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On toroidal rotating drops

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Abstract

The existence of toroidal rotating drops was observed experimentally by Plateau in 1841. In 1983 Gulliver rigorously showed that toroidal solutions of the governing equilibrium equations do indeed exist. In this short note, we settle two questions posed by Gulliver concerning the existence of additional toroidal solutions. We use a general assertion concerning rotationally symmetric surfaces whose meridian curves have inclination angle given as a function of distance from the axis along with explicit estimates for rotating drops.

Key words: rotating drops, mean curvature, Plateau, Delaunay.

In 1843 Joseph Plateau challenged *geometricians* to find rotationally symmetric tori whose mean curvature is an even quadratic function of distance to the axis of rotation:

> *I think it very probable that if calculation could approach the general solution of this great problem, and lead directly to the determination of all the possible figures of equilibrium, the annular figure would be included among them.*

The figures of equilibrium to which Plateau refers are those of rotationally symmetric rotating liquid drops removed from the influence of gravity. Elementary considerations lead one to ordinary differential equations for the meridian curve of such an equilibrium, and solutions of these equations may be understood by considering only portions of meridian which are expressible as graphs $u = u(r)$ with respect to the radial variable r. For these portions, the prescribed mean curvature equation becomes

$$\frac{u''}{(1+u'^2)^{3/2}} + \frac{1}{r}\frac{u'}{\sqrt{1+u'^2}} = -4ar^2 + 2\lambda \qquad (1)$$

where $a = \rho\omega^2/(8\sigma)$ and λ are constants depending on the physical parameters density, angular velocity, surface tension, and enclosed volume. It follows that the solutions form a two parameter family (up to scaling and rigid motion). Scaling so that $a = 1$, we will take

λ and c as the two parameters where c is a constant of integration appearing below. For a more detailed discussion, see [**?**].

Except for certain well defined curves in the (λ, c) parameter space, solutions of equation (1) may be expressed, up to a constant of integration, as

$$u(r) = \int_{r_*}^{r} \frac{-at^4 + \lambda t^2 + c}{\sqrt{t^2 - (-at^4 + \lambda t^2 + c)^2}}\, dt \tag{2}$$

on a suitable interval $[r_*, R_*]$, and the question of existence of toroidal solutions is reduced to finding λ and c for which $u(R_*) = 0$. (If this condition holds, the meridian of the torus is described by joining the graphs of u and $-u$. We are assuming here, of course, that the constant of integration $u(r_*) = 0$; this normalization will be employed throughout this paper, though we note that, due to non-integrability at r_*, the same normalization is not always possible in [**?**].)

R. Gulliver [**?**] showed the following:

> *For each $c \geq 3/16$, there is some $\lambda = \lambda(c)$ for which the corresponding solution is a torus with convex cross section.*
>
> *There is some interval $c \in (0, \epsilon)$ and a smooth function $\lambda = \lambda(c)$ for which the corresponding solution is a (nonconvex) torus.*

Gulliver went on to conjecture that there were toroidal parameter values (λ, c) for every $c > 0$. He also pointed out that only immersed toroidal solutions were possible for $c < 0$ but was unable to verify their existence. We prove the existence of toroidal solutions in both cases, i.e., for all $c \neq 0$. In order to state our result precisely (and prove it), we must first discuss the limits of integration r_* and R_* and their dependence on the parameters λ and c.

Remark Gulliver, in his paper, formulates equation (1) as

$$a + br^2 = 2H = \frac{dv}{dr} + \frac{v}{r}$$

where $v = \sin\psi = du/ds$, ψ is the inclination angle of the meridian, and s is an arclength parameter along the meridian. Gulliver does not explicitly specify the orientation of his arclength parameterization nor his choice of normal (into the drop or out of it) with respect to which he calculates the mean curvature, but by his specification $b > 0$, one can deduce that his formulation is consistent only if the mean curvature is calculated with respect to the inward normal and, hence, if the parameterization is "counter-clockwise."

Since some formal solutions of the equations do not enclose a volume, and hence their meridians do not enclose an area, the notion of "counter-clockwise" does not always make sense. In order to avoid this ambiguity, we have formulated our equation for portions of the meridian on which the normal points upward and, hence, the rotating drop is formally below the meridian locally. It is easily verified that any portions of meridian for which the rotating drop is formally above the meridian are geometric reflections across the line $u = 0$ of those we consider.

For convenience of the reader, a loose translation between Gulliver's parameter notation and ours is as follows:

1. Due to the reversal of normal, Gulliver's H is our $-H$.
2. Gulliver's rotation parameter $b > 0$ is for us $4a$.
3. Gulliver's Lagrange parameter $a \in \mathbb{R}$ is our -2λ.
4. Gulliver's constant of integration C is the same as $-c$.
5. Having made these replacements and setting, without loss of generality, $v = u'/\sqrt{1+u'^2}$, Gulliver's equation (1) translates directly into our equation (1).
6. Gulliver subsequently rescales his rotation parameter b to $4/3$ and gives C the new name $-\gamma^4$ (in harmony with his restriction $C < 0$). We rescale so that the rotation parameter a takes the value 1. Via comparison, one obtains the translation formulae

$$\begin{cases} \lambda = -a\sqrt[3]{3} \\ c = -C/\sqrt[3]{3} = -\gamma^4/\sqrt[3]{3} \end{cases} \tag{3}$$

   with our scaled parameters on the left and Gulliver's on the right. In particular, one sees that the $\gamma \geq (3/8)^{1/3}$ of Gulliver's Theorem 2 corresponds precisely to our $c > 3/16$.
7. The λ, c (or a, C) parameter space has been antipodally reflected through the origin and scaled according to (3).

Inclination Angle and the other Toroidal Solutions

We may rewrite equation (1) as

$$\left(\frac{ru'}{\sqrt{1+u'^2}}\right)' = -4r^3 + 2\lambda r$$

and integrate once to obtain

$$v = \sin\psi = \frac{u'}{\sqrt{1+u'^2}} = -r^3 + \lambda r + c/r,$$

where ψ is the inclination angle of the graph of u with respect to the positive r-axis. For $c \neq 0$, the values r_* and R_* are solutions of the algebraic equations

$$|\sin\psi(r)| = |-r^3 + \lambda r + c/r| = 1 \tag{4}$$

($r_* = 0$ for $c = 0$).

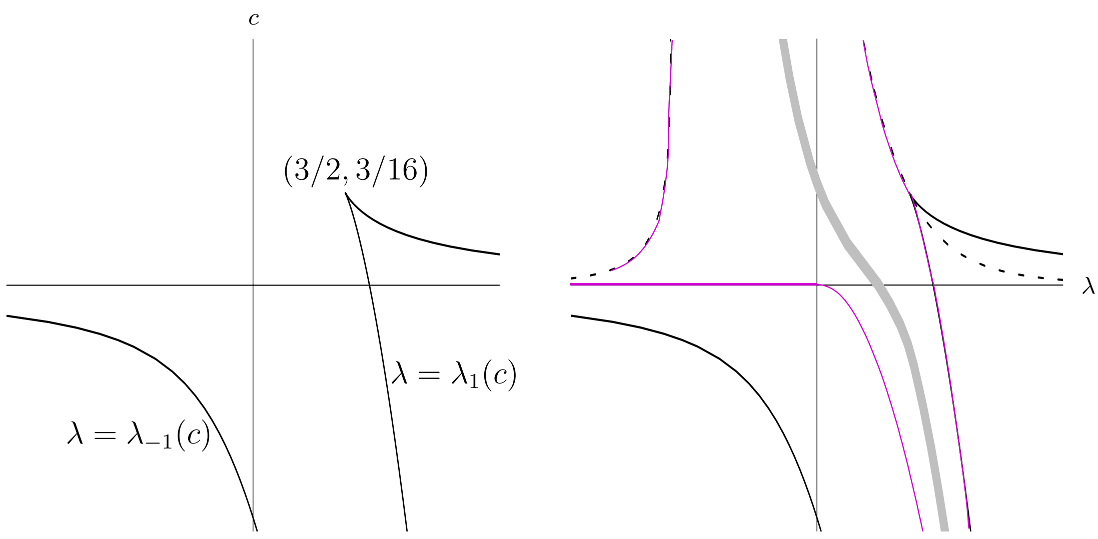


Figure 1: (a) Curves corresponding to double roots of $|\sin\psi| = 1$. (b) Numerically computed torus curve and region of parameter values possibly corresponding to tori.

In general, if we think of λ and c as fixed, we may consider the algebraic expression $v(r) = -r^3 + \lambda r + c/r$ also for values of r for which $|v(r)| > 1$. In this way, the equation $|v(r)| = 1$ appearing in (4) is evidently equivalent to a pair of (quartic) polynomial equations. This guarantees there are only finitely many cases to consider. One must take into account however that the resulting intervals of definition (determined by roots $r_* = r_*(\lambda, c)$ and $R_* = R_*(\lambda, c)$) may depend discontinuously on λ and c. Let us begin, however, with the assumption that λ and c are fixed, and denote by $\mathcal{R} = \mathcal{R}_{\lambda,c}$ the collection of positive roots of $|v(r)| = 1$, counted with multiplicities.

Lemma 1 *Given $r_* \leq R_*$ in $\mathcal{R}$ such that $|v(r)| < 1$ for $r_* < r < R_*$, there is a rotationally symmetric surface whose inclination angle $\psi(r)$ satisfies*

$$\sin\psi(r) = v(r) = -r^3 + \lambda r + c/r \tag{5}$$

for $r_* \leq r \leq R_*$. The surface is unique up to translation in the u direction.

Conversely, each complete rotationally symmetric surface that does not intersect $r = 0$ and whose inclination angle satisfies (5) projects onto an annulus $r_* \leq r \leq R_*$ with $|\sin\psi(r)| < 1$ for $r_* < r < R_*$.

Note that each double root $r_* = R_*$ corresponds to a cylinder. The parameter values for which this is possible lie along three curves in the (λ, c) parameter plane as depicted in Figure 1(a). One can verify the following behavior in the neighborhood of the roots of $|v(r)| = 1$:

Lemma 2 *Let r_* be any real number. If $v \in C^2[r_*, r_* + \epsilon]$ with $|v(r_*)| = 1$ and $|v(r)| < 1$ for $r_* < r \le r_* + \epsilon$, then the improper integral*

$$\left| \int_{r_*}^{r_*+\epsilon} \frac{v}{\sqrt{1-v^2}} dr \right| < \infty$$

if and only if $v'(r_*) \ne 0$. A similar statement holds on intervals of the form $[R_* - \epsilon, R_*]$.

Remark In this result, v may be any function satisfying the hypotheses of the lemma, though we are only interested in applications in which $v = -r^3 + \lambda r + c/r$ as in (4).

The full significance of the curves in Figure 1(a) is explained in [**?**], but for our present purposes we need only verify some isolated facts about two of them. We start with a continuity assertion that was already observed by Gulliver as important for the case $c > 3/16$.

Lemma 3 *For each $\lambda \in \mathbb{R}$ and $c > 3/16$, the set $\mathcal{R} = \mathcal{R}_{\lambda,c}$ consists of exactly two positive roots $r_* = r_*(\lambda, c)$ and $R_* = R_*(\lambda, c)$ with $r_* < R_*$. In this region of the (λ, c)-plane r_* and R_* depend smoothly on λ and c. Consequently, we find that the quantity*

$$u(R_*) = \int_{r_*}^{R_*} \frac{v}{\sqrt{1-v^2}} \, dr$$

depends smoothly on λ and $c > 3/16$. In particular, $u(R_*)$ is a continuous function of λ for fixed $c > 3/16$.

Proof. We consider

$$v(r) = -r^3 + \lambda r + \frac{c}{r}$$

for $r > 0$ and fixed $c > 3/16$. The assertion of the lemma follows from the fact that the equations

$$v(r_*) = -r_*^3 + \lambda r_* + \frac{c}{r_*} = 1 \tag{6}$$

and

$$v(R_*) = -R_*^3 + \lambda R_* + \frac{c}{R_*} = -1 \tag{7}$$

have unique solutions $r_* < R_*$ with $v' < 0$ on $[r_*, R_*]$.

First note that $\lim_{r \to 0} v(r) = +\infty$ and $\lim_{r \to +\infty} v(r) = -\infty$. Therefore, (6) and (7) have at least one solution each. It will be observed that v has a unique inflection point (and v' a unique maximum) at $t_{\max} = \sqrt[4]{c/3}$. If $\lambda \le 2\sqrt{3c}$, then $v' \le 0$ with equality only possible at $r = t_{\max}$ when $\lambda = 2\sqrt{3c}$. In this case, $v(t_{\max}) = 8(c/3)^{3/4} > 1$. Therefore, our assertions concerning (6) and (7) hold.

If $\lambda > 2\sqrt{3c}$, then v' has two zeros; the smaller, which corresponds to a local minimum of v, is given by

$$r_{\min} = \sqrt{\frac{\lambda - \sqrt{\lambda^2 - 12c}}{6}}.$$

Elementary computations show that

$$\frac{dr_{\min}}{d\lambda} = -\frac{r_{\min}}{2\sqrt{\lambda^2 - 12c}} < 0$$

and

$$\frac{d}{d\lambda} v(r_{\min}) = r_{\min} > 0. \tag{8}$$

In this case,

$$\lim_{\lambda \searrow 2\sqrt{3c}} v(r_{\min}) = 8(c/3)^{3/4} > 1.$$

In light of (8), we see that $v(r) > 1$ for $r \leq r_{\max}$ where

$$r_{\max} = \sqrt{\frac{\lambda + \sqrt{\lambda^2 - 12c}}{6}} \tag{9}$$

is the larger zero of v'. In this case too, therefore, our assertions concerning (6) and (7) hold. □

The situation for $c \leq 3/16$ is somewhat more complicated. Nevertheless, we find

Lemma 4 *For each fixed $c \leq 3/16$, there is a unique value $\lambda = \lambda_1(c)$ determined by the equation*

$$v(r_{\max}) = 1,$$

where $r_{\max}$ is given by (9)*; see Figure* 1(a)*.*

For $c \in (0, 3/16]$ and $\lambda < \lambda_1$, the equation

$$|v(r)| = 1 \tag{10}$$

has exactly two positive solutions $r_* < R_*$. For $c = 0$, there is one solution R_*, and we may take $r_* = 0$.

For $c < 0$, there is a unique value $\lambda_{-1} = \lambda_{-1}(c) < \lambda_1$ determined by the equation

$$v(r_{\max}) = -1$$

(*considered as an equation for $\lambda = \lambda_{-1}$*)*. For $c < 0$ and $\lambda_{-1} < \lambda < \lambda_1$ there are again exactly two positive solutions $r_* < R_*$ of* (10)*.*

For $c \in [0, 3/16]$

$$u(R_*) = \int_{r_*}^{R_*} \frac{v}{\sqrt{1 - v^2}}\, dr$$

is a continuous function of λ on $(-\infty, \lambda_1)$; the same function is continuous on $\lambda_{-1} < \lambda < \lambda_1$ for $c < 0$. In all cases,

$$\lim_{\lambda \nearrow \lambda_1} u(R_*) = +\infty. \tag{11}$$

Proof. We observe that (9) is only real valued for $0 \le c \le 3/16$ if $\lambda \ge 2\sqrt{3c}$. For this range of parameters, a calculation similar to that leading to (8) yields

$$\frac{d}{d\lambda} v(r_{\max}) = r_{\max} > 0. \tag{12}$$

Furthermore,

$$\lim_{\lambda \searrow 2\sqrt{3c}} v(r_{\max}) = 8(c/3)^{3/4} \le 1$$

and

$$\lim_{\lambda \nearrow +\infty} v(r_{\max}) = +\infty. \tag{13}$$

Therefore λ_1 is well defined for $0 \le c \le 3/16$. For $c < 0$, conditions (12) and (13) still hold. Furthermore,

$$\lim_{\lambda \searrow -\infty} v(r_{\max}) = -\infty.$$

Therefore, both λ_{-1} and λ_1 are well defined.

Considerations similar to those in the proof of Lemma 1 yield the uniqueness and continuity of r_* and R_* as functions of λ. The continuity of $u(R_*)$ follows as before. It remains to establish (11). To avoid certain technicalities, we restrict to the case $c < 3/16$, but the case $c = 3/16$ may be handled similarly. For $c < 3/16$ and λ sufficiently close to λ_1, we know

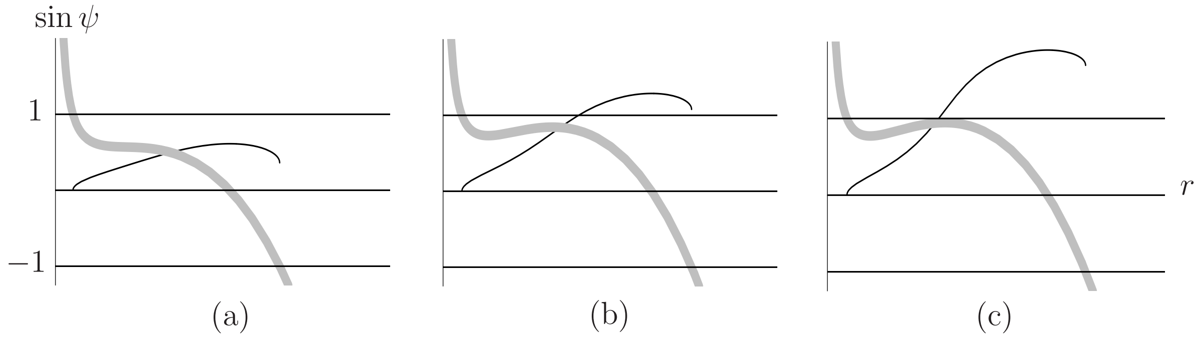


Figure 2: Profiles of $v = \sin\psi(r)$ for $c = 3/32$ shown in gray. The values of λ are (a) 1, (b) 1.5, (c) 1.65. The corresponding (antinodoid) solutions $u(r)$ are also shown on the same graph (black).

that $r_{\max}$ is well defined as described above with $r_{\max}$ and $v(r_{\max})$ increasing as functions of λ; see Figure 2. We set $R_1 = \lim_{\lambda \nearrow \lambda_1} r_{\max}$. Since v' is nonvanishing at r_* and R_* (except in the case $c = 0$ in which $r_* = 0 = \sin\psi(r_*)$ which causes no problem), we find from Lemma 2, that for all ϵ small enough and fixed, the integrals

$$\int_{r_*}^{r_{\max}-\epsilon} \frac{v}{\sqrt{1-v^2}}\, dr \quad \text{and} \quad \int_{r_{\max}+\epsilon}^{R_*} \frac{v}{\sqrt{1-v^2}}\, dr$$

are finite and may be bounded uniformly in λ as $\lambda \nearrow \lambda_1$. Thus,

$$u(R_*) = \int_{r_*}^{R_*} \frac{v}{\sqrt{1-v^2}}\, dr \geq \int_{r_{\max}-\epsilon}^{r_{\max}+\epsilon} \frac{v}{\sqrt{1-v^2}}\, dr - M_\epsilon$$

for some constant M_ϵ.

Expanding $v(r)$ in a power series about $(\lambda, r) = (\lambda_1, R_1)$ on the other hand,

$$v = 1 + R_1(\lambda - \lambda_1) + (\lambda - \lambda_1)(r - R_1) + d(r - R_1)^2 + o(r - R_1)^2$$

where $d = v''(R_1)/2 < 0$. Therefore, we may fix ϵ small enough so that

$$v \geq 1 + [R_1(\lambda - \lambda_1) + d(r - R_1)^2]/2 > 1/2 \qquad \text{for} \quad r_{\max} - \epsilon \leq r \leq r_{\max} + \epsilon$$

uniformly in λ. Thus,

$$\begin{aligned} 1 - v^2 &\leq -R_1(\lambda - \lambda_1) - d(r - R_1)^2 \\ &\qquad - R_1^2(\lambda - \lambda_1)^2/4 - R_1 d(\lambda - \lambda_1)(r - R_1)^2/2 - d^2(r - R_1)^4/4 \\ &\leq -R_1(\lambda - \lambda_1) - d(r - R_1)^2, \end{aligned}$$

and

$$\begin{aligned} \int_{r_{\max}-\epsilon}^{r_{\max}+\epsilon} \frac{v}{\sqrt{1-v^2}}\, dr &\geq \frac{1}{2}\int_{r_{\max}-\epsilon}^{r_{\max}+\epsilon} \frac{1}{\sqrt{-R_1(\lambda - \lambda_1) - d(r - R_1)^2}}\, dr \\ &= \frac{1}{2\sqrt{-d}}\Bigg[\sinh^{-1}\left(\sqrt{\frac{d}{R_1(\lambda - \lambda_1)}}\,(r_{\max} - R_1 + \epsilon)\right) \\ &\qquad - \sinh^{-1}\left(\sqrt{\frac{d}{R_1(\lambda - \lambda_1)}}\,(r_{\max} - R_1 - \epsilon)\right)\Bigg]. \end{aligned}$$

Since $r_{\max} \nearrow R_1$ as $\lambda \nearrow \lambda_1$ and $\epsilon > 0$, we see that

$$\lim_{\lambda \nearrow \lambda_1} \int_{r_{\max}-\epsilon}^{r_{\max}+\epsilon} \frac{v}{\sqrt{1-v^2}}\, dr = +\infty. \qquad \square$$

It remains to obtain a value $\lambda < \lambda_1$ (in the region of continuity) for which $u(R_*) < 0$. For this we use a general relation between the convexity of $v = \sin\psi$ and the height $u(R_*)$. A special case of this result was used implicitly by Gulliver in the case $c > 3/16$.

Lemma 5 (General Convexity-Height Lemma for Rotational Surfaces) *Given* $0 < r_* < R_*$ *and any* v *decreasing from* 1 *to* -1 *on* $[r_*, R_*]$*, if* v *is convex, then*

$$u(R_*) = \int_{r_*}^{R_*} \frac{v}{\sqrt{1-v^2}} dr < 0.$$

Similarly, if v *is concave* ($v'' < 0$)*, then* $u(R_*) > 0$.

Remarks 1 This result is true for any real numbers $r_* < R_*$ and any function v satisfying the hypotheses stated in the lemma. We have used derivatives of v up to order two freely in the proof, but the continuity of v resulting from convexity/concavity is adequate to obtain the result.

2 In the convex case, we say that the resulting surface is *nodoid type*; in the concave case, *antinodoid type.*

Proof of Lemma 5. Assume v is convex. The other case is handled similarly. There is a unique $r = r_{\text{crit}} \in (r_*, R_*)$ such that $v(r_{\text{crit}}) = 0$.

$$u(R_*) = \int_{r_*}^{r_{\text{crit}}} \frac{v}{\sqrt{1-v^2}} dr + \int_{r_{\text{crit}}}^{R_*} \frac{v}{\sqrt{1-v^2}} dr.$$

Again according to the monotonicity, the relation $v(r) = -v(t)$ defines a change of variables, and we obtain

$$u(R_*) = \int_{r_*}^{r_{\text{crit}}} \left(1 - \frac{v'(t)}{v'(r)}\right) \frac{v}{\sqrt{1-v^2}} dt < 0.$$

Notice that over the interval of integration, the second factor is positive; the first is negative by convexity. □

Referring back to the proofs of Lemma 3 and Lemma 4, one finds that for $c > 0$ and $\lambda << 0$, we have $v'(r) < 0$ on $[r_*, R_*]$ and $v(t_{\max}) < -1$ where $t_{\max} = \sqrt[4]{c/3}$ is the unique inflection point. It follows that $v = \sin\psi$ is convex on $[r_*, R_*]$ and $u(R_*) < 0$ by Lemma 5. Thus, by the intermediate value theorem, there is some λ for which $u(R_*) = 0$.

For $c = 0$ and $\lambda \leq 0$, we have $v \leq 0$ and $u(R_*) < 0$ so that the same conclusion holds. Technically, the resulting surface of rotation is not a torus in this case, since $r_* = 0$. However, one does obtain a pinched spheroid which encloses a volume.

For $c < 0$, the function $v = \sin\psi$ has a unique global maximum at the value $r_{\max}$ given in (9) and

$$\lim_{\lambda \searrow -\infty} v(r_{\max}) = -\infty.$$

As mentioned above, the monotonicity (12) holds, and it is clear that all values of λ in the interval for which $-1 < \sin\psi(r_{\max}) \leq 0$ correspond to solutions with $u(R_*) < 0$.

Following through carefully the indicated calculations and applying Lemma 5 in situations similar to those above yields the following bounds for the parameters (λ, c) corresponding to toroidal solutions; see Figure 1(b).

Theorem 1 *For each fixed c there is at least one λ corresponding to a toroidal solution. If $c \geq 3/16$, then*

$$-\sqrt[4]{\frac{3}{c}} - 2\sqrt{\frac{c}{3}} < \lambda < \sqrt[4]{\frac{3}{c}} - 2\sqrt{\frac{c}{3}}.$$

If $0 < c \leq 3/16$, *then*

$$-\sqrt[4]{\frac{3}{c}} - 2\sqrt{\frac{c}{3}} < \lambda < \lambda_1(c).$$

If $c \leq 0$, *then*

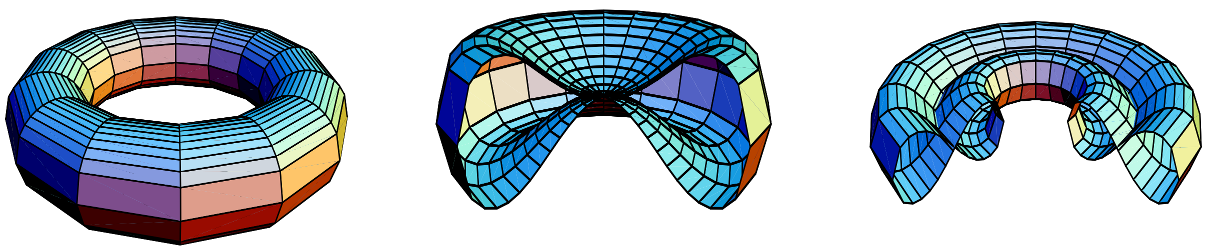
$$2\sqrt{-c} < \lambda < \lambda_1(c).$$

No toroidal solutions can correspond to parameters outside the region defined by these inequalities.

A useful alternative characterization of λ_1 is given in [**?**]. We state it here for convenience.

$$\lambda_1(c) = 3r^2 + c/r^2,$$

where $r = r(c)$ is the larger positive solution of $2r^4 - r + 2c = 0$.



(a) (b) (c)

Figure 3: Toroidal Surfaces (a) Embedded Torus ($c > 0$), (b) Pinched Spheroid ($c = 0$), (c) Immersed Torus ($c < 0$).

As a final remark, we conjecture that there is exactly one value $\lambda_t = \lambda_t(c)$ corresponding to a toroidal solution. These values form a smooth curve in the interior of the region described in Theorem 1; see Figure 1(b) in which the thick gray curve is the numerically calculated curve of toroidal solutions.