rangle . \end{aligned}$$

Then $(\alpha, \gamma) \in \mathcal{Z}_Q^2(\mathfrak{l}, \Phi_{\mathfrak{l}}, \mathfrak{a})$, and $[\alpha, \gamma] \in \mathcal{H}_Q^2(\mathfrak{l}, \Phi_{\mathfrak{l}}, \mathfrak{a})$ is the desired cohomology class. Note that $[\alpha, \gamma]$ does not depend on the choice of s while (α, γ) does.

It will turn out that the data on the right hand side of (4) give a very useful description of the set of isomorphism classes of (\mathfrak{h}, K) -equivariant metric Lie algebras. In fact, the resulting classification scheme, Theorem 2.3 below, is the basis of most of the classification results for metric Lie algebras and symmetric spaces that will be presented in this article.

Let $(\mathfrak{l}, \Phi_{\mathfrak{l}})$ be an (\mathfrak{h}, K) -equivariant Lie algebra. We consider the category $M_{\mathfrak{l}, \Phi_{\mathfrak{l}}}^{ss}$ of semisimple orthogonal $(\mathfrak{l}, \Phi_{\mathfrak{l}})$ -modules, where the morphisms between two modules $\mathfrak{a}_1, \mathfrak{a}_2$ are given by morphism of pairs $(S, U) : ((\mathfrak{l}, \Phi_{\mathfrak{l}}), \mathfrak{a}_1) \rightarrow ((\mathfrak{l}, \Phi_{\mathfrak{l}}), \mathfrak{a}_2)$. We denote the automorphism group of an object \mathfrak{a} of $M_{\mathfrak{l}, \Phi_{\mathfrak{l}}}^{ss}$ by $G_{\mathfrak{l}, \Phi_{\mathfrak{l}}, \mathfrak{a}}$. The natural right action of $G_{\mathfrak{l}, \Phi_{\mathfrak{l}}, \mathfrak{a}}$ on $\mathcal{H}_Q^2(\mathfrak{l}, \Phi_{\mathfrak{l}}, \mathfrak{a})$ leaves $\mathcal{H}_Q^2(\mathfrak{l}, \Phi_{\mathfrak{l}}, \mathfrak{a})_b$ invariant.

Theorem 2.3 (compare [35], Thm. 4.6). *Let \mathcal{L} be a complete set of representatives of isomorphism classes of (\mathfrak{h}, K) -equivariant Lie algebras. For each $(\mathfrak{l}, \Phi_{\mathfrak{l}}) \in \mathcal{L}$ we choose a complete set of representatives $\mathcal{A}_{\mathfrak{l}, \Phi_{\mathfrak{l}}}$ of isomorphism classes of objects in $M_{\mathfrak{l}, \Phi_{\mathfrak{l}}}^{ss}$.*

Then (4) descends to a bijective map from the set of isomorphism classes of (\mathfrak{h}, K) -equivariant metric Lie algebras without simple ideals to the union of orbit spaces

$$\coprod_{(\mathfrak{l}, \Phi_{\mathfrak{l}}) \in \mathcal{L}} \coprod_{\mathfrak{a} \in \mathcal{A}_{\mathfrak{l}, \Phi_{\mathfrak{l}}}} \mathcal{H}_Q^2(\mathfrak{l}, \Phi_{\mathfrak{l}}, \mathfrak{a})_b / G_{\mathfrak{l}, \Phi_{\mathfrak{l}}, \mathfrak{a}} .$$

The inverse of this map sends the orbit of $[\alpha, \gamma] \in \mathcal{H}_Q^2(\mathfrak{l}, \Phi_{\mathfrak{l}}, \mathfrak{a})_b$ to the isomorphism class of $\mathfrak{d}_{\alpha, \gamma}(\mathfrak{l}, \Phi_{\mathfrak{l}}, \mathfrak{a})$.

An (\mathfrak{h}, K) -equivariant metric Lie algebra is indecomposable if and only if the corresponding $G_{\mathfrak{l}, \Phi_{\mathfrak{l}}, \mathfrak{a}}$ -orbit in $\mathcal{H}_Q^2(\mathfrak{l}, \Phi_{\mathfrak{l}}, \mathfrak{a})_b$ consists of indecomposable cohomology classes.

In particular, the theorem says that the (\mathfrak{h}, K) -equivariant metric Lie algebras $\mathfrak{d}_{\alpha, \gamma}(\mathfrak{l}, \Phi_{\mathfrak{l}}, \mathfrak{a})$ exhaust all isomorphism classes of (\mathfrak{h}, K) -equivariant metric Lie algebras without simple ideals.

Its proof is word by word the same as the one for the non-equivariant case that is given in complete detail in [35]. The main ingredient is Theorem 2.2, which actually involves the construction of the standard model $\mathfrak{d}_{\alpha, \gamma}(\mathfrak{l}, \Phi_{\mathfrak{l}}, \mathfrak{a})$. In addition, one has to show that two metric Lie algebras are isomorphic if and only if their associated balanced cohomology classes are mapped to each other by an isomorphism of pairs. This is a rather straightforward consequence of the functoriality of the canonical isotropic ideal $\mathfrak{i}(\mathfrak{g})$ ([35], Prop. 4.2). The statement about indecomposability follows from Prop. 2.16.

Theorem 2.3 provides a complete set of invariants for equivariant metric Lie algebras and therefore structures the set of all isomorphism classes of them in a certain way. At first glance, however, Theorem 2.3 might look rather useless for classification problems. Indeed, the classification scheme seems to involve the classification of all (equivariant) Lie algebras. While this reflects the real difficulty of the problem there are at least two circumstances that, nevertheless, allow for interesting applications of the theorem. First of all, the index of $\mathfrak{d}_{\alpha, \gamma}(\mathfrak{l}, \Phi_{\mathfrak{l}}, \mathfrak{a})$ is at least $\dim \mathfrak{l}$. Thus for classification of equivariant metric Lie algebras of small index only Lie algebras \mathfrak{l} of small dimensions are needed. In general, it is a good strategy to look for certain subclasses of all equivariant metric Lie algebras such that the corresponding ingredients in Theorem 2.3 become manageable.

Secondly, a great many Lie algebras \mathfrak{l} satisfy $\mathcal{H}_Q^2(\mathfrak{l}, \mathfrak{a})_b = \emptyset$ for all semi-simple orthogonal \mathfrak{l} -modules \mathfrak{a} (in view of Proposition 2.12 this also implies $\mathcal{H}_Q^2(\mathfrak{l}, \Phi_{\mathfrak{l}}, \mathfrak{a})_b = \emptyset$ for any equivariant structure $\Phi_{\mathfrak{l}}$ on \mathfrak{l}). We call these Lie algebras non-admissible and the remaining ones, which really occur in Theorem 2.3, admissible. E.g., the two-dimensional non-abelian Lie algebra and Heisenberg Lie algebras of dimension ≥ 5 are non-admissible ([35], Prop. 5.2). It is easy to see that all reductive Lie algebras are admissible and that the class of admissible Lie algebras is closed under forming direct sums. Solvable admissible Lie algebras \mathfrak{l} with $\dim \mathfrak{l}' \leq 2$ are classified in [32] and [35], Section 5. Up to abelian summands, there are only finitely many of them. This result (together with its proof) shows that an a priori classification of Lie algebras (within a certain class) is not really needed for applications of Theorem 2.3. Up to now it is unknown how large the class of admissible Lie algebras really is. Any new structure result for admissible Lie algebras would give highly desirable new insight into the world of (equivariant) metric Lie algebras.

Contrary to \mathcal{L} , the determination of $\mathcal{A}_{\mathfrak{l}, \Phi_{\mathfrak{l}}}$ for given $(\mathfrak{l}, \Phi_{\mathfrak{l}})$ is usually no problem (since only semisimple modules are needed). Also the computation of the cohomology sets $\mathcal{H}_Q^2(\mathfrak{l}, \Phi_{\mathfrak{l}}, \mathfrak{a})_b$ succeeds in many interesting cases (techniques of homological algebra are helpful here). To achieve a true classification, one would have to determine, as a last step, the orbit space of the action of the group $G_{\mathfrak{l}, \Phi_{\mathfrak{l}}, \mathfrak{a}}$ on the set $\mathcal{H}_Q^2(\mathfrak{l}, \Phi_{\mathfrak{l}}, \mathfrak{a})_b$. This may lead to unsolved classification problems again, e.g., to the description of the orbit space $\bigwedge^3 \mathbb{R}^n / GL(n, \mathbb{R})$ for large n (compare [34], Sect. 5). But often $\mathcal{H}_Q^2(\mathfrak{l}, \Phi_{\mathfrak{l}}, \mathfrak{a})_b$ is so small such that things can be carried

out explicitly.

2.5. Classification results for metric Lie algebras. For trivial (\mathfrak{h}, K) , i.e. for $(\mathfrak{h}, K) = (0, \{e\})$ Theorem 2.3 specialises to a classification scheme for metric Lie algebras. As mentioned above for certain classes of metric Lie algebras this can be used to obtain a full classification (in the sense of a list). In particular we can use it to classify metric Lie algebras of small index.

Let us consider the Lorentzian case, i.e. the case where the metric Lie algebra has index 1. We already know examples of Lorentzian metric Lie algebras, namely the oscillator algebras $\mathfrak{osc}(\lambda)$ defined in Section 2.1. For a classification of indecomposable Lorentzian metric Lie algebras we have to restrict to \mathfrak{l} with $\dim \mathfrak{l} \leq 1$ and to Euclidean \mathfrak{l} -modules \mathfrak{a} in our classification scheme. In this way we reproduce the following well-known result, which was originally proved using double extensions.

Theorem 2.4 (Medina [43]). *Each indecomposable non-simple metric Lie algebra of signature $(1, q)$, $q > 0$, is isomorphic to an oscillator algebra $\mathfrak{osc}(\lambda)$ for exactly one $\lambda = (\lambda^1, \dots, \lambda^m) \in (\mathbb{R}^*)^m \cong \mathbb{R}^m$, $q = 2m + 1$, with $\lambda^1 = 1 \leq \lambda^2 \leq \dots \leq \lambda^m$.*

For the study of Lorentzian metric Lie algebras the method of double extensions and the method of quadratic extensions are in some sense equivalent, since every indecomposable non-simple Lorentzian metric \mathfrak{g} Lie algebra has exactly one isotropic ideal, namely its centre $\mathfrak{z}(\mathfrak{g})$. However, as already mentioned, for classification of metric Lie algebras of higher index quadratic extensions are more useful than double extensions, see for example the classification of metric Lie algebras of index 2 in [34] and the classification of metric Lie algebras of index 3 in [35]. Both of them are based on Theorem 2.3.

As a further example that shows that Theorem 2.3 is a useful mean for concrete classification problems let us consider nilpotent metric Lie algebras of small dimension. Favre and Santharoubane classified such Lie algebras up to dimension 7 in [23]. Their proof is based on the double extension method. In [32], Theorem 2.3 has been used to give a classification of nilpotent metric Lie algebras of dimension at most 10. Most of the isomorphism classes are isolated ones, however, in dimension 10 also 1-parameter families occur. Here we will restrict ourselves to nilpotent metric Lie algebras of dimension at most 9.

In the following theorem \mathfrak{a} stands for a pseudo-Euclidean vector space that we consider always as a trivial \mathfrak{l} -module. As usual, let e_1, \dots, e_{p+q} be the standard basis of $\mathbb{R}^{p,q}$. Furthermore, $\sigma^1, \dots, \sigma^l \in \mathfrak{l}^*$ will denote the dual basis of a given basis X_1, \dots, X_l of \mathfrak{l} and $\sigma^{i_1 \dots i_j} := \sigma^{i_1} \wedge \dots \wedge \sigma^{i_j}$.

Theorem 2.5 ([32]). *If $(\mathfrak{g}, \langle \cdot, \cdot \rangle)$ is an indecomposable non-abelian nilpotent metric Lie algebra of dimension at most 9, then it is isomorphic to $\mathfrak{d}_{\alpha, \gamma}(\mathfrak{l}, \mathfrak{a})$ for exactly one of the data in the following list*

1. $\mathfrak{l} = \mathfrak{g}_{4,1} = \{[X_1, X_2] = X_3, [X_1, X_3] = X_4\}$, $\mathfrak{a} \in \{\mathbb{R}^{0,1}, \mathbb{R}^{1,0}\}$,
 $\alpha = \sigma^{14} \otimes e_1$, $\gamma \in \{0, \sigma^{234}, \sigma^{134}, -\sigma^{134}\}$;
2. $\mathfrak{l} = \mathfrak{h}(1) \oplus \mathbb{R} = \{[X_1, X_2] = X_3\} \oplus \mathbb{R} \cdot X_4$, $\mathfrak{a} \in \{\mathbb{R}^{0,1}, \mathbb{R}^{1,0}\}$,

- $(\alpha, \gamma) = (\sigma^{13} \otimes e_1, \sigma^{234});$
3. $\mathfrak{l} = \mathbb{R}^4$, $\mathfrak{a} \in \{\mathbb{R}^{0,1}, \mathbb{R}^{1,0}\}$, $(\alpha, \gamma) = (\sigma^{13} \otimes e_1, \sigma^{234});$
4. $\mathfrak{l} = \mathfrak{h}(1) = \{[X_1, X_2] = X_3\}$,
- (a) $\mathfrak{a} \in \{\mathbb{R}^{0,1}, \mathbb{R}^{1,0}\}$, $(\alpha, \gamma) = (\sigma^{13} \otimes e_1, 0),$
- (b) $\mathfrak{a} \in \{\mathbb{R}^{0,2}, \mathbb{R}^{2,0}, \mathbb{R}^{1,1}\}$, $(\alpha, \gamma) = (\sigma^{13} \otimes e_1 + \sigma^{23} \otimes e_2, 0);$
5. $\mathfrak{l} = \mathbb{R}^3$,
- (a) $\mathfrak{a} = 0$, $(\alpha, \gamma) = (0, \sigma^{123}),$
- (b) $\mathfrak{a} \in \{\mathbb{R}^{0,2}, \mathbb{R}^{2,0}, \mathbb{R}^{1,1}\}$, $(\alpha, \gamma) = (\sigma^{12} \otimes e_1 + \sigma^{13} \otimes e_2, 0),$
- (c) $\mathfrak{a} \in \{\mathbb{R}^{0,3}, \mathbb{R}^{2,1}, \mathbb{R}^{1,2}, \mathbb{R}^{3,0}\}$, $(\alpha, \gamma) = (\sigma^{12} \otimes e_1 + \sigma^{13} \otimes e_2 + \sigma^{23} \otimes e_3, 0);$
6. $\mathfrak{l} = \mathbb{R}^2$, $\mathfrak{a} \in \{\mathbb{R}^{0,1}, \mathbb{R}^{1,0}\}$, $(\alpha, \gamma) = (\sigma^{12} \otimes e_1, 0).$

3. Symmetric spaces

3.1. Symmetric triples and quadratic extensions. In this section we are concerned with \mathbb{Z}_2 -equivariant objects, i.e., (\mathfrak{h}, K) -equivariant modules, Lie algebras, metric Lie algebras etc., where $K = \mathbb{Z}_2$ is the group consisting of two elements and $\mathfrak{h} = 0$. Any equivariant structure $\Phi, \Phi_1 \dots$ is determined by its value on the nontrivial element of \mathbb{Z}_2 , which is an involution. We will denote this involution by $\theta, \theta_1 \dots$. We will keep the notation of Section 2 but with all Φ 's replaced by the corresponding θ 's. Any \mathbb{Z}_2 -module V has a decomposition $V = V_+ \oplus V_-$ into the (± 1) -eigenspaces of θ_V .

As explained in the introduction, one associates with a pseudo-Riemannian symmetric space the Lie algebra of its transvection group together with a natural non-degenerate symmetric bilinear form and an isometric involution on it. This leads to the notion of a symmetric triple.

Definition 3.1.

- (a) A \mathbb{Z}_2 -equivariant Lie algebra (\mathfrak{g}, θ) is called proper if $[\mathfrak{g}_-, \mathfrak{g}_-] = \mathfrak{g}_+$.
- (b) A symmetric triple is a proper \mathbb{Z}_2 -equivariant metric Lie algebra $(\mathfrak{g}, \theta, \langle \cdot, \cdot \rangle)$.
- (c) The index (signature) of a symmetric triple $(\mathfrak{g}, \theta, \langle \cdot, \cdot \rangle)$ is the index (signature) of the symmetric bilinear form $\langle \cdot, \cdot \rangle|_{\mathfrak{g}_-}$.

The Lie algebra of the transvection group of a pseudo-Riemannian symmetric space M of index p carries the structure of a symmetric triple of the same index in a natural way. We call it the symmetric triple of M . The notions of isomorphy and decomposability carry over from general \mathbb{Z}_2 -equivariant metric Lie algebras to symmetric triples. Then we have

Proposition 3.2 (see e.g. [16]). *The assignment which sends each pseudo-Riemannian symmetric space to its symmetric triple induces a bijective map between isometry classes of simply connected symmetric spaces and isomorphism classes of symmetric triples. A symmetric space is indecomposable if and only if its symmetric triple is so.*

All (not necessarily simply connected) symmetric spaces corresponding to a given symmetric triple $(\mathfrak{g}, \theta, \langle \cdot, \cdot \rangle)$ can be easily classified. Let us shortly discuss this classification. For the facts used in this discussion we refer to [40] and [30], Ch.VII, §§8,9.

Let \tilde{G} be the connected and simply connected Lie group with Lie algebra \mathfrak{g} . Then θ integrates to an involutive automorphism $\tilde{\theta} : \tilde{G} \rightarrow \tilde{G}$. The group $\tilde{G}^{\tilde{\theta}}$ is connected. Let $Z(\tilde{G})$ be the center of \tilde{G} , and set $Z_0 := Z(\tilde{G}) \cap \tilde{G}^{\tilde{\theta}}$. Then Z_0 is discrete. Choose a discrete $\tilde{\theta}$ -stable subgroup $Z \subset Z(\tilde{G})$ containing Z_0 and set $G := \tilde{G}/Z$. Then $\tilde{\theta}$ induces an automorphism $\theta : G \rightarrow G$. Its fixed point group G^θ has at most finitely connected components. The connected component G_0^θ satisfies $G_0^\theta \cap Z(G) = \{e\}$. We choose a group G_+ such that $G_0^\theta \subset G_+ \subset G^\theta$ and $G_+ \cap Z(G) = \{e\}$ and define $M := G/G_+$. Let $x_0 = eG_+$ be the base point of the homogeneous space M . Then $\mathfrak{g}_- \cong T_{x_0}M$, and $\langle \cdot, \cdot \rangle|_{\mathfrak{g}_-}$ defines a G_+ -invariant scalar product on $T_{x_0}M$, which extends uniquely to a G -invariant pseudo-Riemannian metric on M . Moreover, $\tilde{\theta}$ induces an involutive isometry θ_{x_0} of M having x_0 as an isolated fixed point. Conjugating θ_{x_0} with elements of G we get involutive isometries θ_x for all $x \in M$. Thus M has the structure of a pseudo-Riemannian symmetric space. Moreover, G is the transvection group of M .

If Z and G_+ run through all possible choices as above, then the resulting spaces M exhaust the isometry classes of pseudo-Riemannian spaces having a symmetric triple isomorphic to $(\mathfrak{g}, \theta, \langle \cdot, \cdot \rangle)$. Two such spaces are isometric if and only if the defining data (Z, G_+) are conjugated by the automorphism group $\text{Aut}(\tilde{G}, \tilde{\theta}, \langle \cdot, \cdot \rangle)$ consisting of all automorphisms of \tilde{G} that respect the involution as well as the pseudo-Riemannian metric on \tilde{G} induced by $\langle \cdot, \cdot \rangle$. The simply connected symmetric space associated with $(\mathfrak{g}, \theta, \langle \cdot, \cdot \rangle)$ arises if we choose $Z = Z_0$, $G_+ = G_0^\theta$.

Thus the classification of symmetric spaces is reduced to the classification of symmetric triples, i.e., proper \mathbb{Z}_2 -equivariant metric Lie algebras. The theory developed in Section 2 associates with every \mathbb{Z}_2 -equivariant metric Lie algebra $(\mathfrak{g}, \theta, \langle \cdot, \cdot \rangle)$ (without simple ideals) via its canonical quadratic extension a quadruple $(\mathfrak{l}, \theta_{\mathfrak{l}}, \mathfrak{a}, [\alpha, \gamma])$, where

- $(\mathfrak{l}, \theta_{\mathfrak{l}})$ is a \mathbb{Z}_2 -equivariant Lie algebra,
- \mathfrak{a} is a semisimple orthogonal $(\mathfrak{l}, \theta_{\mathfrak{l}})$ -module, and
- $[\alpha, \gamma] \in \mathcal{H}_{\mathbb{Q}}^2(\mathfrak{l}, \theta_{\mathfrak{l}}, \mathfrak{a})_b$ is a balanced quadratic cohomology class.

Then $(\mathfrak{g}, \theta, \langle \cdot, \cdot \rangle)$ is isomorphic to the \mathbb{Z}_2 -equivariant metric Lie algebra $\mathfrak{d}_{\alpha, \gamma}(\mathfrak{l}, \theta_{\mathfrak{l}}, \mathfrak{a})$.

Because we want to apply the classification scheme Theorem 2.3 to symmetric triples we have to express the properness condition for $(\mathfrak{g}, \theta, \langle \cdot, \cdot \rangle) \cong \mathfrak{d}_{\alpha, \gamma}(\mathfrak{l}, \theta_{\mathfrak{l}}, \mathfrak{a})$ in Definition 3.1 in terms of $(\mathfrak{l}, \theta_{\mathfrak{l}}, \mathfrak{a}, [\alpha, \gamma])$.

Proposition 3.3 ([36], Sect. 5). *Let $(\mathfrak{l}, \theta_{\mathfrak{l}}, \mathfrak{a}, [\alpha, \gamma])$ be a quadruple as above. By semisimplicity $\mathfrak{a} = \mathfrak{a}^{\mathfrak{l}} \oplus \rho(\mathfrak{l})\mathfrak{a}$, and we have a corresponding decomposition $\alpha = \alpha_0 + \alpha_1$. Then $\mathfrak{d}_{\alpha, \gamma}(\mathfrak{l}, \theta_{\mathfrak{l}}, \mathfrak{a})$ is a symmetric triple if and only if*

(T₁) *the \mathbb{Z}_2 -equivariant Lie algebra $(\mathfrak{l}, \theta_{\mathfrak{l}})$ is proper, and*

(T₂) $\mathfrak{a}_+^{\mathfrak{l}} = \alpha_0(\ker[\cdot, \cdot]_{\mathfrak{l}_-})$.

Definition 3.4. *Let $(\mathfrak{l}, \theta_{\mathfrak{l}})$ be a proper \mathbb{Z}_2 -equivariant Lie algebra, and let \mathfrak{a} be a semisimple orthogonal $(\mathfrak{l}, \theta_{\mathfrak{l}})$ -module. A quadratic extension $(\mathfrak{g}, \mathfrak{i}, i, p)$ of $(\mathfrak{l}, \theta_{\mathfrak{l}})$ by \mathfrak{a} is called admissible if it is balanced and \mathfrak{g} is proper, i.e., a symmetric triple. A cohomology class $[\alpha, \gamma] \in \mathcal{H}_Q^2(\mathfrak{l}, \theta_{\mathfrak{l}}, \mathfrak{a})$ is called admissible, if it is balanced and satisfies (T₂). We denote the set of all admissible quadratic cohomology classes by $\mathcal{H}_Q^2(\mathfrak{l}, \theta_{\mathfrak{l}}, \mathfrak{a})_{\#}$ and its subset of all indecomposable admissible classes by $\mathcal{H}_Q^2(\mathfrak{l}, \theta_{\mathfrak{l}}, \mathfrak{a})_0$.*

By Proposition 3.3 admissible cohomology classes correspond to admissible quadratic extensions.

Let $(\mathfrak{l}, \theta_{\mathfrak{l}})$ be a proper \mathbb{Z}_2 -equivariant Lie algebra. As in Section 2.4 we consider the category $M_{\mathfrak{l}, \theta_{\mathfrak{l}}}^{ss}$ of semisimple orthogonal $(\mathfrak{l}, \theta_{\mathfrak{l}})$ -modules and morphisms of pairs $((\mathfrak{l}, \theta_{\mathfrak{l}}), \mathfrak{a}_1) \rightarrow ((\mathfrak{l}, \theta_{\mathfrak{l}}), \mathfrak{a}_2)$. The automorphism group $G_{\mathfrak{l}, \theta_{\mathfrak{l}}, \mathfrak{a}}$ of an object \mathfrak{a} of $M_{\mathfrak{l}, \theta_{\mathfrak{l}}}^{ss}$ acts on $\mathcal{H}_Q^2(\mathfrak{l}, \theta_{\mathfrak{l}}, \mathfrak{a})_0$ and on $\mathcal{H}_Q^2(\mathfrak{l}, \theta_{\mathfrak{l}}, \mathfrak{a})_{\#}$ from the right.

Combining Proposition 3.3 with Theorem 2.3 we arrive at the following classification scheme for symmetric triples. We prefer to formulate it for indecomposable symmetric triples. One gets the corresponding statement for general symmetric triples if one replaces $\mathcal{H}_Q^2(\mathfrak{l}, \theta_{\mathfrak{l}}, \mathfrak{a})_0$ by $\mathcal{H}_Q^2(\mathfrak{l}, \theta_{\mathfrak{l}}, \mathfrak{a})_{\#}$.

Theorem 3.1 ([36], Section 6). *Let \mathcal{L}_p be a complete set of representatives of isomorphism classes of proper \mathbb{Z}_2 -equivariant Lie algebras. For each $(\mathfrak{l}, \theta_{\mathfrak{l}}) \in \mathcal{L}_p$ we choose a complete set of representatives $\mathcal{A}_{\mathfrak{l}, \theta_{\mathfrak{l}}}$ of isomorphism classes of objects in $M_{\mathfrak{l}, \theta_{\mathfrak{l}}}^{ss}$.*

Then there is a bijective map from the set of isomorphism classes of non-semisimple indecomposable symmetric triples to the union of orbit spaces

$$\coprod_{(\mathfrak{l}, \theta_{\mathfrak{l}}) \in \mathcal{L}_p} \coprod_{\mathfrak{a} \in \mathcal{A}_{\mathfrak{l}, \theta_{\mathfrak{l}}}} \mathcal{H}_Q^2(\mathfrak{l}, \theta_{\mathfrak{l}}, \mathfrak{a})_0 / G_{\mathfrak{l}, \theta_{\mathfrak{l}}, \mathfrak{a}} . \quad (5)$$

The inverse of this map sends the orbit of $[\alpha, \gamma] \in \mathcal{H}_Q^2(\mathfrak{l}, \theta_{\mathfrak{l}}, \mathfrak{a})_0$ to the isomorphism class of $\mathfrak{d}_{\alpha, \gamma}(\mathfrak{l}, \theta_{\mathfrak{l}}, \mathfrak{a})$.

The theorem says in particular that the set

$$\{\mathfrak{d}_{\alpha, \gamma}(\mathfrak{l}, \theta_{\mathfrak{l}}, \mathfrak{a}) \mid (\mathfrak{l}, \theta_{\mathfrak{l}}) \in \mathcal{L}_p, \mathfrak{a} \in \mathcal{A}_{\mathfrak{l}, \theta_{\mathfrak{l}}}, [\alpha, \gamma] \in \mathcal{H}_Q^2(\mathfrak{l}, \theta_{\mathfrak{l}}, \mathfrak{a})_0\}$$

exhausts all isomorphism classes of non-semisimple indecomposable symmetric triples. As discussed at the end of Section 2.4 one does not need the whole set \mathcal{L}_p . Only those $(\mathfrak{l}, \theta_{\mathfrak{l}})$ such that the Lie algebra \mathfrak{l} is admissible really occur.

We will explain in Section 3.3 how Theorem 3.1 can be used in order to give a full classification of all symmetric triples of index at most 2. Before doing this we want to discuss the implications of the theory of quadratic extensions for the geometry of symmetric spaces.

3.2. The geometry of quadratic extensions. Let M be a pseudo-Riemannian symmetric space without semisimple local factors. As discussed in the last section, its (local) geometry is completely determined by its symmetric triple and, moreover, this symmetric triple carries the additional structure of an admissible quadratic extension. It is this structure that leads to the classification scheme Theorem 3.1. But what does this structure mean for the geometry of M ? This is the question we are going to discuss now. In particular, it will turn out that any pseudo-Riemannian symmetric space without semisimple local factors M comes with a distinguished fibration $q : M \rightarrow N$ over an *affine* symmetric space N such that all fibres are flat.

Let us first recall the notion of an affine symmetric space. A connected manifold with connection (M, ∇) is called an affine symmetric space if for each $x \in M$ there is an involutive affine transformation θ_x of (M, ∇) such that x is an isolated fixed point of θ_x . Note that θ_x , if it exists, is uniquely determined by ∇ . Forgetting about the metric and only remembering the Levi-Civita connection we can consider any pseudo-Riemannian symmetric space as an affine symmetric space. There are, however, many affine symmetric spaces that do not admit any symmetric pseudo-Riemannian metric. In exactly the same way as in the pseudo-Riemannian case one constructs the group of transvections of (M, ∇) , which acts transitively on M . Its Lie algebra comes with an involution but without scalar product. We will call this Lie algebra with involution the symmetric pair of M .

Definition 3.5. *A symmetric pair is a proper \mathbb{Z}_2 -equivariant Lie algebra (\mathfrak{g}, θ) satisfying $\mathfrak{z}(\mathfrak{g}) \subset \mathfrak{g}_-$.*

It follows from the effectivity of the action of the transvection group that the symmetric pair of M is indeed a symmetric pair in the sense of this definition. Moreover, if $(\mathfrak{g}, \theta, \langle \cdot, \cdot \rangle)$ is a symmetric triple, then (\mathfrak{g}, θ) is a symmetric pair. Indeed, if $X \in \mathfrak{z}(\mathfrak{g})$, then $\langle X, [Y, Z] \rangle = \langle [X, Y], Z \rangle = 0$ holds for all $Y, Z \in \mathfrak{g}_-$, i.e., $X \in [\mathfrak{g}_-, \mathfrak{g}_-]^\perp = \mathfrak{g}_-$. We have the following analogue of Proposition 3.2.

Proposition 3.6. *The assignment which sends each affine symmetric space to its symmetric pair induces a bijective map between affine diffeomorphism classes of simply connected affine symmetric spaces and isomorphism classes of symmetric pairs.*

Also the description of all affine symmetric spaces corresponding to a given symmetric pair proceeds in the same way as in the pseudo-Riemannian case. In practice, affine symmetric spaces often arise as follows: Let $\theta : G \rightarrow G$ be an involutive automorphism of a connected Lie group, and let $G_+ \subset G$ be a closed θ -stable subgroup having the same identity component as the fixed point group G^θ . Then θ induces an involutive diffeomorphism θ_{x_0} of $M := G/G_+$, and there is a unique G -invariant connection ∇ on M making θ_{x_0} affine. Then (M, ∇) is an affine symmetric space. Note that G might be different from the transvection group of M . In fact, the transvection group is a subgroup of a quotient of G .

If $q : M_1 \rightarrow M_2$ is an affine map between two affine symmetric spaces, then $q \circ \theta_x = \theta_{q(x)} \circ q$ for all $x \in M_1$. If q is surjective, then it follows that the fibres

$q^{-1}(m)$, $m \in M_2$, are totally geodesic (possibly disconnected) affine symmetric submanifolds of M_1 and that q induces a surjective homomorphism between the transvection groups of M_1 and M_2 . A submanifold Y of a pseudo-Riemannian manifold (M, g) is called coisotropic, if $T_y Y^{\perp_{g_y}} \subset T_y Y$ for all $y \in Y$. The following notion is a geometric counterpart of the notion of a quadratic extension (in the context of symmetric triples).

Definition 3.7. *Let N be an affine symmetric space. A special affine fibration over N is a surjective affine map $q : M \rightarrow N$, where M is a pseudo-Riemannian symmetric space and the fibres of q are flat, coisotropic, and connected.*

Let N be an affine symmetric space with corresponding symmetric pair $(\mathfrak{l}, \theta_{\mathfrak{l}})$. Its cotangent bundle T^*N carries the structure of a pseudo-Riemannian symmetric space such that its symmetric triple is $\mathfrak{d}_{0,0}(\mathfrak{l}, \theta_{\mathfrak{l}}, 0)$. Then the bundle projection $p : T^*N \rightarrow N$ is the simplest example of a special affine fibration over N .

Recall that $\mathfrak{d}_{0,0}(\mathfrak{l}, \theta_{\mathfrak{l}}, 0)$ is a quadratic extension of $(\mathfrak{l}, \theta_{\mathfrak{l}})$. We now want to construct special affine fibrations that correspond to quite general quadratic extensions in the same way as $p : T^*N \rightarrow N$ corresponds to $\mathfrak{d}_{0,0}(\mathfrak{l}, \theta_{\mathfrak{l}}, 0)$. Any proper \mathbb{Z}_2 -equivariant Lie algebra $(\mathfrak{l}, \theta_{\mathfrak{l}})$ gives rise to a symmetric pair $(\mathfrak{l}_0, \theta_{\mathfrak{l}_0})$, where $\mathfrak{l}_0 := \mathfrak{l}/(\mathfrak{z}(\mathfrak{l}) \cap \mathfrak{l}_+)$ and $\theta_{\mathfrak{l}_0}$ is induced by $\theta_{\mathfrak{l}}$.

Proposition 3.8. *Let $(\mathfrak{l}, \theta_{\mathfrak{l}})$ be a proper \mathbb{Z}_2 -equivariant Lie algebra. Let $(\mathfrak{g}, \theta, \langle \cdot, \cdot \rangle)$ be a symmetric triple equipped with the structure $(\mathfrak{g}, \mathfrak{i}, \mathfrak{i}, p)$ of a quadratic extension of $(\mathfrak{l}, \theta_{\mathfrak{l}})$ by some orthogonal $(\mathfrak{l}, \theta_{\mathfrak{l}})$ -module. Let M be a pseudo-Riemannian symmetric space with symmetric triple $(\mathfrak{g}, \theta, \langle \cdot, \cdot \rangle)$. We assume in addition that at least one of the following two conditions is satisfied:*

(a) M is simply connected.

(b) $\mathfrak{z}(\mathfrak{g}) \subset \mathfrak{i}^{\perp}$.

Then there is an affine symmetric space N , unique up to isomorphism, with associated symmetric pair $(\mathfrak{l}_0, \theta_{\mathfrak{l}_0})$ and a unique special affine fibration $q : M \rightarrow N$ such that

$$dQ_e = p_0 . \quad (6)$$

Here Q is the homomorphism of transvection groups induced by q and p_0 is the composition of natural maps $\mathfrak{g} \rightarrow \mathfrak{g}/\mathfrak{i} \xrightarrow{p} \mathfrak{l} \rightarrow \mathfrak{l}_0$. The symmetric space N can be written as a homogeneous space $N = L/L_+$, where L, L_+ are certain Lie groups with Lie algebras $\mathfrak{l}_+ \subset \mathfrak{l}$.

If $N = L/L_+$, then the transvection group of N equals $L_0 = L/(Z(L) \cap L_+)$. Note that $Z(L) \cap L_+$ acts trivially on N . Thus the group L (and the Lie algebra \mathfrak{l}) that corresponds to the data of the quadratic extension arises as a central extension of the geometrically visible transvection group L_0 (of the Lie algebra \mathfrak{l}_0).

The proof of the proposition will be given in the appendix. The idea behind it is very simple. Let G be the transvection group of M , and let $J \subset G$ be the analytic subgroup corresponding to the ideal $\mathfrak{i}^{\perp} \subset \mathfrak{g}$. Then J acts on M , and we would like to set q to be the projection onto the orbit space $N = J \backslash M$. That

the orbits are flat and coisotropic is a simple consequence of the properties of \mathfrak{i}^\perp . The main problem is to show that the orbit space is a manifold (the orbits have to be closed, in particular). It is this point, where we need Condition (a) or (b). Without these conditions, it is not difficult to construct examples with non-closed J -orbits. For them the resulting geometric structure will be a foliation only, not a fibration.

We are mainly interested in admissible, hence balanced quadratic extensions. They always satisfy Condition (b). Indeed, using Equation (1) we find $\mathfrak{i} = \mathfrak{i}(\mathfrak{g}) \subset R(\mathfrak{g}) \subset \mathfrak{g}'$. Forming orthogonal complements yields $\mathfrak{z}(\mathfrak{g}) \subset \mathfrak{i}^\perp$. Thus admissible quadratic extensions give rise to special affine fibrations.

Corollary 3.9. *Let M be a pseudo-Riemannian symmetric space without semisimple local factors. Let $(\mathfrak{g}, \theta, \langle \cdot, \cdot \rangle)$ be the corresponding symmetric triple. Then the canonical quadratic extension associated with $(\mathfrak{g}, \theta, \langle \cdot, \cdot \rangle)$ (in the sense of Corollary 2.8) defines a special affine fibration $q : M \rightarrow N$ over an affine symmetric space N .*

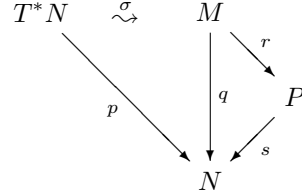
We call this fibration the *canonical fibration* of M . Its base N is an important invariant of the pseudo-Riemannian space M . Since there are a lot of non-admissible Lie algebras (see the end of Section 2.4) not all affine symmetric spaces N can appear as the base of the canonical fibration of some pseudo-Riemannian symmetric space M .

Let $q : M \rightarrow N$ be a special affine fibration. Let us collect some of its basic properties.

1. $q : M \rightarrow N$ is locally trivial, i.e., it is a fibre bundle with flat affine symmetric fibres. More precisely, there exist $k, l \in \mathbb{N}_0$, an open covering $\{U_i\}$ of N , and diffeomorphisms $\Phi_i : q^{-1}(U_i) \rightarrow U_i \times \mathbb{R}^k \times (S^1)^l$ such that for all $x \in U_i$ the restriction of Φ_i to $q^{-1}(x)$ is an affine diffeomorphism from the fibre onto $\{x\} \times \mathbb{R}^k \times (S^1)^l$. If $l = 0$ we call the fibration q very nice. Note that a very nice fibration is not a vector bundle, in general. There is no distinguished zero section.
2. M is simply connected if and only if N is simply connected and q is very nice.
3. The fibres of q are foliated by the null spaces of the restriction of the metric to the fibre. The leaves are totally geodesic subspaces of dimension $n = \dim N$. If the leaves are closed we call q nice. Of course, very nice implies nice.
4. We look at the cotangent bundle T^*N as bundle of abelian groups. There is a natural action σ of the bundle T^*N on the bundle $q : M \rightarrow N$ by translations on the fibres. Its orbits are precisely the null leaves described in 3.
5. If q is nice, then the space of null leaves (= the orbit space of the T^*N -action) is itself an affine symmetric space which is fibred over N . We obtain a factorisation $q = s \circ r$, where $r : M \rightarrow P$ and $s : P \rightarrow N$ are affine maps which are fibre bundles with flat affine symmetric fibres. Moreover,

the fibres of s come with a non-degenerate metric. They are quotients of pseudo-Euclidean spaces by a (often trivial) discrete group of translations.

One should observe the analogy to the properties of quadratic extensions. We summarize the structure of a nice special affine fibration by the following diagram.



Using Lemma 6.3 it can be shown that the canonical fibration of every indecomposable symmetric space M is nice (in fact, the absence of flat local factors that are not global factors is sufficient). Thus any such space M carries a canonical structure of the kind indicated by the diagram. The data $(\mathfrak{l}, \theta_{\mathfrak{l}}, \mathfrak{a}, [\alpha, \gamma])$ appearing in the classification scheme Thm. 3.1 should be regarded as a complete set of (local) invariants describing this structure. It would be an interesting project to work out the precise geometric meaning of each of these invariants.

We conclude this section with a certain converse of Proposition 3.8 saying that all special affine fibrations come from quadratic extensions. Let us denote the special affine fibration constructed in Proposition 3.8 by $q(M, \mathfrak{i}, i, p)$.

Proposition 3.10. *Let $q : M \rightarrow N$ be a special affine fibration. Then there exists a structure of a quadratic extension $(\mathfrak{g}, \mathfrak{i}, i, p)$ on the symmetric triple $(\mathfrak{g}, \theta, \langle \cdot, \cdot \rangle)$ of M such that $q = q(M, \mathfrak{i}, i, p)$.*

We remark that the quadratic extension $(\mathfrak{g}, \mathfrak{i}, i, p)$ is not completely determined by q . What is uniquely determined is $\mathfrak{i}_- \subset \mathfrak{g}_-$. Indeed, if we identify $\mathfrak{g}_- = T_{x_0}M$, then $(\mathfrak{i}_-)^{\perp} \subset \mathfrak{g}_-$ has to be the tangent space to the fibre of q . We then have to choose $\mathfrak{i}_+ \subset \mathfrak{g}_+$ subject to the conditions

- (a) $\mathfrak{i} = \mathfrak{i}_+ \oplus \mathfrak{i}_- \subset \mathfrak{g}$ is an isotropic ideal,
- (b) $\mathfrak{i}^{\perp}/\mathfrak{i}$ is abelian.

The proposition says that such a choice is always possible. Indeed, one can show that

$$\mathfrak{i}_+ := \{X \in [\mathfrak{g}_-, (\mathfrak{i}_-)^{\perp}] \mid [X, (\mathfrak{i}_-)^{\perp}] = 0\} .$$

always satisfies (a) and (b). Nevertheless, \mathfrak{i}_+ is not uniquely determined by \mathfrak{i}_- , (a), and (b), in general.

3.3. Symmetric spaces of index one and two. In this section we will comment on some classification results for symmetric triples of small index. First we want to reformulate the classification of indecomposable non-semisimple Lorentzian symmetric triples by Cahen and Wallach [17] in terms of quadratic

extensions. Indeed, this result follows easily from Theorem 3.1. To see this we just have to check which elements of (5) correspond to Lorentzian symmetric triples. Clearly, $\mathfrak{d}_{\alpha,\gamma}(\mathfrak{l}, \theta_{\mathfrak{l}}, \mathfrak{a})$ is Lorentzian if and only if $a + \dim \mathfrak{l} = 1$, where a is the index of $\langle \cdot, \cdot \rangle_{\mathfrak{a}}$ restricted to \mathfrak{a}_- . Hence, either $\mathfrak{l} = 0$ and $\mathfrak{a} = \mathfrak{a}_- = \mathbb{R}^{1,0}$ or $(\mathfrak{l}, \theta_{\mathfrak{l}}) = (\mathbb{R}, -\text{id})$ and $\langle \cdot, \cdot \rangle_{\mathfrak{a}}$ restricted to \mathfrak{a}_- is positive definite. The first case is trivial. Let us consider the second one, i.e. suppose $(\mathfrak{l}, \theta_{\mathfrak{l}}) = (\mathbb{R}, -\text{id})$. Take $\mathfrak{a} = \mathbb{R}^{p,p} \oplus \mathbb{R}^{0,2q}$, $p, q \geq 0$. Let e_1, \dots, e_{2p} be the standard basis of $\mathbb{R}^{p,p}$ and let e'_1, \dots, e'_{2q} be the standard basis of $\mathbb{R}^{0,2q}$. We define an involution $\theta_{\mathfrak{a}}$ on \mathfrak{a} by $\mathfrak{a}_+ = \text{span}\{e_1, \dots, e_p, e'_1, \dots, e'_q\}$ and $\mathfrak{a}_- = \text{span}\{e_{p+1}, \dots, e_{2p}, e'_{q+1}, \dots, e'_{2q}\}$. For $\lambda = (\lambda^1, \dots, \lambda^p) \in (\mathfrak{l}^*)^p \cong \mathbb{R}^p$ and $\mu = (\mu^1, \dots, \mu^q) \in (\mathfrak{l}^*)^q \cong \mathbb{R}^q$ we define a representation $\rho_{\lambda,\mu}$ of \mathfrak{l} on \mathfrak{a} by

$$\rho_{\lambda,\mu}(L)(e_i) = \lambda^i(L)e_{i+p}, \quad \rho_{\lambda,\mu}(L)(e_{i+p}) = \lambda^i(L)e_i, \quad (7)$$

$$\rho_{\lambda,\mu}(L)(e'_j) = \mu^j(L)e'_{j+q}, \quad \rho_{\lambda,\mu}(L)(e'_{j+q}) = -\mu^j(L)e'_j \quad (8)$$

for $L \in \mathfrak{l}$, $i = 1, \dots, p$ and $j = 1, \dots, q$. Then $\mathfrak{a}_{\lambda,\mu} := (\rho_{\lambda,\mu}, \mathfrak{a})$ is an orthogonal $(\mathfrak{l}, \theta_{\mathfrak{l}})$ -module and we can define

$$\mathfrak{d}(p, q, \lambda, \mu) := \mathfrak{d}_{0,0}(\mathfrak{l}, \theta_{\mathfrak{l}}, \mathfrak{a}_{\lambda,\mu}).$$

It is not hard to prove that every semisimple orthogonal $(\mathbb{R}, -\text{id})$ -module for which the -1 -eigenspace of the involution is positive definite is of the kind defined above. Since \mathfrak{l} is one-dimensional we have $\mathcal{H}_Q^2(\mathfrak{l}, \theta_{\mathfrak{l}}, \mathfrak{a}) = \{(0, 0)\}$ and indecomposability and admissibility conditions are easy to handle. Now Theorem 3.1 gives:

Theorem 3.2 (Cahen/Wallach [17]). *Every indecomposable non-semisimple Lorentzian symmetric triple is either one-dimensional or isomorphic to exactly one of the symmetric triples $\mathfrak{d}(p, q, \lambda, \mu)$, $p, q \geq 0$, $p + q > 0$, $(\lambda, \mu) \in M_{p,q}$, where*

$$M_{p,q} := \left\{ (\lambda, \mu) \in \mathbb{R}^p \oplus \mathbb{R}^q \left| \begin{array}{l} 0 < \lambda^1 \leq \lambda^2 \leq \dots \leq \lambda^p, \quad 0 < \mu^1 \leq \mu^2 \leq \dots \leq \mu^q, \\ \lambda^1 = 1 \text{ if } p > 0, \quad \mu^1 = 1 \text{ else} \end{array} \right. \right\}.$$

Cahen and Wallach [17] constructed explicit models for all simply connected Lorentzian symmetric spaces. Let us describe the simply connected Lorentzian symmetric space M associated with the symmetric triple $\mathfrak{d} := \mathfrak{d}(p, q, \lambda, \mu)$. Since \mathfrak{d} is isomorphic to the semidirect product of a Heisenberg algebra by \mathbb{R} it is not hard to see that the simply connected group G with Lie algebra \mathfrak{d} is isomorphic to $\mathfrak{d} = \mathfrak{l}^* \oplus \mathfrak{a} \oplus \mathfrak{l}$ with group multiplication

$$(Z, A, L)(\bar{Z}, \bar{A}, \bar{L}) = (Z + \bar{Z} + \frac{1}{2} [e^{-\text{ad}\bar{L}}(A), \bar{A}], e^{-\text{ad}\bar{L}}(A) + \bar{A}, L + \bar{L}).$$

Here $[\cdot, \cdot]$ and ad are the operations in \mathfrak{d} . The analytic subgroup G_+ of G with Lie algebra \mathfrak{d}_+ then equals \mathfrak{a}_+ . The projection $G \rightarrow G/G_+$ has the global section

$$s : G/G_+ \longrightarrow G, \quad (Z, A_+ + A_-, L) \cdot G_+ \longmapsto (Z + \frac{1}{2}[A_+, A_-], A_-, L),$$

where $A_+ \in \mathfrak{a}_+$ and $A_- \in \mathfrak{a}_-$. In particular, we can identify G/G_+ with \mathfrak{d}_- (as a set). Let $(z, a_1, \dots, a_p, a'_1, \dots, a'_q, l)$ denote the coordinates of a vector in

$\mathfrak{d}_- \cong \mathfrak{l}^* \oplus \mathfrak{a}_- \oplus \mathfrak{l} \cong \mathbb{R} \oplus \mathbb{R}^{p+q} \oplus \mathbb{R}$. In these coordinates, the metric on G/G_+ determined by the symmetric triple \mathfrak{d} is given by

$$2dzdl + \sum_{i=1}^p da_i^2 + \sum_{j=1}^q da'_j{}^2 + \left(\sum_{i=1}^p \lambda_i^2 a_i^2 - \sum_{j=1}^q \mu_j^2 a'_j{}^2 \right) dl^2.$$

Now let us turn to symmetric triples of index two. First classification results for this case were already obtained by Cahen and Parker in [15] and [16]. In [15] symmetric spaces of index two with solvable transvection group were studied. However, the classification presented there turned out to be incomplete. In his diploma thesis [46] Th. Neukirchner elaborated the claimed results, found the gaps and gave a revised classification of indecomposable symmetric spaces of index two that have a solvable transvection group. We tried to reproduce this result using our classification scheme and observed that also Neukirchner's classification is not quite correct. Besides minor errors a series of spaces is missing and some of the normal forms contain too much parameters. In [36] we give the corrected classification result. We don't want to recall this result to its full extent here. However, let us give a rough classification of indecomposable non-semisimple symmetric triples of signature $(2, n)$, which shows how our classification scheme works.

Following Thm. 3.1 we have first to determine all proper \mathbb{Z}_2 -equivariant Lie algebras $(\mathfrak{l}, \theta_{\mathfrak{l}})$ for which $\dim \mathfrak{l}_- \leq 2$ holds. For \mathfrak{l} besides $\mathbb{R}, \mathbb{R}^2, \mathfrak{su}(2), \mathfrak{sl}(2, \mathbb{R})$ and $\mathfrak{h}(1)$ the following Lie algebras appear:

$$\mathfrak{n}(2) = \{[X, Y] = Z, [X, Z] = -Y\}, \quad \mathfrak{r}_{3,-1} = \{[X, Y] = Z, [X, Z] = Y\}.$$

The involutions $\theta_{\mathfrak{l}}$ are described in Table 1. The table also lists the associated simply connected affine symmetric spaces $N = L/L_+$ for all $(\mathfrak{l}, \theta_{\mathfrak{l}})$ (see Section 3.2 for the geometric meaning of N). The Lie algebra $\mathfrak{n}(2)$ is isomorphic to $\mathfrak{so}(2) \times \mathbb{R}^2$, $(\mathfrak{n}(2), \theta_{\mathfrak{l}})$ is a symmetric pair and the associated simply-connected affine symmetric space equals the universal covering $\widetilde{\mathcal{L}}(2)$ of the space of (affine) lines in \mathbb{R}^2 . Analogously, $\mathfrak{r}_{3,-1}$ is isomorphic to $\mathfrak{so}(1, 1) \times \mathbb{R}^{1,1}$, $(\mathfrak{r}_{3,-1}, \theta_{\mathfrak{l}})$ is also a symmetric pair and the associated simply-connected affine symmetric space equals the space $\mathcal{L}(1, 1)$ of time-like (affine) lines in $\mathbb{R}^{1,1}$. Note that we have two non-conjugate involutions on $\mathfrak{sl}(2, \mathbb{R})$. The two associated simply-connected symmetric spaces are the hyperbolic plane H^2 and the universal covering $\widetilde{S}^{1,1}$ of the unit sphere $S^{1,1} := \{x \in \mathbb{R}^{1,2} \mid \langle x, x \rangle_{1,2} = 1\}$ in $\mathbb{R}^{1,2}$.

For all these $(\mathfrak{l}, \theta_{\mathfrak{l}})$ we have to determine a set $\mathcal{A}_{\mathfrak{l}, \theta_{\mathfrak{l}}}$ as described in Thm. 3.1. More exactly, since we want to classify only indecomposable symmetric spaces of signature $(2, n)$ we may restrict ourselves to the subset $\mathcal{A}_{\mathfrak{l}, \theta_{\mathfrak{l}}}^n \subset \mathcal{A}_{\mathfrak{l}, \theta_{\mathfrak{l}}}$ of all $\mathfrak{a} \in \mathcal{A}_{\mathfrak{l}, \theta_{\mathfrak{l}}}$ for which $\mathcal{H}_{\mathbb{Q}}^2(\mathfrak{l}, \theta_{\mathfrak{l}}, \mathfrak{a})_0$ is not empty and for which $(2, n) = \text{sign } \mathfrak{a}_- + (\dim \mathfrak{l}_-, \dim \mathfrak{l}_-)$. For all $(\mathfrak{l}, \theta_{\mathfrak{l}})$ listed above $\mathcal{A}_{\mathfrak{l}, \theta_{\mathfrak{l}}}^n$ consists of finitely many families $(\rho_1, \mathfrak{a}_1), \dots, (\rho_k, \mathfrak{a}_k)$ of $(\mathfrak{l}, \theta_{\mathfrak{l}})$ -representations, where for each family (ρ_j, \mathfrak{a}_j) the space \mathfrak{a}_j is fixed and ρ_j depends in a certain sense on d_{ρ_j} continuous parameters for some $d_{\rho_j} \in \mathbb{N}_0$. We list the spaces $\mathfrak{a}_1, \dots, \mathfrak{a}_k$ explicitly in the table, where we use the following notation: $\mathfrak{a}_+^{p,q} := (\mathbb{R}^{p,q}, \text{id})$, $\mathfrak{a}_-^{p,q} := (\mathbb{R}^{p,q}, -\text{id})$. For the families ρ_1, \dots, ρ_k we omit

a detailed description, we only give the number d_{ρ_j} of parameters for each ρ_j . Next we have to compute $\mathcal{H}_Q^2(\mathfrak{l}, \theta_l, \mathfrak{a}_j)/G_{\mathfrak{l}, \theta_l, \mathfrak{a}_j}$ for each element of the family (ρ_j, \mathfrak{a}_j) , $j = 1, \dots, k$. We do not give the result in the table, either. However, we give the number $d_{\mathcal{H}}$ of continuous parameters on which $\mathcal{H}_Q^2(\mathfrak{l}, \theta_l, \mathfrak{a}_j)/G_{\mathfrak{l}, \theta_l, \mathfrak{a}_j}$ depends for a generic element in the family (ρ_j, \mathfrak{a}_j) . Here $d_\rho = 0$ or $d_{\mathcal{H}} = 0$ means that the corresponding set is discrete (in fact, it is finite). Note that the values of d_ρ and $d_{\mathcal{H}}$ for $N = \mathbb{R}^2$ in the table are not correct for small n , namely for those n for which the value of d_ρ given in the table would be negative. We don't consider these special cases in the table.

In order to not present only vague data here let us study one case in more detail. We consider the case, where \mathfrak{l} is the Heisenberg algebra $\mathfrak{h}(1) = \{[X, Y] = Z\}$. In Example 2.14 we computed $\mathcal{H}_Q^2(\mathfrak{l}, \mathfrak{a})_b$ for $\mathfrak{l} = \mathfrak{h}(1)$ and any semisimple orthogonal $\mathfrak{h}(1)$ -modul \mathfrak{a} . We identified $\mathcal{H}_Q^2(\mathfrak{l}, \mathfrak{a})_b$ with a certain subset $Z_{\mathfrak{l}, b}$ of $C^2(\mathfrak{l}, \mathfrak{a})$. Now let \mathfrak{a} be a semisimple orthogonal (\mathfrak{l}, θ_l) -module for θ_l given by $\mathfrak{l}_+ = \mathbb{R} \cdot Z$, $\mathfrak{l}_- = \text{span}\{X, Y\}$. Then $Z_{\mathfrak{l}, b}$ is invariant under the morphism of pairs (θ_l, θ_a) . Using Prop. 2.12 we see that $\mathcal{H}_Q^2(\mathfrak{l}, \theta_l, \mathfrak{a})_b$ corresponds bijectively to $\{\alpha \in Z_{\mathfrak{l}, b} \mid \alpha(Z, \mathfrak{l}) \subset \mathfrak{a}_-^{\mathfrak{l}}\}$. Furthermore, since \mathfrak{l} is indecomposable, $\mathcal{H}_Q^2(\mathfrak{l}, \theta_l, \mathfrak{a})_0$ corresponds bijectively to $\{\alpha \in Z_{\mathfrak{l}, b} \mid \alpha(Z, \mathfrak{l}) = \mathfrak{a}_-^{\mathfrak{l}}\}$. Summarising we see that

$$\begin{aligned} Z_{\mathfrak{l}, 0} &:= \{\alpha \in C^2(\mathfrak{l}, \mathfrak{a}) \mid \alpha(X, Y) = 0, \alpha(Z, \mathfrak{l}) = \mathfrak{a}_-^{\mathfrak{l}} \neq 0\} \longrightarrow \mathcal{H}_Q^2(\mathfrak{l}, \theta_l, \mathfrak{a})_0 \quad (9) \\ \alpha &\longmapsto [\alpha, 0] \end{aligned}$$

is a bijection. Let us now determine a suitable set $\mathcal{A}_{\mathfrak{l}, \theta_l}^n$. Let (ρ, \mathfrak{a}) be an orthogonal (\mathfrak{l}, θ_l) -module such that $\mathcal{H}_Q^2(\mathfrak{l}, \theta_l, \mathfrak{a})_0 \neq \emptyset$. In particular, ρ is semisimple, which implies $\rho(Z) = 0$ since $R(\mathfrak{l}) = \mathbb{R} \cdot Z$. Hence, ρ can be considered as a semisimple representation of the abelian Lie algebra \mathfrak{l}_- , which is determined by its weights. Moreover, we know from (9) that $\mathfrak{a}^{\mathfrak{l}} = \mathfrak{a}_-^{0,1}$ or $\mathfrak{a}^{\mathfrak{l}} = \mathfrak{a}_-^{0,2}$. For $\lambda = (\lambda^1, \dots, \lambda^p) \in (\mathfrak{l}_-^* \setminus 0)^p$ and $\mu = (\mu^1, \dots, \mu^q) \in (\mathfrak{l}_-^* \setminus 0)^q$ we define a representation $\rho_{\lambda, \mu}$ of \mathfrak{l}_- on $\mathfrak{a}_+^{p,q} \oplus \mathfrak{a}_-^{0,p+q}$ by (7) and (8), where we identify $\mathfrak{a}_+^{p,q} \oplus \mathfrak{a}_-^{0,p+q}$ with $\mathbb{R}^{p,p} \oplus \mathbb{R}^{0,2q}$. Now we define

$$\begin{aligned} \mathfrak{a}_{1, \lambda \mu} &:= (\rho_{\lambda, \mu} \oplus \rho_0, \mathfrak{a}_+^{p,q} \oplus \mathfrak{a}_-^{0,p+q} \oplus \mathfrak{a}_-^{0,1}), \\ \mathfrak{a}_{2, \lambda \mu} &:= (\rho_{\lambda, \mu} \oplus \rho_0, \mathfrak{a}_+^{p,q} \oplus \mathfrak{a}_-^{0,p+q} \oplus \mathfrak{a}_-^{0,2}), \end{aligned}$$

where ρ_0 denotes the trivial representation of dimension one and two, respectively. It is easy to see that (ρ, \mathfrak{a}) is equivalent to one of these representations. Next we have to decide which of these representations are isomorphic as objects of $M_{\mathfrak{l}, \theta_l}^{ss}$. The automorphism group of (\mathfrak{l}, θ_l) equals

$$\text{Aut}(\mathfrak{l}, \theta_l) = \left\{ \begin{pmatrix} A & 0 \\ 0 & u \end{pmatrix} \mid A \in GL(2, \mathbb{R}), \det A = u \right\},$$

where the automorphisms are written with respect to the basis X, Y, Z . Let \mathfrak{S}_p denote the symmetric group of degree p . We define an action of $(\mathfrak{S}_p \times (\mathbb{Z}_2)^p) \times GL(2, \mathbb{R})$ on $(\mathfrak{l}_-^* \setminus 0)^p$ by $(\lambda_1, \dots, \lambda_p) \cdot \sigma \cdot \varepsilon \cdot A = (\varepsilon_1 A^* \lambda_{\sigma(1)}, \dots, \varepsilon_p A^* \lambda_{\sigma(p)})$ for

N	$\mathfrak{l}, \mathcal{A}_{\mathfrak{l}, \theta}^n$	d_ρ	$d_{\mathcal{H}}$
\mathbb{R}^1	$\mathfrak{l} = \mathfrak{l}_- = \mathbb{R}$ 1. $(\rho_{1,p}, \mathfrak{a}_+^{p,n-p} \oplus \mathfrak{a}_-^{1,n-1}), p = 0, \dots, n-1$ 2. $(\rho_{2,p}, \mathfrak{a}_+^{p,n-p} \oplus \mathfrak{a}_-^{1,n-1}), p = 1, \dots, n$ 3. $(\rho_{3,p}, \mathfrak{a}_+^{p,n-p} \oplus \mathfrak{a}_-^{1,n-1}), p = 1, \dots, n-1$	$n-1$ $n-1$ $n-1$	0 0 0
\mathbb{R}^2	$\mathfrak{l} = \mathfrak{l}_- = \mathbb{R}^2$ 1. $(\rho_{1,p}, \mathfrak{a}_+^{p,n-2-p} \oplus \mathfrak{a}_-^{0,n-2}), p = 0, \dots, n-2$ (occurs only for $n \geq 5$) 2. $(\rho_{2,p}, \mathfrak{a}_+^{p,n-1-p} \oplus \mathfrak{a}_-^{0,n-2}), p = 0, \dots, n-2$ 3. $(\rho_{3,p}, \mathfrak{a}_+^{p,n-1-p} \oplus \mathfrak{a}_-^{0,n-2}), p = 1, \dots, n-1$ $\mathfrak{l} = \mathfrak{h}(1) = \{[X, Y] = Z\}, \mathfrak{l}_+ = \mathbb{R} \cdot Z, \mathfrak{l}_- = \text{span}\{X, Y\}$ 1. $(\rho_{1,p}, \mathfrak{a}_+^{p,n-3-p} \oplus \mathfrak{a}_-^{0,n-2}), p = 0, \dots, n-3$ 2. $(\rho_{2,p}, \mathfrak{a}_+^{p,n-4-p} \oplus \mathfrak{a}_-^{0,n-2}), p = 0, \dots, n-4$	$2n-8$ $2n-8$ $2n-8$ $2n-10$ $2n-12$	0 1 1 2 3
$\widetilde{\mathcal{L}(2)}$	$\mathfrak{l} = \mathfrak{n}(2), \mathfrak{l}_+ = \mathbb{R} \cdot Z, \mathfrak{l}_- = \text{span}\{X, Y\}$ 1. $(\rho_{1,p}, \mathfrak{a}_+^{p,n-2-p} \oplus \mathfrak{a}_-^{0,n-2}), p = 0, \dots, n-2$ 2. $(\rho_{2,p}, \mathfrak{a}_+^{p,n-3-p} \oplus \mathfrak{a}_-^{0,n-2}), p = 0, \dots, n-3$ 3. $(\rho_{3,p}, \mathfrak{a}_+^{p,n-2-p} \oplus \mathfrak{a}_-^{0,n-2}), p = 0, \dots, n-3$ 4. $(\rho_{4,p}, \mathfrak{a}_+^{p,n-3-p} \oplus \mathfrak{a}_-^{0,n-2}), p = 0, \dots, n-4$	$n-2$ $n-3$ $n-3$ $n-4$	0 0 0 1
$\mathcal{L}(1,1)$	$\mathfrak{l} = \mathfrak{t}_{3,-1}, \mathfrak{l}_+ = \mathbb{R} \cdot Z, \mathfrak{l}_- = \text{span}\{X, Y\}$ 1. $(\rho_{1,p}, \mathfrak{a}_+^{p,n-2-p} \oplus \mathfrak{a}_-^{0,n-2}), p = 0, \dots, n-2$ 2. $(\rho_{2,p}, \mathfrak{a}_+^{p,n-3-p} \oplus \mathfrak{a}_-^{0,n-2}), p = 0, \dots, n-3$ 3. $(\rho_{3,p}, \mathfrak{a}_+^{p,n-2-p} \oplus \mathfrak{a}_-^{0,n-2}), p = 0, \dots, n-3$ 4. $(\rho_{4,p}, \mathfrak{a}_+^{p,n-3-p} \oplus \mathfrak{a}_-^{0,n-2}), p = 0, \dots, n-4$	$n-2$ $n-3$ $n-3$ $n-4$	0 0 0 1
$\widetilde{S^{1,1}}$	$\mathfrak{l} = \mathfrak{sl}(2, \mathbb{R}) = \{[H, X] = 2Y, [H, Y] = 2X,$ $[X, Y] = 2H\}, \mathfrak{l}_+ = \mathbb{R} \cdot H, \mathfrak{l}_- = \text{span}\{X, Y\}$ $(\rho, \mathfrak{a}_+^{n-2, n-2} \oplus \mathfrak{a}_-^{0, n-2})$	0	1
H^2	$\mathfrak{l} = \mathfrak{sl}(2, \mathbb{R}), \mathfrak{l}_+ = \mathbb{R} \cdot X, \mathfrak{l}_- = \text{span}\{H, Y\}$ $(\rho_{klm}^{pqr}, \mathfrak{a}_+^{n-2+p-q, 0} \oplus \mathfrak{a}_-^{0, n-2}), k \in \mathbb{N}^p, l \in \mathbb{N}^q, m \in \mathbb{N}^r,$ $k_1 \leq k_2 \leq \dots \leq k_p, l_1 \leq \dots \leq l_q, m_1 \leq \dots \leq m_r$ $ k + l + 2 m + q = n - 2$	0	1
S^2	$\mathfrak{l} = \mathfrak{su}(2) = \{[H, X] = 2Y, [H, Y] = -2X,$ $[X, Y] = 2H\}, \mathfrak{l}_+ = \mathbb{R} \cdot H, \mathfrak{l}_- = \text{span}\{X, Y\}$ $(\rho_{klm}^{pqr}, \mathfrak{a}_+^{0, n-2+p-q} \oplus \mathfrak{a}_-^{0, n-2}),$ p, q, r, k, l, m as in the previous entry	0	1

Table 1. Non-semisimple indecomposable symmetric triples of signature $(2, n)$

$(\lambda_1, \dots, \lambda_p) \in (\mathfrak{l}^* \setminus 0)^p$, $\sigma \in \mathfrak{S}_p$, $(\varepsilon_1, \dots, \varepsilon_p) \in (\mathbb{Z}_2)^p$ and $A \in GL(2, \mathbb{R})$. We define the sets

$$\Lambda_p := (\mathfrak{l}^* \setminus 0)^p / (\mathfrak{S}_p \times (\mathbb{Z}_2)^p), \quad \Lambda_{p,q} = \Lambda_p \times \Lambda_q.$$

Denoting the isomorphism relation in $M_{\mathfrak{l}, \theta_{\mathfrak{l}}}^{ss}$ by \cong we obtain $\mathfrak{a}_{1, \lambda \mu} \not\cong \mathfrak{a}_{2, \lambda' \mu'}$ and

$$\mathfrak{a}_{i, \lambda \mu} \cong \mathfrak{a}_{i, \lambda' \mu'} \Leftrightarrow p = p', q = q' \text{ and } [\lambda, \mu] = [\lambda', \mu'] \in \Lambda_{p,q} / GL(2, \mathbb{R})$$

for $i = 1, 2$. Hence, the elements of the following $2n - 5$ families of $(\mathfrak{l}, \theta_{\mathfrak{l}})$ -modules constitute a suitable set $\mathcal{A}_{\mathfrak{l}, \theta_{\mathfrak{l}}}^n$:

$$\begin{aligned} (\rho_{1,p}, \mathfrak{a}_+^{p, n-3-p} \oplus \mathfrak{a}_-^{0, n-2}) &:= \{ \mathfrak{a}_{1, \lambda \mu} \mid [\lambda, \mu] \in \Lambda_{p, n-3-p} / GL(2, \mathbb{R}) \}, \\ (\rho_{2, \tilde{p}}, \mathfrak{a}_+^{\tilde{p}, n-4-\tilde{p}} \oplus \mathfrak{a}_-^{0, n-2}) &:= \{ \mathfrak{a}_{2, \lambda \mu} \mid [\lambda, \mu] \in \Lambda_{\tilde{p}, n-4-\tilde{p}} / GL(2, \mathbb{R}) \}, \end{aligned}$$

where $p = 0, \dots, n-3$, $\tilde{p} = 0, \dots, n-4$. In particular, for each family $\rho_{1,p}$, $p = 0, \dots, n-3$, we have $d_{\rho} = 2n - 10$ if $n \geq 5$ and $d_{\rho} = 0$ for $n = 3, 4$. Similarly, for each $\rho_{2, \tilde{p}}$, $\tilde{p} = 0, \dots, n-4$, we have $d_{\rho} = 2n - 12$ if $n \geq 6$ and $d_{\rho} = 0$ for $n = 4, 5$.

Finally, let us determine $\mathcal{H}_Q^2(\mathfrak{l}, \theta_{\mathfrak{l}}, \mathfrak{a}) / G_{\mathfrak{l}, \theta_{\mathfrak{l}}, \mathfrak{a}}$ for $\mathfrak{a} \in \mathcal{A}_{\mathfrak{l}, \theta_{\mathfrak{l}}}^n$. Take $\mathfrak{a} = \mathfrak{a}_{1, \lambda \mu}$. If $n \geq 5$, then in the generic case $(\lambda(X), \mu(X)) \in \mathbb{R}^{n-3}$ and $(\lambda(Y), \mu(Y)) \in \mathbb{R}^{n-3}$ are linearly independent and we get $\mathcal{H}_Q^2(\mathfrak{l}, \theta_{\mathfrak{l}}, \mathfrak{a}) / G_{\mathfrak{l}, \theta_{\mathfrak{l}}, \mathfrak{a}} \cong Z_{1,0} / \mathbb{Z}_2$. Hence, $d_{\mathcal{H}} = 2$. In the non-generic case $G_{\mathfrak{l}, \theta_{\mathfrak{l}}, \mathfrak{a}}$ becomes larger and we get $d_{\mathcal{H}} = 0$. For $n = 3, 4$ we have $d_{\mathcal{H}} = 0$. Now take $\mathfrak{a} = \mathfrak{a}_{2, \lambda \mu}$. If $n \geq 6$, then in the generic case $(\lambda(X), \mu(X)) \in \mathbb{R}^{n-4}$ and $(\lambda(Y), \mu(Y)) \in \mathbb{R}^{n-4}$ are linearly independent and we have $\mathcal{H}_Q^2(\mathfrak{l}, \theta_{\mathfrak{l}}, \mathfrak{a}) / G_{\mathfrak{l}, \theta_{\mathfrak{l}}, \mathfrak{a}} \cong Z_{1,0} / O(2)$. Hence, $d_{\mathcal{H}} = 3$. In the non-generic case we get $d_{\mathcal{H}} = 1$. For $n = 5$ we have $d_{\mathcal{H}} = 1$ and for $n = 4$ we get $d_{\mathcal{H}} = 0$.

4. Special geometric structures on symmetric spaces

4.1. Examples of geometric structures. We are now going to discuss pseudo-Riemannian symmetric spaces that are equipped with certain geometric structures coming from complex and quaternionic geometry.

Let (M, g) be a pseudo-Riemannian manifold. The Levi-Civita connection induces a connection on the bundle $\mathfrak{so}(TM) \subset \text{End}(TM)$ of endomorphisms that are skew-symmetric w.r.t. g . A Kähler structure on (M, g) is a parallel section I of $\mathfrak{so}(TM)$ satisfying $I^2 = -\text{id}_{TM}$. In particular, I is an integrable almost complex structure, and thus (M, I) is a complex manifold. A *pseudo-Hermitian symmetric space* is a tuple (M, g, I) , where (M, g) is a pseudo-Riemannian symmetric space and I is a Kähler structure on (M, g) . A hyper-Kähler structure on (M, g) is a pair of Kähler structures (I, J) satisfying $IJ = -JI$. A quaternionic Kähler structure on M arises if we weaken the parallelity conditions on I, J : it consists of a 3-dimensional parallel subbundle $E \subset \mathfrak{so}(TM)$ that can be locally spanned by almost complex structures I, J , and $K := IJ = -JI$. We have the corresponding

notions of a *hyper-Kähler symmetric space* (M, g, I, J) and a *quaternionic Kähler symmetric space* (M, g, E) . In particular, hyper-Kähler symmetric spaces form a subclass of all quaternionic Kähler symmetric spaces.

In the pseudo-Riemannian world all these structures have their “para”-versions. If we replace the condition $I^2 = -\text{id}_{TM}$ for a Kähler structure by $I^2 = \text{id}_{TM}$ we are led to the notion of a para-Kähler structure. A para-Kähler structure on (M, g) is equivalent to a parallel splitting $TM = TM_+ \oplus TM_-$ into totally isotropic subbundles. Thus para-Kähler structures can exist for metrics of neutral signature (m, m) , only. A pair (I, J) , where I is a Kähler structure and J is a para-Kähler structure such that $IJ = -JI$, is called a hypersymplectic structure (sometimes also para-hyper-Kähler structure). Note that then $K := IJ = -JI$ is a second para-Kähler structure on M . A parallel subbundle $E \subset \mathfrak{so}(TM)$ locally spanned by (not necessarily parallel) sections I, J, K of this kind is called a para-quaternionic Kähler structure. There are the corresponding notions of *para-Hermitian*, *hypersymplectic*, and *para-quaternionic Kähler symmetric spaces*.

Recall the notion of an (\mathfrak{h}, K) -module (V, Φ_V) from Section 2.2. As usual, (V, Φ_V) is called irreducible if it has no proper submodules. Let $\widehat{(\mathfrak{h}, K)}$ be the set of equivalence classes of irreducible (\mathfrak{h}, K) -modules. Let (V, Φ_V) be arbitrary and fix $\pi \in \widehat{(\mathfrak{h}, K)}$. The π -isotypic component $V(\pi) \subset V$ is the sum of all irreducible submodules of V belonging to the equivalence class π . If (V, Φ_V) is semisimple, then

$$V = \bigoplus_{\pi \in \widehat{(\mathfrak{h}, K)}} V(\pi).$$

In particular, we can consider V as a $\widehat{(\mathfrak{h}, K)}$ -graded vector space.

Definition 4.1. Let $\Pi \subset \widehat{(\mathfrak{h}, K)}$. An (\mathfrak{h}, K) -module (V, Φ_V) is called Π -graded, if it is semisimple and $V(\pi) = 0$ for all $\pi \notin \Pi$.

Looking at the underlying (\mathfrak{h}, K) -module structure we can speak of Π -graded (\mathfrak{h}, K) -equivariant (metric) Lie algebras, Π -graded $(\mathfrak{l}, \Phi_{\mathfrak{l}})$ -modules, etc. In order to save words we call a Π -graded (\mathfrak{h}, K) -equivariant (metric) Lie algebra $(\mathfrak{l}, \Phi_{\mathfrak{l}})$ simply a (metric) *Lie algebra with Π -grading*. Thus, the term Π -grading stands for the whole equivariant structure $\Phi_{\mathfrak{l}}$, not only for the decomposition into isotypic components.

Let us explain the examples of Π -gradings relevant for the geometric structures discussed above. In all these cases we can take $\mathfrak{h} = 0$. Another interesting Π -grading for $(\mathfrak{h}, K) = (\mathbb{R}, \mathbb{Z}_2)$ will appear in Section 5.1.

- $K = U(1)$, $\Pi = \{1, \sigma\}$. Here 1 stands for the one-dimensional trivial representation, and σ for the standard representation of $U(1)$ on $\mathbb{C} \cong \mathbb{R}^2$. A Π -grading of this kind is called *complex grading*.
- $K = \mathbb{R}^*$, $\Pi = \{1, \sigma, \sigma^*\}$. Here σ stands for the standard representation of the multiplicative group \mathbb{R}^* on \mathbb{R}^1 , and σ^* denotes the dual of σ . The corresponding Π -grading is called *para-complex grading*.

- $K = Sp(1)$, $\Pi = \{1, \sigma\}$. Here σ stands for the standard representation of $Sp(1)$ on $\mathbb{H} \cong \mathbb{R}^4$. The corresponding Π -grading is called *quaternionic grading*.
- $K = SL(2, \mathbb{R})$, $\Pi = \{1, \sigma\}$. Here σ stands for the 2-dimensional standard representation of $SL(2, \mathbb{R})$. The corresponding Π -grading is called *para-quaternionic grading*.

In all cases above we have a natural embedding of $\mathbb{Z}_2 \cong O(1) \hookrightarrow K$ such that $\sigma(w) = -\text{id}$ for the non-trivial element $w \in \mathbb{Z}_2$. Therefore objects V with such a Π -grading are special \mathbb{Z}_2 -equivariant objects. Thus they come with a splitting $V = V_+ \oplus V_-$. We call such a Π -grading of a Lie algebra \mathfrak{g} *proper* if $[\mathfrak{g}_-, \mathfrak{g}_-] = \mathfrak{g}_+$. In particular, metric Lie algebras with a proper Π -grading of this kind are symmetric triples, which are equipped with an additional structure.

Definition 4.2. *A pseudo-Hermitian (para-Hermitian, hyper-Kähler, hypersymplectic) symmetric triple $(\mathfrak{g}, \Phi, \langle \cdot, \cdot \rangle)$ is a metric Lie algebra $(\mathfrak{g}, \langle \cdot, \cdot \rangle)$ with proper complex (para-complex, quaternionic, para-quaternionic) grading Φ .*

Then we have the following variant of Proposition 3.2.

Proposition 4.3. *There is a bijective map between isometry classes of simply connected “geostruc” symmetric spaces and isomorphism classes of “geostruc” symmetric triples, where “geostruc” stands for pseudo-Hermitian, para-Hermitian, hyper-Kähler, or hypersymplectic.*

Note that there is no analogous correspondence for (para-)quaternionic Kähler symmetric spaces. Let us explain the correspondence for hyper-Kähler symmetric spaces. The other cases are similar. Let (M, g, I, J) be a hyper-Kähler symmetric space with base point x_0 , and let $(\mathfrak{g}, \theta, \langle \cdot, \cdot \rangle)$ be the associated symmetric triple. Then $I, J, K := IJ$ span a Lie algebra $\mathfrak{k} \cong \mathfrak{sp}(1)$ which acts orthogonally on $\mathfrak{g}_- \cong T_{x_0}M$. This action commutes with the one of \mathfrak{g}_+ . We extend the \mathfrak{k} -action to \mathfrak{g} by the trivial action on \mathfrak{g}_+ . For $X, Y \in \mathfrak{g}_-, Z \in \mathfrak{g}_+$, and $Q \in \mathfrak{k}$ we compute

$$\begin{aligned} \langle [QX, Y] + [X, QY], Z \rangle &= \langle QX, [Y, Z] \rangle + \langle X, [QY, Z] \rangle \\ &= \langle QX, [Y, Z] \rangle + \langle X, Q[Y, Z] \rangle = 0. \end{aligned}$$

It follows that \mathfrak{k} acts by derivations on \mathfrak{g} . Integrating the resulting homomorphism from $\mathfrak{sp}(1)$ into antisymmetric derivations of \mathfrak{g} we obtain the desired homomorphism $\Phi : Sp(1) \rightarrow \text{Aut}(\mathfrak{g})$. Vice versa, if $(\mathfrak{g}, \Phi, \langle \cdot, \cdot \rangle)$ is a hyper-Kähler symmetric triple and (M, g) is the simply connected symmetric space with symmetric triple $(\mathfrak{g}, \Phi(-1), \langle \cdot, \cdot \rangle)$, then $I = \Phi(i)|_{\mathfrak{g}_-}$ and $J = \Phi(j)|_{\mathfrak{g}_-}$ are \mathfrak{g}_+ -invariant anticommuting complex structures on $\mathfrak{g}_- \cong T_{x_0}M$ respecting the metric. They induce a G -invariant hyper-Kähler structure on M .

It is now easy to specify the classification scheme Thm. 2.3 for (\mathfrak{h}, K) -equivariant metric Lie algebras to pseudo-Hermitian (para-Hermitian, ...) symmetric triples (compare also Thm. 3.1). We don't want to write down the complete results. We only remark that any indecomposable non-semisimple “geostruc” symmetric triple is isomorphic to some $\mathfrak{d}_{\alpha, \gamma}(l, \Phi_l, \mathfrak{a})$, where

- (O1) $(\mathfrak{l}, \Phi_{\mathfrak{l}})$ is a Lie algebra with proper Π -grading, Π choosen according to the geometric structure,
- (O2) \mathfrak{a} is a Π -graded semisimple orthogonal $(\mathfrak{l}, \Phi_{\mathfrak{l}})$ -module, and
- (O3) $[\alpha, \gamma] \in \mathcal{H}_Q^2(\mathfrak{l}, \Phi_{\mathfrak{l}}, \mathfrak{a})_b$ is indecomposable and satisfies Condition (T_2) in Proposition 3.3.

4.2. Pseudo-Hermitian symmetric spaces. We have seen in the previous section that in order to classify indecomposable pseudo-Hermitian symmetric spaces one would have to classify the objects (O1)–(O3) for $\Phi_{\mathfrak{l}}$ being a complex grading. This task is of a similar complexity as the classification of all symmetric spaces discussed in Section 3. However, there are structural restrictions coming from $U(1)$ -equivariance and making some aspects of the theory simpler then for general symmetric spaces. In addition, we will see that only very few symmetric spaces of index 2 (as listed in Section 3.3, Table 1) admit a Kähler structure.

We treat pseudo-Hermitian symmetric spaces together with para-Hermitian ones because of the similarity of their behaviour. Indeed, Kähler and para-Kähler structures could be viewed as different real forms of only one complexified structure as the common complexification of $U(1)$ and \mathbb{R}^* is \mathbb{C}^* .

Proposition 4.4. *Let $(\mathfrak{l}, \Phi_{\mathfrak{l}})$ be a Lie algebra with proper complex or para-complex grading. Then the radical $\mathfrak{r} \subset \mathfrak{l}$ is nilpotent and acts trivially on every semisimple $(\mathfrak{l}, \Phi_{\mathfrak{l}})$ -module. In particular, every solvable pseudo-Hermitian or para-Hermitian symmetric triple is nilpotent.*

Proof. Differentiating $\Phi_{\mathfrak{l}}$ we can consider $(\mathfrak{l}, \Phi_{\mathfrak{l}})$ as a \mathfrak{k} -equivariant Lie algebra, where \mathfrak{k} is the Lie algebra of $U(1)$ or \mathbb{R}^* , respectively. Properness of $(\mathfrak{l}, \Phi_{\mathfrak{l}})$ implies that $\mathfrak{l}^{\mathfrak{k}} \subset \mathfrak{l}'$. Now we apply Lemma 6.1. \square

Proposition 4.4 has also the following consequence: For fixed $(\mathfrak{l}, \Phi_{\mathfrak{l}})$, the set of isomorphism classes of (para-)complex graded semisimple orthogonal $(\mathfrak{l}, \Phi_{\mathfrak{l}})$ -modules is discrete. Indeed, such an orthogonal $(\mathfrak{l}, \Phi_{\mathfrak{l}})$ -module is essentially determined by the action of the (semisimple) Levi factor of \mathfrak{l} on it.

The smallest possible nonzero index of a pseudo-Hermitian symmetric triple is two. For this case the objects (O1)–(O3) can be classified completely. Thanks to Proposition 4.4 this is considerably simpler than to classify all symmetric triples of index 2 (compare Section 3.3).

Theorem 4.1 (compare [36], Section 7.3). *If $(\mathfrak{g}, \Phi, \langle \cdot, \cdot \rangle)$ is an indecomposable pseudo-Hermitian symmetric triple of signature $(2, 2q)$ that is neither semisimple nor abelian, then $(\mathfrak{g}, \Phi, \langle \cdot, \cdot \rangle)$ is isomorphic to $\mathfrak{d}_{\alpha, \gamma}(\mathfrak{l}, \Phi_{\mathfrak{l}}, \mathfrak{a})$ for exactly one of the data in the following list:*

1. $q = 1 : \mathfrak{l} = \mathbb{R}^2 \cong \mathbb{C}$, $\Phi_{\mathfrak{l}} = \sigma$ is the standard action of $U(1)$,
 - (a) $\mathfrak{a} = \mathbb{R}$ with the standard scalar product, $\rho, \Phi_{\mathfrak{a}}$ trivial, $\alpha(z_1, z_2) = \text{Im } \bar{z}_1 z_2$, $z_1, z_2 \in \mathbb{C}$, $\gamma = 0$;
 - (b) all data as in (a) except for the opposite sign of the scalar product;

2. $q = 2 : \mathfrak{l} = \mathfrak{h}(1)$, $\Phi_{\mathfrak{l}} = \sigma \oplus 1$ via $\mathfrak{l} \cong \mathbb{C} \oplus \mathfrak{z}(\mathfrak{l})$ as a vector space,

$\mathfrak{a} = \mathbb{C}$ with the real standard scalar product, ρ trivial, $\Phi_{\mathfrak{a}} = \sigma$,
 $\alpha(z, x) = z \cdot x$, $\alpha(z_1, z_2) = 0$, $z, z_1, z_2 \in \mathbb{C}$, $x \in \mathfrak{z}(\mathfrak{l}) \cong \mathbb{R}$, $\gamma = 0$;
3. $q = 1 + p$, $p \geq 0 : \mathfrak{l} = \mathfrak{su}(2)$, $\Phi_{\mathfrak{l}} = \text{Ad} \circ i_1$, where $i_1 : U(1) \rightarrow U(2)$ is the standard embedding into the left upper corner,

$\mathfrak{a} = (\mathfrak{a}_1)^{p-r} \oplus (\mathfrak{a}_2)^r$, $0 \leq r \leq p$, where
 $\mathfrak{a}_1 = \mathbb{C}^2$ with positive definite standard scalar product, ρ_1 is the standard representation of $\mathfrak{su}(2)$, $\Phi_{\mathfrak{a}_1} = i_1$, and
 $\mathfrak{a}_2 = \mathfrak{su}(2)$ with $-B$, B the Killing form, $\rho_2 = \text{ad}$, $\Phi_{\mathfrak{a}_2} = \text{Ad} \circ i_1$;
 $\alpha = 0$, $\gamma(X_1, X_2, X_3) = cB([X_1, X_2], X_3)$, $c \in \mathbb{R}$.
4. $q = 1 + p$, $p \geq 0 : \mathfrak{l} = \mathfrak{sl}(2, \mathbb{R})$, $\Phi_{\mathfrak{l}}(z) = \text{Ad}(i_2(\sqrt{z}))$, $z \in U(1)$, where $i_2 : U(1) \cong SO(2) \rightarrow SL(2, \mathbb{R})$ is the natural embedding,

$\mathfrak{a} = (\mathfrak{a}_1)^{p-r} \oplus (\mathfrak{a}_2)^r$, $0 \leq r \leq p$, where
 $\mathfrak{a}_1 = \mathbb{R}^2 \otimes \mathbb{C}$ with the scalar product given by $-\omega_{\mathbb{R}^2} \otimes \omega_{\mathbb{C}}$, $\omega_{\mathbb{R}^2}, \omega_{\mathbb{C}}$ being the standard symplectic forms of the factors, ρ_1 is the complexified standard representation of $\mathfrak{sl}(2, \mathbb{R})$, $\Phi_{\mathfrak{a}_1}(z) = (i_2 \otimes \sigma)(\sqrt{z})$, and
 $\mathfrak{a}_2 = \mathfrak{sl}(2, \mathbb{R})$ with the Killing form B , $\rho_2 = \text{ad}$, $\Phi_{\mathfrak{a}_2}(z) = \text{Ad}(i_2(\sqrt{z}))$;
 $\alpha = 0$, $\gamma(X_1, X_2, X_3) = cB([X_1, X_2], X_3)$, $c \in \mathbb{R}$.

The bases $N = L/L_+$ of the canonical fibrations of the simply connected pseudo-Hermitian symmetric spaces corresponding to 1.-4. are $\mathbb{C}, \mathbb{C}, S^2 \cong \mathbb{C}P^1$, and H^2 , respectively. The complex gradings of \mathfrak{l} correspond to the natural complex structures of these spaces. Note that the data defined in 4. do not depend on the choice of the square root of z . The classification of complex graded semisimple orthogonal $(\mathfrak{l}, \Phi_{\mathfrak{l}})$ -modules in cases 3. and 4. can be found already in [16], Ch. V, Prop. 3.3.

We remark that the statement of Theorem 4.1 differs slightly from the corresponding statement in [36]. In [36] we determined all symmetric triples of signature $(2, 2q)$ that admit the structure of a pseudo-Hermitian symmetric triple, whereas here we have determined isomorphism classes of pseudo-Hermitian symmetric triples. Comparing both results we find that all of these symmetric triples admit exactly one complex grading (up to isomorphism).

In a similar manner one can classify para-Hermitian symmetric triples of small index. All para-Hermitian symmetric spaces of signature $(1, 1)$ are locally isomorphic to either the flat space $R^{1,1}$ or to the one-sheeted hyperboloid $S^{1,1} \subset \mathbb{R}^{1,2}$, which is semisimple. The complexifications of the pseudo-Hermitian symmetric triples of signature $(2, 2q)$ listed in Theorem 4.1 admit real forms that are para-Hermitian symmetric triples of signature $(q+1, q+1)$. Concerning index 2, we obtain:

Proposition 4.5. *There are exactly two isolated isomorphism classes and one 1-parameter family of isomorphism classes of indecomposable para-Hermitian symmetric triples of index at most 2 that are neither semisimple nor abelian, namely $\mathfrak{d}_{\alpha, \gamma}(\mathfrak{l}, \Phi_{\mathfrak{l}}, \mathfrak{a})$ for:*

1. $\mathfrak{l} = \mathbb{R}^2 \cong \mathbb{R}^1 \oplus (\mathbb{R}^1)^*$, $\Phi_{\mathfrak{l}} = \sigma \oplus \sigma^*$, $\mathfrak{a} = \mathbb{R}$ with the standard scalar product, $\rho, \Phi_{\mathfrak{a}}$ trivial, α induced by the dual pairing, $\gamma = 0$;
2. all data as in 1. except for the opposite sign of the scalar product of \mathfrak{a} ;
3. $\mathfrak{l} = \mathfrak{sl}(2, \mathbb{R})$, $\Phi_{\mathfrak{l}} = \text{Ad} \circ i_3$, where $i_3 : \mathbb{R}^* \rightarrow GL(2, \mathbb{R})$ is the standard embedding into the left upper corner, $\mathfrak{a} = \{0\}$, $\alpha = 0$,
 $\gamma(X_1, X_2, X_3) = cB([X_1, X_2], X_3)$, $c \in \mathbb{R}$, B being the Killing form.

Note that the family in 3. corresponds to a 1-parameter family of para-Hermitian metrics on the symmetric space $T^*S^{1,1}$.

As in the cases of metric Lie algebras and general symmetric triples there seems to be no hope for a complete classification of pseudo-Hermitian and para-Hermitian triples without index restrictions. The reason is that the Lie algebra structure of nilpotent pseudo-Hermitian and para-Hermitian symmetric triples can be arbitrarily complicated. E.g., a basic invariant of a nilpotent Lie algebra \mathfrak{g} is its nilindex, which is by definition the smallest non-negative integer k such that $\mathfrak{g}^{k+1} = \{0\}$. Nilpotent Lie algebras of nilindex k are sometimes also called k -step nilpotent Lie algebras. The following series of examples shows that there are nilpotent pseudo-Hermitian symmetric triples with an arbitrary large nilindex. This is in sharp contrast to the theory of hyper-Kähler symmetric triples discussed in the next section, see Theorem 4.4.

Example 4.6. For each $m \in \mathbb{N}_0$ we define a pseudo-Hermitian symmetric triple $(\mathfrak{g}(m), \Phi, \langle \cdot, \cdot \rangle)$ as follows: As a vector space with complex grading we set

$$\mathfrak{g}(m) = \mathbb{C}^{m+1} \oplus \mathbb{R}^m, \quad \Phi = \sigma^{m+1} \oplus 1^m.$$

Let E_i , $i = 1, \dots, m+1$, Z_k , $k = 1, \dots, m$, be the standard basis vectors of \mathbb{C}^{m+1} and \mathbb{R}^m , respectively. Set $F_j := iE_j$. Then $\{E_i, F_j, Z_k\}$ is a basis of the real vector space $\mathfrak{g}(m)$. The nonzero Lie brackets between the basis vectors are defined as

$$[E_i, F_j] = Z_{i+j-1}, \quad [Z_k, E_i] = F_{i+k}, \quad [Z_k, F_j] = -E_{k+j}.$$

Here $Z_l = E_q = F_q = 0$ for $l > m$, $q > m+1$. Finally, the scalar product is given by $\mathbb{C}^{m+1} \perp \mathbb{R}^m$ and

$$\langle E_i, F_j \rangle = 0, \quad \langle E_i, E_j \rangle = \langle F_i, F_j \rangle = \delta_{i+j, m+2}, \quad \langle Z_k, Z_l \rangle = \delta_{k+l, m+1}.$$

It is easy to check that $(\mathfrak{g}(m), \Phi, \langle \cdot, \cdot \rangle)$ is indeed a pseudo-Hermitian symmetric triple. It is indecomposable and nilpotent of nilindex $2m+1$. Note that $\mathfrak{g}(1)$ and $\mathfrak{g}(2)$ appear in Theorem 4.1 under 1.(a) and 2., respectively. Observe that the holonomy algebras $\mathfrak{g}(m)_+ = \mathbb{R}^m$ are abelian. However, there exist nilpotent pseudo-Hermitian symmetric triples having even holonomy algebras of arbitrary large nilindex.

4.3. Quaternionic Kähler and hyper-Kähler symmetric spaces.

Let (M, g, E) be a pseudo-Riemannian manifold of dimension $4n > 4$ with quaternionic or para-quaternionic Kähler structure E . Then (M, g) is Einstein. We have to distinguish between two cases: If the scalar curvature of (M, g) is non-zero, then (M, g) is indecomposable and E has no nontrivial parallel section. Otherwise E can be spanned by parallel sections, i.e., (M, g) carries a hyper-Kähler or hyper-symplectic structure, respectively. For these facts we refer to [3] and [11]. The latter reference deals only with the Riemannian case, but the arguments work in the indefinite case as well. For symmetric spaces we have:

Proposition 4.7 (Alekseevsky/Cortés [3]). *Let (M, g, E) be a (para-)quaternionic Kähler symmetric space of non-zero scalar curvature. Then the transvection group G of (M, g) is simple.*

Indeed, the Ricci curvature of (M, g) is essentially given by the Killing form of the Lie algebra \mathfrak{g} of G . The Einstein property now implies that the Killing form is non-degenerate. It follows that G is semisimple. Simplicity of G follows from the indecomposability of (M, g) . Complete lists of these spaces can be found in [3] for the quaternionic and in [19] for the para-quaternionic case.

The above discussion and Proposition 4.3 reduces the classification of (para-)quaternionic symmetric spaces to the classification of hyper-Kähler (hypersymplectic) symmetric triples. First of all we have the following counterpart of Prop. 4.4 and Prop. 4.7.

Proposition 4.8 ([37], Prop. 2.1). *Let $(\mathfrak{l}, \Phi_{\mathfrak{l}})$ be a Lie algebra with proper quaternionic or para-quaternionic grading. Then \mathfrak{l} is nilpotent. In particular, every hyper-Kähler or hypersymplectic symmetric triple is nilpotent.*

Proof. We can consider $(\mathfrak{l}, \Phi_{\mathfrak{l}})$ as an $(\mathfrak{k}, \mathbb{Z}_2)$ -equivariant Lie algebra, where \mathfrak{k} is $\mathfrak{sp}(1)$ or $\mathfrak{sl}(2, \mathbb{R})$, respectively. Properness implies that this $(\mathfrak{k}, \mathbb{Z}_2)$ -equivariant Lie algebra satisfies the assumptions of Lemma 6.2, which says that \mathfrak{l} has to be nilpotent. \square

That hyper-Kähler and hypersymplectic symmetric triples are solvable has been already observed in [2]. Recently, we obtained an important sharpening of Proposition 4.8. We shall discuss it at the end of the present section, see Theorem 4.4. First we want to give an overview on the results on hyper-Kähler symmetric triples obtained by Alekseevsky, Cortés, and the authors in [2],[18], and [37].

Following our classification scheme we have to study the objects (O1)–(O3) for $\Phi_{\mathfrak{l}}$ being a quaternionic grading. Let us begin, however, with an alternative approach to hyper-Kähler symmetric triples due to Alekseevsky/Cortés [2] that provides additional information. Let (E, ω) be a complex symplectic vector space. Any $S \in S^4 E$ defines a complex linear subspace $\mathfrak{h}_S \subset \mathfrak{sp}(E, \omega) \cong S^2 E$ by

$$\mathfrak{h}_S = \text{span}\{S_{v,w} \in S^2 E \mid v, w \in E\} ,$$

where $S_{v,w}$ is the contraction of S with v and w via the symplectic form ω . If

$$S \in (S^4 E)^{\mathfrak{h}_S} , \tag{10}$$

then $\mathfrak{h}_S \subset \mathfrak{sp}(E, \omega)$ is a Lie subalgebra and, moreover, there is a natural Lie bracket on $\mathfrak{g}_S := \mathfrak{h}_S \oplus (\mathbb{H} \otimes_{\mathbb{C}} E)$ such that $\mathfrak{h}_S \subset \mathfrak{g}_S$ is a subalgebra and \mathfrak{h}_S acts on $\mathbb{H} \otimes_{\mathbb{C}} E$ by the natural action on the second factor. The remaining part of the commutator maps $\mathbb{H} \otimes_{\mathbb{C}} E \times \mathbb{H} \otimes_{\mathbb{C}} E$ to \mathfrak{h}_S as follows:

$$[p \otimes v, q \otimes w] = \omega_{\mathbb{H}}(p, q)S_{v,w}, \quad p, q \in \mathbb{H}, v, w \in E,$$

where $\omega_{\mathbb{H}}$ is the alternating complex bilinear 2-form on \mathbb{H} such that $\omega_{\mathbb{H}}(1, j) = 1$. Equation (10) is a system of quadratic equations for $S \in S^4 E$. It admits families of particularly simple solutions, namely all $S \in S^4 E_+ \subset S^4 E$, where $E_+ \subset E$ is a Lagrangian subspace. Let us call solutions of this kind tame. If S is tame, then the Lie algebra \mathfrak{h}_S is abelian.

Let J be a quaternionic structure on E such that $J^* \omega = \bar{\omega}$. Then J induces a real structure on each of the spaces $S^4 E$, $S^2 E \cong \mathfrak{sp}(E, \omega)$, and $\mathbb{H} \otimes_{\mathbb{C}} E$. We denote all these structures by the the same symbol τ . If $S \in (S^4 E)^\tau$ satisfies (10), then the real Lie algebra

$$\mathfrak{g}_{J,S} := (\mathfrak{g}_S)^\tau = (\mathfrak{h}_S)^\tau \oplus (\mathbb{H} \otimes_{\mathbb{C}} E)^\tau$$

carries a canonical structure of a hyper-Kähler symmetric triple. The $Sp(1)$ -action on $\mathfrak{g}_{J,S}$ is given by $\sigma \otimes 1$ on $(\mathbb{H} \otimes_{\mathbb{C}} E)^\tau$ and the trivial action on $(\mathfrak{h}_S)^\tau$. This construction produces all hyper-Kähler symmetric triples.

Proposition 4.9 (Aleksseevsky/Cortés [2]). *Let $(\mathfrak{g}, \Phi, \langle \cdot, \cdot \rangle)$ be a hyper-Kähler symmetric triple, then there exist data (E, ω, J, S) as above, $S \in S^4 E$ being a τ -invariant solution of (10), such that $(\mathfrak{g}, \Phi, \langle \cdot, \cdot \rangle) \cong \mathfrak{g}_{J,S}$ as a hyper-Kähler symmetric triple. The tuple (E, ω, J, S) is uniquely determined by $(\mathfrak{g}, \Phi, \langle \cdot, \cdot \rangle)$ up to complex linear isomorphisms.*

Thus the classification of hyper-Kähler symmetric triples is equivalent to the classification of all τ -invariant solutions of (10). Recall that there is a family of easy-to-find solutions called tame. Unfortunately, the claim in [2] (repeated in [3] and [1]) that all solutions of (10) are tame is false. Recall that tame τ -invariant solutions S produce hyper-Kähler symmetric triples with abelian holonomy algebra $\mathfrak{g}_+ = (\mathfrak{h}_S)^\tau$. Indeed, in the beginning of 2005 we found examples of hyper-Kähler symmetric triples with non-abelian holonomy (see [37] and Example 4.12 below) and noticed that the contradiction relies on a sign mistake in [2]. It appears that it is impossible to find *all* solutions of (10) in a straightforward way. In the recent paper [18] Cortés reconsiders the situation and is able to prove the following:

Proposition 4.10 (Cortés [18], Thm. 10). *Let S be a solution of (10). If \mathfrak{h}_S is abelian, then S is tame.*

Thus, Propositions 4.7 and 4.9 provide a classification of all hyper-Kähler symmetric triples with abelian holonomy. In order to compare it with the other results of this paper we want to give the precise formulation of this classification in terms of our standard models $\mathfrak{d}_{\alpha, \gamma}(l, \Phi_l, \mathfrak{a})$.

In the following Examples 4.11 and 4.12 we consider quaternionic vector spaces V in two different ways: as complex vector spaces equipped with a quaternionic

structure J and as $Sp(1)$ -modules. Symmetric powers are symmetric powers of complex vector spaces. Then J induces a real structure τ on the complex vector spaces $S^{2k}E$. The \mathfrak{l} -action on all appearing $(\mathfrak{l}, \Phi_{\mathfrak{l}})$ -modules \mathfrak{a} is trivial.

Example 4.11. We fix $n \in \mathbb{N}$ and $S \in (S^4\mathbb{H}^{n^*})^{\tau}$. Let $\mathfrak{l} = \mathfrak{l}_- = \mathbb{H}^n$ be abelian. The polynomial S defines a symmetric bilinear form b_S on the real vector space $(S^2\mathbb{H}^n)^{\tau}$. We set $\mathfrak{a}_S = (\mathfrak{a}_S)_+ := (S^2\mathbb{H}^n)^{\tau}/\text{rad}(b_S)$ and equip \mathfrak{a}_S with the scalar product induced by b_S . We define $\alpha_S \in C^2(\mathbb{H}^n, \mathfrak{a}_S)^{Sp(1)}$ by

$$\alpha_S(v, w) = vJ(w) - wJ(v) \pmod{\text{rad}(b_S)} \in \mathfrak{a}_S, \quad v, w \in \mathbb{H}^n.$$

Then $(\alpha_S, 0) \in \mathcal{Z}_Q^2(\mathbb{H}^n, \sigma^n, \mathfrak{a}_S)$. Moreover, α_S satisfies Condition (T_2) in Proposition 3.3, and $\mathfrak{d}_{\alpha_S, 0}(\mathbb{H}^n, \sigma^n, \mathfrak{a}_S)$ is a hyper-Kähler symmetric triple. It has abelian holonomy \mathfrak{a}_S and signature $(4n, 4n)$. One can check that $\mathfrak{d}_{\alpha_S, 0}(\mathbb{H}^n, \sigma^n, \mathfrak{a}_S)$ is isomorphic to $\mathfrak{g}_{J, S}$ for $(E, \omega) = \mathbb{H}^{n^*} \oplus \mathbb{H}^n$, ω induced by the dual pairing. Note that $\mathbb{H}^{n^*} \subset E$ is a Lagrangian subspace, and thus S is a tame solution of (10).

We call $S \in (S^4\mathbb{H}^{n^*})^{\tau}$ indecomposable if $S \neq 0$ and $S \notin S^4V^* \oplus S^4W^*$ for all non-trivial decompositions $\mathbb{H}^n = V \oplus W$ into two quaternionic subspaces. We denote the set of all indecomposable S by $(S^4\mathbb{H}^{n^*})_0^{\tau}$. There is a natural right action of the group $GL(n, \mathbb{H})$ on $(S^4\mathbb{H}^{n^*})_0^{\tau}$. Moreover, if S is indecomposable, then the cohomology class $[\alpha_S, 0] \in \mathcal{H}_Q^2(\mathbb{H}^n, \sigma^n, \mathfrak{a}_S)$ is balanced and indecomposable. Now we have the following consequence of Propositions 4.7 and 4.9.

Theorem 4.2 (Alekseevsky, Cortés). *The assignment*

$$(S^4\mathbb{H}^{n^*})_0^{\tau} \ni S \mapsto \mathfrak{d}_{\alpha_S, 0}(\mathbb{H}^n, \sigma^n, \mathfrak{a}_S)$$

yields a bijection between the union of the orbit spaces $(S^4\mathbb{H}^{n^})_0^{\tau}/GL(n, \mathbb{H})$, $n \in \mathbb{N}$, and the set of isomorphism classes of indecomposable non-abelian hyper-Kähler symmetric spaces with abelian holonomy.*

Note that this rather satisfactory classification result is not a classification in the sense of a list since for large n the orbit spaces $(S^4\mathbb{H}^{n^*})_0^{\tau}/GL(n, \mathbb{H})$ are not explicitly known.

Example 4.12. Fix $n \in \mathbb{N}$, $p \in \mathbb{N}_0$. We define a Lie algebra (\mathfrak{l}^n, Φ_n) with proper quaternionic grading as follows: $\mathfrak{l}_-^n := \mathbb{H}^n$, $\mathfrak{l}_+^n := (S^2\mathbb{H}^n)^{\tau}$, $\mathfrak{l}_+^n = \mathfrak{z}(\mathfrak{l}^n)$,

$$[v, w]_{\mathfrak{l}} := vJ(w) - wJ(v) \in \mathfrak{l}_+^n, \quad v, w \in \mathbb{H}^n.$$

We set $\mathfrak{a}_n = (\mathfrak{a}_n)_- := S^3\mathbb{H}^n$. The quaternionic structure of \mathbb{H}^n induces one on $S^3\mathbb{H}^n$. Thus \mathfrak{a}_n is a quaternionic vector space. The standard $Sp(1)$ -invariant complex Hermitian form on \mathbb{H}^n of signature $(2p, 2(n-p))$ induces a Hermitian form on \mathfrak{a}_n . We equip \mathfrak{a}_n with the real part of this Hermitian form and denote the resulting orthogonal (\mathfrak{l}^n, Φ_n) -module by $\mathfrak{a}_{n,p}$. We define $\alpha_n \in C^2(\mathfrak{l}^n, \mathfrak{a}_n)^{Sp(1)}$ by

$$\alpha_n(v, L) = vL \in S^3\mathbb{H}^n, \quad v \in \mathbb{H}^n, L \in (S^2\mathbb{H}^n)^{\tau}, \quad \alpha_n(\mathfrak{l}_-^n, \mathfrak{l}_-^n) = \alpha_n(\mathfrak{l}_+^n, \mathfrak{l}_+^n) = 0.$$

The above mentioned Hermitian form on \mathbb{H}^n induces a natural identification of $\mathfrak{l}_+^n = (S^2\mathbb{H}^n)^\tau$ with the Lie algebra $\mathfrak{sp}(p, n-p)$ and a scalar product $\langle \cdot, \cdot \rangle_p$ on \mathfrak{l}_+^n . We denote the resulting Lie bracket on \mathfrak{l}_+^n by $[\cdot, \cdot]_p$. Eventually, we define a 3-form $\gamma_p \in C^3(\mathfrak{l}_+^n) \subset C^3(\mathfrak{l}^n)^{Sp(1)}$ by

$$\gamma_p(L_1, L_2, L_3) := \langle [L_1, L_2]_p, L_3 \rangle_p, \quad L_i \in \mathfrak{l}_+^n.$$

Then $(\alpha_n, \gamma_p) \in \mathcal{Z}_Q^2(\mathfrak{l}^n, \Phi_n, \mathfrak{a}_{n,p})$. Moreover, the associated cohomology class is balanced, indecomposable, and satisfies (T_2) . Thus $\mathfrak{d}_{\alpha_n, \gamma_p}(\mathfrak{l}^n, \Phi_n, \mathfrak{a}_{n,p})$ is an indecomposable hyper-Kähler symmetric triple. Because of the form of γ_p its holonomy algebra $(\mathfrak{l}_+^n)^* \oplus \mathfrak{l}_+^n$ is non-abelian.

These hyper-Kähler symmetric triples are natural generalisations of Example 1 in [37], which is isomorphic to $\mathfrak{d}_{\alpha_1, \gamma_0}(\mathfrak{l}^1, \Phi_1, \mathfrak{a}_{1,0})$ and has signature $(4, 12)$. The paper [37] contains further examples with non-abelian holonomy.

The smallest possible index of a non-abelian hyper-Kähler symmetric triple is 4. In the following classification result we use the notation of Examples 4.11, 4.12.

Theorem 4.3 ([37], Thm. 7.4). *Let $(\mathfrak{g}, \Phi, \langle \cdot, \cdot \rangle)$ be a non-abelian indecomposable hyper-Kähler symmetric triple of signature $(4, 4q)$. Then $q = 1$ or $q = 3$, and $(\mathfrak{g}, \Phi, \langle \cdot, \cdot \rangle)$ is isomorphic to exactly only one of the following triples:*

$$q=1: \quad \mathfrak{d}_{\alpha_{S_\lambda}, 0}(\mathbb{H}, \sigma, \mathfrak{a}_{S_\lambda}), \quad \lambda \in [-2, 2], \quad S_\lambda(z + wj) := z^4 + \lambda z^2 w^2 + w^4, \quad z, w \in \mathbb{C};$$

$$q=3: \quad \mathfrak{d}_{\alpha_1, \gamma_0}(\mathfrak{l}^1, \Phi_1, \mathfrak{a}_{1,0}).$$

The proof of the theorem in [37] is based on Thm. 2.3, i.e., it classifies directly the relevant objects (O1)–(O3). It does not rely on Theorem 4.2.

There is a completely parallel theory for hypersymplectic symmetric triples. E.g., in the Alekseevsky-Cortés construction one has simply to replace the real structure τ induced by J by a real structure coming from real structures on both factors \mathbb{H} and E . In other words, one can work from the beginning with $\mathbb{R}^2 \otimes_{\mathbb{R}} E_0$ instead of $\mathbb{H} \otimes_{\mathbb{C}} E$, where E_0 is a real symplectic vector space. Hypersymplectic triples with abelian holonomy can be classified as in Theorem 4.2, see [18, 19, 1]. In addition, there is a variant of Example 4.12 producing hypersymplectic symmetric triples with non-abelian holonomy (one has to replace \mathbb{H}^n by $\mathbb{R}^2 \otimes \mathbb{R}^{2n}$, the parameter p disappears).

Note that all examples of hyper-Kähler symmetric triples presented so far have nilindex at most 5. As the following theorem and its corollary show, this is not an accident.

Theorem 4.4 ([47]). *Let $(\mathfrak{l}, \Phi_{\mathfrak{l}})$ be a Lie algebra with proper quaternionic or para-quaternionic grading. Then the nilindex of \mathfrak{l} is at most 6.*

Corollary 4.13. *Let $(\mathfrak{g}, \Phi, \langle \cdot, \cdot \rangle)$ be a non-abelian hyper-Kähler or hypersymplectic symmetric triple. As usual, let $\mathfrak{l} = \mathfrak{g}/\mathfrak{i}(\mathfrak{g})^\perp$, where $\mathfrak{i}(\mathfrak{g}) \subset \mathfrak{g}$ is the canonical isotropic*

ideal. Then there are only two possibilities for the nilindices of \mathfrak{g} , \mathfrak{g}_+ , and \mathfrak{l} as listed in the following table:

	holonomy	\mathfrak{g}	\mathfrak{g}_+	\mathfrak{l}
1.	abelian	3	1	1
2.	non-abelian	5	2	2

Proof. Let us denote the nilindex of a nilpotent Lie algebra \mathfrak{h} by $n(\mathfrak{h})$. It is easy to see that $n(\mathfrak{g})$ is odd and that $2n(\mathfrak{g}_+) < n(\mathfrak{g})$ for any nilpotent symmetric pair $\mathfrak{g} = \mathfrak{g}_+ \oplus \mathfrak{g}_-$. Thus in our case $n(\mathfrak{g}) = 3$ or $n(\mathfrak{g}) = 5$ by Thm. 4.4. The possibility that at the same time $n(\mathfrak{g}) = 5$ and $n(\mathfrak{g}_+) = 1$ is excluded by Thm. 4.2. The corollary now follows from the inequality $2n(\mathfrak{l}) \leq n(\mathfrak{g}) \leq 2n(\mathfrak{l}) + 1$, which holds for any nilpotent metric Lie algebra and can be derived from the definition of $i(\mathfrak{g})$. \square

Corollary 4.14. *The base N of the canonical fibration (see Section 3.2) of a hyper-Kähler or hypersymplectic symmetric space M is flat.*

Proof. By Corollary 4.13 the Lie algebra \mathfrak{l} is at most two-step nilpotent. It follows that $N = L/L_+$ is flat. \square

Corollary 4.13 shows that the structure of hyper-Kähler and hypersymplectic symmetric triples is strongly restricted. Therefore the goal of a full classification of these triples might be not unrealistic. The (lengthy) proof of Theorem 4.4, which we don't want to explain here, gives further inside into the structure of these triples: it computes certain universal Lie algebras \mathfrak{L}_n with proper (para-)quaternionic grading such that any Lie algebra with proper (para-)quaternionic grading in n generators is a quotient of \mathfrak{L}_n .

We conclude this section with the following open question. Is the dimension of every hyper-Kähler symmetric space without flat local factors divisible by 8? Of course, this is true for all known examples. Note that indecomposable hypersymplectic symmetric spaces exist in all dimensions that are multiples of 4.

5. Further applications

5.1. Extrinsic symmetric spaces. Let us consider a non-degenerate connected submanifold $M \subset \mathbb{R}^{p,q}$. For $x \in M$ let s_x be the reflection of $\mathbb{R}^{p,q}$ at the normal space $T_x^\perp M$ of M at x , i.e., s_x is an affine isometry and $s_x|_{T_x M} = -\text{id}$, $s_x|_{T_x^\perp M} = \text{id}$. Here and throughout the section we consider $T_x M$ and $T_x^\perp M$ as affine subspaces of $\mathbb{R}^{p,q}$. Then M is called extrinsic symmetric if $s_x(M) = M$ holds for each point $x \in M$. Extrinsic symmetric spaces in $\mathbb{R}^{p,q}$ are exactly those complete submanifolds whose second fundamental form is parallel. Extrinsic symmetric spaces in the Euclidean space are well understood. A classification in this case follows from Ferus' results discussed below and the classification of symmetric

R -spaces due to Kobayashi and Nagano. The case of a pseudo-Euclidean ambient space seems to be more involved.

In Section 3.1 we discussed the correspondence between pseudo-Riemannian symmetric spaces and symmetric triples. Here we will see that there is a similar correspondence for (a certain class of) extrinsic symmetric spaces. While symmetric triples are special (namely proper) \mathbb{Z}_2 -equivariant metric Lie algebras the algebraic objects that we will use here are certain $(\mathbb{R}, \mathbb{Z}_2)$ -equivariant metric Lie algebras. The group $\mathbb{Z}_2 = \{1, -1\}$ acts on \mathbb{R} by multiplication. Hence for an $(\mathbb{R}, \mathbb{Z}_2)$ -equivariant metric Lie algebra $(\mathfrak{g}, \Phi, \langle \cdot, \cdot \rangle)$ we can regard Φ as a pair (D, θ) that consists of a derivation $D \in \text{Der}(\mathfrak{g})$ and an involution $\theta \in \text{Aut}(\mathfrak{g})$ such that $D\theta = -\theta D$. Assume that for such an $(\mathbb{R}, \mathbb{Z}_2)$ -equivariant metric Lie algebra $D^3 = -D$ holds. Then the eigenvalues of D are in $\{i, -i, 0\}$. We put

$$\mathfrak{g}^+ := \ker D, \quad \mathfrak{g}^- := \text{span}\{X \in \mathfrak{g} \mid D^2(X) = -X\}$$

and define an involution τ_D on \mathfrak{g} by $\tau_D : \mathfrak{g} \rightarrow \mathfrak{g}$, $\tau_D|_{\mathfrak{g}^+} = \text{id}$, $\tau_D|_{\mathfrak{g}^-} = -\text{id}$. Obviously τ_D and θ commute, hence \mathfrak{g}_+ and \mathfrak{g}_- are invariant under τ_D . We introduce the notation

$$\mathfrak{g}_+^\pm := \mathfrak{g}_+ \cap \mathfrak{g}^\pm, \quad \mathfrak{g}_-^\pm := \mathfrak{g}_- \cap \mathfrak{g}^\pm, \quad \mathfrak{g}_\pm^\pm := \mathfrak{g}_\pm \cap \mathfrak{g}^\pm. \quad (11)$$

Definition 5.1. *An extrinsic symmetric triple is an $(\mathbb{R}, \mathbb{Z}_2)$ -equivariant metric Lie algebra $(\mathfrak{g}, \Phi, \langle \cdot, \cdot \rangle)$, $\Phi = (D, \theta)$, for which*

- (i) *the derivation D is inner and satisfies $D^3 = -D$,*
- (ii) *the \mathbb{Z}_2 -equivariant metric Lie algebras $(\mathfrak{g}, \theta, \langle \cdot, \cdot \rangle)$ and $(\mathfrak{g}_+, \tau_D|_{\mathfrak{g}_+}, \langle \cdot, \cdot \rangle|_{\mathfrak{g}_+})$ are proper (i.e. symmetric triples).*

Two extrinsic symmetric triples are called isomorphic if they are isomorphic as $(\mathbb{R}, \mathbb{Z}_2)$ -equivariant metric Lie algebras.

Remark 5.2. We consider the subset

$$\Pi = \{(\mathbb{R}, \Phi_1), (\mathbb{R}, \Phi_2), (\mathbb{R}^2, \Phi_3)\} \subset \widehat{(\mathbb{R}, \mathbb{Z}_2)},$$

where $\Phi_1(r) = \Phi_2(r) = 0$, $\Phi_1(z) = 1$, $\Phi_2(z) = z$, and

$$\Phi_3(r) = \begin{pmatrix} 0 & -r \\ r & 0 \end{pmatrix}, \quad \Phi_3(z) = \begin{pmatrix} 1 & 0 \\ 0 & z \end{pmatrix}$$

for $r \in \mathbb{R}$ and $z \in \mathbb{Z}_2$. In other words, Π contains exactly those representations of $(\mathbb{R}, \mathbb{Z}_2)$ that integrate to one of the following representation of $O(2)$: the one-dimensional trivial representation, the one-dimensional representation via the determinant or the two-dimensional standard representation. Then an $(\mathbb{R}, \mathbb{Z}_2)$ -equivariant (metric) Lie algebra $(\mathfrak{g}, \Phi, \langle \cdot, \cdot \rangle)$, $\Phi = (D, \theta)$, satisfying $D^3 = -D$ is the same as a Π -graded (metric) Lie algebra.

Let $(\mathfrak{g}, \Phi, \langle \cdot, \cdot \rangle)$, $\Phi = (D, \theta)$, be an extrinsic symmetric triple and choose $\xi \in \mathfrak{g}_-$ such that $D = \text{ad}(\xi)$. We consider the subgroup $G_+ := \langle \exp(\text{ad}X)|_{\mathfrak{g}_-} \mid X \in \mathfrak{g}_+ \rangle$ of $O(\mathfrak{g}_-)$ and define

$$M_{\mathfrak{g}, \xi} := G_+ \cdot \xi \subset \mathfrak{g}_-.$$

Proposition 5.3 ([33]). *1. For any extrinsic symmetric triple $(\mathfrak{g}, \Phi, \langle \cdot, \cdot \rangle)$ and for each $\xi \in \mathfrak{g}_-$ with $D = \text{ad}(\xi)$ the submanifold $M_{\mathfrak{g}, \xi} \subset \mathfrak{g}_-$ is an extrinsic symmetric space. The abstract symmetric space $M_{\mathfrak{g}, \xi}$ with the induced metric is associated with the symmetric triple $(\mathfrak{g}_+, \tau_D|_{\mathfrak{g}_+}, \langle \cdot, \cdot \rangle|_{\mathfrak{g}_+})$.*

2. Let $(\mathfrak{g}_i, \Phi_i, \langle \cdot, \cdot \rangle_i)$, $i = 1, 2$, be extrinsic symmetric triples and let $M_{\mathfrak{g}_i, \xi_i}$ be constructed as above. Then there exists an affine isometry $f : (\mathfrak{g}_1)_- \rightarrow (\mathfrak{g}_2)_-$ mapping $M_{\mathfrak{g}_1, \xi_1}$ to $M_{\mathfrak{g}_2, \xi_2}$ if and only if $(\mathfrak{g}_1, \Phi_1, \langle \cdot, \cdot \rangle_1)$ and $(\mathfrak{g}_2, \Phi_2, \langle \cdot, \cdot \rangle_2)$ are isomorphic.

Now let us turn to a construction that may be considered as a converse of Prop. 5.3, 1. Given an extrinsic symmetric space $M \subset \mathbb{R}^{p,q}$ satisfying certain additional assumptions this construction yields an extrinsic symmetric triple $(\mathfrak{g}, \Phi, \langle \cdot, \cdot \rangle)$ and an element $\xi \in \mathfrak{g}_-$ such that $M = M_{\mathfrak{g}, \xi}$. Let us first discuss these additional assumptions.

Definition 5.4. *A submanifold $M \subset \mathbb{R}^{p,q}$ is called full if it is not contained in any proper affine subspace of $\mathbb{R}^{p,q}$. It is called normal if the intersection of the normal spaces of all points of M is not empty, i.e. $\bigcap_{x \in M} T_x^\perp M \neq \emptyset$.*

To be full is not really a restriction for submanifolds M of the Euclidean space \mathbb{R}^n since we may always consider the smallest affine subspace that contains M . Contrary to that, there are many submanifolds of the pseudo-Euclidean space $\mathbb{R}^{p,q}$ that are contained in an affine subspace that is degenerate with respect to the inner product but not in a proper non-degenerate one. This is also the case if one restricts oneself to extrinsic symmetric spaces. As far as normality is concerned, Ferus proved in [24] that extrinsic symmetric spaces in the Euclidean space decompose into a product of an affine subspace and a normal extrinsic symmetric space. This seems to be not true for a pseudo-Euclidean ambient space.

Let us now describe the construction. It was developed by Ferus [25, 26] who used it to prove that every full and normal extrinsic symmetric space in \mathbb{R}^n is a standard imbedded symmetric R-space. In our language this means that any such extrinsic symmetric space arises from an extrinsic symmetric triple as described in Proposition 5.3, 1. Here we will present the more elementary description of this construction given by Eschenburg and Heintze [22]. We will also include the necessary modifications for the pseudo-Riemannian situation discussed in [33, 38].

Let $M \subset \mathbb{R}^{p,q}$ be a full and normal extrinsic symmetric space. Fix a point $x_0 \in M$. Since M is normal we may assume that $0 \in \mathbb{R}^{p,q}$ is contained in the intersection of normal spaces $\bigcap_{x \in M} T_x^\perp M$. This implies that $s_x \in O(p, q)$ for all $x \in M$. We define the transvection group of $M \subset \mathbb{R}^{p,q}$ by

$$K := \langle s_x \circ s_y \mid x, y \in M \rangle \subset O(p, q).$$

Obviously, K is isomorphic to the transvection group of the abstract symmetric space M . Let $\mathfrak{k} \subset \mathfrak{so}(p, q)$ be the Lie algebra of K . Since it is isomorphic to the Lie algebra of the transvection group of the abstract symmetric space M it is the underlying Lie algebra of a symmetric triple $(\mathfrak{k}, \theta_{\mathfrak{k}}, \langle \cdot, \cdot \rangle_{\mathfrak{k}})$. Besides \mathfrak{k} we can associate with $M \subset \mathbb{R}^{p, q}$ the following metric Lie algebra \mathfrak{g} . As a vector space \mathfrak{g} equals

$$\mathfrak{g} = \mathfrak{k} \oplus V, \quad V = \mathbb{R}^{p, q}.$$

Using the standard scalar product $\langle \cdot, \cdot \rangle_{p, q}$ on V we define a scalar product on \mathfrak{g} by $\langle \cdot, \cdot \rangle = \langle \cdot, \cdot \rangle_{\mathfrak{k}} \oplus \langle \cdot, \cdot \rangle_{p, q}$. Furthermore, we define a bracket $[\cdot, \cdot] : \mathfrak{g} \times \mathfrak{g} \rightarrow \mathfrak{g}$ by

- (i) $[\cdot, \cdot]$ restricted to $\mathfrak{k} \times \mathfrak{k}$ equals the Lie bracket of \mathfrak{k} ,
- (ii) $[A, v] = A(v)$ for $A \in \mathfrak{k}$ und $v \in V$,
- (iii) $[\cdot, \cdot]$ restricted to $V \times V$ is given by the condition $[\cdot, \cdot]_{V \times V} : V \times V \rightarrow \mathfrak{k}$ and by $\langle A, [x, y] \rangle = \langle [A, x], y \rangle$ for all $A \in \mathfrak{k}$ and $x, y \in V$.

Using the fullness of $M \subset \mathbb{R}^{p, q}$ one can prove that this bracket satisfies the Jacobi identity. Moreover, we have an involution θ on \mathfrak{g} given by $\mathfrak{g}_+ = \mathfrak{k}$, $\mathfrak{g}_- = V$ and an inner derivation $D := \text{ad}(x_0)$. We obtain:

Theorem 5.1 (Ferus [25, 26], see also [22, 33, 38]). *Let $M \subset \mathbb{R}^{p, q}$ be a full and normal extrinsic symmetric space. Let $\mathfrak{g}, \langle \cdot, \cdot \rangle$ and $\Phi = (D, \theta)$ be as constructed above. Then $(\mathfrak{g}, \Phi, \langle \cdot, \cdot \rangle)$ is an extrinsic symmetric triple and $M = M_{\mathfrak{g}, x_0}$.*

In [38] Kim discusses this construction also for the case of a non-normal extrinsic symmetric space.

Remark 5.5. Let $(\mathfrak{g}, \Phi, \langle \cdot, \cdot \rangle)$, $\Phi = (D, \theta)$, be an extrinsic symmetric triple and suppose $D = \text{ad}(\xi)$, $\xi \in \mathfrak{g}_-$. Then $M_{\mathfrak{g}, \xi}$ is normal by construction. It is full if and only if

$$[\mathfrak{g}_+, \mathfrak{g}_-] = \mathfrak{g}_+^+ \quad (12)$$

(see (11) for the notation used here). We call an extrinsic symmetric triple satisfying (12) full.

Having explained the relation between extrinsic symmetric spaces and extrinsic symmetric triples we now want to apply our classification scheme to extrinsic symmetric triples. Recall that we can understand an extrinsic symmetric triple as a Π -graded metric Lie algebra satisfying additional conditions, where Π is given as in Remark 5.2. We can proceed as in Section 4.1 and we see that any extrinsic symmetric triple is isomorphic to some $\mathfrak{d}_{\alpha, \gamma}(\mathfrak{l}, \Phi_{\mathfrak{l}}, \mathfrak{a})$, where now

- (O1) $(\mathfrak{l}, \Phi_{\mathfrak{l}})$ is a Lie algebra with Π -grading such that the \mathbb{Z}_2 -equivariant Lie algebras $(\mathfrak{l}, \theta_{\mathfrak{l}})$ and $(\mathfrak{l}_+, \tau_{D_{\mathfrak{l}}} |_{\mathfrak{l}_+})$ are proper,
- (O2) \mathfrak{a} is a Π -graded semisimple orthogonal $(\mathfrak{l}, \Phi_{\mathfrak{l}})$ -module,
- (O3) $[\alpha, \gamma] \in \mathcal{H}_Q^2(\mathfrak{l}, \Phi_{\mathfrak{l}}, \mathfrak{a})_b$ is indecomposable and satisfies besides (T_2) two further conditions (T_2^+) and (A_0^+) (see [33]), which ensure that $\mathfrak{d}_{\alpha, \gamma}(\mathfrak{l}, \Phi_{\mathfrak{l}}, \mathfrak{a})$ satisfies the properness condition (ii) in Def. 5.1,

(O4) the derivation $D = -D_{\mathfrak{l}}^* \oplus D_{\mathfrak{a}} \oplus D_{\mathfrak{l}}$ is inner.

Condition (O4) is equivalent to $\exists l \in \mathfrak{l}_- \exists a \in \mathfrak{a}_- \subset C^0(\mathfrak{l}, \mathfrak{a}) \exists z \in \mathfrak{l}_-^* \subset C^1(\mathfrak{l})$:

$$D_{\mathfrak{l}} = \text{ad}_{\mathfrak{l}}(l), \quad D_{\mathfrak{a}} = \rho(l), \quad da = i(l)\alpha, \quad dz = \langle a \wedge \alpha \rangle + i(l)\gamma.$$

Combining (O1) – (O4) with Theorem 2.3 we obtain a classification scheme for extrinsic symmetric triples.

Example 5.6 (Full and normal extrinsic Cahen-Wallach spaces). We want to answer the question which indecomposable non-semisimple non-flat Lorentzian symmetric spaces can be embedded into a pseudo-Euclidean space as full and normal extrinsic symmetric spaces. Recall from Theorem 3.2 that every indecomposable non-semisimple non-flat Lorentzian symmetric space is associated with a symmetric triple of the kind $\mathfrak{d}(p, q, \lambda, \mu)$, $p, q \geq 0$, $p + q > 0$, $(\lambda, \mu) \in M_{p, q}$.

Theorem 5.2 (cf. [33]). *The symmetric triple $\mathfrak{d}(p, q, \lambda, \mu)$, $p, q \geq 0$, $p + q > 0$, $(\lambda, \mu) \in M_{p, q}$ admits an associated symmetric space $M^{1, n+1}$, $n = p + q$, that can be embedded as a full and normal extrinsic symmetric space if and only if*

1. $q = 0$ and $\lambda = (1, \dots, 1)$ or
2. $p = 0$ and $\mu = (1, \dots, 1)$.

In both cases the ambient space is $\mathbb{R}^{2, n+2}$ and there is a one-parameter family of non-isomorphic embeddings.

To verify the theorem we have to determine all extrinsic symmetric triples $(\mathfrak{g}, \Phi, \langle \cdot, \cdot \rangle)$, $\Phi = (D, \theta)$, that satisfy (12) and $(\mathfrak{g}_+, \tau_D|_{\mathfrak{g}_+}, \langle \cdot, \cdot \rangle_{\mathfrak{g}_+}) \cong \mathfrak{d}(p, q, \lambda, \mu)$ with $(\lambda, \mu) \in M_{p, q}$ (see Section 3.3). In particular, since \mathfrak{g}_+ is indecomposable and non-reductive the Lie algebra \mathfrak{g} is indecomposable and non-semisimple. This implies $\mathfrak{g} \cong \mathfrak{d} := \mathfrak{d}_{\alpha, \gamma}(\mathfrak{l}, \Phi_{\mathfrak{l}}, \mathfrak{a})$ for a suitable $(\mathbb{R}, \mathbb{Z}_2)$ -equivariant Lie algebra $(\mathfrak{l}, \Phi_{\mathfrak{l}})$, an orthogonal $(\mathfrak{l}, \Phi_{\mathfrak{l}})$ -module \mathfrak{a} and some $(\alpha, \gamma) \in \mathcal{Z}_Q^2(\mathfrak{l}, \Phi_{\mathfrak{l}}, \mathfrak{a})$. In particular, we have $\mathfrak{g}_+ \cong \mathfrak{d}_+ = \mathfrak{d}_{\alpha|_{\mathfrak{l}_+}, \gamma|_{\mathfrak{l}_+}}(\mathfrak{l}_+, \Phi_{\mathfrak{l}}|_{\mathfrak{l}_+}, \mathfrak{a}_+)$. Hence $\dim \mathfrak{l}_+^+ = \dim \mathfrak{l}_-^+ = 1$ and $[\mathfrak{l}_+^+, \mathfrak{l}_+^+] = 0$. Equation (12) implies $[\mathfrak{l}_+^+, \mathfrak{l}_-^+] = \mathfrak{l}_+^+$. Since $\mathfrak{l}_+^+ \neq 0$ we also obtain $\dim \mathfrak{l}_-^+ = 1$. Now it is not hard to see that \mathfrak{l} is isomorphic to one of the Lie algebras

1. $\mathfrak{sl}(2, \mathbb{R}) = \{[H, X] = 2Y, [H, Y] = 2X, [X, Y] = 2H\}$
2. $\mathfrak{su}(2) = \{[H, X] = 2Y, [H, Y] = -2X, [X, Y] = 2H\}$

and that in both cases $\Phi_{\mathfrak{l}} = (D_{\mathfrak{l}}, \theta_{\mathfrak{l}})$ is given by $\mathfrak{l}_+ = \mathbb{R} \cdot H$, $\mathfrak{l}_- = \text{span}\{X, Y\}$ and $D_{\mathfrak{l}} = (1/2) \cdot \text{ad}X$. Since $H^2(\mathfrak{l}, \mathfrak{a}) = 0$ for $\mathfrak{l} \in \{\mathfrak{sl}(2, \mathbb{R}), \mathfrak{su}(2)\}$ we may assume $\alpha = 0$. Since \mathfrak{d} is indecomposable we get $\mathfrak{a}^{\mathfrak{l}} = 0$. Because of $D^3 = -D$ the eigenvalues of $\rho(X)$ are in $\{0, 2i, -2i\}$. Consequently, \mathfrak{a} is the direct sum of submodules that are all equivalent to the adjoint representation of \mathfrak{l} . Finally, we obtain the following result, which proves Theorem 5.2 and gives all embeddings explicitly.

Proposition 5.7 (cf. [33]). *If $M^{1, n+1} \hookrightarrow \mathbb{R}^{r, s}$ is a full and normal extrinsic symmetric space of Lorentz signature and if $M^{1, n+1}$ is solvable and indecomposable, then $(r, s) = (2, n + 2)$ and we are in one of the following cases:*

1. $M^{1,n+1}$ is associated to the symmetric triple $\mathfrak{d}(n, 0, \lambda, 0)$, $\lambda = (1, \dots, 1) \in \mathbb{R}^n$ and $M^{1,n+1} \subset \mathbb{R}^{2,n+2}$ is extrinsic isometric to $M_{\mathfrak{d},\xi}$ for $\mathfrak{d} = \mathfrak{d}_{\alpha,\gamma}(\mathfrak{l}, \Phi_{\mathfrak{l}}, \mathfrak{a})$ with

$$\begin{aligned} \mathfrak{l} &= \mathfrak{sl}(2, \mathbb{R}), \quad \Phi_{\mathfrak{l}} \text{ as above, } \mathfrak{a} = \bigoplus_{i=1}^n (\text{ad}, \mathfrak{l}, B_{\mathfrak{l}}, \Phi_{\mathfrak{a}}), \quad \Phi_{\mathfrak{a}} = (D_{\mathfrak{l}}, -\theta_{\mathfrak{l}}), \\ \alpha &= 0, \quad \gamma = cB_{\mathfrak{l}}(\cdot, [\cdot, \cdot]), \quad c \in \mathbb{R}, \end{aligned}$$

and $\xi = (1/2) \cdot X$. Here $B_{\mathfrak{l}}$ is the Killing form of \mathfrak{l} .

2. $M^{1,n+1}$ is associated to the symmetric triple $\mathfrak{d}(0, n, 0, \mu)$, $\mu = (1, \dots, 1) \in \mathbb{R}^n$ and $M^{1,n+1} \subset \mathbb{R}^{2,n+2}$ is extrinsic isometric to $M_{\mathfrak{d},\xi}$ for $\mathfrak{d} = \mathfrak{d}_{\alpha,\gamma}(\mathfrak{l}, \Phi_{\mathfrak{l}}, \mathfrak{a})$ with

$$\begin{aligned} \mathfrak{l} &= \mathfrak{su}(2), \quad \Phi_{\mathfrak{l}} \text{ as above, } \mathfrak{a} = \bigoplus_{i=1}^n (\text{ad}, \mathfrak{l}, -B_{\mathfrak{l}}, \Phi_{\mathfrak{a}}), \quad \Phi_{\mathfrak{a}} = (D_{\mathfrak{l}}, -\theta_{\mathfrak{l}}), \\ \alpha &= 0, \quad \gamma = cB_{\mathfrak{l}}(\cdot, [\cdot, \cdot]), \quad c \in \mathbb{R}, \end{aligned}$$

and $\xi = (1/2) \cdot X$. Again $B_{\mathfrak{l}}$ denotes the Killing form of \mathfrak{l} .

We remark that there are decomposable solvable non-flat Lorentzian symmetric spaces that can be embedded as full and normal extrinsic symmetric spaces such that the embedding is indecomposable, i.e., the associated extrinsic symmetric triple is indecomposable.

In [33] we classify all extrinsic symmetric triples without simple ideals that are associated with a Lorentzian extrinsic symmetric space. Of course, that classification contains all extrinsic symmetric triples that we obtained above. However, it contains also extrinsic symmetric triples whose associated extrinsic symmetric spaces are not full.

Remark 5.8. Let $(\mathfrak{g}, \Phi, \langle \cdot, \cdot \rangle)$, $\Phi = (D, \theta)$, be an extrinsic symmetric triple. Exponentiating D defines a complex grading Φ_0 on $(\mathfrak{g}, \langle \cdot, \cdot \rangle)$. Equation (12) implies that this grading is proper if $(\mathfrak{g}, \Phi, \langle \cdot, \cdot \rangle)$ is full. Thus, forgetting about θ , a full extrinsic symmetric triple can be considered as a pseudo-Hermitian symmetric triple such that the complex grading is given by inner automorphisms. If $(\mathfrak{g}_+, \tau_D|_{\mathfrak{g}_+}, \langle \cdot, \cdot \rangle_{\mathfrak{g}_+})$ has signature (p, q) , then the pseudo-Hermitian symmetric triple $(\mathfrak{g}, \Phi_0, \langle \cdot, \cdot \rangle)$ has signature $(2p, 2q)$. Moreover, Prop. 4.4 implies that \mathfrak{g} has a non-trivial Levi factor and that the radical of \mathfrak{g} is nilpotent. In view of these facts, the reader should compare Prop. 5.7 with Thm. 4.1.

5.2. Manin triples. Manin triples are algebraic objects that are associated with Poisson-Lie groups. A Poisson-Lie group is a Lie group equipped with a Poisson bracket that satisfies a compatibility condition with the group multiplication. The infinitesimal object associated with such a Poisson-Lie group G is the Lie algebra \mathfrak{g} of G together with a 1-cocycle $\gamma : \mathfrak{g} \rightarrow \mathfrak{g} \wedge \mathfrak{g}$ that satisfies co-Jacobi identity, i.e., $\gamma^* : \mathfrak{g}^* \wedge \mathfrak{g}^* \rightarrow \mathfrak{g}^*$ is a Lie bracket on \mathfrak{g}^* . Such a pair (\mathfrak{g}, γ) is called Lie bialgebra. Given such a Lie bialgebra (\mathfrak{g}, γ) it is easy to see that there exists a unique Lie algebra structure on the vector space $\mathfrak{g} \oplus \mathfrak{g}^*$ such that the inner product on $\mathfrak{g} \oplus \mathfrak{g}^*$ defined by the dual pairing is invariant. What we obtain is an example of a so-called Manin triple.

Definition 5.9. A Manin triple $(\mathfrak{g}, \mathfrak{h}_1, \mathfrak{h}_2)$ consists of a metric Lie algebra \mathfrak{g} and two complementary isotropic subalgebras \mathfrak{h}_1 and \mathfrak{h}_2 .

Here we only consider finite-dimensional Lie bialgebras and Manin triples. In this case the above described assignment associating a Manin triple to a given Lie bialgebra is a one-to-one correspondence. For a detailed introduction to this subject we refer to [41].

Manin triples $(\mathfrak{g}, \mathfrak{h}_1, \mathfrak{h}_2)$ for which \mathfrak{g} is a complex reductive Lie algebra were classified by Delorme [20]. Having learned about the difficulties in classifying non-reductive metric Lie algebras it is not surprising that there is little known about non-reductive Manin triples. However, there are some results in low dimensions. Figueroa-O'Farrill [27] studied Manin triples in the context of conformal field theory and the $N = 2$ Sugawara constructions. By rather heavy calculations he achieved a classification of complex six-dimensional Manin triples. A more conceptual proof of his result, which also yields a classification in the real case is due to Gomez [28].

Unfortunately, the method of quadratic extensions seems to be not adequate for the description of Manin triples. The difficulty is to handle both isotropic subalgebras at the same time. However, we can describe Manin pairs.

Definition 5.10. A Manin pair $(\mathfrak{g}, \mathfrak{h})$ consists of a metric Lie algebra of signature (n, n) and an n -dimensional isotropic subalgebra $\mathfrak{h} \subset \mathfrak{g}$.

Applying the method of quadratic extensions to Manin pairs we obtain the following result.

Proposition 5.11. Let \mathfrak{l} be a Lie algebra and $\mathfrak{l}' \subset \mathfrak{l}$ a subalgebra. Let \mathfrak{a} be an orthogonal \mathfrak{l} -module of signature (m, m) and $\mathfrak{a}' \subset \mathfrak{a}$ an m -dimensional isotropic \mathfrak{l}' -invariant subspace. Let $(\alpha, \gamma) \in \mathcal{Z}_Q^2(\mathfrak{l}, \mathfrak{a})$ satisfy $\alpha(\mathfrak{l}', \mathfrak{l}') \subset \mathfrak{a}'$ and $\gamma(\mathfrak{l}', \mathfrak{l}', \mathfrak{l}') = 0$. Then $\mathfrak{d} := \mathfrak{d}_{\alpha, \gamma}(\mathfrak{l}, \mathfrak{a})$ has signature $(m + \dim \mathfrak{l}, m + \dim \mathfrak{l})$ and $(\mathfrak{d}, \text{Ann}(\mathfrak{l}') \oplus \mathfrak{a}' \oplus \mathfrak{l}')$ is a Manin pair, where $\text{Ann}(\mathfrak{l}') \subset \mathfrak{l}^*$ denotes the annihilator of \mathfrak{l}' .

Conversely, every Manin pair $(\mathfrak{g}, \mathfrak{h})$ for which \mathfrak{g} does not contain simple ideals is isomorphic to some pair $(\mathfrak{d}, \text{Ann}(\mathfrak{l}') \oplus \mathfrak{a}' \oplus \mathfrak{l}')$ constructed in this way, where, moreover, \mathfrak{d} is balanced.

Proof. The first part of the proposition is easy to check. The second part relies on the functorial assignment (4) and the fact that the section $s : \mathfrak{l} \rightarrow \mathfrak{g}$ defining $(\alpha, \gamma) \in \mathcal{Z}_Q^2(\mathfrak{l}, \mathfrak{a})$ can be chosen in the following way. Decompose \mathfrak{h} as a vector space into $\mathfrak{h} = \mathfrak{h}_1 \oplus \mathfrak{h}_2 \oplus \mathfrak{h}_3$, where $\mathfrak{h}_1 = \mathfrak{h} \cap \mathfrak{i}(\mathfrak{g})$ and \mathfrak{h}_2 is a complement of \mathfrak{h}_1 in $\mathfrak{h} \cap \mathfrak{i}(\mathfrak{g})^\perp$. Then choose $s : \mathfrak{l} \rightarrow \mathfrak{g}$ as described in Section 2.4 such that it satisfies in addition $\mathfrak{h}_3 \subset s(\mathfrak{l})$ and $s(\mathfrak{l}) \perp \mathfrak{h}_2$. \square

Combining this with Theorem 2.3 we obtain a classification scheme for those Manin pairs $(\mathfrak{g}, \mathfrak{h})$ for which \mathfrak{g} does not have simple ideals. Let us remark that now the absence of simple ideals is a real restriction contrary to the case of metric Lie algebras without distinguished subalgebra.

In small dimensions Prop. 5.11 is a helpful tool not only for classification of Manin pairs but also for classification of Manin triples, since a complementary isotropic subalgebra of $\mathfrak{h} \subset \mathfrak{g}$ (if it exists) can be determined by hand. E.g.,

using this method it is easy to recover the classification in dimension six without extensive calculations.

6. Appendix: Some lemmas and proofs

6.1. Implications of (\mathfrak{h}, K) -equivariance. We first consider the case of the trivial group $K = \{e\}$.

Lemma 6.1. *Let $(\mathfrak{l}, \Phi_{\mathfrak{l}})$ be an \mathfrak{h} -equivariant Lie algebra such that $\mathfrak{l}^{\mathfrak{h}} \subset \mathfrak{l}'$. Let $\mathfrak{r} \subset \mathfrak{l}$ be the radical of the Lie algebra \mathfrak{l} . Then \mathfrak{r} is nilpotent and acts trivially on every semisimple $(\mathfrak{l}, \Phi_{\mathfrak{l}})$ -module.*

Proof. Recall the notion of the nilpotent radical $R(\mathfrak{g})$ of a Lie algebra \mathfrak{g} from Section 2.2. The ideal $R(\mathfrak{g})$ is nilpotent and acts trivially on any semisimple \mathfrak{g} -module. We consider the Lie algebra $\mathfrak{h} \ltimes \mathfrak{l}$, where the action of \mathfrak{h} on \mathfrak{l} is given by $\Phi_{\mathfrak{l}}$. A semisimple $(\mathfrak{l}, \Phi_{\mathfrak{l}})$ -module can be considered as a semisimple $\mathfrak{h} \ltimes \mathfrak{l}$ -module. It is therefore sufficient to show that $\mathfrak{r} \subset R(\mathfrak{h} \ltimes \mathfrak{l})$.

Formula (1) implies that $\Phi_{\mathfrak{l}}(\mathfrak{h})\mathfrak{r} + \mathfrak{r} \cap \mathfrak{l}' \subset R(\mathfrak{h} \ltimes \mathfrak{l})$. Since \mathfrak{h} acts semisimply on \mathfrak{l} and $\mathfrak{r} \subset \mathfrak{l}$ is \mathfrak{h} -invariant we have $\mathfrak{r} = \mathfrak{r}^{\mathfrak{h}} \oplus \Phi_{\mathfrak{l}}(\mathfrak{h})\mathfrak{r}$. By assumption $\mathfrak{r}^{\mathfrak{h}} \subset \mathfrak{r} \cap \mathfrak{l}'$. We conclude that $\mathfrak{r} \subset R(\mathfrak{h} \ltimes \mathfrak{l})$. The lemma follows. \square

We now look at the case $K = \mathbb{Z}_2$. Any $(\mathfrak{h}, \mathbb{Z}_2)$ -equivariant Lie algebra has a decomposition $\mathfrak{l} = \mathfrak{l}_+ \oplus \mathfrak{l}_-$ w.r.t. the \mathbb{Z}_2 -action.

Lemma 6.2. *Let \mathfrak{h} be a semisimple Lie algebra. Let $(\mathfrak{l}, \Phi_{\mathfrak{l}})$ be an $(\mathfrak{h}, \mathbb{Z}_2)$ -equivariant Lie algebra such that (a) $\mathfrak{l}_+ = \mathfrak{l}^{\mathfrak{h}}$, and (b) $\mathfrak{l}_+ = [\mathfrak{l}_-, \mathfrak{l}_-]$. Then \mathfrak{l} is nilpotent.*

Proof. Let \mathfrak{r} be the radical of \mathfrak{l} . Then the semisimple Lie algebra $\mathfrak{s} := \mathfrak{l}/\mathfrak{r}$ inherits an $(\mathfrak{h}, \mathbb{Z}_2)$ -equivariant structure $\Phi_{\mathfrak{s}}$. Since any derivation of \mathfrak{s} is inner we may identify $\Phi_{\mathfrak{s}}(\mathfrak{h})$ with a semisimple subalgebra $\mathfrak{k} \subset \mathfrak{s}$ acting on \mathfrak{s} by the adjoint representation. $\Phi_{\mathfrak{s}}(\mathfrak{h})$ respects the decomposition $\mathfrak{s} = \mathfrak{s}_+ \oplus \mathfrak{s}_-$. Thus $\mathfrak{k} \subset \mathfrak{s}_+$. By Condition (a) we have $\mathfrak{s}_+ = \mathfrak{s}^{\mathfrak{k}}$. Thus \mathfrak{k} is abelian, hence zero. Applying (a) again we obtain $\mathfrak{s} = \mathfrak{s}_+$, thus $\mathfrak{s}_- = \{0\}$. Now Condition (b) yields $\mathfrak{s} = 0$. Thus \mathfrak{l} is solvable. Now we consider \mathfrak{l} as an \mathfrak{h} -equivariant solvable Lie algebra and apply Lemma 6.1. \square

6.2. Proof of Proposition 2.12. Any semisimple (\mathfrak{h}, K) -module V has a canonical decomposition $V = V^{(\mathfrak{h}, K)} \oplus V^1$, where V^1 is the sum of all irreducible submodules of V carrying a non-trivial (\mathfrak{h}, K) -action. We denote the corresponding components of any $v \in V$ by v^0, v^1 . Tensor products of semisimple (\mathfrak{h}, K) -modules are again semisimple. In particular, the natural (\mathfrak{h}, K) -actions on $C^p(\mathfrak{l}, \mathfrak{a})$, $C^q(\mathfrak{l})$ are semisimple.

Let us first prove injectivity of the natural map $\mathcal{H}_Q^2(\mathfrak{l}, \Phi_{\mathfrak{l}}, \mathfrak{a}) \rightarrow \mathcal{H}_Q^2(\mathfrak{l}, \mathfrak{a})^{(\mathfrak{h}, K)}$. Let $(\alpha_i, \gamma_i) \in \mathcal{Z}_Q^2(\mathfrak{l}, \Phi_{\mathfrak{l}}, \mathfrak{a}) = \mathcal{Z}_Q^2(\mathfrak{l}, \mathfrak{a})^{(\mathfrak{h}, K)}$, $i = 1, 2$, be two cocycles that represent the same element in $\mathcal{H}_Q^2(\mathfrak{l}, \mathfrak{a})$, i.e., there exists $(\tau, \sigma) \in C^1(\mathfrak{l}, \mathfrak{a}) \oplus C^2(\mathfrak{l})$ such that

$$(\alpha_2, \gamma_2) = (\alpha_1 + d\tau, \gamma_1 + d\sigma + \langle (\alpha_1 + \frac{1}{2}d\tau) \wedge \tau \rangle) .$$

The invariance of α_i implies that $d\tau = (d\tau)^0 = d\tau^0$. It follows that

$$(d\tau \wedge \tau)^0 = (d\tau^0 \wedge \tau)^0 = d\tau^0 \wedge \tau^0 .$$

This and the invariance of γ_i and α_1 implies

$$\gamma_2 = \gamma_1 + d\sigma^0 + \langle (\alpha_1 + \frac{1}{2}d\tau^0) \wedge \tau^0 \rangle .$$

Thus $(\alpha_2, \gamma_2) = (\alpha_1, \gamma_1)(\tau^0, \sigma^0)$. Since $(\tau^0, \sigma^0) \in \mathcal{C}_Q^1(\mathfrak{l}, \Phi_{\mathfrak{l}}, \mathfrak{a})$ the cocycles (α_i, γ_i) , $i = 1, 2$, represent the same class in $\mathcal{H}_Q^2(\mathfrak{l}, \Phi_{\mathfrak{l}}, \mathfrak{a})$. This shows injectivity.

Surjectivity is a little bit more involved. Let $(\alpha, \gamma) \in \mathcal{Z}_Q^2(\mathfrak{l}, \mathfrak{a})$ such that $[\alpha, \gamma] \in \mathcal{H}_Q^2(\mathfrak{l}, \mathfrak{a})^{(\mathfrak{h}, K)}$. This means that for all $X \in \mathfrak{h}$, $k \in K$ there exist elements $(\tau_X, \sigma_X), (\tau_k, \sigma_k) \in C^1(\mathfrak{l}, \mathfrak{a}) \oplus C^2(\mathfrak{l})$ such that

$$k\alpha = \alpha + d\tau_k, \quad k\gamma = \gamma + d\sigma_k + \langle (\alpha + \frac{1}{2}d\tau_k) \wedge \tau_k \rangle, \quad (13)$$

$$X\alpha = d\tau_X, \quad X\gamma = d\sigma_X + \langle \alpha \wedge \tau_X \rangle. \quad (14)$$

We have to show that $[\alpha, \gamma]$ can be represented by an (\mathfrak{h}, K) -invariant cocycle.

We first consider the case that K is connected. Then (\mathfrak{h}, K) -invariance is the same as $\mathfrak{k} \times \mathfrak{h}$ -invariance, where \mathfrak{k} is the Lie algebra of K which acts by derivations on \mathfrak{h} . Since the action of $\mathfrak{k} \times \mathfrak{h}$ on \mathfrak{l} and \mathfrak{a} is semisimple it factors over a reductive quotient of $\mathfrak{k} \times \mathfrak{h}$. Thus without loss of generality we may assume that K is trivial and $\mathfrak{h} = \mathfrak{s} \oplus \mathfrak{t}$, where \mathfrak{s} is semisimple and \mathfrak{t} is abelian. Let C be the Casimir operator of \mathfrak{s} . For any semisimple \mathfrak{h} -module V there exists an element $X_V \in \mathfrak{t}$ such that the operator $D_V := X_V + C$ acts invertibly on any \mathfrak{h} -submodule of V^1 . In particular, $\text{im } D_V = V^1$, $\ker D_V = V^{\mathfrak{h}}$. Now we fix $D := D_V$ for the semisimple \mathfrak{h} -module $V = C^2(\mathfrak{l}, \mathfrak{a}) \oplus C^1(\mathfrak{l}, \mathfrak{a}) \oplus C^3(\mathfrak{l}) \oplus C^2(\mathfrak{l})$.

The operator D is of the form $D = X + \sum Y_i Z_i$ for some $X, Y_i, Z_i \in \mathfrak{h}$. Now (14) implies that there exists an element $\tau_D \in C^1(\mathfrak{l}, \mathfrak{a})$ such that $D\alpha = d\tau_D$. Thus $D\alpha \in B^2(\mathfrak{l}, \mathfrak{a}) \cap V^1$, where $B^2(\mathfrak{l}, \mathfrak{a}) \subset C^2(\mathfrak{l}, \mathfrak{a})$ denotes the \mathfrak{h} -submodule of coboundaries. Therefore there exists a coboundary $d\tau$ such that $D\alpha = Dd\tau$. We now consider the quadratic cocycle $(\alpha', \gamma') := (\alpha, \gamma)(-\tau, 0)$. Then $D\alpha' = 0$, thus α' is \mathfrak{h} -invariant. The cocycle (α', γ') satisfies (14) again. We claim that

$$D\gamma' = d\sigma'_D + \langle \alpha' \wedge \tau'_D \rangle \quad (15)$$

for some quadratic cochain (τ'_D, σ'_D) satisfying $d\tau'_D = 0$. It is obvious that (15) is valid for D replaced by $X \in \mathfrak{h}$. We have to show that it also holds for D replaced by a second order monomial YZ for $Y, Z \in \mathfrak{h}$. Thus we can assume (15) for Z and apply Y . Note that $Y\alpha' = 0$ by \mathfrak{h} -invariance. Setting $\sigma'_{YZ} := Y\sigma'_Z$ and $\tau'_{YZ} := Y\tau'_Z$ we obtain

$$YZ\gamma' = dY\sigma'_Z + Y\langle \alpha' \wedge \tau'_Z \rangle = d\sigma'_{YZ} + \langle \alpha' \wedge \tau'_{YZ} \rangle .$$

This justifies (15). We denote by $B^3(\mathfrak{l}) \subset C^3(\mathfrak{l})$ and $Z^1(\mathfrak{l}, \mathfrak{a}) \subset C^1(\mathfrak{l}, \mathfrak{a})$ the corresponding submodules of coboundaries and cocycles. Then (15) says

$$D\gamma' \in (B^3(\mathfrak{l}) + \langle \alpha' \wedge Z^1(\mathfrak{l}, \mathfrak{a}) \rangle) \cap V^1 =: W^1 .$$

Thus we can solve

$$D\gamma' = D(d\sigma' + \langle \alpha' \wedge \tau' \rangle)$$

in W^1 , in particular with $d\tau' = 0$. Now we set $(\alpha'', \gamma'') := (\alpha', \gamma')(-\tau', -\sigma')$. Note that $\alpha'' = \alpha'$, thus $D\alpha'' = 0$. We have $D\gamma'' = D\gamma' - Dd\sigma' - D\langle \alpha' \wedge \tau' \rangle = 0$. Thus (α'', γ'') is \mathfrak{h} -invariant and $[\alpha, \gamma] = [\alpha'', \gamma'']$. This implies surjectivity of the natural map $\mathcal{H}_Q^2(\mathfrak{l}, \Phi_{\mathfrak{l}}, \mathfrak{a}) \rightarrow \mathcal{H}_Q^2(\mathfrak{l}, \mathfrak{a})^{(\mathfrak{h}, K)}$ in the case of connected K .

If K is disconnected and $[\alpha, \gamma] \in \mathcal{H}_Q^2(\mathfrak{l}, \mathfrak{a})^{(\mathfrak{h}, K)}$ we can assume by the above that $(\alpha, \gamma) \in \mathcal{Z}_Q^2(\mathfrak{l}, \mathfrak{a})^{(\mathfrak{h}, K_0)}$, where $K_0 \subset K$ is the identity component. We now work in the vector space $V = [C^2(\mathfrak{l}, \mathfrak{a}) \oplus C^1(\mathfrak{l}, \mathfrak{a}) \oplus C^3(\mathfrak{l}) \oplus C^2(\mathfrak{l})]^{(\mathfrak{h}, K_0)}$, which carries an action of the finite group $G = K/K_0$. By injectivity of the canonical map $\mathcal{H}_Q^2(\mathfrak{l}, \Phi_{\mathfrak{l}}|_{(\mathfrak{h}, K_0)}, \mathfrak{a}) \rightarrow \mathcal{H}_Q^2(\mathfrak{l}, \mathfrak{a})^{(\mathfrak{h}, K_0)}$, the Equations (13) are valid with k replaced by $g \in G$ and with τ_g, σ_g invariant under (\mathfrak{h}, K_0) . We now proceed similar as in the connected case. We will use the projection operator $P = \frac{1}{|G|} \sum_{g \in G} g$. We set $\tau := \frac{1}{|G|} \sum_{g \in G} \tau_g$ and $(\alpha', \gamma') := (\alpha, \gamma)(\tau, 0)$. Then we have $\alpha' = P\alpha$, i.e., α is G -invariant. Equation (13) for (α', γ') provides for $g \in G$ certain (\mathfrak{h}, K_0) -invariant elements (τ'_g, σ'_g) with $d\tau'_g = 0$. We now define

$$\tau' := \frac{1}{|G|} \sum_{g \in G} \tau'_g, \quad \sigma' := \frac{1}{|G|} \sum_{g \in G} \sigma'_g$$

and set $(\alpha'', \gamma'') := (\alpha', \gamma')(\tau', \sigma')$. Then $\alpha'' = \alpha'$ is G -invariant. Using $d\tau'_g = 0$ and (13) we eventually obtain

$$\gamma'' = \gamma' + d\sigma' + \langle \alpha' \wedge \tau' \rangle = P\gamma'.$$

Thus (α'', γ'') is G -invariant. In other words, $(\alpha'', \gamma'') \in \mathcal{Z}_Q^2(\mathfrak{l}, \mathfrak{a})^{(\mathfrak{h}, K)}$. Since $[\alpha, \gamma] = [\alpha'', \gamma''] \in \mathcal{H}_Q^2(\mathfrak{l}, \mathfrak{a})$ this finishes the proof of surjectivity of the canonical map $\mathcal{H}_Q^2(\mathfrak{l}, \Phi_{\mathfrak{l}}, \mathfrak{a}) \rightarrow \mathcal{H}_Q^2(\mathfrak{l}, \mathfrak{a})^{(\mathfrak{h}, K)}$. \square

6.3. Proof of Proposition 3.8. We first assume Condition (b). Let G be the transvection group of M , and let $J \subset G$ be the analytic subgroup corresponding to the ideal $\mathfrak{i}^\perp \subset \mathfrak{g}$. By the definition of a quadratic extension (Definition 2.5) the Lie algebra $\mathfrak{i}^\perp/\mathfrak{i} \cong \mathfrak{a}$ is abelian, i.e., $[\mathfrak{i}^\perp, \mathfrak{i}^\perp] \subset \mathfrak{i}$. Moreover,

$$\langle [\mathfrak{i}^\perp, \mathfrak{i}], \mathfrak{g} \rangle = \langle \mathfrak{i}^\perp, [\mathfrak{i}, \mathfrak{g}] \rangle \subset \langle \mathfrak{i}^\perp, \mathfrak{i} \rangle = \{0\}.$$

It follows that \mathfrak{i}^\perp is 2-step nilpotent:

$$[\mathfrak{i}^\perp, [\mathfrak{i}^\perp, \mathfrak{i}^\perp]] = 0. \quad (16)$$

The following lemma together with Condition (b) implies that $J \subset G$ is closed.

Lemma 6.3. *Let G be a connected Lie group with Lie algebra \mathfrak{g} . Let $\mathfrak{j} \subset \mathfrak{g}$ be a nilpotent ideal containing the center $\mathfrak{z}(\mathfrak{g})$. Then the analytic subgroup $J \subset G$ corresponding to \mathfrak{j} is closed.*

Proof. The assertion of the lemma is well-known, if $\mathfrak{j} \subset \mathfrak{g}$ is the maximal nilpotent ideal (see e.g. [14], Ch. III §9). In this case J is called the nilradical of G . We look at the adjoint group $G_1 := \text{Ad}_{\mathfrak{g}}(G) \cong G/Z(G)$ acting on the Lie algebra \mathfrak{g} . Let N_1 be the nilradical of G_1 . Then $N_1 \subset G_1$ is closed. By Engel's Theorem, there is a basis of \mathfrak{g} such that all elements of N_1 are represented by upper triangular matrices with 1's on the diagonal with respect to this basis. It follows that the exponential is a diffeomorphism from the Lie algebra \mathfrak{n}_1 of N_1 to N_1 . Now $\text{Ad}_{\mathfrak{g}}(J) \subset N_1$. Via the exponential we see that $\text{Ad}_{\mathfrak{g}}(J) \cong \text{ad}_{\mathfrak{g}}(\mathfrak{j})$ is closed in $N_1 \cong \mathfrak{n}_1$. Hence $\text{Ad}_{\mathfrak{g}}(J)$ is closed in G_1 . It follows that $J \cdot Z(G)$ is closed in G . Since $\mathfrak{z}(\mathfrak{g}) \subset \mathfrak{j}$ the group J is the identity component of $J \cdot Z(G)$. Therefore, J is closed as well. \square

We set $L := G/J$. We identify the Lie algebra of L via p with \mathfrak{l} . Let $Q_L : G \rightarrow L$ be the natural projection. Let $G_+ \subset G$ be the stabilizer of the base point x_0 , and let θ be the corresponding involution of G . Then $G_+ \subset G^\theta$. Since J is θ -stable θ induces an involution $\theta_L : L \rightarrow L$. Then $L_+ := Q_L(G_+)$ is contained in L^{θ_L} and has Lie algebra \mathfrak{l}_+ . Since L^{θ_L} has the same Lie algebra we see that $L_+ \subset L$ is closed. Then $N := L/L_+$ is an affine symmetric space and the map $q : M \cong G/G_+ \rightarrow N$ induced by Q_L is affine and surjective. The transvection group of N is equal to $L_0 = L/(Z(L) \cap L_+)$. The natural map Q between the transvection groups of M and N is therefore the composition of Q_L with the projection of L to L_0 . It is now evident that Condition (6) is satisfied.

The fibres of q are connected since they are precisely the orbits of the connected group J . Let us show that they are coisotropic and flat. By homogeneity it is sufficient to look at tangent space $T_{x_0}M$ at the base point which can be identified with \mathfrak{g}_- . Then $(\mathfrak{i}^\perp)_-$ corresponds to the tangent space of the orbit. Since $\mathfrak{i}_- \subset (\mathfrak{i}^\perp)_-$ the orbits are coisotropic. For $X, Y, Z \in \mathfrak{g}_- \cong T_{x_0}M$ the curvature tensor is given by $R(X, Y)Z = -[[X, Y], Z]$. Flatness of the orbits now follows from (16).

The uniqueness of N and q is a simple consequence of Condition (6) and the required connectedness of the fibres.

If Condition (b) is not satisfied, but M is simply connected, we make the following modifications of the above proof. First of all it is more convenient to work with the universal covering group of the transvection group of M . We now denote this universal cover by \tilde{G} . The involution θ of \mathfrak{g} induces one of \tilde{G} which we again denote by θ . \tilde{G} acts on M . Let $G_+ \subset \tilde{G}$ be the stabilizer of the base point $x_0 \in M$. The Lie algebra of G_+ is \mathfrak{g}_+ . Since M and \tilde{G} are simply connected G_+ is connected. It follows that $G_+ \subset \tilde{G}^\theta$. (This is the crucial point where the simple connectedness of M enters the proof. If M were not simply connected we would have $\tilde{G}^\theta \subsetneq G_+$.) Let $\tilde{p} : \mathfrak{g} \rightarrow \mathfrak{l}$ be the homomorphism induced by p . Let L be the connected and simply connected group with Lie algebra \mathfrak{l} . The surjection \tilde{p} integrates to a surjection $Q_L : \tilde{G} \rightarrow L$. We set $J := \ker Q_L$ which is closed in \tilde{G} . The group J has Lie algebra \mathfrak{i}^\perp . Moreover, J is connected since $L \cong \tilde{G}/J$ and \tilde{G} are simply connected. With these changes understood the proof runs as in the first case. Lemma 6.3 is no longer needed.

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