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An Analytic Approach to Purely Nonlocal Bellman Equations Arising in Models of Stochastic Control

by

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Abstract

Given a bounded domain $\Omega \subset \mathbb{R}^d$ and two integro-differential operators L^1, L^2 of the form $L^j u(x) = \mathrm{p.\,v.} \int_\Omega (u(x) - u(y)) k^j(x, y, x - y) dy$ we study the fully nonlinear Bellman equation

$$\max_{j=1,2} \left\{ L^j u(x) + a^j(x) u(x) - f^j(x) \right\} = 0 \quad \text{in } \Omega \,, \tag{0.1}$$

with Dirichlet boundary conditions. Here, $a^j, f^j : \Omega \to \mathbb{R}$ are nonnegative functions. We prove the existence of a nonnegative function $u : \Omega \to \mathbb{R}$ which satisfies (0.1) almost everywhere. The main difficulty arises through the nonlocality of L^j and the absence of regularity near the boundary.

Keywords: Bellman equation, fully nonlinear equation, integro-differential operator, Markov jump process, stochastic control

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1 Introduction

Over the last years there has been an increasing interest in the study of nonlocal operators generating Markov jump processes. It turns out that a theory for linear equations, analogous to the one developed in [KS80] for diffusion operators can be developed for jump processes, see [BK05a, BK05b, Sil05]. From both points of view, theory and application, it is very interesting to study fully nonlinear Bellman equations with such operators. Some results concerning the martingale problem and viscosity solutions have been achieved using, at least partly, probabilistic methods in [MP91, MP94, MP96]. So far, analytical methods were successful only in the case of jump-diffusions [GL84], [AT96], i.e. when a dominating diffusion is present or the equation is not fully nonlinear, [MR97]. In conclusion, a satisfactory analytical approach to fully nonlinear nonlocal equations has not been established yet. It is the aim of this work to make a first step in this direction by employing tools similar to those used in [EF79, BE79]. For local diffusion operators, results on Hölder regularity for linear equations were crucial in setting up a theory of fully nonlinear equations, see [Eva83, CIL92, CC95, Kry97] and it would be highly desirable to investigate fully nonlinear nonlocal equations in a similar fashion.

Several kinds of nonlinear equations including nonlocal operators of the same type as the ones considered in this paper have been studied in the area of financial mathematics. Since neither the equations nor the techniques are related to our problem we do not discuss these results here but refer the interested reader to the references mentioned in the introduction of [JK05].

Let $\Omega \subset \mathbb{R}^d$ be a bounded domain with C^{d+1}-boundary. The aim of this work is to show the existence of nonnegative solutions $u\colon \Omega \to \mathbb{R}_+$ to the following equation:

$$\begin{aligned} \max_{j=1,2}\{L^j u(x) + a^j(x)u(x) - f^j(x)\} &= 0 \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{1.1}$$

where L^1, L^2 are nonlocal integro-differential operators of order $\alpha \in (1, 2)$. Roughly speaking, the operators under consideration are similar to restrictions of pseudo-differential operators of order α with variable coefficients and generators of Markov processes with jumps.

Let us define an operator L as follows.

$$Lu(x) = \text{p. v.} \int_\Omega (u(x) - u(y))k(x, y, x - y)dy \tag{1.2}$$
$$= \lim_{\varepsilon \to 0} \int_{\Omega \setminus B_\varepsilon(x)} (u(x) - u(y))k(x, y, x - y)dy,$$

where $x \in \Omega$ and $\Omega \subset \mathbb{R}^d$ is a bounded domain with C^{d+1}-boundary. Moreover, $k \colon \mathbb{R}^d \times \mathbb{R}^d \times (\mathbb{R}^d \setminus \{0\}) \to \mathbb{R}$ is $(d+2)$-times continuously differentiable and satisfies the following conditions:

$$k(x, y, z) = k(y, x, -z)\,, \tag{1.3}$$
$$|\partial_x^\beta \partial_y^\gamma \partial_z^\delta k(x, y, z)| \leq C_{\beta,\gamma,\delta} |z|^{-d-\alpha-|\delta|}\,, \tag{1.4}$$
$$c_0 |z|^{-d-\alpha} \leq k(x, y, z) \leq C_0 |z|^{-d-\alpha}\,. \tag{1.5}$$

for all $x, y, z \in \mathbb{R}^d$, $z \neq 0$ and $\beta, \gamma, \delta \in \mathbb{N}_0^d$ with $|\beta| + |\gamma| + |\delta| \leq d + 2$ where $\alpha \in (1, 2)$ is the order of the operator.

An example for $k(\cdot, \cdot, \cdot)$ is given by $k(x, y, z) = b(x, y)|z|^{-d-\alpha}$ and $b \in C_b^{d+2}(\mathbb{R}^d)$. Note that the definition of the operator L depends on Ω. Formally, in the case $\Omega = \mathbb{R}^d$ and $k(x, y, z) = |z|^{-d-\alpha}$ one has $L = const \times (-\Delta)^{\frac{\alpha}{2}}$. On one hand, $(-\Delta)^{\frac{\alpha}{2}}$ is a fractional power of the Laplace operator, on the other hand it is the generator of so called α-stable processes which explains partly our motivation. For a bounded domain Ω the operator L has the same form as the generator of a censored stable process [BBC03].

Our main result reads as follows.

Theorem 1.1 *Let $\Omega \subset \mathbb{R}^d$ be a bounded domain with C^{d+1}-boundary and $\alpha \in (1, 2)$ be fixed. Assume L^1 and L^2 are defined as in (1.2) for two kernels $k^1(\cdot, \cdot, \cdot)$, $k^2(\cdot, \cdot, \cdot)$ that both satisfy assumptions (1.3) through (1.5). Let $\mathbf{a} \in L^\infty(\Omega; \mathbb{R}^2)$ and $\mathbf{f} \in L^2(\Omega; \mathbb{R}^2)$ be nonnegative. Then there exists a nonnegative function $u \in H_0^{\alpha/2}(\Omega) \cap H_{loc}^\alpha(\Omega)$ satisfying*

$$\begin{aligned} \max_{j=1,2} \{L^j u(x) + a^j(x)u(x) - f^j(x)\} &= 0 \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega. \end{aligned} \tag{1.6}$$

Moreover, for any open set $\Omega' \Subset \Omega$

$$\|u\|_{H^{\frac{\alpha}{2}}(\Omega)} + \|u\|_{H^\alpha(\Omega')} \leq C\,,$$

where C depends on Ω', Ω, c_0, C_0, $C_{\beta,\gamma,\delta}$, $\|\mathbf{a}\|_{L^\infty(\Omega;\mathbb{R}^2)}$, $\|\mathbf{f}\|_{L^2(\Omega;\mathbb{R}^2)}$.

Remark 1.2 *In the above theorem, we do not focus on weakest possible regularity assumptions for $\partial\Omega$ and k_j. Analogously to Bellman equations with local diffusion operators one expects solutions to be more regular than stated in Theorem 1.1, see a related remark in [Sil05]. In the case of the integral operators above solutions will have some limited regularity near the boundary, even for linear equations with smooth data. This can be seen in the following example. Let $u \in H_0^{\frac{\alpha}{2}}(-1,1) \cap H^{\alpha}_{loc}(-1,1)$, $\alpha \in (1,2)$, be a nonnegative weak solution of*

$$Lu(x) = 1, \quad x \in (-1,1), \quad u|_{x=\pm 1} = 0,$$

where L is as above with kernel $k(x,y,z) = |z|^{-1-\alpha}$. Then u cannot be in $C^{\beta}([-1,1])$ for $\beta > \alpha$. This is proved by contradiction: If $u \in C^{\beta}([-1,1])$, then $Lu(x) = 1$ for all $x \in (-1,1)$ implies that $u'(\pm 1) = 0$. Hence $u(x) = O(|x+1|^{\beta})$ as $x \to -1$ and

$$1 = \lim_{x\to -1} Lu(x) = -\int_{-1}^{1} |y+1|^{-1-\alpha} u(y)\, dy < 0$$

since $u(x) \geq 0$, which is a contradiction. Since u is also a solution of the Bellman equation for $L^j = L$ and $f^j \equiv 1$, the solutions of the Bellman equation will in general not be in $C^{\beta}(\overline{\Omega})$ either.

The paper is organized as follows: Section 2 contains preliminaries such as definitions of function spaces and notation. In section 3 we discuss the linear nonlocal operators L^j. We study their mapping properties and estimates of commutators with localization functions. Bilinear forms corresponding to L^j are investigated in section 4. Section 5 contains the proof of Theorem 1.1.

2 Preliminaries

In the following $\langle .,. \rangle$ denotes the duality product between a Banach space X and its dual X' and $(.,.)$ denotes the L^2-scalar product.

Let $\Omega = \mathbb{R}^d$, $\Omega = \mathbb{R}^d_+$, or let Ω be a bounded domain with C^{d+1}-boundary. Then $H^s(\Omega)$, $s \in [0,d]$ denotes the usual L^2-Sobolev-Slobodeckii space normed by

$$\|u\|^2_{H^m(\Omega)} = \sum_{|\alpha|\leq m} \|D^{\alpha}_x u\|^2_{L^2(\Omega)}$$

if $s = m \in \mathbb{N}_0$ and

$$\|u\|^2_{H^s(\Omega)} = \sum_{|\alpha|\le[s]} \|D^\alpha_x u\|^2_{L^2(\Omega)} + \sum_{|\alpha|=[s]} \int_\Omega\int_\Omega \frac{|D^\alpha u(x) - D^\alpha u(y)|^2}{|x-y|^{d+2(s-[s])}} dxdy$$

if $s \notin \mathbb{N}_0$, see for example [Ada75]. Moreover, $H^s_0(\Omega)$ denotes the closure of $C^\infty_0(\Omega)$ in $H^s(\Omega)$ for $s \notin \frac{1}{2} + \mathbb{N}_0$ and $H^{-s}(\Omega)$ the dual of $H^{-s}_0(\Omega)$, i.e. $H^{-s}(\Omega) = H^s_0(\Omega)'$.

We note that, if $s \in (0,1), s \neq \frac{1}{2}$ and Ω is a bounded domain with a C^1-boundary, then

$$\|u\|^2_{\dot{H}^s(\Omega)} = \int_\Omega\int_\Omega \frac{|u(x)-u(y)|^2}{|x-y|^{d+2s}} dxdy$$

is a norm on $H^s_0(\Omega)$, which is equivalent to the norm defined before. Here, $\dot{H}^s(\Omega)$ denotes the corresponding homogeneous space. The fact can be easily proven by contradiction using that $H^s_0(\Omega)$ is compactly embedded in $L^2(\Omega)$ and that $\|u\|_{\dot{H}^s(\Omega)} = 0$ if and only if $u \equiv$ *const*.

We use bold letters like $\mathbf{v}$ for vector valued functions such as $\mathbf{v} = (v^1, v^2) \in H^s(\Omega;\mathbb{R}^2)$. We say that a vector is nonnegative if all of its components are nonnegative. Moreover, if $f\colon \Omega \to \mathbb{R}$, then $f_+(x) := \max(f(x), 0)$, $f_-(x) = \min(f(x), 0)$.

Finally, if $f \in L^1(\mathbb{R}^d)$, then the Fourier transform of f is defined by

$$\hat{f}(\xi) = \mathcal{F}_{x\mapsto\xi}[f](\xi) = \int_{\mathbb{R}^d} e^{-ix\cdot\xi} f(x)dx, \qquad \xi \in \mathbb{R}^d,$$

and the inverse Fourier transform is denoted by $\mathcal{F}^{-1}$.

3 Properties of the integral operators

In this section we study properties of the integral operator L as defined by (1.2) and the operator $\mathcal{L}$ defined as follows.

$$\begin{aligned} \mathcal{L}u(x) &= \mathrm{p.\,v.} \int_{\mathbb{R}^d} (u(x) - u(y))k(x,y,x-y)dy \\ &= \lim_{\varepsilon\to 0} \int_{\mathbb{R}^d\setminus B_\varepsilon(x)} (u(x)-u(y))k(x,y,x-y)dy\,. \end{aligned} \tag{3.1}$$

Note that L equals $\mathcal{L}$ when $\Omega = \mathbb{R}^d$. For both, L and $\mathcal{L}$ we require the kernel k to belong to the following class:

Definition 3.1 *The class $\mathcal{K}^\alpha(R)$, $R, \alpha \geq 0$, consists of all functions $k\colon \mathbb{R}^d \times \mathbb{R}^d \times \left(\mathbb{R}^d \setminus \{0\}\right) \to \mathbb{R}$ which are $(d+2)$-times continuously differentiable such that $k(x,x,-z) = k(x,x,z)$, $k(x,y,z) = \widetilde{k}(z)$ if $|x|, |y| \geq R$, and*

$$|\partial_x^\beta \partial_y^\gamma \partial_z^\delta k(x,y,z)| \leq C|z|^{-d-\alpha-|\delta|} \quad \text{for all } x,y,z \in \mathbb{R}^d, z \neq 0 \tag{3.2}$$

for all $|\beta| + |\gamma| + |\delta| \leq d+2$ and some constant $C > 0$. Finally, for $k \in \mathcal{K}^\alpha$, $\|k\|_{\mathcal{K}^\alpha}$ denotes the least constant such that (3.2) holds for all $|\beta| + |\gamma| + |\delta| \leq d+2$ and

$$|k|_{\mathcal{K}^\alpha} := \sup_{x,y,z \in \mathbb{R}^d, z \neq 0} |z|^{d+\alpha} |k(x,y,z)|.$$

Remark 3.2 *Obviously, $\mathcal{K}^\alpha(R)$ equipped with $\|\cdot\|_{\mathcal{K}^\alpha}$ is a Banach space. The weaker norm $|\cdot|_{\mathcal{K}_\alpha}$ will be used, when "freezing coefficients" in Lemma 3.7.*

Remark 3.3 *There are many other classes of kernels similar to $\mathcal{K}^\alpha(R)$ one could consider in the following and obtain similar results. In particular, the smoothness assumptions w.r.t. to x, y are not optimal and the assumption $k(x,y,z) = \widetilde{k}(z)$ if $|x|, |y| \geq R$ could be weakened considerably.*

Lemma 3.4 *Let $k \in \mathcal{K}^\alpha(R)$, $\alpha \in (1,2)$, $R > 0$, and let $\mathcal{L}$ be as in (3.1). Then for all $u \in H^\alpha(\mathbb{R}^d)$ the right-hand side of (3.1) converges in $L^2(\mathbb{R}^d)$ and*

$$\|\mathcal{L}u\|_{L^2(\mathbb{R}^d)} \leq C\left(|k|_{\mathcal{K}^\alpha} + R^d \|k\|_{\mathcal{K}^\alpha}\right) \|u\|_{H^\alpha(\mathbb{R}^d)} + C\|k\|_{\mathcal{K}^\alpha} \|u\|_{H^{\frac{\alpha}{2}}(\mathbb{R}^d)}$$

for all $u \in H^\alpha(\mathbb{R}^d)$. Here C depends only on d and α.

Proof: First of all, let

$$\mathcal{L}_\varepsilon u(x) := \int_{\mathbb{R}^d \setminus B_\varepsilon(x)} (u(x) - u(y)) k(x,y,x-y)\,dy.$$

Then

$$\begin{aligned}
\mathcal{L}_\varepsilon u(x) &= \int_{\mathbb{R}^d \setminus B_\varepsilon(x)} (u(x) - u(y) + \nabla u(x) \cdot (y-x)) k(x,x,x-y)\,dy \\
&\quad + \int_{\mathbb{R}^d \setminus B_\varepsilon(x)} (u(x) - u(y))(k(x,y,x-y) - k(x,x,x-y))\,dy \\
&\equiv \mathcal{L}_\varepsilon^A u(x) + \mathcal{L}_\varepsilon^B u(x)
\end{aligned}$$

where we have used that

$$\int_{\mathbb{R}^d\setminus B_\varepsilon(x)}(x-y)k(x,x,x-y)dy = \int_{\mathbb{R}^d\setminus B_\varepsilon(0)} hk(x,x,h)dh = 0$$

since $k(x,x,-z) = k(x,x,z)$. From the form above it can be easily checked that $\lim_{\varepsilon\to 0}\mathcal{L}_\varepsilon u(x)$ exists for all $x\in\mathbb{R}^d$ and converges in $L^2(\mathbb{R}^d)$ if $u\in C_0^\infty(\mathbb{R}^d)$. Note that by the assumptions on k we have

$$|k(x,y,x-y)-k(x,x,x-y)|\leq C\|k\|_{\mathcal{K}^\alpha}|x-y|^{-d-\alpha+1}$$

and $\alpha<2$. Hence it is sufficient to prove that

$$\|\mathcal{L}_\varepsilon u\|_{L^2(\mathbb{R}^d)}\leq C(|k|_{\mathcal{K}^\alpha}+R^d\|k\|_{\mathcal{K}^\alpha})\|u\|_{H^\alpha(\mathbb{R}^d)}+C|k|_{\mathcal{K}^\alpha}\|u\|_{H^{\frac{\alpha}{2}}(\mathbb{R}^d)}$$

uniformly in $\varepsilon>0$ and for all $u\in C_0^\infty(\mathbb{R}^d)$.

Concerning $\mathcal{L}_\varepsilon^B$ we obtain by direct estimates

$$\begin{aligned}
&\left\|\mathcal{L}_\varepsilon^B u\right\|_{L^2(\mathbb{R}^d)}^2\\
&\leq\ C\|k\|_{\mathcal{K}^\alpha}^2\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\frac{|u(x)-u(y)|^2}{|x-y|^{d+\alpha}}dy\int_{\mathbb{R}^d}|x-y|^{-d-\alpha+2}(1+|x-y|)^{-2}dydx\\
&\leq\ C\|k\|_{\mathcal{K}^\alpha}^2\|u\|_{H_2^{\frac{\alpha}{2}}(\mathbb{R}^d)}^2
\end{aligned}$$

where we have used that

$$|k(x,y,x-y)-k(x,x,x-y)|\leq C\|k\|_{\mathcal{K}^\alpha}|x-y|^{-d-\alpha+1}(1+|x-y|)^{-1}.$$

For $\mathcal{L}_\varepsilon^A$ we use Fourier transformation:

$$\begin{aligned}
\mathcal{L}_\varepsilon^A u(x) &= \int_{\mathbb{R}^d\setminus B_\varepsilon(0)}(u(x)-u(x+h))k(x,x,h)dh\\
&= \int_{\mathbb{R}^d}e^{ix\cdot\xi}\int_{\mathbb{R}^d\setminus B_\varepsilon(0)}(1-e^{ih\cdot\xi})k(x,x,h)dh\,\hat{u}(\xi)\frac{d\xi}{(2\pi)^d}\\
&= \int_{\mathbb{R}^d}e^{ix\cdot\xi}p_\varepsilon(x,\xi)\hat{u}(\xi)\frac{d\xi}{(2\pi)^d}=p_\varepsilon(x,D_x)u,
\end{aligned}$$

where

$$\begin{aligned}
p_\varepsilon(x,\xi) &:= \int_{\mathbb{R}^d\setminus B_\varepsilon(0)}(1-e^{ih\cdot\xi})k(x,x,h)dh\\
&= \int_{\mathbb{R}^d\setminus B_\varepsilon(0)}(1-\cos h\cdot\xi)k(x,x,h)dh
\end{aligned}$$

since $k(x,x,-h) = k(x,x,h)$. Therefore

$$|p_\varepsilon(x,\xi)| \leq |k|_{\mathcal{K}^\alpha}|\xi|^\alpha \int_{\mathbb{R}^d}\left(1-\cos s\cdot\frac{\xi}{|\xi|}\right)|s|^{-d-\alpha}ds \leq C|k|_{\mathcal{K}^\alpha}|\xi|^\alpha$$

since $0 \leq 1-\cos s\cdot\frac{\xi}{|\xi|} \leq C\min(1,s^2)$, $0 \leq k(x,x,h) \leq |k|_{\mathcal{K}^\alpha}|h|^{-d-\alpha}$, and $1<\alpha<2$. In the same way one proves that

$$|\partial_x^\beta p_\varepsilon(x,\xi)| \leq C\|k\|_{\mathcal{K}^\alpha}|\xi|^\alpha \tag{3.3}$$

for all $|\beta| \leq d+1$.

Now, if $k(x,y,z) = \widetilde{k}(z)$ for *all* $x,y\in\mathbb{R}^d$, then $p_\varepsilon(x,\xi) = p_\varepsilon(\xi)$ and

$$\begin{aligned}\|\mathcal{L}_\varepsilon^A u\|_2^2 &= \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d}|p_\varepsilon(\xi)\hat{u}(\xi)|^2 d\xi \\ &\leq C|k|_{\mathcal{K}^\alpha}^2\frac{1}{(2\pi)^d}\int_{\mathbb{R}^d}|\xi|^{2\alpha}|\hat{u}(\xi)|^2d\xi \leq C|k|_{\mathcal{K}^\alpha}^2\|u\|_{H^\alpha(\mathbb{R}^d)}^2\end{aligned}$$

Moreover, if $k(x,x,z)=0$ for $|x|\geq R$, then

$$\mathcal{F}\left[\mathcal{L}_\varepsilon^A u\right](\xi) = \int_{\mathbb{R}^d}\hat{p}_\varepsilon(\xi-\eta,\eta)\hat{u}(\eta)\frac{d\eta}{(2\pi)^d},$$

where $\hat{p}_\varepsilon(\xi,\eta) := \mathcal{F}_{x\mapsto\xi}[p(.,\eta)]$ satisfies

$$\left|\xi^\beta\hat{p}_\varepsilon(\xi,\eta)\right| \leq CR^d\|k\|_{\mathcal{K}^\alpha}|\eta|^\alpha \qquad \text{for all } |\beta|\leq d+1$$

because of (3.3) and therefore

$$|\hat{p}_\varepsilon(\xi,\eta)| \leq CR^d\|k\|_{\mathcal{K}^\alpha}|\eta|^\alpha(1+|\xi|)^{-d-1}.$$

Thus

$$\|\mathcal{L}_\varepsilon^A u\|_{L^2(\mathbb{R}^d)} \leq CR^d\|k\|_{\mathcal{K}^\alpha}\|u\|_{H^\alpha(\mathbb{R}^d)}$$

in that case by Young's inequality.

Finally, the general case follows easily from the two cases above by decomposing $k(x,y,z) = k'(x,y,z)+\widetilde{k}(z)$, where $k'\in\mathcal{K}^\alpha(R)$ is supported in $\{(x,y)\in\mathbb{R}^{2d} : |x|\leq R \text{ or } |y|\leq R\}$. ∎

Now, we use the result above to obtain mapping properties of L in the case of a bounded domain Ω.

Lemma 3.5 *Let L be as in (1.2), where $k \in \mathcal{K}^\alpha(R)$, $\alpha \in (1,2)$, $R > 0$ and $\Omega \subset \mathbb{R}^d$ is a bounded domain. Then for all $u \in H^\alpha_{loc}(\Omega) \cap L^2(\Omega)$ the right-hand side of (1.2) converges in $L^2(\Omega')$. Moreover, for all $\Omega, \Omega', \Omega''$ satisfying $\Omega' \Subset \Omega'' \Subset \Omega$*

$$\|Lu\|_{L^2(\Omega')} \le C\|k\|_{\mathcal{K}^\alpha} \left(\|u\|_{H^\alpha(\Omega'')} + \|u\|_{L^2(\Omega)} \right),$$

where C is independent of u and k.

Proof: Let $\psi \in C_0^\infty(\Omega)$ such that $\psi \equiv 1$ on Ω''. Moreover, let

$$L_\varepsilon u(x) := \int_{\Omega \setminus B_\varepsilon(x)} (u(x) - u(y)) k(x, y, x-y) dy$$

for $x \in \Omega'$ and $0 < \varepsilon \le \mathrm{dist}(\Omega', \partial\Omega'')$. Then

$$\begin{aligned} L_\varepsilon u(x) = & \int_{\mathbb{R}^d \setminus B_\varepsilon(x)} ((\psi u)(x) - (\psi u)(y)) k(x, y, x-y) dy \\ & - u(x) \int_{\Omega^c} k(x, y, x-y)\, dy - \int_{\Omega \setminus B_\varepsilon(x)} (1 - \psi(y)) u(y) k(x, y, x-y) dy \\ & =: \mathcal{L}_\varepsilon(\psi u)(x) + I_1(x) + I_2(x)\,. \end{aligned}$$

Lemma 3.4 can be applied in order to estimate the $L^2(\mathbb{R}^d)$-norm of $\mathcal{L}_\varepsilon u(x)$ since $\psi u \in H^\alpha(\mathbb{R}^d)$. For $x \in \Omega'$ and $y \in \Omega^c$ the function $k(x, y, x-y)$ is bounded. For $x \in \Omega'$ and $y \in \Omega$ one has $|(1 - \psi(y)) k(x, y, x-y)| \le C(\Omega', \psi) |k|_{\mathcal{K}^\alpha}$. These observations together imply

$$\|I_1(x)\|_{L^2(\Omega')} + \|I_2(x)\|_{L^2(\Omega')} \le C|k|_{\mathcal{K}^\alpha} \|u\|_{L^2(\Omega)}\,,$$

which proves the lemma. ∎

The following lemma will be an essential ingredient throughout the article since it shows that the commutator of L and $\mathcal{L}$ with a suitably smooth cut-off function φ is an operator of lower order. It is the basis for "localizing" the nonlocal operators L.

Lemma 3.6 *Let $k \in \mathcal{K}^\alpha(R)$, $\alpha \in (1,2), R > 0$ and let Ω be a bounded domain. Let L be as in (1.2) and $\mathcal{L}$ be as in (3.1). Let $\varphi \in C_b^\beta(\mathbb{R}^d)$ with $\beta > \alpha$. Then*

$$\begin{aligned} \|[L, \varphi]u\|_{L^2(\Omega)} &\le C|k|_{\mathcal{K}^\alpha} \|\varphi\|_{C^\beta(\overline{\Omega})} \|u\|_{H^{\frac{\alpha}{2}}(\Omega)}\,, \\ \|[\mathcal{L}, \varphi]v\|_{L^2(\mathbb{R}^d)} &\le C|k|_{\mathcal{K}^\alpha} \|\varphi\|_{C_b^\beta(\mathbb{R}^d)} \|v\|_{H^{\frac{\alpha}{2}}(\mathbb{R}^d)}\,, \end{aligned}$$

for all $u \in H^\alpha_{loc}(\Omega) \cap H^{\frac{\alpha}{2}}(\Omega)$, $v \in H^\alpha(\mathbb{R}^d)$. *Here* $[A,\varphi]u = A(\varphi u) - \varphi(Au)$ *with* $A = L, \mathcal{L}$. *Moreover,*

$$\int_\Omega \big([L,\varphi]u(x)\big)v(x)dx = -\int_\Omega u(x)\big([L,\varphi]v(x)\big)dx \tag{3.4}$$

for all $u, v \in H_0^{\frac{\alpha}{2}}(\Omega)$ *after an extension of* $[L,\varphi]$ *from* $C_0^\infty(\Omega)$ *to* $H^{\frac{\alpha}{2}}(\Omega)$.

Proof: For u as in the statement of the lemma, we have

$$\begin{aligned} [L,\varphi]u(x) &= \text{p. v.} \int_\Omega k(x,y,x-y)(\varphi(x)-\varphi(y))u(y)dy \\ &= \text{p. v.} \int_\Omega (u(x)-u(y))k'(x,y,x-y)dy + u(x)L\varphi(x) \end{aligned}$$

for almost all $x \in \Omega$, where

$$k'(x,y,z) = k(x,y,z)(\varphi(y)-\varphi(x)).$$

By the same arguments as in the proof of Lemma 3.4 one see that $\|L\varphi\|_{L^\infty(\Omega)} \leq C|k|_{\mathcal{K}^\alpha}\|\varphi\|_{C^\beta}$. Moreover, since $\varphi \in C^\beta(\overline{\Omega})$, $\beta > \alpha > 1$,

$$|k'(x,y,x-y)| \leq C|k|_{\mathcal{K}^\alpha}\|\varphi\|_{C^\beta}|x-y|^{-d-\alpha+\beta'}(1+|x-y|).$$

Hence as in the proof of Lemma 3.4

$$\begin{aligned} &\|[\varphi,L]u\|^2_{L^2(\Omega)} \\ &\leq C|k|^2_{\mathcal{K}^\alpha}\|\varphi\|^2_{C^\beta}\int_\Omega\int_\Omega \frac{|u(x)-u(y)|^2}{|x-y|^{d+\alpha}}dydx \int_{\mathbb{R}^d}|h|^{-d-\alpha+2\beta'}(1+|h|)^{-2\beta'}dh \\ &\leq C|k|^2_{\mathcal{K}^\alpha}\|\varphi\|^2_{C^\beta}\|u\|^2_{H^{\frac{\alpha}{2}}(\Omega)}. \end{aligned}$$

The proof for $\mathcal{L}$ is the same. The last statement easily follows from the explicite form of $[L,\varphi]$. ∎

Lemma 3.7 *Let* $k_j \in \mathcal{K}^\alpha(R)$, $\alpha \in (1,2), R > 0$, $j = 1,2$, *satisfy*

$$k_j(x,y,z) \geq c_0|z|^{-d-\alpha} \qquad \text{for all } x,y,z \in \mathbb{R}^d, z \neq 0.$$

Moreover, let $\mathcal{L}^j$, $j = 1,2$, *be the associated operators defined as in (3.1). Then there are* $C, C' > 0$ *such that*

$$\|u\|^2_{H^\alpha(\mathbb{R}^d)} \leq C\int_{\mathbb{R}^d}\mathcal{L}^1u(x)\overline{\mathcal{L}^2u(x)}dx + C'\|u\|^2_{L^2(\mathbb{R}^d)} \tag{3.5}$$

for all $u \in H^\alpha(\mathbb{R}^d)$. *Finally,* $\lambda + \mathcal{L}^j \colon H^\alpha(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ *is invertible for every* $\lambda \geq \lambda_0$ *for some* $\lambda_0 > 0$.

Proof: First we assume that $k_j(x,y,z) = k_j(z)$ do not depend on x, y. Then

$$\mathcal{F}[\mathcal{L}^j u](\xi) = \lim_{\varepsilon\to 0} \left(p^j_\varepsilon(\xi)\hat{u}(\xi)\right),$$

where

$$0 \leq p^j_\varepsilon(\xi) = \int_{\mathbb{R}^d\setminus B_\varepsilon(0)} (1-\cos h\cdot\xi)k_j(h)dh \leq C\|k_j\|_{\mathcal{K}^\alpha}|\xi|^\alpha,$$

cf. proof of Lemma 3.4. Hence $\lim_{\varepsilon\to 0}\left(p^j_\varepsilon(\xi)\hat{u}(\xi)\right) = p^j(\xi)\hat{u}(\xi)$ in $L^2(\mathbb{R}^d)$, where $p^j(\xi) = \lim_{\varepsilon\to 0} p^j_\varepsilon(\xi)$ satisfies

$$\begin{aligned} p^j(\xi) &\geq c_0\int_{\mathbb{R}^d}(1-\cos h\cdot\xi)|h|^{-d-\alpha}dh \\ &= c_0|\xi|^\alpha\int_{\mathbb{R}^d}\left(1-\cos s\cdot\frac{\xi}{|\xi|}\right)|s|^{-d-\alpha}ds \geq c_0'|\xi|^\alpha \end{aligned}$$

Therefore

$$\begin{aligned} \|u\|^2_{H^\alpha(\mathbb{R}^d)} &= C\int_{\mathbb{R}^d}(1+|\xi|^2)^\alpha\hat{u}(\xi)\overline{\hat{u}(\xi)}d\xi \\ &\leq C\int_{\mathbb{R}^d}|\xi|^\alpha\hat{u}(\xi)\overline{|\xi|^\alpha\hat{u}(\xi)} + C'\|u\|^2_{L^2(\mathbb{R}^d)} \\ &\leq C\int_{\mathbb{R}^d}p^1(\xi)\hat{u}(\xi)\overline{p^2(\xi)\hat{u}(\xi)} + C'\|u\|^2_{L^2(\mathbb{R}^d)} \\ &= C\int_{\mathbb{R}^d}\mathcal{L}^1u(x)\overline{\mathcal{L}^2u(x)}dx + C'\|u\|^2_{L^2(\mathbb{R}^d)}, \end{aligned}$$

where we have used

$$c_0'|\xi|^\alpha|\operatorname{Re}\hat{u}(\xi)| \leq p^j(\xi)|\operatorname{Re}\hat{u}(\xi)|$$

and the same for $\operatorname{Im}\hat{u}(\xi)$. Note that the constants C, C' above depend only on c_0, d, and α. Moreover, in the present case it is easy to see that $\lambda + \mathcal{L}^j$ is invertible for $\lambda > 0$ and its inverse is given by

$$\left(\lambda+\mathcal{L}^j\right)^{-1}f = \mathcal{F}^{-1}\left[(\lambda+p^j(\xi))^{-1}\hat{f}(\xi)\right] \qquad \text{for } f\in L^2(\mathbb{R}^d)$$

and satisfies

$$\lambda\left\|\left(\lambda+\mathcal{L}^j\right)^{-1}f\right\|_{L^2(\mathbb{R}^d)} + \left\|\left(\lambda+\mathcal{L}^j\right)^{-1}f\right\|_{H^\alpha(\mathbb{R}^d)} \leq C\|f\|_{L^2(\mathbb{R}^d)} \tag{3.6}$$

for all $f\in L^2(\mathbb{R}^d)$ uniformly in $\lambda\geq 1$.

Next we consider the case that $k_j \in \mathcal{K}^\alpha(R_0)$ such that $\left|k_j - \widetilde{k}_j\right|_{\mathcal{K}^\alpha} \leq \varepsilon \leq 1$ for some $\widetilde{k}_j(z) \in \mathcal{K}^\alpha(R)$ independent of x, y satisfying the assumptions of the lemma with the same c_0 with $\varepsilon > 0$ to be chosen later. Denoting the operators associated by $\widetilde{k}_j$ by $\widetilde{\mathcal{L}}^j$ we obtain

$$\begin{aligned}\|u\|^2_{H^\alpha(\mathbb{R}^d)} &\leq C\int_{\mathbb{R}^d} \widetilde{\mathcal{L}}^1 u(x)\overline{\widetilde{\mathcal{L}}^2 u(x)}dx + C'\|u\|^2_{L^2(\mathbb{R}^d)}\\ &\leq C\int_{\mathbb{R}^d} \mathcal{L}^1 u(x)\overline{\mathcal{L}^2 u(x)}dx + C'\|u\|^2_{L^2(\mathbb{R}^d)}\\ &\quad +(\varepsilon + R_0^d)C''\|u\|^2_{H^\alpha(\mathbb{R}^d)} + C'''\max_{j=1,2}\|k_j - \widetilde{k}_j\|^2_{\mathcal{K}^\alpha}\|u\|^2_{H^{\frac{\alpha}{2}}(\mathbb{R}^d)},\end{aligned}$$

where we applied Lemma 3.4 to $\mathcal{L} = \widetilde{\mathcal{L}}^j - \mathcal{L}^j$. Choosing $\varepsilon = \varepsilon(c_0, d, \alpha) > 0$ and $R_0 = R_0(c_0, d, \alpha) > 0$ small enough we obtain

$$\begin{aligned}\|u\|^2_{H^\alpha(\mathbb{R}^d)} &\leq C\int_{\mathbb{R}^d} \mathcal{L}^1 u(x)\overline{\mathcal{L}^2 u(x)}dx\\ &\quad +C'\|u\|^2_{L^2(\mathbb{R}^d)} + C''\max_{j=1,2}\|k_j - \widetilde{k}_j\|^2_{\mathcal{K}^\alpha}\|u\|^2_{H^{\frac{\alpha}{2}}(\mathbb{R}^d)}\end{aligned}$$

for all $k_j \in \mathcal{K}^\alpha(R_0)$ as above. Furthermore, since $\lambda - \widetilde{\mathcal{L}}^j$ is invertible and

$$\begin{aligned}&\|(\widetilde{\mathcal{L}}^j - \mathcal{L}^j)(\lambda - \mathcal{L}^j)^{-1}f\|_{L^2(\mathbb{R}^d)}\\ &\leq C(\varepsilon + R_0^d)\|(\lambda - \mathcal{L}^j)^{-1}f\|_{H^\alpha(\mathbb{R}^d)} + C\|(\lambda - \mathcal{L}^j)^{-1}f\|_{H^{\frac{\alpha}{2}}(\mathbb{R}^d)}\\ &\leq C\left(\varepsilon + R_0^d + |\lambda|^{\frac{1}{2}}\right)\|f\|_{L^2(\mathbb{R}^d)}\end{aligned}$$

due to Lemma 3.4 and (3.6), the $\mathcal{L}(L^2)$-norm of $\left(\widetilde{\mathcal{L}}^j - \mathcal{L}^j\right)(\lambda - \mathcal{L}^j)^{-1}$ is arbitrarily small if $\varepsilon, R_0 > 0$ are chosen sufficiently small and $\lambda \leq \lambda_0$ is sufficiently large. Hence $\lambda + \mathcal{L}^j$ is invertible and satisfies

$$\lambda\left\|\left(\lambda + \mathcal{L}^j\right)^{-1} f\right\|_{L^2(\mathbb{R}^d)} + \left\|\left(\lambda + \mathcal{L}^j\right)^{-1} f\right\|_{H^\alpha(\mathbb{R}^d)} \leq C\|f\|_{L^2(\mathbb{R}^d)} \tag{3.7}$$

for all $f \in L^2(\mathbb{R}^d)$, $\lambda \geq \lambda_0$.

Now, let $k_j \in \mathcal{K}^\alpha(R)$ be as in the assumptions of the lemma. Then we can choose finitely many balls $B_r(x_l)$, $l = 1, \ldots, N$, $r \leq R_0$ such that $B_R(0) \subseteq \bigcup_{l=1}^N B_{\frac{r}{2}}(x_l)$ and

$$|k_j(x, y, z) - k_j(x_l, x_l, z)| \leq \varepsilon|z|^{-d-\alpha} \qquad \text{for all } x, y \in B_r(x_l), \tag{3.8}$$

where ε and R_0 are as above. Furthermore, let φ_l, $l = 0, \dots, N$, be smooth functions such that $(\varphi_l)^2$, $l = 0, \dots, N$ is a partition of unity on $\mathbb{R}^d$ with $\operatorname{supp} \varphi_l \subset B_r(x_l)$ and $\varphi_l \equiv 1$ on $B_{\frac{r}{2}}(x_l)$ for $j = l, \dots, N$ and let $\psi_l \in C_0^\infty(B_r(x_l))$ be such that $\psi_l \equiv 1$ on $\operatorname{supp} \varphi_l$. Moreover, let

$$k_j^l(x,y,z) := \psi_l(x)\psi_l(y)k_j(x,y,z) + (1 - \psi_l(x)\psi_l(y))k_j(x_l, x_l, z)$$

for $l = 1, \dots, N$, and $k_j^0(x,y,z) = \widetilde{k}_j(z)$, where $\widetilde{k}_j(z) = k_j(x,y,z)$ for $|x|, |y| \geq R$. Then $|k_j^l - k_j^l(x_l, x_l, .)|_{\mathcal{K}^\alpha} \leq \varepsilon$ by (3.8). Let $\mathcal{L}_l^j$ denote the operator with kernel k_j^l. Hence we can use the statement for the cases proved so far to conclude that

$$\begin{aligned}
\|u\|^2_{H^\alpha(\mathbb{R}^d)} &\leq C_N \sum_{l=0}^N \|\varphi_j^2 u\|^2_{H^\alpha(\mathbb{R}^d)} \\
&\leq C_N \sum_{l=0}^N \left(\int_{\mathbb{R}^d} \mathcal{L}_l^1(\varphi_l^2 u) \overline{\mathcal{L}_l^2(\varphi_l^2 u)} dx + (1 + \max_{j=1,2} \|k_j\|_{\mathcal{K}^\alpha}) \|\varphi_l^2 u\|_{H^{\frac{\alpha}{2}}(\mathbb{R}^d)} \right) \\
&\leq C_N \left(\sum_{l=0}^N \int_{\mathbb{R}^d} \varphi_l \mathcal{L}_l^1(\varphi_l u) \overline{\varphi_l \mathcal{L}_l^2(\varphi_l u)} dx + (1 + \max_{j=1,2} \|k_j\|_{\mathcal{K}^\alpha}) \|u\|_{H^{\frac{\alpha}{2}}(\mathbb{R}^d)} \right) \\
&\leq C_N \left(\sum_{l=0}^N \int_{\mathbb{R}^d} \varphi_l \mathcal{L}^1(\varphi_l u) \overline{\varphi_l \mathcal{L}^2(\varphi_l u)} dx + (1 + \max_{j=1,2} \|k_j\|_{\mathcal{K}^\alpha}) \|u\|_{H^{\frac{\alpha}{2}}(\mathbb{R}^d)} \right) \\
&\leq C_N \left(\sum_{l=0}^N \int_{\mathbb{R}^d} \varphi_l^4 \mathcal{L}^1 u(x) \overline{\mathcal{L}^2 u(x)} dx + (1 + \max_{j=1,2} \|k_j\|_{\mathcal{K}^\alpha}) \|u\|_{H^{\frac{\alpha}{2}}(\mathbb{R}^d)} \right) \\
&\leq C_N \left(\int_{\mathbb{R}^d} \mathcal{L}^1 u(x) \overline{\mathcal{L}^2 u(x)} dx + (1 + \max_{j=1,2} \|k_j\|_{\mathcal{K}^\alpha}) \|u\|_{H^{\frac{\alpha}{2}}(\mathbb{R}^d)} \right),
\end{aligned}$$

where we have used the Lemma 3.6 and

$$\varphi_l(x) \mathcal{L}_l^j(\varphi_l u)(x) = \varphi_l(x) \mathcal{L}^j(\varphi_l u)(x) + g_{j,l}(x) u(x)$$

for a bounded function $g_{j,l}(x)$. Using $\|u\|_{H^{\frac{\alpha}{2}}} \leq C_\varepsilon \|u\|_{L^2} + \varepsilon \|u\|_{H^\alpha}$ for suitable small ε finishes the proof.

Finally, we prove the invertibility of $\lambda + \mathcal{L}^j$. First of all by (3.5) for $\mathcal{L}^1 = \mathcal{L}^2 = \mathcal{L}^j$ and $(\mathcal{L}^j u, u) = \mathcal{E}^j(u,u) \geq 0$

$$\begin{aligned}
\|\lambda + \mathcal{L}^j u\|^2_{L^2(\mathbb{R}^d)} &= \int_{\mathbb{R}^d} \mathcal{L}^j u(x) \overline{\mathcal{L}^j(x)} dx + 2\lambda (\mathcal{L}^j u, u) + \lambda^2 \|u\|^2_{L^2(\mathbb{R}^d)} \\
&\geq \frac{1}{C} \|u\|^2_{H^\alpha(\mathbb{R}^d)} + \left(\lambda^2 - \frac{C'}{C} \right) \|u\|^2_{L^2(\Omega)}.
\end{aligned}$$

This implies that the range of $\lambda + \mathcal{L}^j$ is closed and $\lambda + \mathcal{L}^j$ is injective for every $\lambda \geq \lambda_0$ for some $\lambda_0 > 0$ large enough. Hence $\lambda + \mathcal{L}^j$ is a semi-Fredholm operator for every $\lambda \geq \lambda_0$. Therefore it is sufficient to prove that $\lambda + \mathcal{L}^j$ is invertible for some $\lambda \geq \lambda_0$ because of the homotopy invariance of the Fredholm index.

Using the cut-off functions above, we construct an approximate resolvent $R_\lambda \colon L^2(\mathbb{R}^d) \to H^\alpha(\mathbb{R}^d)$ by

$$R_\lambda^j f = \sum_{l=1}^N \varphi_l^2 (\lambda + \mathcal{L}_l^j)^{-1} f, \qquad f \in L^2(\mathbb{R}^d),$$

Then we calculate similarly as above that

$$\begin{aligned}
(\lambda + \mathcal{L}^j) R_\lambda f &= \sum_{l=1}^N \varphi_l (\lambda + \mathcal{L}^j) \varphi_l (\lambda + \mathcal{L}_l^j)^{-1} f + S R_\lambda f \\
&= \sum_{l=1}^N \varphi_l (\lambda + \mathcal{L}_l^j) \varphi_l (\lambda + \mathcal{L}_l^j)^{-1} f + S' R_\lambda f \\
&= \sum_{l=1}^N \varphi_l^2 (\lambda + \mathcal{L}_l^j)(\lambda + \mathcal{L}_l^j)^{-1} f + S'' R_\lambda f \\
&= \sum_{l=1}^N \varphi_l^2 f + S' R_\lambda f = f + S'' R_\lambda f,
\end{aligned}$$

where $S, S', S'' \colon H_0^{\frac{\alpha}{2}}(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ are bounded operator by Lemma 3.6. Because of (3.7), we obtain by interpolation

$$\|(\lambda + \mathcal{L}_l^j)^{-1} f\|_{H^{\frac{\alpha}{2}}(\mathbb{R}^d)} \leq C |\lambda|^{-\frac{1}{2}} \|f\|_{L^2(\mathbb{R}^d)},$$

uniformly in $\lambda \geq \lambda_0$, $f \in L^2(\mathbb{R}^d)$ for some $\lambda_0 > 0$. Thus we see that $S'' R_\lambda = O(|\lambda|^{-\frac{1}{2}})$ in $\mathcal{L}(L^2(\mathbb{R}^d))$ as $\lambda \to \infty$. Hence $\lambda + \mathcal{L}^j$ is invertible for sufficiently large $\lambda \geq \lambda_0 > 0$. ■

Lemma 3.8 *Let $k_j \in \mathcal{K}^\alpha(R)$, $\alpha \in (1,2)$, $j = 1, 2$, $R > 0$ and let $\mathcal{L}^j$ be defined as in (3.1) with k replaced by k_j. Then*

$$\left|(\mathcal{L}^1 u, \mathcal{L}^2 v) - (\mathcal{L}^2 u, \mathcal{L}^1 v)\right| \leq C \left(\|u\|_{H^{\frac{\alpha}{2}}(\mathbb{R}^d)} \|v\|_{H^\alpha(\mathbb{R}^d)} + \|u\|_{H^\alpha(\mathbb{R}^d)} \|v\|_{H^{\frac{\alpha}{2}}(\mathbb{R}^d)}, \right)$$

where C depends on k^1, k^2 but is independent of $u, v \in H^\alpha(\mathbb{R}^d)$.

Proof: First, we consider the case that $k_j(x,y,z) = k_j(z)$ is independent of $x, y \in \mathbb{R}^d$. Then the statement is trivial since $k_j(-z) = k_j(z)$, $\mathcal{L}^1$ and $\mathcal{L}^2$ commute and therefore

$$(\mathcal{L}^1 u, \mathcal{L}^2 v) = (\mathcal{L}^2\mathcal{L}^1 u, v) = (\mathcal{L}^1\mathcal{L}^2 u, v) = (\mathcal{L}^2 u, \mathcal{L}^1 v)$$

for all $u, v \in C_0^\infty(\mathbb{R}^d)$. Note that $\mathcal{L}^j u = \mathcal{F}^j[p^j(\xi)\hat{u}(\xi)]$ as seen in the proof of Lemma 3.7. More generally, if $k_j(z)$ is a complex valued function satisfying $k_j(-z) = k_j(z)$ and $|k_j(z)| \leq C|z|^{-d-\alpha}$, then

$$(\mathcal{L}^1 u, \mathcal{L}^2 v) = (\bar{\mathcal{L}}^2\mathcal{L}^1 u, v) = (\mathcal{L}^1\bar{\mathcal{L}}^2 u, v) = (\bar{\mathcal{L}}^2 u, \bar{\mathcal{L}}^1 v)$$

where $\bar{\mathcal{L}}^j$ denotes the operator with $k_j(z)$ replaced by $\overline{k_j(z)}$. (This will be needed in the following.)

Secondly, let $k_j(x,y,z) = 0$ if $|x| + |y| \geq R$. As in the proof of Lemma 3.4.

$$\begin{aligned}
\mathcal{L}^j u &= \text{p. v.} \int_{\mathbb{R}^d} (u(x) - u(y))k_j(x,x,x-y)dy \\
&\quad + \int_{\mathbb{R}^d} (u(x) - u(y))(k_j(x,y,x-y) - k_j(x,x,x-y))dy \\
&\equiv \mathcal{L}_1^j u + \mathcal{L}_2^j u,
\end{aligned}$$

where $\|\mathcal{L}_2^j u\|_{L^2(\mathbb{R}^d)} \leq C\|u\|_{H^{\frac{\alpha}{2}}(\mathbb{R}^d)}$. Hence it is sufficient to consider the case that $k_j(x,y,z)$ is independent of y. Moreover, we assume for simplicity that $R < \pi$. Then we can use a Fourier series expansion in $[-\pi,\pi]^d$ to decompose k_j as

$$k_j(x,z) = \sum_{l\in\mathbb{Z}^d} e^{ix\cdot l} k_l^j(z) \qquad \text{for all } x \in [-\pi,\pi]^d, z \neq 0,$$

where $k_l^j(-z) = k_l^j(z)$, $\overline{k_l^j(z)} = k_{-l}^j(z)$ since $k_j(x,z)$ is real, and

$$\begin{aligned}
|k_l^j(z)| &\leq C(1+|l|)^{-d-2}\|k_j(.,z)\|_{C^{d+2}(\mathbb{R}^d)} \\
&\leq C(1+|l|)^{-d-2}\|k_j\|_{\mathcal{K}^\alpha}|z|^{-d-\alpha}.
\end{aligned}$$

Furthermore, let $\psi \in C_0^\infty(\mathbb{R}^d))$ such that $\psi \equiv 1$ in $B_R(0)$, $\operatorname{supp}\psi \subseteq [-\pi,\pi]^d$, and $\psi(-x) = \psi(x)$. Hence

$$\mathcal{L}^j u(x) = \sum_{l\in\mathbb{Z}^d} \varphi_l(x)\mathcal{L}_l^j u(x),$$

where $\mathcal{L}_l^j$ denotes the operator defined as in (3.1) with kernel $k_l^j(x-y)$ and $\varphi_l(x) = \psi(x)e^{ix\cdot l}$. Now

$$\begin{aligned}(\mathcal{L}^1 u, \mathcal{L}^2 v) &= \sum_{l\in\mathbb{Z}^d} (\varphi_l \mathcal{L}_l^1 u, \mathcal{L}^2 v) \\ &= \sum_{l\in\mathbb{Z}^d} (\mathcal{L}_l^1 u, \mathcal{L}^2(\varphi_{-l} v)) + \sum_{l\in\mathbb{Z}^d} (\mathcal{L}_l^1 u, [\mathcal{L}^2, \varphi_{-l}]v)),\end{aligned}$$

where

$$\begin{aligned}\sum_{l\in\mathbb{Z}^d} \left|(\mathcal{L}_l^1 u, [\mathcal{L}^2, \varphi_{-l}]v))\right| &\leq C \sum_{l\in\mathbb{Z}^d} (1+|l|)^{-d-2} \|u\|_{H^\alpha(\mathbb{R}^d)} \|\varphi_{-l}\|_{C^\beta(\mathbb{R}^d)} \|v\|_{H^{\frac{\alpha}{2}}(\mathbb{R}^d)} \\ &\leq \sum_{l\in\mathbb{Z}^d} (1+|l|)^{-d-2+\beta} \|u\|_{H^\alpha(\mathbb{R}^d)} \|v\|_{H^{\frac{\alpha}{2}}(\mathbb{R}^d)} \\ &\leq C \|u\|_{H^\alpha(\mathbb{R}^d)} \|v\|_{H^{\frac{\alpha}{2}}(\mathbb{R}^d)}\end{aligned}$$

by Lemma 3.6 with $\beta \in (\frac{\alpha}{2}, 1)$. Using the latter argument again, we calculate

$$\begin{aligned}(\mathcal{L}^1 u, \mathcal{L}^2 v) &= \sum_{l,l'\in\mathbb{Z}^d} (\mathcal{L}_l^1(\varphi_{-l'} u), \mathcal{L}_{l'}^2(\varphi_{-l} v)) + R(u,v) \\ &= \sum_{l,l'\in\mathbb{Z}^d} (\mathcal{L}_{-l'}^2(\varphi_{-l'} u), \mathcal{L}_{-l}^1(\varphi_{-l} v)) + R(u,v) \\ &= \sum_{l,l'\in\mathbb{Z}^d} (\varphi_{l'} \mathcal{L}_{l'}^2 u, \varphi_l \mathcal{L}_l^1 v) + R'(u,v), \\ &= (\mathcal{L}^2 u, \mathcal{L}^1 v) + R'(u,v)\end{aligned}$$

where

$$|R(u,v)| + |R'(u,v)| \leq C\left(\|u\|_{H^\alpha(\mathbb{R}^d)} \|v\|_{H^{\frac{\alpha}{2}}(\mathbb{R}^d)} + \|u\|_{H^{\frac{\alpha}{2}}(\mathbb{R}^d)} \|v\|_{H^\alpha(\mathbb{R}^d)}\right).$$

Similar calculations as above can be used to prove the statement in the case that $k^1(x,y,z) = k^1(z)$ is independent of x, y and $k^2(x,y,z) = 0$ if $|x| + |y| \geq R$.

Finally, general $k_j \in \mathcal{K}^\alpha(R)$ can be decomposed in

$$k_j(x,y,z) = \widetilde{k}_j(z) + k_j'(x,y,z),$$

where $k_j'(x,y,z) = 0$ if $|x|+|y| \geq R$. Applying the cases proved so far finishes the proof. ■

4 Bilinear forms associated to the integral operators

Let Ω be a bounded domain, $\Omega = \mathbb{R}^d$ or $\Omega = \mathbb{R}^d_+$. In this section we study the bilinear form associated to the integral operator L, which is

$$\mathcal{E}(v,w) = \frac{1}{2}\int\limits_{\Omega}\int\limits_{\Omega} \big(v(y)-v(x)\big)\big(w(y)-w(x)\big)k(x,y,x-y)\,dx\,dy, \tag{4.1}$$

where $v,w \in H_0^{\frac{\alpha}{2}}(\Omega)$, $k\colon \mathbb{R}^d\times\mathbb{R}^d\times\big(\mathbb{R}^d\setminus\{0\}\big)\to(0,\infty)$ is a measurable function such that $k(x,y,z)=k(y,x,-z)$, and

$$c_0|z|^{-d-\alpha} \le k(x,y,z) \le C_0|z|^{-d-\alpha} \tag{4.2}$$

for almost all $x,y,z\in\mathbb{R}^d$.

Straight from the definition we get that $\mathcal{E}$ is a coercive bounded symmetric bilinear form on $H_0^{\frac{\alpha}{2}}(\Omega)$:

$$\begin{aligned} |\mathcal{E}(v,w)| &\le C\|v\|_{H^{\frac{\alpha}{2}}(\Omega)}\|w\|_{H^{\frac{\alpha}{2}}(\Omega)} \\ |\mathcal{E}(v,v)| &\ge C'\|v\|^2_{H^{\frac{\alpha}{2}}(\Omega)} \end{aligned} \tag{4.3}$$

for all $v,w\in H_0^{\frac{\alpha}{2}}(\Omega)$, where C,C' depend only on d,s,α and c_0. Note that for these properties of the bilinear form less conditions on the kernel k are required than for the kernel of the operator L. As an immediate consequence of the lemma by Lax-Milgram we obtain:

Corollary 4.1 *Let $\mathcal{E}$ and k be as above. Then for every $f\in H^{-\frac{\alpha}{2}}(\Omega)$ there is a unique $u\in H_0^{\frac{\alpha}{2}}(\Omega)$ such that*

$$\mathcal{E}(u,\varphi) = \langle f,\varphi\rangle \qquad \text{for all } \varphi\in H_0^{\frac{\alpha}{2}}(\Omega). \tag{4.4}$$

Note that the result above is still valid if $k(x,y,x-y)$ is replaced by a function $\widetilde{k}(x,y)$, symmetric in x,y, satisfying

$$c_0|x-y|^{-d-\alpha} \le |\widetilde{k}(x,y)| \le \frac{1}{c_0}|x-y|^{-d-\alpha}.$$

The special form of the kernel is used for the following connection between the bilinear form $\mathcal{E}$ and the integral operator L:

Lemma 4.2 *Let $k \in \mathcal{K}^\alpha(R)$, $R > 0$, $\alpha \in (1,2)$, and let $\mathcal{E}$ and L be defined as in (4.1) and (1.2), respectively. Then*

$$\mathcal{E}(v,\varphi) = \int_\Omega Lv(x)\varphi(x)dx \qquad \text{for all } \varphi \in C_0^\infty(\Omega)$$

provided that $v \in H^\alpha_{loc}(\Omega) \cap H^{\frac{\alpha}{2}}(\Omega)$ if Ω is a bounded domain or $\Omega = \mathbb{R}^d_+$ and $v \in H^\alpha(\Omega)$ if $\Omega = \mathbb{R}^d$. Moreover, if $\psi \in C^1_b(\overline{\Omega})$, then

$$\mathcal{E}(v,\psi w) = \mathcal{E}(\psi v, w) + ([\psi, L]v, w) \qquad \text{for all } v, w \in H_0^{\frac{\alpha}{2}}(\Omega). \tag{4.5}$$

Proof: Using the symmetries of the kernel k we easily calculate

$$\begin{aligned}
\mathcal{E}(v,\varphi) &= \lim_{\varepsilon\to 0}\frac{1}{2}\int_{\{|x-y|\geq\varepsilon\}}(v(y)-v(x))(\varphi(y)-\varphi(x))k(x,y,x-y)d(x,y)\\
&= \lim_{\varepsilon\to 0}\int_\Omega\int_{\Omega\setminus B_\varepsilon(x)}(v(x)-v(y))k(x,y,x-y)dy\varphi(x)dx\\
&= \int_\Omega Lv(x)\varphi(x)dx,
\end{aligned}$$

where we used Lemma 3.5 in order to exchange the order of $\lim_{\varepsilon\to 0}$ and integration with respect to x.

Finally, since $C_0^\infty(\Omega)$ is dense in $H_0^{\frac{\alpha}{2}}(\Omega)$, it is sufficient to prove (4.5) for $v, w \in C_0^\infty(\Omega)$ for which the statement immediately follows from the first part and Lemma 3.6. ■

Using the latter relation we obtain the following result on higher regularity of solutions of (4.4):

Lemma 4.3 *Let $u \in H_0^{\frac{\alpha}{2}}(\Omega)$ be the solution of (4.4) and $f \in H^{-\frac{\alpha}{2}}(\Omega)$. If $\Omega = \mathbb{R}^d$ and additionally $f \in L^2(\mathbb{R}^d)$, then $u \in H^\alpha(\mathbb{R}^d)$ and $\|u\|_{H^\alpha(\mathbb{R}^d)} \leq C\|f\|_{L^2(\mathbb{R}^d)}$. Moreover, if Ω is a bounded domain or $\Omega = \mathbb{R}^d_+$ and additionally $f \in L^2_{loc}(\Omega)$, then $u \in H^\alpha_{loc}(\Omega)$ and for every $\Omega' \Subset \Omega'' \Subset \Omega$*

$$\|u\|_{H^\alpha(\Omega')} \leq C(\Omega',\Omega'')\left(\|f\|_{L^2(\Omega'')} + \|f\|_{H^{-\frac{\alpha}{2}}(\Omega)}\right).$$

Proof: First let $\Omega = \mathbb{R}^d$. Moreover, let $v \in H^\alpha(\mathbb{R}^d)$ be the unique solution of $(1+\mathcal{L})v = f + u$, which exists due to Lemma 3.7. Recall the definition of the operator $\mathcal{L}$ 3.1. By Lemma 4.2 v solves

$$(v,\varphi)+\mathcal{E}(v,\varphi)=(f+u,\varphi)=(u,\varphi)+\mathcal{E}(u,\varphi).$$

Hence $u=v\in H^\alpha(\mathbb{R}^d)$ by the coerciveness of $\mathcal{E}$.

Next let Ω be a bounded domain or let $\Omega=\mathbb{R}^d_+$. Then for every $\psi,\eta\in C_0^\infty(\Omega)$ with $\psi\equiv 1$ on $\operatorname{supp}\eta$,

$$\begin{aligned}\mathcal{E}(u,\eta\varphi)&=\int_\Omega(\psi u)(x)L(\eta\varphi)(x)dx-\int_\Omega u[\psi,L](\eta\varphi)dx\\&=\int_{\mathbb{R}^d}(\psi u)(x)\mathcal{L}(\eta\varphi)(x)dx-\int_\Omega u[\psi,L](\eta\varphi)dx+\int_{\mathbb{R}^d}gu\varphi dx\,,\end{aligned}$$

where

$$g(x)=\frac{\psi(x)}{\varphi(x)}\big\{L(\eta\varphi)(x)-\mathcal{L}(\eta\varphi)(x)\big\}=-\psi(x)\eta(x)\int_{\Omega^c}k(x,y,x-y)\,dy\,.$$

Note that $g\in L^\infty(\mathbb{R}^d)$ depends only on ψ,η and k. We obtain further

$$\begin{aligned}\mathcal{E}(u,\eta\varphi)&=\int_{\mathbb{R}^d}\eta u(x)\mathcal{L}\varphi dx-\int_\Omega u[\psi,L](\eta\varphi)dx\\&\quad-\int_{\mathbb{R}^d}\psi u[\eta,\mathcal{L}]\varphi dx+\int_{\mathbb{R}^d}gu\varphi dx\\&=\mathcal{E}_{\mathbb{R}^d}(\eta u,\varphi)+I(u,\varphi)+\int_{\mathbb{R}^d}gu\varphi dx\end{aligned}$$

for all $\varphi\in C_0^\infty(\mathbb{R}^d)$. Here

$$I(u,\varphi)=-\int_\Omega u[\psi,L](\eta\varphi)dx-\int_{\mathbb{R}^d}\psi u[\eta,\mathcal{L}]\varphi\,dx\,,$$

and we make use of the notation: $\mathcal{E}_{\mathbb{R}^d}(v,w)=\int_{\mathbb{R}^d}v(x)\mathcal{L}(w)(x)dx$. Altogether, $v=\eta u$ solves

$$\mathcal{E}_{\mathbb{R}^d}(\eta u,\varphi)=\langle f,\eta\varphi\rangle+I(u,\varphi)+\int_{\mathbb{R}^d}gu\varphi dx,$$

where the right-hand side defines a bounded functional on $L^2(\mathbb{R}^d)$ because of (3.4). Thus $\eta u\in H^\alpha(\mathbb{R}^d)$ and

$$\|\eta u\|_{H^\alpha(\mathbb{R}^d)}\leq C(\psi,\eta)\left(\|\eta f\|_{L^2(\mathbb{R}^d)}+\|u\|_{H^{\frac{\alpha}{2}}(\Omega)}\right).$$

Since $\eta\in C_0^\infty(\Omega)$ is arbitrary, this implies the statement of the lemma. ∎

5 Proof of Theorem 1.1

The strategy of our main proof is as follows. An equivalent formulation to (1.6) is the following:

$$\left\{\begin{array}{rcll} L^1u(x)+a^1(x)u(x)-f^1(x) & \leq & 0 & \text{in } \Omega, \\ L^2u(x)+a^2(x)u(x)-f^2(x) & \leq & 0 & \text{in } \Omega, \\ \big(L^1u+a^1u-f^1\big)\big(L^2u+a^2u-f^2\big) & = & 0 & \text{in } \Omega, \\ u & = & 0 & \text{on } \partial\Omega. \end{array}\right. \tag{5.1}$$

where the first three lines are supposed to hold almost everywhere in Ω. We define penalty functions $\beta_\varepsilon : \mathbb{R} \to \mathbb{R}$ as smooth versions of the function $t \mapsto t_+$. More precisely, for $\varepsilon > 0$ we assume that β_ε satisfies

$$\beta_\varepsilon \in C^\infty(\mathbb{R}), \qquad \beta_\varepsilon\big|_{(-\infty,0]} \equiv 0\,, \qquad \beta_\varepsilon \text{ monotone}\,, \tag{5.2}$$

$$\beta_\varepsilon(t) \leq t \quad \forall t > 0 \qquad |\beta_\varepsilon(t) - t| \leq \varepsilon^2 \quad \forall\, t > 0\,. \tag{5.3}$$

As a consequence of the definition we obtain

$$r\beta_\varepsilon(t) \leq \beta_\varepsilon(rt) + \varepsilon^2(1+r) \qquad \forall r, t \geq 0\,. \tag{5.4}$$

We will obtain the solution u of (1.6) as a limit of approximating solutions $\mathbf{u}_\varepsilon = (u^1_\varepsilon, u^2_\varepsilon) \to (u, u)$ that satisfy the following equations:

$$L^1u^1_\varepsilon + a^1u^1_\varepsilon - f^1 + \varepsilon^{-1}\beta_\varepsilon\big(u^1_\varepsilon - u^2_\varepsilon\big) = 0 \quad \text{a.e. in } \Omega\,, \tag{5.5}$$

$$L^2u^2_\varepsilon + a^2u^2_\varepsilon - f^2 + \varepsilon^{-1}\beta_\varepsilon\big(u^2_\varepsilon - u^1_\varepsilon\big) = 0 \quad \text{a.e. in } \Omega\,, \tag{5.6}$$

$$u^1_\varepsilon, u^2_\varepsilon = 0 \quad \text{on } \partial\Omega\,, \tag{5.7}$$

where (5.7) is understood in the sense of traces. Note that this step of our proof is similar to the one in [EF79]. We shall also mention [Hel01] where a similar strategy was applied to a local Bellman equation with additional nonlinearities.

Definition 5.1 *For $j = 1, 2$ and functions $v, w \in H^{\frac{\alpha}{2}}(\Omega)$ set*

$$\mathcal{E}^j(v,w) = \frac{1}{2}\int\limits_\Omega\int\limits_\Omega \big(v(y)-v(x)\big)\big(w(y)-w(x)\big)k_j(x,y,x-y)\,dx\,dy\,.$$

We say, a function $\mathbf{u}_\varepsilon \in H_0^{\frac{\alpha}{2}}(\Omega;\mathbb{R}^2)$ *is a weak solution of the system (5.5)-(5.7) if the following set of equations holds true for all test functions* $\boldsymbol{\varphi} \in C_0^\infty(\Omega;\mathbb{R}^2)$*:*

$$\mathcal{E}^1(u_\varepsilon^1,\varphi^1) + \int\limits_\Omega \Big(a^1 u_\varepsilon^1 - f^1 + \varepsilon^{-1}\beta_\varepsilon(u_\varepsilon^1 - u_\varepsilon^2)\Big)\varphi^1\, dx = 0\,, \tag{5.8}$$

$$\mathcal{E}^2(u_\varepsilon^2,\varphi^2) + \int\limits_\Omega \Big(a^2 u_\varepsilon^2 - f^2 + \varepsilon^{-1}\beta_\varepsilon(u_\varepsilon^2 - u_\varepsilon^1)\Big)\varphi^2\, dx = 0\,. \tag{5.9}$$

Remark 5.2 $\big(\mathcal{E}^j, C_0^\infty(\Omega)\big)$ *is closable in* $L^2(\Omega)$*. Let* $\mathcal{F}$ *be the closure of* $C_0^\infty(\Omega)$ *under the scalar product* $\mathcal{E}^j(\cdot,\cdot) + (\cdot,\cdot)_{L^2(\Omega)}$*. By the assumption (4.2)* $\mathcal{F} = H_0^{\frac{\alpha}{2}}(\Omega)$*. Moreover, we note that*

$$\mathcal{E}^j(u_+,u_-) = \frac{1}{2}\int_\Omega\int_\Omega (u_+(x)-u_+(y))(u_-(x)-u_-(y))k_j(x,y,x-y)dxdy \geq 0$$

since the integrand is nonnegative. In particular, this yields $\mathcal{E}^j(u,u_+) \geq 0$ *and* $\mathcal{E}^j(u_+,u_+) \leq \mathcal{E}^j(u,u)$*. Therefore the tuple* $\big(\mathcal{E}^j,\mathcal{F}\big)$ *is a regular Dirichlet form on* $L^2(\Omega)$ *[FŌT94]. Therefore, there exists an associated symmetric Hunt process taking values in* Ω*.*

We start by proving the existence of solutions to the approximating problem.

Lemma 5.3 *Consider* $\mathbf{a},\mathbf{f},\Omega$ *as in Theorem 1.1. Then, for any* $\varepsilon > 0$ *there exists a solution* $\mathbf{u}_\varepsilon \in H_0^{\frac{\alpha}{2}}(\Omega;\mathbb{R}^2)$ *satisfying (5.8), (5.9).*

Proof: Since the operators $\mathbf{u} \mapsto \varepsilon^{-1}\beta_\varepsilon(u^1-u^2)$, $\mathbf{u} \mapsto \varepsilon^{-1}\beta_\varepsilon(u^2-u^1)$ are compact perturbations of L^1 and L^2 the existence of $\mathbf{u}_\varepsilon = (u_\varepsilon^1, u_\varepsilon^2)$ satisfying (5.8), (5.9) can be proved by various means. Here, we apply the existence result of Leray-Lions, see Theorem 5.12.1 in [Mor66]. For $\mathbf{u},\mathbf{v} \in H_0^{\frac{\alpha}{2}}(\Omega;\mathbb{R}^2)$ define $\mathfrak{A}(\mathbf{u},\mathbf{v}) = \big(\mathfrak{A}^1(\mathbf{u},\mathbf{v}), \mathfrak{A}^2(\mathbf{u},\mathbf{v})\big) \in H^{-\frac{\alpha}{2}}(\Omega;\mathbb{R}^2)$ where for all $\boldsymbol{\varphi} \in H_0^{\frac{\alpha}{2}}(\Omega;\mathbb{R}^2)$

$$\langle \mathfrak{A}^1(\mathbf{u},\mathbf{v}),\varphi^1\rangle = \mathcal{E}^1(v_\varepsilon^1,\varphi^1) + \int\limits_\Omega \Big(a^1 u_\varepsilon^1 + \varepsilon^{-1}\beta_\varepsilon(u_\varepsilon^1 - u_\varepsilon^2)\Big)\varphi^1\, dx\,, \tag{5.10}$$

and analogously for $\langle \mathfrak{A}^2(\mathbf{u},\mathbf{v}),\varphi^2\rangle$. Since $\langle \mathfrak{A}(\mathbf{v},\mathbf{v}),\mathbf{v}\rangle \geq C\|\mathbf{v}\|^2_{H_0^{\frac{\alpha}{2}}(\Omega;\mathbb{R}^2)} - \frac{c}{\varepsilon}\|\mathbf{v}\|^2_{L^2(\Omega;\mathbb{R}^2)}$ for some constant $C>0$ the coercivity condition

$$\lim_{\|v\|\to\infty} \frac{\langle \mathfrak{A}(\mathbf{v},\mathbf{v}),\mathbf{v}\rangle}{\|\mathbf{v}\|} = +\infty$$

holds true. It is now easy to check conditions (i) through (iv) of Theorem 5.12.1 in [Mor66]. (ii) is fulfilled because the $\mathcal{E}^i$ are coercive. (i), (iii) and (iv) can be checked by using three tools: the definition of $\mathfrak{A}(\mathbf{u},\mathbf{v})$, growth condition (5.3) and the compact embedding $H_0^{\frac{\alpha}{2}}(\Omega) \hookrightarrow L^2(\Omega)$. Therefore, we can apply theorem and obtain the existence of $\mathbf{u}_\varepsilon$. ■

It will be crucial to our consideration to prove that $\mathbf{u}_\varepsilon$ is uniformly bounded in $H_0^{\frac{\alpha}{2}}(\Omega;\mathbb{R}^2)$. This result will derive with the help of the following lemma.

Lemma 5.4 *Let $f^1, f^2 \in H^m(\Omega)$ and $a^1, a^2 \in C_b^m(\mathbb{R}^d)$ be nonnegative functions, where $m = [\frac{d}{2}]+1$. Assume $\mathbf{u}_\varepsilon = (u_\varepsilon^1, u_\varepsilon^2) \in H_0^{\frac{\alpha}{2}}(\Omega;\mathbb{R}^2)$ solves (5.8), (5.9). Then*

1. *$u_\varepsilon \in C^{\frac{\alpha-1}{2}}(\overline{\Omega})$.*
2. *For each $j=1,2$ we have $u_\varepsilon^j(x) > 0$ for all $x \in \Omega$ unless $u_\varepsilon^j \equiv 0$.*

Proof: First we consider the case of a half-space $\Omega = \mathbb{R}^d_+$. We will prove that $u_\varepsilon^j \in H_0^{\frac{\alpha}{2}}(\mathbb{R}_+; H^m(\mathbb{R}^{d-1}))$ by approximating tangential derivatives by difference quotients. Then

$$u_\varepsilon^j \in H_0^{\frac{\alpha}{2}}(\mathbb{R}_+; H^m(\mathbb{R}^{d-1})) \hookrightarrow C^{\frac{\alpha-1}{2}}(\mathbb{R}_+; H^m(\mathbb{R}^{d-1})) \hookrightarrow C^{\frac{\alpha-1}{2}}(\overline{\mathbb{R}^d_+}),$$

by [Sim90, Corollary 26].

We denote $\tau_{i,s}f(x) = f(x+se_i)$, $\Delta^+_{i,h}f(x) = \tau_{i,h}f(x) - f(x)$, $\Delta^-_{i,h}f(x) = f(x) - \tau_{i,-h}f(x)$, $h>0$, $i=1,\dots,d-1$, where e_i is the i-th canonical unit vector. Replacing φ^j by $-h^{-s}\Delta^-_{i,h}\varphi^j$ in (5.8), (5.9), we obtain that $\mathbf{v}_h = h^{-s}\Delta^+_{i,h}\mathbf{u}_\varepsilon$, $s\in[0,1]$ solves

$$\begin{aligned}
\mathcal{E}^1(v_h^1,\varphi^1) &= -\mathcal{E}^1_{i,h}(\tau_{i,h}u_\varepsilon^1,\varphi^1) - \left(f^1 - a^1u_\varepsilon^1 - \varepsilon^{-1}\beta_\varepsilon(u_\varepsilon^1 - u_\varepsilon^2), h^{-s}\Delta^-_{i,h}\varphi^1\right)\\
\mathcal{E}^2(v_h^2,\varphi^2) &= -\mathcal{E}^2_{i,h}(\tau_{i,h}u_\varepsilon^2,\varphi^2) - \left(f^2 - a^2u_\varepsilon^2 - \varepsilon^{-1}\beta_\varepsilon(u_\varepsilon^2 - u_\varepsilon^1), h^{-s}\Delta^-_{i,h}\varphi^2\right)
\end{aligned}$$

for all $\varphi \in C_0^\infty(\Omega)$, where $\mathcal{E}^j_{i,h}$ is the bilinear form with kernel $h^{-s}(k_j(x+he_i, y+he_i, z) - k_j(x,y,z))$, where we note that by (1.4) the latter kernel is bounded by $C|z|^{-d-\alpha}$ uniformly in $h>0$. First let $s \in (\frac{1}{2}, \frac{\alpha}{2})$. Then choosing $\boldsymbol{\varphi} = \mathbf{v}_h$ and using (4.3) we conclude

$$\|\mathbf{v}_h\|^2_{H^{\frac{\alpha}{2}}(\mathbb{R}^d_+)} \leq C\left(\|\mathbf{f}\|_{L^2(\mathbb{R}^d_+)} + \|\mathbf{u}_\varepsilon\|_{L^2(\mathbb{R}^d_+)}\right)\left(\|h^{-s}\Delta^-_{i,h}\mathbf{v}_h\|_{L^2(\mathbb{R}^d_+)} + \|\mathbf{v}_h\|_{H^{\frac{\alpha}{2}}(\mathbb{R}^d_+)}\right)$$

Now we use that

$$\|h^{-s}\Delta^\pm_{i,h} w\|_{L^2(\mathbb{R}^d_+)} \leq C\|w\|_{H^s(\mathbb{R}^d_+)} \leq C\|w\|_{H^{\frac{\alpha}{2}}(\mathbb{R}^d_+)},$$

which is obtained by an easy interpolation argument. Hence

$$\sup_{i=1,\ldots,d-1}\left\|h^{-2s}\left(\Delta^+_{i,h}\right)^2\mathbf{u}_\varepsilon\right\|_{L^2(\mathbb{R}^d_+)} \leq \|\mathbf{v}_h\|_{H^{\frac{\alpha}{2}}(\mathbb{R}^d_+)} \leq C\left(\|\mathbf{f}\|_{L^2(\mathbb{R}^d_+)} + \|\mathbf{u}_\varepsilon\|_{L^2(\mathbb{R}^d_+)}\right),$$

which implies that $\mathbf{u}_\varepsilon$ is uniformly bounded in

$$L^2(\mathbb{R}_+; B^{2s}_{2,\infty}(\mathbb{R}^{d-1})) \hookrightarrow L^2(\mathbb{R}_+; H^1(\mathbb{R}^{d-1})),$$

cf. [BL76, Theorem 6.2.5]. Using this bound we choose $s=1$ in the definition of $\mathbf{v}_h$ and obtain

$$\|\mathbf{v}_h\|^2_{H^{\frac{\alpha}{2}}(\mathbb{R}^d_+)} \leq C\left(\|\mathbf{f}\|_{H^1(\mathbb{R}^d_+)} + \left(\|a\|_{C^1_b(\mathbb{R}^d)} + 1\right)\|\mathbf{u}_\varepsilon\|_{L^2(\mathbb{R}_+;H^1(\mathbb{R}^{d-1}))}\right)\|\mathbf{v}_h\|_{H^{\frac{\alpha}{2}}(\mathbb{R}^d_+)}$$

Hence $\mathbf{v}_h = h^{-1}\Delta^+_{i,h}\mathbf{u}_\varepsilon$, $h>0$, is uniformly bounded in $H^{\frac{\alpha}{2}}(\mathbb{R}^d_+;\mathbb{R}^2)$ and therefore $\partial_{x_i}\mathbf{u}_\varepsilon \in H^{\frac{\alpha}{2}}(\mathbb{R}^d_+;\mathbb{R}^2)$. Repeating these arguments m-times shows that $D^\alpha_{x'}\mathbf{u}_\varepsilon \in H^{\frac{\alpha}{2}}(\mathbb{R}^d_+;\mathbb{R}^2)$ for all $\alpha \in \mathbb{N}_0^{n-1}$, $|\alpha| \leq m$.

In order to prove the statement for a bounded domain Ω, it is sufficient to show that for every $x \in \overline{\Omega}$ and for some open neighborhood U of x $\mathbf{u}_\varepsilon$ is in $C^{\frac{\alpha-1}{2}}(\overline{\Omega}\cap U)$. Let U_0 be an open neighborhood of x and $F\colon \mathbb{R}^d \to \mathbb{R}^d$ be a C^{d+1}-diffeomorphism which maps $U_0 \cap \overline{\Omega}$ onto $\overline{\mathbb{R}^d_+} \cap V_0$ for some open set V_0. Moreover, let $\psi \in C_0^\infty(U_0)$ with $\psi \equiv 1$ on some neighborhood $U_1 \Subset U_0$ of x, let V_1 be an open set such that $V_1 \cap \overline{\mathbb{R}^d_+} = F(U_1 \cap \overline{\Omega})$ and let $F^*g(x) = g(F(x))$ denotes the pull-back of g by F. Now we obtain for $\varphi^j \in C_0^\infty(\mathbb{R}^d_+)$, $j=1,2$, that $v^j_\varepsilon := F^{*,-1}(\psi u_\varepsilon) \in H_0^{\frac{\alpha}{2}}(\mathbb{R}^d_+)$ solves

$$\begin{aligned}
\widetilde{\mathcal{E}}^1(v^1_\varepsilon, \varphi^1) &= \mathcal{E}^1(\psi u^1_\varepsilon, F^*(\varphi^1)) \\
&= \mathcal{E}^1(u^1_\varepsilon, \psi F^*(\varphi^1)) + ([L^1,\psi]u^1_\varepsilon, F^*(\varphi^1)) \\
&= (f^1 - a^1 u^1_\varepsilon - \varepsilon^{-1}\beta_\varepsilon(u^1_\varepsilon - u^2_\varepsilon), \psi F^*(\varphi^1)) + ([L^1,\psi]u^1_\varepsilon, F^*(\varphi^1))
\end{aligned}$$

and a similar expression for v_ε^2 where

$$\begin{aligned}
\widetilde{\mathcal{E}}^j(u,v) &= \int_{\mathbb{R}^d_+}\int_{\mathbb{R}^d_+}(u(x)-u(y))(v(x)-v(y))\tilde{k}_j(x,y,x-y)dxdy\\
\tilde{k}_j(x,y,z) &= k_j(F(x),F(y),A(x,y)z)\det DF(x)\det DF(y),\\
A(x,y) &= \int_0^1 DF((1-s)y+sx)ds.
\end{aligned}$$

Moreover, $\widetilde{L}^j$ denotes the associated integral operator. It is not difficult to prove that $\tilde{k}_j\in\mathcal{K}^\alpha(R')$ for some $R'=R'(R,F)$. Now all terms on the right-hand side of the equation above define a functional on $L^2(\mathbb{R}^d_+)$ due to Lemma 3.6. Hence $v_\varepsilon^j\in L^2(\mathbb{R}_+;H^1(\mathbb{R}^{d-1}))$ by the arguments in the case $\mathbb{R}^d_+$. In particular this implies that

$$\partial_{x_i}g^j(x)\equiv\partial_{x_i}F^{*,-1}\left[\psi\left(f^j-a^ju_\varepsilon^j-\varepsilon^{-1}\beta_\varepsilon(u_\varepsilon^{j\pm1}-u_\varepsilon^{j\mp1})\right)\right]\in L^2(\mathbb{R}^d_+\cap V_1)$$

for $i=1,\dots,d-1$ since $v_\varepsilon^j(x)=F^{*,-1}(u_\varepsilon^j)(x)$ for $x\in V_1$. Now choosing another $\psi\in C_0^\infty(U_1)$ such that $\psi\equiv1$ on an open neighborhood $U_2\Subset U_1$ of x one obtains that

$$\widetilde{\mathcal{E}}^j(v_\varepsilon^j,\varphi^j)=(g^j,\varphi^j)-(\eta u_\varepsilon^j,[L^j,\psi]F^*(\varphi^j))+([L^j,\psi](1-\eta)u_\varepsilon^j,F^*(\varphi^j))$$

for all $\varphi\in C_0^\infty(\mathbb{R}^d_+)$, where $\eta\in C_0^\infty(U_1)$ with $\eta\equiv1$ on $\operatorname{supp}\psi$, $\partial_{x_i}F^{*,-1}(\eta u_\varepsilon^j)=\partial_{x_i}(F^{*,-1}(\eta)v_\varepsilon^j)\in L^2(\mathbb{R}^d_+)$, and $[\psi,L^j](1-\eta)$ is a Hilbert Schmidt operator. Hence choosing $\varphi^j=h^{-2}\Delta_{i,h}^2v_\varepsilon^j$ one obtains by the same arguments as in the half-space case that $\partial_{x_i}v_\varepsilon^j\in H_0^{\frac{\alpha}{2}}(\mathbb{R}^d_+)$, $i=1,\dots,d-1$. Repeating this argument on a sequence of open neighborhoods $U_{k+1}\Subset U_k$ of x one proves that $\partial_{x'}^\alpha v_\varepsilon^j\in H_0^{\frac{\alpha}{2}}(\mathbb{R}^d_+)$ for all $\alpha'\in\mathbb{N}_0^{d-1}$, $|\alpha|\leq m$. This implies the Hölder continuity of $\mathbf{u}_\varepsilon$ in a neighborhood of x.

We prove the second part of the lemma by contradiction. Without loss of generality let $\inf_{x\in\Omega}u_\varepsilon^1(x)\leq\inf_{x\in\Omega}u_\varepsilon^2(x)$. Now assume that u_ε^1 attains its minimum at $x_0\in\Omega$ and that $\inf_{x\in\Omega}u_\varepsilon^1(x)\leq0$. Then

$$L_\varepsilon^1u_\varepsilon^1(x_0)=\mathrm{p.\,v.}\int_\Omega(u_\varepsilon^1(x_0)-u_\varepsilon^1(y))k_1(x_0,y,x_0-y)dy<0$$

unless $u_\varepsilon^1\equiv0$ since $k_1(x,y,z)>0$ for $z\neq0$. Hence, if $u_\varepsilon^1\not\equiv0$,

$$0>L_\varepsilon^1u_\varepsilon^1(x_0)=f^1(x_0)-a^1(x_0)u_\varepsilon^1(x_0)-\varepsilon^{-1}\beta_\varepsilon(u_\varepsilon^1(x_0)-u_\varepsilon^2(x_0))\geq0$$

since $a^1,f^1\geq0$, $u_\varepsilon^1(x_0)\leq0$, and $\beta_\varepsilon(u_\varepsilon^1(x_0)-u_\varepsilon^2(x_0))=0$, which is a contradiction. Hence $u_\varepsilon^1(x)>0$ in Ω unless $u_\varepsilon^1\equiv0$. Therefore $\inf_{x\in\Omega}u_\varepsilon^j(x)=0$.

Using the same argumentation as above shows that $u_2(x) > 0$ in Ω unless $u_2 \equiv 0$. ■

Before we can show that the limit u of $\mathbf{u}_\varepsilon$ solves equation (1.6) we need to establish bounds that are uniform in $\varepsilon > 0$.

Lemma 5.5 *Let* $\mathbf{f} \in L^2(\Omega;\mathbb{R}^2)$ *and* $\mathbf{a} \in L^\infty(\mathbb{R}^d;\mathbb{R}^2)$ *be nonnegative. Assume* $\mathbf{u}_\varepsilon = (u_\varepsilon^1, u_\varepsilon^2) \in H_0^{\frac{\alpha}{2}}(\Omega;\mathbb{R}^2)$ *is nonnegative and solves (5.8), (5.9). Then*

$$\|\mathbf{u}_\varepsilon\|_{H_0^{\frac{\alpha}{2}}(\Omega;\mathbb{R}^2)} + \varepsilon^{-\frac{1}{2}} \|u_\varepsilon^1 - u_\varepsilon^2\|_{L^2(\Omega)} \le C\left(\|\mathbf{f}\|_{L^2(\Omega;\mathbb{R}^2)} + 1\right), \tag{5.11}$$

where $c > 0$ *is a constant independent of* $\varepsilon \in (0,1]$.

Proof: Choosing $\varphi = (u_\varepsilon^1, u_\varepsilon^2)$ as a test function in (5.8), (5.9) and applying (1.5) proves

$$\mathcal{E}(u_\varepsilon^j, u_\varepsilon^j) + \varepsilon^{-1} \int_\Omega \beta_\varepsilon(u_\varepsilon^j - u_\varepsilon^{j\pm 1}) u_\varepsilon^j \, dx \le \int_\Omega f^j u_\varepsilon^j \, dx, \tag{5.12}$$

where $1 \pm 1 = 2, 2 \pm 1 = 1$. Note that (5.3) implies for almost all x

$$\begin{aligned}\beta_\varepsilon\big(u_\varepsilon^j(x) - u_\varepsilon^{j\pm1}(x)\big)u_\varepsilon^j(x) &\ge \beta_\varepsilon\big(u_\varepsilon^j(x) - u_\varepsilon^{j\pm1}(x)\big)\big(u_\varepsilon^j(x) - u_\varepsilon^{j\pm1}(x)\big)\\ &\ge \frac{1}{2}\big(u_\varepsilon^j(x) - u_\varepsilon^{j\pm1}(x)\big)_+^2 - \frac{\varepsilon^4}{2}.\end{aligned}$$

Applying this inequality to (5.12) and using Hölder's inequality for term on the right hand side in (5.12) one obtains (5.11). ■

Lemma 5.6 *Consider* Ω, $\mathbf{f}$ *and* $\mathbf{a}$ *as in Theorem 1.1. Then for every* $\Omega' \Subset \Omega$ *there is a constant* $C(\Omega,\Omega')$ *such that for all* $\varepsilon \in (0,1]$ *and nonnegative solutions* $\mathbf{u}_\varepsilon \in H_0^{\frac{\alpha}{2}}(\Omega) \cap H_{loc}^\alpha(\Omega)$ *of (5.8), (5.9)*

$$\|\mathbf{u}_\varepsilon\|_{H^\alpha(\Omega';\mathbb{R}^2)} \le C(\Omega')\left(\|\mathbf{f}\|_{L^2(\Omega;\mathbb{R}^2)} + 1\right). \tag{5.13}$$

Proof: Let $\psi \in C_0^\infty(\Omega)$ be such that $\psi \equiv 1$ on Ω'. Set $\mathbf{v}_\varepsilon = \psi \mathbf{u}_\varepsilon = (\psi u_\varepsilon^1, \psi u_\varepsilon^2) \in H^\alpha(\mathbb{R}^d;\mathbb{R}^2)$. Then $\mathbf{v}_\varepsilon$ is a solution of

$$\mathcal{L}^1 v_\varepsilon^1 + a^1 v_\varepsilon^1 - \widetilde{f}_\varepsilon^1 + \varepsilon^{-1}\beta_\varepsilon\big(v_\varepsilon^1 - v_\varepsilon^2\big) = 0 \quad \text{a.e. in } \mathbb{R}^d, \tag{5.14}$$

$$\mathcal{L}^2 v_\varepsilon^2 + a^2 v_\varepsilon^2 - \widetilde{f}_\varepsilon^2 + \varepsilon^{-1}\beta_\varepsilon\big(v_\varepsilon^2 - v_\varepsilon^1\big) = 0 \quad \text{a.e. in } \mathbb{R}^d, \tag{5.15}$$

where $\widetilde{f}^j_\varepsilon$ is defined by

$$\widetilde{f}^j_\varepsilon(x) = \psi(x) f^j(x) + [L^j, \psi] u^j_\varepsilon(x) + \psi(x) u^j_\varepsilon(x) \int\limits_{\mathbb{R}^d \setminus \Omega} k_j(x, y, x-y)\, dy$$
$$+ \varepsilon^{-1} \beta_\varepsilon(\psi(u^j_\varepsilon - u^{j\pm 1}_\varepsilon)) - \psi(x) \varepsilon^{-1} \beta_\varepsilon(u^j_\varepsilon - u^{j\pm 1}_\varepsilon) \quad \text{for } x \in \Omega$$

and $\widetilde{f}^j_\varepsilon(x) = \mathcal{L}^j v^j_\varepsilon(x)$ for $x \notin \Omega$. Note that $\mathcal{L}^1 v^1_\varepsilon(x) \in L^2(\mathbb{R}^d \setminus \Omega)$ since $\operatorname{supp} v^j_\varepsilon \subseteq \operatorname{supp} \psi \Subset \Omega$ and that

$$\varepsilon^{-1} \left| \psi(x) \beta_\varepsilon(u^j_\varepsilon - u^{j\pm 1}_\varepsilon) - \beta_\varepsilon(\psi(x)(u^j_\varepsilon - u^{j\pm 1}_\varepsilon)) \right| \le \|\psi\|_{L^\infty(\mathbb{R}^d)} + 1$$

due to (5.4). Moreover, $\psi(x) \int\limits_{\mathbb{R}^d \setminus \Omega} k_j(x, y, x-y)\, dy \le C(\psi)$ where $C(\psi)$ depends on ψ and on the constants appearing in the conditions on k_j. Together with Lemma 3.6 and Lemma 5.5 we obtain

$$\|\widetilde{f}^j_\varepsilon\|_{L^2(\mathbb{R}^d)} \le C(\psi) \left(\|f^j\|_{L^2(\Omega)} + \|u^j_\varepsilon\|_{H^{\frac{\alpha}{2}}(\Omega)} + 1 \right) \le C(\psi) \left(\|\mathbf{f}\|_{L^2(\Omega;\mathbb{R}^2)} + 1 \right) ,$$

which allows us to interpret (5.14), (5.15) as a global version of (5.5), (5.6). We will show that with some positive constant C independent of $\varepsilon > 0$ the following estimate holds:

$$\|\psi \mathbf{u}_\varepsilon\|_{H^\alpha(\mathbb{R}^d;\mathbb{R}^2)} = \|\mathbf{v}_\varepsilon\|_{H^\alpha(\mathbb{R}^d;\mathbb{R}^2)} \le C \left(\|\mathbf{f}\|_{L^2(\Omega,\mathbb{R}^2)} + 1 \right) . \tag{5.16}$$

Since $\psi \in C^\infty_0(\Omega)$ can be chosen arbitrarily, (5.13) follows. In order to prove (5.16) let us multiply both sides of (5.14) by $\mathcal{L}^2(v^1_\varepsilon - v^2_\varepsilon)(x)$ and integrate over $\mathbb{R}^d$. We obtain

$$\begin{aligned} &\left(\mathcal{L}^1 v^1_\varepsilon, \mathcal{L}^2 v^1_\varepsilon\right) + \varepsilon^{-1} \left(\beta_\varepsilon(v^1_\varepsilon - v^2_\varepsilon), \mathcal{L}^2(v^1_\varepsilon - v^2_\varepsilon)\right) \\ &\quad = \left(\mathcal{L}^1 v^1_\varepsilon, \mathcal{L}^2 v^2_\varepsilon\right) + \left(\widetilde{f}^1_\varepsilon - a^1 v^1_\varepsilon, \mathcal{L}^2(v^1_\varepsilon - v^2_\varepsilon)\right) . \end{aligned} \tag{5.17}$$

The main idea is to use Lemma 3.7 in order to estimate the first term on the left hand side from below. For the other terms we note that

$$\begin{aligned} &\varepsilon^{-1} \left(\beta_\varepsilon(v^1_\varepsilon - v^2_\varepsilon), \mathcal{L}^2(v^1_\varepsilon - v^2_\varepsilon)\right) \\ &\quad = \varepsilon^{-1} \left((v^1_\varepsilon - v^2_\varepsilon)_+, \mathcal{L}^2(v^1_\varepsilon - v^2_\varepsilon)\right) + \varepsilon^{-1} \left(\beta_\varepsilon(v^1_\varepsilon - v^2_\varepsilon) - (v^1_\varepsilon - v^2_\varepsilon)_+, \mathcal{L}^2(v^1_\varepsilon - v^2_\varepsilon)\right) \\ &\quad \ge -\left(1, |\mathcal{L}^2(v^1_\varepsilon - v^2_\varepsilon)|\right) \ge -C |\Omega|^{\frac{1}{2}} \|\mathbf{v}_\varepsilon\|_{H^\alpha(\mathbb{R}^d;\mathbb{R}^2)} . \end{aligned}$$

Here we used $(u_+, \mathcal{L}^2 u) = \mathcal{E}^2(u_+, u) \ge 0$, cf. Remark 5.2, and assumption (5.3). Furthermore,

$$\begin{aligned} &\left(\widetilde{f}^1_\varepsilon - a^1 v^1_\varepsilon, \mathcal{L}^2(v^1_\varepsilon - v^2_\varepsilon)\right) \\ &\quad \le C \Big\{ \|\mathbf{f}\|_{L^2(\mathbb{R}^d;\mathbb{R}^2)} + 1 + \|\mathbf{a}\|_{L^\infty(\mathbb{R}^d;\mathbb{R}^2)} \|\mathbf{v}_\varepsilon\|_{L^2(\mathbb{R}^d;\mathbb{R}^2)} \Big\} \|\mathbf{v}_\varepsilon\|_{H^\alpha(\mathbb{R}^d;\mathbb{R}^2)} . \end{aligned}$$

The remaining term $\left(\mathcal{L}^1 v_\varepsilon^1, \mathcal{L}^2 v_\varepsilon^2\right)$ seems to be of the same order as $\left(\mathcal{L}^1 v_\varepsilon^1, \mathcal{L}^2 v_\varepsilon^1\right)$; but it is not. In fact, by trivial additions and subtractions we can use equation (5.14), (5.15) and obtain

$$\begin{aligned}
&\left(\mathcal{L}^1 v_\varepsilon^1, \mathcal{L}^2 v_\varepsilon^2\right) \\
&\quad= \left(\mathcal{L}^1 v_\varepsilon^1 + a^1 v_\varepsilon^1 - \widetilde{f}_\varepsilon^1, \mathcal{L}^2 v_\varepsilon^2\right) - \left(a^1 v_\varepsilon^1 - \widetilde{f}_\varepsilon^1, \mathcal{L}^2 v_\varepsilon^2\right) \\
&\quad= \underbrace{\left(\mathcal{L}^1 v_\varepsilon^1 + a^1 v_\varepsilon^1 - \widetilde{f}_\varepsilon^1, \mathcal{L}^2 v_\varepsilon^2 + a^2 v_\varepsilon^2 - \widetilde{f}_\varepsilon^2\right)}_{=0} - \left(a^1 v_\varepsilon^1 - \widetilde{f}_\varepsilon^1, \mathcal{L}^2 v_\varepsilon^2\right) \\
&\qquad - \left(\mathcal{L}^1 v_\varepsilon^1 + a^1 v_\varepsilon^1 - \widetilde{f}_\varepsilon^2, a^2 v_\varepsilon^2 - \widetilde{f}_\varepsilon^2\right) \\
&\quad\le C\Big\{\|\mathbf{f}_\varepsilon\|_{L^2(\mathbb{R}^d;\mathbb{R}^2)} + \|\mathbf{a}\|_{L^\infty(\mathbb{R}^d;\mathbb{R}^2)}\|\mathbf{v}_\varepsilon\|_{L^2(\mathbb{R}^d;\mathbb{R}^2)}\Big\}\|\mathbf{v}_\varepsilon\|_{H^\alpha(\mathbb{R}^d;\mathbb{R}^2)} \\
&\qquad + C\Big\{\|\mathbf{a}\|^2_{L^\infty(\mathbb{R}^d;\mathbb{R}^2)}\|\mathbf{v}_\varepsilon\|^2_{L^2(\mathbb{R}^d;\mathbb{R}^2)} + \|\widetilde{\mathbf{f}}_\varepsilon\|^2_{L^2(\mathbb{R}^d;\mathbb{R}^2)}\Big\}\,.
\end{aligned}$$

Altogether, using Lemma 3.7 equality (5.17) implies

$$\begin{aligned}
\|v_\varepsilon^1\|^2_{H^\alpha(\mathbb{R}^d;\mathbb{R}^2)} &\le C\Big\{\|\mathbf{a}\|^2_{L^\infty(\mathbb{R}^d;\mathbb{R}^2)}\|\mathbf{v}_\varepsilon\|^2_{L^2(\mathbb{R}^d;\mathbb{R}^2)} + \|\widetilde{\mathbf{f}}_\varepsilon\|^2_{L^2(\mathbb{R}^d;\mathbb{R}^2)}\Big\} \\
&\quad + C\Big\{\|\mathbf{v}_\varepsilon\|_{L^2(\mathbb{R}^d;\mathbb{R}^2)} + \|\mathbf{a}\|_{L^\infty(\mathbb{R}^d;\mathbb{R}^2)}\|\mathbf{v}_\varepsilon\|_{L^2(\mathbb{R}^d;\mathbb{R}^2)} + 1\Big\}\|\mathbf{v}_\varepsilon\|_{H^\alpha(\mathbb{R}^d;\mathbb{R}^2)}.
\end{aligned}$$

Multiplying (5.15) by $\mathcal{L}^1(v_\varepsilon^2 - v_\varepsilon^1)$ and applying the same strategy proves the same estimate for v_ε^2. An application of Young's inequality on the right hand side and using (5.11) finally gives

$$\begin{aligned}
\|\mathbf{v}_\varepsilon\|_{H^\alpha(\mathbb{R}^d;\mathbb{R}^2)} &\le C\left(\|\mathbf{a}\|_{L^\infty(\mathbb{R}^d;\mathbb{R}^2)}\|\mathbf{v}_\varepsilon\|_{L^2(\mathbb{R}^d;\mathbb{R}^2)} + \|\widetilde{\mathbf{f}}_\varepsilon\|_{L^2(\mathbb{R}^d;\mathbb{R}^2)} + 1\right)\,, \\
&\le C\left(\|\mathbf{f}\|_{L^2(\Omega;\mathbb{R}^2)} + 1\right)
\end{aligned}$$

which proves (5.16). The proof of the lemma is complete. ■

Lemma 5.7 *Consider Ω, $\mathbf{f}$ and $\mathbf{a}$ as in Theorem 1.1. For any sequence $(\mathbf{u}_{\varepsilon_n}), \varepsilon_n \to_{n\to\infty} 0$, of nonnegative solutions $\mathbf{u}_{\varepsilon_n} \in H_0^{\frac{\alpha}{2}}(\Omega) \cap H^\alpha_{loc}(\Omega)$ to (5.8), (5.9) and any $\Omega' \Subset \Omega$ there exists a subsequence $\{\mathbf{u}_{\varepsilon_k}\} \subset \{\mathbf{u}_{\varepsilon_n}\}$ such that*

$$\left\|u^1_{\varepsilon_k} - u^2_{\varepsilon_k}\right\|_{H^\alpha(\Omega')} \to_{k\to\infty} 0, \qquad \textit{where } \varepsilon_k \to_{k\to\infty} 0\,.$$

Proof: For simplicity, we write ε instead of ε_n. Let $\mathbf{u}_\varepsilon \in H_0^{\frac{\alpha}{2}}(\Omega;\mathbb{R}^2) \cap H^\alpha_{\mathrm{loc}}(\Omega,\mathbb{R}^2)$ be a sequence of nonnegative solutions to (5.8), (5.9). Let $\Omega' \Subset \Omega$.

We start off analogously to the proof of Lemma 5.6. We set $\mathbf{v}_\varepsilon = \psi \mathbf{u}$ and note that $\mathbf{v}_\varepsilon$ solves (5.14), (5.15) where $\widetilde{\mathbf{f}}_\varepsilon$ is uniformly bounded in L^2. As we did above, let us multiply both sides of (5.14) by $\mathcal{L}^2(v_\varepsilon^1 - v_\varepsilon^2)(x)$ and integrate over $\mathbb{R}^d$. Equality (5.17) can be written as

$$\begin{aligned}\big(\mathcal{L}^1 v_\varepsilon^1, \mathcal{L}^2(v_\varepsilon^1 - v_\varepsilon^2)\big) &= -\varepsilon^{-1}\big(\beta_\varepsilon(v_\varepsilon^1 - v_\varepsilon^2), \mathcal{L}^2(v_\varepsilon^1 - v_\varepsilon^2)\big)\\ &\quad + \big(\widetilde{f}_\varepsilon^1 - a^1 v_\varepsilon^1, \mathcal{L}^2(v_\varepsilon^1 - v_\varepsilon^2)\big)\ .\end{aligned} \tag{5.18}$$

Using assumption (5.3) we observe that

$$\begin{aligned}&-\varepsilon^{-1}\big(\beta_\varepsilon(v_\varepsilon^1 - v_\varepsilon^2), \mathcal{L}^2(v_\varepsilon^1 - v_\varepsilon^2)\big)\\ &\quad = -\varepsilon^{-1}\big((v_\varepsilon^1 - v_\varepsilon^2)_+, \mathcal{L}^2(v_\varepsilon^1 - v_\varepsilon^2)\big) + \varepsilon^{-1}\big((v_\varepsilon^1 - v_\varepsilon^2)_+ - \beta_\varepsilon(v_\varepsilon^1 - v_\varepsilon^2), \mathcal{L}^2(v_\varepsilon^1 - v_\varepsilon^2)\big)\\ &\quad \le \big(\varepsilon, |\mathcal{L}^2(v_\varepsilon^1 - v_\varepsilon^2)|\big) \le C\varepsilon \|v_\varepsilon^1 - v_\varepsilon^2\|_{H^\alpha} \le C\varepsilon\,,\end{aligned}$$

where we used Lemma 5.6 and $(u_+, \mathcal{L}^2 u) = \mathcal{E}^2(u_+, u) \ge 0$.

In order to estimate the right-hand side of (5.18) let $g = f^1 - a^1 v_\varepsilon^1$. Then for every $\varepsilon' > 0$ there is some $g_{\varepsilon'} \in C_0^\infty(\Omega)$ such that $\|g_{\varepsilon'} - g\|_{L^2} \le \varepsilon'$. Then

$$\big|\big(g_{\varepsilon'}, \mathcal{L}^2(v_\varepsilon^1 - v_\varepsilon^2)\big)\big| = \big|\mathcal{E}^2(g_{\varepsilon'}, v_\varepsilon^1 - v_\varepsilon^2)\big| \le C\|g_{\varepsilon'}\|_{H^{\frac{\alpha}{2}}}\|v_\varepsilon^1 - v_\varepsilon^2\|_{H^{\frac{\alpha}{2}}} \le C(\varepsilon')o(1)$$

as $\varepsilon \to 0$ for a subsequence because of Lemma 5.6 and Rellich's theorem. On the other hand,

$$\big|\big(g - g_{\varepsilon'}, \mathcal{L}^2(v_\varepsilon^1 - v_\varepsilon^2)\big)\big| \le C(\psi)\varepsilon'\left(\|\mathbf{f}\|_{L^2(\Omega;\mathbb{R}^2)} + 1\right)$$

by Lemma 5.6. Together, we obtain for a subsequence

$$\big(\mathcal{L}^1 v_\varepsilon^1, \mathcal{L}^2(v_\varepsilon^1 - v_\varepsilon^2)\big) \le C(\varepsilon')o(1) + C\varepsilon' \quad \text{as } \varepsilon \to 0. \tag{5.19}$$

Working with (5.15) instead of (5.14) analogously leads to

$$\big(\mathcal{L}^2 v_\varepsilon^2, \mathcal{L}^1(v_\varepsilon^2 - v_\varepsilon^1)\big) \le C(\varepsilon')o(1) + C\varepsilon' \quad \text{as } \varepsilon \to 0. \tag{5.20}$$

Together with Lemma 3.8 estimate (5.20) implies for a subsequence

$$\big(\mathcal{L}^1 v_\varepsilon^2, \mathcal{L}^2(v_\varepsilon^2 - v_\varepsilon^1)\big) \le C(\varepsilon')o(1) + C\varepsilon' \quad \text{as } \varepsilon \to 0, \tag{5.21}$$

where we again use Lemma 5.6 and Rellich's theorem. Summation of (5.19) and (5.21) implies together with Lemma 3.7

$$\|v_\varepsilon^1 - v_\varepsilon^2\|_{H^\alpha} \le C\big(\mathcal{L}^1(v_\varepsilon^1 - v_\varepsilon^2), \mathcal{L}^2(v_\varepsilon^1 - v_\varepsilon^2)\big) \le C(\varepsilon')o(1) + C\varepsilon'. \tag{5.22}$$

By first choosing $\varepsilon' > 0$ small and then letting $\varepsilon \to 0$ for a suitable subsequence shows that $\|v_\varepsilon^1 - v_\varepsilon^2\|_{H^\alpha}$ is arbitrarily small if $\varepsilon > 0$ is small enough. Remember that $v_\varepsilon = \psi \mathbf{u}_\varepsilon$ where $\psi \in C_0^\infty(\Omega)$ satisfies $\psi \equiv 1$ on Ω'. Therefore, the proof of Lemma 5.7 is complete. ∎

Proof of Theorem 1.1: First of all, during the approximation we can assume with loss of generality that $\mathbf{f}, \mathbf{a}$ are smooth. Otherwise, we replace for each $\varepsilon > 0$ $\mathbf{f}, \mathbf{a}$ by $\mathbf{f}_\varepsilon, \mathbf{a}_\varepsilon$ which are smooth and converge strongly in $L^2(\Omega; \mathbb{R}^2)$, $L^\infty(\Omega; \mathbb{R}^2)$, resp., to $\mathbf{a}, \mathbf{f}$ as $\varepsilon \to 0$. Now for any $\varepsilon > 0$ by Lemma 5.3 there is a solution $\mathbf{u}_\varepsilon \in H_0^{\frac{\alpha}{2}}(\Omega; \mathbb{R}^2)$ of the approximate system (5.5)-(5.7) which is in $H^\alpha_{\mathrm{loc}}(\Omega; \mathbb{R}^2)$ by Lemma 4.3 and which is nonnegative by Lemma 5.4. By the Lemmas 5.5, 5.6 and 5.7 $\mathbf{u}_\varepsilon$ converges weakly in $H_0^{\frac{\alpha}{2}}(\Omega; \mathbb{R}^2)$ and strongly in $H^\alpha(\Omega'; \mathbb{R}^2)$ for any $\Omega' \Subset \Omega$ to some $\mathbf{u} \in H_0^{\frac{\alpha}{2}}(\Omega; \mathbb{R}^2) \cap H^\alpha_{loc}(\Omega; \mathbb{R}^2)$ for a suitable subsequence. By (5.11) $\|u_\varepsilon^1 - u_\varepsilon^2\|_{L^2(\Omega)} \to 0$ as $\varepsilon \to 0$. Hence $u^1 \equiv u^2 =: u$. By the construction

$$L^j u_\varepsilon^j(x) + a^j u_\varepsilon^j(x) - f^j(x) \leq 0 \qquad \text{a.e. in } \Omega$$

for $j = 1, 2$, which yields

$$L^j u(x) + a^j u(x) - f^j(x) \leq 0 \qquad \text{a.e. in } \Omega.$$

It remains to prove the third equation of (5.1). By (5.5)-(5.6)

$$\begin{aligned}
&\int_\Omega \varphi(x)\big(L^1 u_\varepsilon^1(x) + a^1 u_\varepsilon^1(x) - f^1(x)\big)\big(L^2 u_\varepsilon^2(x) + a^2 u_\varepsilon^2(x) - f^2(x)\big)\, dx \\
&\quad = \int_\Omega \varphi(x) \varepsilon^{-2} \beta_\varepsilon(u_\varepsilon^1(x) - u_\varepsilon^2(x)) \beta_\varepsilon(u_\varepsilon^2(x) - u_\varepsilon^1(x)) dx = 0
\end{aligned}$$

for all $\varphi \in C_0^\infty(\Omega)$. Hence

$$\begin{aligned}
0 &= \lim_{\varepsilon \to 0} \int_\Omega \varphi(L^1 u_\varepsilon^1 + a^1 u_\varepsilon^1 - f^1)(L^2 u_\varepsilon^2 + a^2 u_\varepsilon^2 - f^2)\, dx \\
&= \lim_{\varepsilon \to 0} \int_\Omega \varphi L^1 u_\varepsilon^1 L^2 u_\varepsilon^2 \, dx + \int_\Omega \varphi L^1 u (a^2 u - f^2)\, dx \\
&\quad + \int_\Omega \varphi (a^1 u - f^1) L^2 u \, dx + \int_\Omega \varphi (a^1 u - f^1)(a^2 u - f^2)\, dx
\end{aligned}$$

for all $\varphi \in C_0^\infty(\Omega)$ by the strong convergence of u_ε^j in $H^\alpha_{loc}(\Omega)$. Now

$$\int_\Omega \varphi(x) L^1 u(x) L^2 u(x) dx \leq \lim_{\varepsilon \to 0} \int_\Omega \varphi(x) L^1 u_\varepsilon^1(x) L^2 u_\varepsilon^2(x) dx$$

for all $\varphi \in C_0^\infty(\Omega)$, $\varphi \geq 0$, which follows from the fact that

$$\lim_{\varepsilon \to 0} \int_\Omega \varphi(x) L^1 u_\varepsilon^1(x) L^2 u_\varepsilon^2(x) dx = \lim_{\varepsilon \to 0} \int_\Omega \varphi(x) L^1 u_\varepsilon^1(x) L^2 u_\varepsilon^1(x) dx$$

since $u_\varepsilon^1 - u_\varepsilon^2 \to 0$ in $H^\alpha_{loc}(\Omega)$ as $\varepsilon \to 0$ and the fact that

$$F(v) := \int_\Omega \varphi(x) L^1 v(x) L^2 v(x) dx$$

is a convex functional on $H^\alpha(\Omega') \cap H_0^{\frac{\alpha}{2}}(\Omega)$, where $\operatorname{supp} \varphi \subseteq \Omega' \Subset \Omega$. Summing up

$$\int_\Omega \varphi(L^1 u + a^1 u - f^1)(L^2 u + a^2 u - f^2) dx \leq 0$$

for all $\varphi \in C_0^\infty(\Omega), \varphi \geq 0$. On the other hand, by (5.5)-(5.6) the integrand above is nonnegative almost everywhere, which shows the third equation of (5.1) and finishes the proof. ■

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