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für Mathematik
in den Naturwissenschaften
Leipzig

Equivariant Plateau Problems

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Preprint no.: 17 2006

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13th February 2006

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Abstract: Let (M, Q) be a compact, three dimensional manifold of strictly negative sectional curvature. Let (Σ, P) be a compact, orientable surface of hyperbolic type (i.e. of genus at least two). Let $\theta : \pi_1(\Sigma, P) \to \pi_1(M, Q)$ be a homomorphism. Generalising a recent result of Gallo, Kapovich and Marden concerning necessary and sufficient conditions for the existence of complex projective structures with specified holonomy to manifolds of non-constant negative curvature, we obtain necessary conditions on θ for the existence of a so called θ-equivariant Plateau problem over Σ, which is equivalent to the existence of a strictly convex immersion $i : \Sigma \to M$ which realises θ (i.e. such that $\theta = i_*$).

Key Words: Kleinian groups, Fuchsian groups, Plateau problem, complex projective structures, immersions

AMS Subject Classification: 57M50 (30F10, 30F40, 32G15)

1 - Introduction.

In this paper, we study means of obtaining constant curvature realisations of homomorphisms of fundamental groups of surfaces into fundamental groups of compact, three dimensional of strictly negative sectional curvature.

Let (M, Q) be a pointed, compact three dimensional Riemannian manifold of strictly negative sectional curvature. Let (Σ, P) be a closed (orientable) surface of hyperbolic type (i.e. of genus at least 2). Let $(\tilde{M}, \tilde{Q})$ be the universal cover of (M, Q). Let $\theta : \pi_1(\Sigma, P) \to \pi_1(M, Q)$ be a representation of $\pi_1(\Sigma, P)$ in $\pi_1(M, Q)$. We observe that it is in many ways preferable to work in the framework of pointed manifolds. Firstly, $\pi_1(M, Q)$ is canonically defined in terms of (M, Q), and, secondly, the action of $\pi_1(M, Q)$ over $(\tilde{M}, \tilde{Q})$ is well defined.

$\tilde{M}$ is a Hadamard manifold. We denote by $\partial_\infty \tilde{M}$ its ideal boundary, and we make the following definition:

Definition 1.1

*Let $(\tilde{\Sigma}, \tilde{P})$ be the universal cover of (Σ, P). Let $\theta : \pi_1(\Sigma, P) \to \pi_1(M, Q)$ be a homomorphism. A θ-**equivariant Plateau problem** over Σ is a function $\varphi : \tilde{\Sigma} \to \partial_\infty \tilde{M}$ such that, for all $\gamma \in \pi_1(\Sigma, P)$:*

$$\varphi \circ \gamma = \theta(\gamma) \circ \varphi.$$

The interest of the study of such objects is best illustrated by the case where M is of constant sectional curvature equal to -1. In this case, $\tilde{M} = \mathbb{H}^3$, and there exists a canonical identification of $\partial_\infty \tilde{M}$ with $\hat{\mathbb{C}}$. Using the existence and uniqueness theorems of [7], one may use θ-equivariant Plateau problems in order to obtain constant curvature realisations of θ. Formally:

Lemma 1.2

Suppose that M has constant sectional curvature equal to -1. Suppose that there exists a θ-equivariant Plateau problem, φ, over Σ. Then, for all $k \in (0, 1)$, there exists a convex immersion $i : \Sigma \to M$ of constant Gaussian curvature equal to k such that:

$$\theta = i_*.$$

The group $\pi_1(M, Q)$ acts canonically on $\tilde{M} \cup \partial_\infty \tilde{M}$. We underline that $\pi_1(M, Q)$ acts on $\tilde{M} \cup \partial_\infty \tilde{M}$ **from the right**. Thus, throughout this entire paper, composition is to be read **from left to right**. A subgroup Γ of $\pi_1(M, Q)$ is said to be **non-elementary** if and only if it has no fixed points in $\tilde{M} \cup \partial_\infty \tilde{M}$, and we will say that it is **elementary** otherwise. In our case, the only elementary subgroups of $\pi_1(M, Q)$ are the trivial group, and isomorphic copies of $\mathbb{Z}$.

The group $\pi_1(M, Q)$ acts faithfully over $\partial_\infty \tilde{M}$, and may thus be considered as a subgroup of $\mathrm{Homeo}_0(\partial_\infty \tilde{M})$, the connected component of the group of homeomorphisms of $\partial_\infty \tilde{M}$

which preserves the identity. Let $\widetilde{\mathrm{Homeo}}_0(\partial_\infty\tilde{M})$ be the universal cover of $\partial_\infty\tilde{M}$, and let $\pi:\widetilde{\mathrm{Homeo}}_0(\partial_\infty\tilde{M})\rightarrow\partial_\infty\tilde{M}$. We have the following short exact sequence:

$$0\rightarrow\mathbb{Z}_2\xrightarrow{i}\widetilde{\mathrm{Homeo}}_0(\partial_\infty\tilde{M})\xrightarrow{\pi}\mathrm{Homeo}_0(\partial_\infty\tilde{M})\rightarrow 0.$$

For Γ an arbitrary group, and for $\varphi:\Gamma\rightarrow\mathrm{Homeo}_0(\partial_\infty\tilde{M})$ a homomorphism, we define a **lifting** of φ in $\widetilde{\mathrm{Homeo}}_0(\partial_\infty\tilde{M})$ to be a homomorhism $\hat{\varphi}:\Gamma\rightarrow\widetilde{\mathrm{Homeo}}_0(\partial_\infty\tilde{M})$ such that:

$$\pi\circ\hat{\varphi}=\varphi.$$

We say that φ **lifts** if such a lifting may be found.

Returning to the special case where M is of constant sectional curvature equal to -1, we recall an existence result, obtained by Gallo, Kapovich and Marden in [4], which, when translated into our framework, may be expressed as follows:

Theorem 1.3 [Gallo, Kapovich, Marden, 2000]

Suppose that M is of constant sectional curvature equal to -1. Then, there exists a θ-equivariant Plateau problem φ over Σ if and only if θ is non-elementary and lifts.

The requirement that θ be non-elementary arises from the fact that φ defines a $\mathbb{P}SL(2,\mathbb{C})$ structure over the surface Σ. If θ were elementary, then it would either be a subset of the rotation group, in which case it would define an $SO(3)$ structure over Σ, or it would be a subset of the affine group, in which case it would define an affine structure over Σ. Neither of these are possible, since they would induce non-negative curvature metrics over Σ, which are excluded by the Gauss-Bonnet theorem. Combining this result with lemma 1.2, we obtain the following result:

Lemma 1.4

Suppose that M has constant sectional curvature equal to -1. For all $k\in(0,1)$, there exists a convex, immersion $i:\Sigma\rightarrow M$ of constant Gaussian curvature equal to k such that:

$$\theta=i_*,$$

if and only if θ is non-elementary and lifts.

The objective of this paper is to obtain an analogue of this result in the more general case where M is only of strictly negative sectional curvature. The main result of this paper is the following:

Theorem 1.5

Suppose that (M,Q) is a pointed, compact manifold of strictly negative sectional curvature. Let (Σ,P) be a pointed, compact surface of hyperbolic type (i.e. of genus at least two). Let $\theta:\pi_1(\Sigma,P)\rightarrow\pi_1(M,Q)$ be a homomorphism. Suppose that θ is non-elementary and may be lifted to a homomorphism $\hat{\theta}$ of $\pi_1(\Sigma,P)$ into the group $\widetilde{\mathrm{Homeo}}_0(\partial_\infty\tilde{M})$. Then there exists an equivariant Plateau problem for θ.

This result permits us to obtain a convex realisation of θ:

Theorem 1.6

If θ is non-elementary and lifts, then there exists a convex immersion $i : \Sigma \rightarrow M$ such that:

$$\theta = i_*.$$

Theorem 1.5 is, in many respects, a generalisation to non-constant curvature of theorem 1.3, and will be proven using essentially the same techniques, following the observation that many of the algebraic results used in [4], which thus depend on the structure of $\mathbb{P}SL(2,\mathbb{C}) = \mathrm{Isom}(\mathbb{H}^3)$, may be expressed in a purely topological manner, and may thus be applied to $\mathrm{Isom}(\widetilde{M})$. In order to understand the algebraic obstruction to the construction of a θ-equivariant Plateau problem, we require a more subtle understanding of the algebraic properties of objects associated to Schottky groups. This study will occupy the better part of the second half of this paper.

The following sections of this paper are organised as follows:

In the second section, we study the topological properties of isometries of Hadamard manifolds, recalling the North/South dynamics of hyperbolic isometries, and we obtain algebraic results associated to Dehn twists of surfaces.

In the third section, we study Schottky groups in Hadamard manifolds of pinched sectional curvature, proving existence results for the construction of such groups.

In the fourth section, we use Dehn twists to construct a trouser decomposition of Σ in such a manner that the θ-image of the fundamental group of each trouser is a Schottky group. We essentially follow the directions laid down in [4], although, in our case, the results are easier to obtain since we do not have to consider parabolic maps.

In the fifth section, we construct invariant domains over Schottky groups, and obtain various algebraic results concerning these domains.

In the last section, we show how invariant domains may be used to construct θ-equivariant Plateau problems over trousers, and we then show how to glue these functions together to obtain a proof of theorem 1.5.

In the appendix, we prove that all the Schottky groups that we will be studying are equivalent up to homeomorphism. In particular, they are all equivalent to any given classical Schottky subgroup of $\mathbb{P}SL(2,\mathbb{C})$.

I would like to thank François Labourie for having drawn my attention to this problem, Kevin Costello for a brief introduction to characteristic classes (although they don't explicitely appear in this paper), and Frédéric Leroux, for having introduced me to the results of Friberg. I would also like to thank the Max Planck Institute for Mathematics in the Sciences in Leipzig for providing the comfort required to write the english version of this paper.

$\diamond$

2 - Dehn Twists Viewed Algebraicly.

2.1 Introduction.

Throughout this section, M will denote a three dimensional Hadamard manifold. Let (Σ, P_0) be a pointed compact surface and let $\theta : \pi_1(\Sigma, P_0) \rightarrow \mathrm{Isom}(M)$ be a representation of the fundamental group of (Σ, P_0). We prove algebraic results for hyperbolic isometries of Hadamard manifolds which may be associated to Dehn twists of (Σ, P_0).

The first result concerns the effect on a curve γ of the n'th iterated power of a Dehn twist about a Jordan curve η which γ intersects once. If we denote $\beta = \theta(\gamma)$ and $\alpha = \theta(\eta)$, then the desired result concerning Dehn twists may be expressed in the following terms:

Lemma 2.1

Let $\alpha : M \rightarrow M$ be hyperbolic, and let p_+ and p_- be the attractive and repulsive fixed points of α respectively. Let β be an isometry of M such that $p_\pm \cdot \beta \neq p_\mp$. For all $K > 0$, there exists $N \in \mathbb{N}$ such that for $n \geqslant N$, the isometry $\beta\alpha^{\pm n}$ is hyperbolic and:

$$\|\beta\alpha^{\pm n}\| \geqslant K.$$

We may also show where the fixed points of $\beta\alpha^n$ lie:

Corollary 2.2

With the same hypotheses as in the first part of the preceeding lemma, the attractive and repulsive fixed points of $\beta\alpha^n$ tend towards $p_\pm$ and $p_\mp \cdot \beta^{-1}$ respectively as n tends to $+\infty$.

The second result concerns the n-fold iterated effect of a Dehn twist about a Jordan curve η on a composed curve $\nu_1\nu_2$ where each of ν_1 and ν_2 intersect η exactly once, but in different senses. If we denote $\beta = \theta(\nu_1)$, $\alpha = \theta(\nu_2)$ and $\gamma = \theta(\eta)$, then the desired result may be expressed in the following terms:

Lemma 2.3

Let $\gamma : M \rightarrow M$ be hyperbolic. Let α and β be isometries of M with no fixed points in common with γ. Let $K > 0$ be a positive real number. There exists $N > 0$ such that for all $|n| \geqslant N$:

(i) $\gamma^n\alpha\gamma^{-n}\beta$ is hyperbolic,

(ii) $\|\gamma^n\alpha\gamma^{-n}\beta\| > K$,

(iii) $\gamma^n\alpha\gamma^{-n}\beta$ has neither fixed point in common with either α or β.

In the second part of this section, we describe models for Hadamard manifolds which are most appropriate to our uses. In the third part we review the classification of isometries of Hadamard manifolds into elliptic, parabolic and hyperbolic type. In the fourth part, we study the minimal displacement of such isometries. In the fifth part, we recall what is referred to as the north/south dynamics of these isometries, and in the final part, we provide proofs of lemmata 2.1 and 2.3 and corollary 2.2.

2.2 Models for M and $\partial_\infty M$.

It is well known that a Hadamard manifoldis canonically diffeomorphic (under the action of the exponential mapping) to its tangent space over any given point. Similarly, its ideal boundary is canonically homeomorphic to the unit sphere in the tangent space over any given point. In the sequel, we will require analogous results for normal bundles over geodesics. First, we have:

Lemma 2.4

Let M be a Hadamard manifold and let γ be a geodesic in M. Let N_γ be the normal bundle over γ. In otherwords:

$$N_\gamma = \left\{X \in T_{\gamma(t)}M | t \in \mathbb{R}, X_p \perp \partial_t\gamma(t)\right\}.$$

The restriction of the exponential mapping to N_γ is a diffeomorphism between N_γ and M.

Remark: In fact, in the statement of the preceeding lemma, one could replace γ by any complete totally geodesic submanifold.

Next, if the sectional curvature of M is bounded above by $-\epsilon < 0$, then we obtain the corresponding result for the ideal boundary of M:

Lemma 2.5

Let M be a Hadamard manifold of sectional curvature bounded above by $-\epsilon < 0$. Let γ be a geodesic in M and let Σ_γ be the normal sphere bundle over γ contained in N_γ. In otherwords:

$$\Sigma_\gamma = \{X \in N_\gamma | \|X\| = 1\}.$$

Let us define $\partial_\infty\gamma$ by:

$$\partial_\infty\gamma = \{\gamma(-\infty), \gamma(\infty)\}.$$

The restriction of the Gauss-Minkowski mapping defines a homeomorphism between Σ_γ and $\partial_\infty M \setminus \partial_\infty\gamma$.

2.3 A Classification of Isometries of M.

Let us suppose that the sectional curvature of M is bounded above by $-\epsilon < 0$. Let $\alpha : M \to M$ be an isometry of M. The mapping α may be extended to a homeomorphism of $M \cup \partial_\infty M$. Brouwer's fixed point theorem then informs us that α has at least one fixed point in M. We may thus classify the isometries of M according to their fixed points, in a similar fashion as for $\text{Isom}(\mathbb{H}^3) \cong \text{Aut}(\hat{\mathbb{C}})$.

Indeed, if α has a fixed point p in M, then the application $T_p\alpha$ is a rotation of T_pM. The mapping α is entirely defined by this rotation and is said to be **elliptic**. If α does not have any fixed points in M, and only one fixed point in $\partial_\infty M$, then the application is said to be **parabolic**. This case will be of no interest to us in the sequel, since it will later be excluded by the compactness of the manifolds that we will study. If α does not have any fixed points in M, and exactly two fixed points, p_1 and p_2 in $\partial_\infty M$, then it is said to be **hyperbolic**. Since M is a visibility manifold, there exists an oriented geodesic γ such that $\gamma(-\infty) = p_1$ and $\gamma(+\infty) = p_2$. Moreover, α sends γ onto another oriented geodesic parallel to γ. However, since the sectional curvature of M is bounded above by $-\epsilon < 0$, any two parallel geodesics in M coincide. It thus follows that α sends γ onto itself. Since α is an isometry, there exists $T_0 \in \mathbb{R}$ such that, for all $t \in \mathbb{R}$:

$$(\alpha \circ \gamma)(t) = \gamma(t + T_0).$$

The following result permits us to show that any isometry of M must belong to one of these three categories:

Lemma 2.6

Let M be a three dimensional Hadamard manifold of sectional curvature bounded above by $-\epsilon < 0$. Let α be an isometry of M. If α is different to the identity, then α has at most 2 distinct fixed points in $\partial_\infty M$.

Proof: We show that an isometry of M having at least three fixed points is necesarily the identity. Let $p_1, p_2, p_3 \in \partial_\infty M$ be the fixed points of α. Let γ be the unique geodesic in M such that $\gamma(-\infty) = p_1$ and $\gamma(+\infty) = p_2$. Let Σ_γ be the normal sphere bundle over γ. Let $\overrightarrow{n} : \Sigma_\gamma \rightarrow \partial_\infty M \setminus \partial_\infty \gamma$ be the Gauss-Minkowski mapping. By lemma 2.5, there exists a unique $X \in \Sigma_\gamma$ such that $\overrightarrow{n}(X) = p_3$. Since α is an isometry, it commutes with $\overrightarrow{n}$. Moreover, since it fixes p_1 and p_2, it sends γ, and thus Σ_γ, onto itself. Consequently, by uniqueness:

$$T\alpha(X) = X.$$

It follows that $\pi(X)$ as a fixed point of α in M. Since $T_{\pi(X)}\alpha$ is a rotation in three dimensions, and since it preserves $\partial_t\gamma$ and X, it must be the identity. The α is thus the identity, and the result now follows. $\square$

2.4 The Dynamics of Isometries of M.

If α is an isometry of M, we define $\|\alpha\|$, which we call the **minimal displacement** of α, by:

$$\|\alpha\| = \operatorname*{Inf}_{x \in M} d(x, x \cdot \alpha).$$

We may equally consider $D_\alpha : M \rightarrow \mathbb{R}$, the **displacement function** of α, defined by:

$$D_\alpha(x) = d(x, x \cdot \alpha).$$

With these definitions, we obtain the following trivial result:

Lemma 2.7

If α is elliptic with fixed point $p_0 \in M$, then D_α takes its minimum value over M at p_0, and:

$$\|\alpha\| = 0.$$

We obtain an analogous result for hyperbolic applications:

Lemma 2.8

Let α be hyperbolic. Let γ be the (oriented) geodesic in M preserved by α and let $T_0 \in \mathbb{R}$ be such that, for all $t \in \mathbb{R}$:

$$\gamma(t) \cdot \alpha = \gamma(t + T_0).$$

The function D_α takes its minimum value over M at every point of γ, and:

$$\|\alpha\| = |T_0|\,.$$

Proof: By definition, for all $p \in \gamma$:

$$D_\alpha(p) = |T_0|\,.$$

Let $\pi : M \to \gamma$ be the orthogonal projection onto γ. This projection reduces distance. In otherwords, for $p, q \in M$:

$$d(\pi(p), \pi(q)) \leqslant d(p, q).$$

Since α is an isometry of M which preserves γ, it commutes with π:

$$\alpha \circ \pi = \pi \circ \alpha.$$

Thus, for all $p \in M$:

$$\begin{aligned} d(p, p \cdot \alpha) &\geqslant d(\pi(p), \pi(p \cdot \alpha)) \\ &= d(\pi(p), \pi(p) \cdot \alpha) \\ &= T_0. \end{aligned}$$

The result now follows. □

In the sequel, we will work with actions of fundamental groups of compact manifolds of strictly negative sectional curvature on the universal covers of these manifolds. In such subgroups $\Gamma \subseteq \mathrm{Isom}(M)$, every element that is different to the identity is necessarily hyperbolic, and thus an application $\alpha \in \Gamma$ is hyperbolic if and only if $\|\alpha\| > 0$.

2.5 North-South Dynamics.

We now describe what is called the "North-South Dynamics" of hyperbolic isometries of a Hadamard manifold:

Lemma 2.9 *North-South Dynamics.*

Let $p_\pm$ be two distinct points in $\partial_\infty M$ and let $U_\pm$ be neighbourhoods in $M \cup \partial_\infty M$ of $p_\pm$. There exists $K > 0$ such that, if $\alpha : M \to M$ is a hyperbolic isometry having p_+ and p_- as attractive and repulsive fixed points respectively, and if $\|\alpha\| > K$, then $(U_-)^C \cdot \alpha \subseteq U_+$.

Proof: Let γ be the unique geodesic in M such that:

$$\gamma(\pm\infty) = p_\pm.$$

Let N_γ be the normal bundle over γ in TM. By lemma 2.4, the exponential mapping of M generates a diffeomorphism between N_γ and M. For $t_0 \in \mathbb{R}$, the parallel transport along γ generates a canonical isometry of vector bundles $i : \mathbb{R} \times N_{\gamma(t_0)} \to N_\gamma$. We define $\tilde{\alpha} : \mathbb{R} \times N_{\gamma(t_0)} \to \mathbb{R} \times N_{\gamma(t_0)}$ such that the following diagram commutes:

$$\begin{tikzcd}
\mathbb{R} \oplus N_{\gamma(t_0)} \arrow[r, "\mathrm{Exp}\circ i"] \arrow[d, "\tilde{\alpha}"] & M \arrow[d, "\alpha"] \\
\mathbb{R} \oplus N_{\gamma(t_0)} \arrow[r, "\mathrm{Exp}\circ i"] & M
\end{tikzcd}$$

Since α commutes with parallel transport along γ, we find that there exists a rotation R of $N_{\gamma(t_0)}$ and $T_0 \in \mathbb{R}$ such that:

$$\tilde{\alpha}(t, v) = (t + T_0, Rv).$$

Next, there exists $T_\pm \in \mathbb{R}$ such that:

$$\begin{array}{ll}(\mathrm{Exp} \circ i)(\{(t, v) | t \geqslant T_+\}) & \subseteq U_+, \\ (\mathrm{Exp} \circ i)(\{(t, v) | t \leqslant T_-\}) & \subseteq U_-.\end{array}$$

We define K such that $K \geqslant T_+ - T_-$, and the result follows. $\square$

We also obtain a converse of this result which permits us to show when a given isometry is hyperbolic. Before giving the statement of this lemma, we recall that if γ is an oriented Jordan curve contained in a sphere Σ_2, we define the **interior** of γ, which we denote by $\mathrm{Int}(\gamma)$, to be the connected component of $\Sigma_2 \setminus \gamma$ lying to its left hand side. We then define the **exterior** of γ, which we denote by $\mathrm{Ext}(\gamma)$ to be the other connected component of $\Sigma_2 \setminus \gamma$. We now obtain the following result:

Lemma 2.10

Let $C_\pm$ be two disjoint Jordan curves in $\partial_\infty M$. We suppose that each of these curves lies in the exterior of the other. If $\alpha : M \to M$ is an isometry such that $\mathrm{Ext}(C_-) \cdot \alpha = \mathrm{Int}(C_+)$, then α is hyperbolic. Moreover, if p_+ and p_- are the attractive and repulsive fixed points respectively of α, then:

$$p_\pm \in \mathrm{Int}(C_\pm).$$

Proof: Since $\mathrm{Int}(C_+) \subseteq \mathrm{Ext}(C_-)$, it follows that α sends $\overline{\mathrm{Ext}(C_-)}$ into itself. Brouwer's fixed point theorem now tells us that there exists a fixed point q of α in $\overline{\mathrm{Ext}(C_-)}$. Moreover, since $q \cdot \alpha = q$, we find that $q \in \overline{\mathrm{Int}(C_+)} \subseteq \mathrm{Ext}(C_-)$. Consequently, applying α again, we obtain $q \in \mathrm{Int}(C_+)$. Applying the same reasoning to α^{-1}, we find that there exists another fixed point q' of α in $\mathrm{Int}(C_-)$. It thus follows that α has two fixed points in $\partial_\infty M$. Consequently, α is either the identity, elliptic, or hyperbolic.

Since $C_- \cdot \alpha = C_+$ and $C_- \cap C_+ = \emptyset$, it follows that α is different from the identity. Let γ be the geodesic going from q' to q:

$$\gamma(-\infty) = q', \gamma(+\infty) = q.$$

Let Σ_γ be the normal sphere bundle over γ. By lemma 2.5, the Gauss-Minkowski mapping, $\overrightarrow{n}$, generates a homeomorphism between Σ_γ and $\partial_\infty M \setminus \partial_\infty \gamma$. For $t_0 \in \mathbb{R}$, parallel transport permits us to construct a canonical isomorphism $i : \mathbb{R} \times \Sigma_{\gamma(t_0)} \to \Sigma_\gamma$. Let $\tilde{\alpha}$ be the homeomorphism of $\mathbb{R} \times \Sigma_{\gamma(t_0)}$ defined such that the following diagram commutes:

$$\begin{tikzcd}
\mathbb{R} \oplus \Sigma_{\gamma(t_0)} \arrow[r, "\overrightarrow{n} \circ i"] \arrow[d, "\tilde{\alpha}"] & \partial_\infty M \setminus \partial_\infty \gamma \arrow[d, "\alpha"] \\
\mathbb{R} \oplus \Sigma_{\gamma(t_0)} \arrow[r, "\overrightarrow{n} \circ i"] & \partial_\infty M \setminus \partial_\infty \gamma
\end{tikzcd}$$

We now define $\tilde{C}_\pm \subseteq \mathbb{R} \times \Sigma_{\gamma(t_0)}$ such that:

$$(\overrightarrow{n} \circ i)(\tilde{C}_\pm) = C_\pm.$$

We see that $\tilde{\alpha}(\tilde{C}_-) = \tilde{C}_+$. If $\pi : \Sigma_\gamma \to \mathbb{R}$ is the canonical projection, since $\tilde{C}_+$ and $\tilde{C}_-$ do not intersect, we find that:

$$\mathrm{Sup}(\pi(\tilde{C}_-)) \quad < \mathrm{Sup}(\pi(\tilde{C}_+)).$$

Since α commutes with parallel transport, there exists a rotation, R, and a real number, T_0, such that, for all $(t, v) \in \Sigma_\gamma$:

$$\tilde{\alpha}(t, v) = (t + T_0, Rv).$$

Consequently:

$$\pi \circ \tilde{\alpha} = T_0 + \pi.$$

By considering the restriction of π to $\tilde{C}_-$, we obtain $T_0 > 0$, and the result now follows. □

Moreover, we can estimate the norm of a hyperbolic isometry:

Lemma 2.11

Let $D_\pm$ be two disjoint closed oriented discs in $M \cup \partial_\infty M$ such that $C_\pm = \partial D_\pm$ are two disjoint Jordan curves in $\partial_\infty M$. If $\alpha : M \to M$ is an isometry such that $\mathrm{Ext}(D_-) \cdot \alpha = \mathrm{Int}(D_+)$, then α is hyperbolic, and:

$$\|\alpha\| \geqslant d(D_-, D_+).$$

Proof: By the preceeding lemma, α is hyperbolic. Moreover, if p_+ and p_- are the attractive and repulsive fixed points of α respectively, then $p_+ \in \mathrm{Int}(C_+)$ and $p_- \in \mathrm{Int}(C_-)$. Let γ be the geodesic joining p_- to p^+:

$$\gamma(\pm\infty) = p_\pm.$$

Then, there exists $T_0 \in \mathbb{R}$ such that, for all $t \in \mathbb{R}$:

$$\gamma(t) \cdot \alpha = \gamma(t + T_0).$$

Next, $p_\pm = \gamma(\pm\infty)$ is an interior points of $\mathrm{Int}(C_\pm)$. It thus follows that there exists $t_0 \in \mathbb{R}$ such that:

$$\gamma(t_0) \in D_-.$$

Since, $\gamma(t_0) \cdot \alpha$ lies in D_+, we have:

$$\begin{aligned} \|\alpha\| &= \|T_0\| \\ &= d(\gamma(t_0), \gamma(t_0) \cdot \alpha) \\ &\geqslant d(D_-, D_+). \end{aligned}$$

The result now follows. □

2.6 Proofs of Main Results of This Section.

We are now in a position to prove the main results of this section. To begin with, we have:

Lemma 2.1

Let $\alpha : M \to M$ be hyperbolic, and let p_+ and p_- be the attractive and repulsive fixed points of α respectively. Let β be an isometry of M such that $p_\pm \cdot \beta \neq p_\mp$. For all $K > 0$, there exists $N \in \mathbb{N}$ such that for $n \geqslant N$, the isometry $\beta\alpha^{\pm n}$ is hyperbolic and:

$$\|\beta\alpha^{\pm n}\| \geqslant K.$$

Proof: Suppose first that $p_+ \cdot \beta \neq p_-$. We define $q = p_- \cdot \beta^{-1}$. By hypothesis, $q \neq p_+$. Let $D_1, D_2 \subseteq \tilde{M} \cup \partial_\infty \tilde{M}$ be plunged oriented disks such that ∂D_1 and ∂D_2 are Jordan curves in $\partial_\infty \tilde{M}$ and $\mathrm{Int}(D_1)$ and $\mathrm{Int}(D_2)$ are neigbourhoods of q and p_+ in $\tilde{M} \cup \partial_\infty \tilde{M}$ respectively. We may suppose, moreover, that:

$$d(D_1, D_2) > K.$$

$\mathrm{Int}(D_1 \cdot \beta)$ is a neighbourhood of p_- in $\tilde{M} \cup \partial_\infty \tilde{M}$. It thus follows by the principal of North/South dynamics (lemma 2.9) that there exists $N \in \mathbb{N}$ such that, for $n \geqslant N$:

$$\begin{aligned} & \mathrm{Ext}(D_1 \cdot \beta) \cdot \alpha^n &&\subseteq \mathrm{Int}(D_2) \\ \Rightarrow\ & \mathrm{Ext}(D_1) \cdot (\beta\alpha^n) &&\subseteq \mathrm{Int}(D_2). \end{aligned}$$

By lemma 2.11, the mapping $\beta\alpha^n$ is hyperbolic and $\|\beta\alpha^n\| \geqslant K$. The first case now follows. Next, suppose that:

$$p_- \cdot \beta \neq p_+.$$

In particular $p_+ \cdot \beta^{-1} \neq p_-$. It follows that there exists $N \in \mathbb{N}$ such that, for $n \geqslant N$, the mapping $\beta^{-1}\alpha^n$ is hyperbolic, and:

$$\|\beta^{-1}\alpha^n\| \geqslant K.$$

It now follows by conjugation that $\alpha^n\beta^{-1}$ is hyperbolic. Consequently, by inversion, we find that $\beta\alpha^{-n}$ is too. Moreover:

$$\|\beta\alpha^{-n}\| = \|\alpha^n\beta^{-1}\| = \|\beta^{-1}(\alpha^n\beta^{-1})\beta\| = \|\beta^{-1}\alpha^n\| \geqslant K.$$

The second case now follows. □

Corollary 2.2 follows almost immediately:

Corollary 2.2

With the same hypotheses as in the first part of the preceeding lemma, the attractive and repulsive fixed points of $\beta\alpha^n$ tend towards $p_\pm$ and $p_\mp \cdot \beta^{-1}$ respectively as n tends to $+\infty$.

Proof: For sufficiently large n, let $q_{n,+}$ and $q_{n,-}$ be the attractive and repulsive fixed points respectively of $\beta\alpha^n$. In the proof of the preceeding lemma, we showed that $q_{n,+}{}^* \in \partial_\infty\mathrm{Int}(D_2)$ and $q_{n,-} \in \partial_\infty\mathrm{Int}(D_1)$. Since these two open sets may be chosen arbitrarily close to p_+ and $p_- \cdot \beta^{-1}$ respectively, the first case now follows. The second case also follows by a similar reasoning. □

Finally, we prove:

Lemma 2.3

Let $\gamma : M \to M$ be hyperbolic. Let α and β be isometries of M with no fixed points in common with γ. Let $K > 0$ be a positive real number. There exists $N > 0$ such that for all $|n| \geqslant N$:

(i) $\gamma^n\alpha\gamma^{-n}\beta$ is hyperbolic,

(ii) $\|\gamma^n\alpha\gamma^{-n}\beta\| > K$,

(iii) $\gamma^n\alpha\gamma^{-n}\beta$ has neither fixed point in common with either α or β.

Proof: Let p_+ and p_- be the attractive and repulsive fixed points of γ respectively. We define $q = p_- \cdot \beta$. Let $D_1, D_2 \subseteq \tilde{M} \cup \partial_\infty\tilde{M}$ be plunged oriented disks such that ∂D_1 and ∂D_2 are Jordan curves in $\partial_\infty M$ and $\mathrm{Int}(D_1)$ and $\mathrm{Int}(D_2)$ are neighbourhoods of p_- and q in $\tilde{M} \cup \partial_\infty\tilde{M}$ respectively. Since $q \neq p_*$, we may suppose moreover that:

$$d(D_1, D_2) > K.$$

Since $p_+ \neq p_+ \cdot \alpha$, there exist disjoint neighbourhoods U_1 and U_2 of p_+ and $p_- \cdot \alpha$ respectively in $\tilde{M} \cup \partial_\infty\tilde{M}$. By restricting U_1 if necessary, we may suppose moreover that $U_1 \cdot \alpha \subseteq U_2$. Let V be a neighbourhood of p_- in $\tilde{M} \cup \partial_\infty\tilde{M}$ such that:

$$V \cdot \beta \subseteq \mathrm{Int}(D_2).$$

By the principal of North/South dynamics (lemma 2.9), there exists $N \in \mathbb{N}$ such that, for $n \geqslant N$:

$$\begin{array}{ll} \mathrm{Ext}(D_1) \cdot \gamma^n & \subseteq U_1, \\ U_1^C \cdot \gamma^{-n} & \subseteq V. \end{array}$$

Consequently, for $n \geqslant N$:

$$\begin{array}{ll} \mathrm{Ext}(D_1) \cdot \gamma^n \alpha \gamma^{-n} \beta & \subseteq U_1 \cdot \alpha \gamma^{-n} \beta \\ & \subseteq U_2 \cdot \gamma^{-n} \beta \\ & \subseteq U_1^C \cdot \gamma^{-n} \beta \\ & \subseteq V \cdot \beta \\ & \subseteq \mathrm{Int}(D_2). \end{array}$$

Thus, by lemma 2.11, the mapping $\gamma^n \alpha \gamma^{-n} \beta$ is hyperbolic and:

$$\| \gamma^n \alpha \gamma^{-n} \beta \| > K.$$

We now follow the reasoning of Gallo, Kapovich and Marden in order to prove the result concerning the fixed points. Let $q_{n,+}$ and $q_{n,-}$ be the attractive and repulsive fixed points points respectively of $\gamma^n \alpha \gamma^{-n} \beta$. As in the proof of corollary 2.2, we may show that $(q_{n,+})_{n \in \mathbb{N}}$ tends towards $p_- \cdot \beta$ and $(q_{n,-})_{n \in \mathbb{N}}$ tends towards p_-. Since neither $p_- \cdot \beta$ nor p_- is a fixed point of β, it follows that, for sufficiently large n, $\gamma^n \alpha \gamma^{-n} \beta$ does not have a fixed point in common with β.

Let us suppose that there exists $(k_n)_{n \in \mathbb{N}}$ such that $(k_n)_{n \in \mathbb{N}} \uparrow \infty$ and that $\gamma^{k_n} \alpha \gamma^{-k_n} \beta$ has a fixed point in common with α for all n. Since α, being different to the identity, has at most 2 fixed points, by taking a sub-sequence if necessary, we may suppose that there exists p_0 which is a fixed point of α and of $\gamma^{k_n} \alpha \gamma^{-k_n} \beta$ for all n. In particular, for all n:

$$p_0 \in \{ q_{\pm,n} \}.$$

Thus, by taking limits as n tends to infinity, we obtain:

$$p_0 \in \{ p_- \cdot \beta, p_- \}.$$

Let us now suppose that $p_0 = p_-$. Since p_0 is a fixed point of α, we have:

$$\begin{array}{lll} & p_- \cdot (\gamma^{k_n} \alpha \gamma^{-k_n} \beta) & = p_- \\ \Rightarrow & p_- \cdot \beta & = p_-. \end{array}$$

This is absurd. It thus follows that $p_0 = p_- \cdot \beta$. However, for all n:

$$\begin{array}{lll} & (p_- \cdot \beta) \cdot (\gamma^{-k_n} \alpha \gamma^{k_n} \beta) & = p_- \cdot \beta \\ \Rightarrow & (p_- \cdot \beta) \cdot (\gamma^{k_n} \alpha) & = p_-. \end{array}$$

Thus, in particular, for $n \neq m$:

$$\begin{array}{lll} & (p_- \cdot \beta) \cdot (\gamma^{k_n} \alpha) & = (p_- \cdot \beta)(\gamma^{k_m} \alpha) \\ \Rightarrow & (p_- \cdot \beta) & = (p_- \cdot \beta) \cdot \gamma^{k_m - k_n}. \end{array}$$

Letting $k_m - k_n$ tend towards infinity, we now find that $p_- \cdot \beta$ must be a fixed point of γ. By hypothesis, $p_- \cdot \beta \neq p_-$. Consequently, $p_0 = p_- \cdot \beta = p_+$, and so $p_+ \cdot \alpha = p_+$, which is absurd. The result for $n \geqslant 0$ thus follows, and the result for $n \leqslant 0$ follows by a similar reasoning. □

◇

3 - Schottky Groups.

3.1 Introduction.

Throughout this section, M will denote a three dimensional Hadamard manifold. The construction that we will carry out in the sequel makes considerable use of Schottky groups. We define such groups as follows:

Definition 3.1

Let C_a^-, C_a^+, C_b^- and C_b^+ be disjoint Jordan curves in the sphere S_2 oriented such that each one of these curves is situated in the exterior of the three others. We denote the group of homeomorphisms of S_2 which preserve orientation by $\mathrm{Homeo}_0(S_2)$. We will adopt the convention that this group acts on S_2 **from the right**. *A* **Schottky Group** *is a subgroup Γ of $\mathrm{Homeo}_0(S_2)$ generated by two elements a and b such that:*

$$\mathrm{Ext}(C_a^-) \cdot a = \mathrm{Int}(C_a^+), \qquad \mathrm{Ext}(C_b^-) \cdot b = \mathrm{Int}(C_b^+).$$

We will refer to the curves $(C_a^\pm, C_b^\pm)$ as **generating circles** *for the Schottky group Γ with respect to the pair of generators (a, b).*

When we study Schottky subgroups of the group of homeomorphisms of $\partial_\infty M$, we may replace the generating circles in this definition with oriented disks $D_a^\pm$ and $D_b^\pm$ in M whose boundaries are Jordan curves in $\partial_\infty M$. Such disks will later be refered to as **generating disks** of the Schottky group Γ with respect to the generator (a, b).

The principal result of this section is the following:

Lemma 3.2

Let $K \geqslant k > 0$ be real numbers and let M be a Hadamard manifold with curvature pinched between $-K$ and $-k$. Let α be a hyperbolic isometry of M having p_+ and p_- as attractive and repulsive fixed points respectively. Let p_0 be a point distinct from $p_\pm$. For every $B > 0$, there exists a neighbourhood Ω of p_0 in $\partial_\infty M$ such that if $q_\pm \in \Omega$ and if β is a hyperbolic isometry of M having $q_\pm$ as fixed points such that $\|\beta\| > B$, then the subgroup $\langle \alpha, \beta \rangle$ of $\mathrm{Isom}(M)$ generated by α and β is a Schottky group.

This lemma has the following useful corollary:

Corollary 3.3

Let $K \geqslant k > 0$ be real numbers and let M be a Hadamard manifold with curvature pinched between $-K$ and $-k$. Let Γ be a subgroup of $\mathrm{Isom}(M)$ which only contains hyperbolic elements (and the identity). Let γ be a hyperbolic element of Γ and let $p_\pm$ be the attractive and repulsive fixed points of γ respectively. If α, β are elements of Γ having no fixed points in common with γ, then there exists $N > 0$ such that, for $|n| \geqslant N$, the subgroup of $\mathrm{Isom}(M)$ generated by $\gamma^n \alpha \gamma^{-n}$ and by β is a Schottky group.

Proof: Since α does not fix p_-, we find that the fixed points of $\gamma^{-n}\alpha\gamma^n$ tend towards p_+ as n tends towards $+\infty$. By hypothesis, p_+ is not a fixed point of β. Moreover, α is not equal to the identity and is thus hyperbolic. Next, we see that:

$$\|\gamma^{-n}\alpha\gamma^n\| = \|\alpha\| > 0.$$

By the preceeding lemma, there exists $N \geqslant 0$ such that, for $n \geqslant N$, the subgroup of $\mathrm{Isom}(M)$ generated by $\gamma^{-n}\alpha\gamma^n$ and β is a Schottky group. A similar reasoning permits us to prove the result for n negative, and the result now follows. □

We also obtain the following result:

Lemma 3.4

Let $k > 0$ be a real number. Let M be a Hadamard manifold of sectional curvature bounded above by $-k$. Let $p_{1,\pm}, p_{2,\pm}$ be distinct points in $\partial_\infty M$. There exists $B > 0$ such that, if α_1 and α_2 are hyperbolic isometries of M such that $p_{k,\pm}$ are fixed points of α_k for each k and if $\|\alpha_1\|, \|\alpha_2\| > B$, then the subgroup of $\mathrm{Isom}(M)$ generated by α_1 and α_2 is a Schottky group.

In order to prove these results, we require upper and lower bounds of lengths and angles of triangles in Hadamard manifolds of pinched curvature. We obtain these using classical comparison techniques for such manifolds arising from the Topogonov theorems and related results. We refer the reader to [2] for details.

In the second part of this section, we review the properties of various function defined over triangles in constant curvature Hadamard manifolds. In the third part we recall Topogonov's comparison theorems and the distance contracting/dilating property of canonical maps between triangles and their corresponding comparison triangles. In the fourth part, we study degenerate triangles having one point at infinity. In the fifth part, we take this one step further, studying degenerate triangles having two points at infinity. In the sixth part, we study another useful bound. In the seventh part, we show how these results may be used to control the positions of complete geodesics in the manifolds. Finally, in the eighth section, we bring all these results together in order to prove lemmata 3.2 and 3.4.

3.2 Comparison Triangles.

We study various functions arising from geodesic triangles in Hadamard manifolds. For all $\lambda \geqslant 0$, let M_λ be the 2 dimensional Hadamard manifold with constant sectional curvature equal to $-\lambda$. M_λ will act as a comparison space for objects in any given Hadamard manifold.

Let ΔABC be a triangle in M_λ (by triangle, we mean any subset of M_λ made up of three vertices A, B and C and the geodesic arcs joining these vertices. This set inherits the structure of a metric space from M_λ). Next, we denote by α, β and γ the sizes of the angles at the vertices A, B and C respectively and we denote by a, b and c the lengths of the edges BC, AC and AB respectively.

We begin by studying the edge AB. By using, for example, the Poincaré disc model of $M_{-1} = \mathbb{H}^2$ and by dilating it if necessary, we find that the length of c is uniquely defined by the triplet (a, b, γ). By keeping a and b constant, we thus obtain a function $\tilde{c}_\lambda : \gamma \mapsto c$ (see figure 3.1).

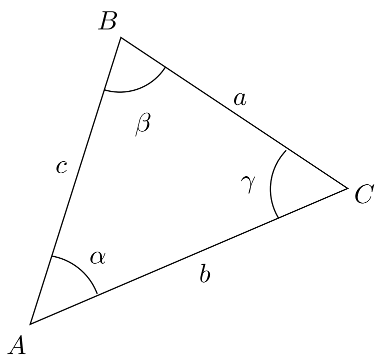

Figure 3.1

We obtain the following result concerning the properties of $\tilde{c}_\lambda$:

Lemma 3.5

$\tilde{c}_\lambda$ has the following properties:

(i) $\tilde{c}_\lambda$ is continuous,

(ii) $\tilde{c}_\lambda(0) = |b - a|$ and $\tilde{c}_\lambda(\pi) = a + b$, and

(iii) $\tilde{c}_\lambda$ is strictly increasing (and thus injective).

Remark: In particular, by continuity, for all a, b and c satisfying the triangle inequality, there exists a triangle ΔABC in M_λ having a, b and c as lengths of edges.

Next, we observe that the angle β is uniquely defined by the triplet (a, b, γ). Keeping a and b constant, we thus obtain a function $\tilde{\beta}_\lambda : \gamma \mapsto \beta$. We obtain the following result concerning the properties of $\tilde{\beta}_\lambda$:

Lemma 3.6

The function $\tilde{\beta}_\lambda$ has the following properties:

(i) $\tilde{\beta}_\lambda$ is continuous,

(ii) if $b > a$, then $\tilde{\beta}_\lambda(0) = \pi$ and $\tilde{\beta}_\lambda(\pi) = 0$,

(iii) If $b > a$, then the function $\tilde{\beta}_\lambda$ is strictly decreasing (and thus injective) over $[0, \pi]$ and,

(iv) the function $\tilde{\beta}_\lambda$ is strictly decreasing (and thus injective) over the interval $[\pi/2, \pi]$.

The interested reader may find a more detailed study of these functions and proofs of these results in appendix B of [8].

3.3 Topogonov's Theorems and Canonical Homeomorphisms.

Now, let $K \geqslant k \geqslant 0$ and let M be a Hadamard manifold with sectional curvature pinched between $-K$ and $-k$. Let P be a point in M, let P_{-k} be a point in M_{-k} and let P_{-K} be a point in M_{-K}. Let $I_{-k} : T_P M \to T_{P_{-k}} M_{-k}$ and $I_{-K} : T_P M \to T_{P_{-K}} M_{-K}$ be isometries. If we define g_{-k}, g and g_{-K} to be the metrics over M_{-k}, M and M_{-K} respectively, then Rauch's comparison theorem tells us that:

$$I^*_{-K}\mathrm{Exp}^* g_{-K} \geqslant \mathrm{Exp}^* g \geqslant I^*_{-k}\mathrm{Exp}^* g_{-k}.$$

This theorem permits us to obtain Topogonov's theorems. Firstly, we have:

Theorem 3.7 [Topogonov I]

Let ΔABC be a triangle in M and let $\Delta A'B'C'$ be a comparison triangle in M_{-k} such that $A'B' = AB$ and $A'C' = AC$ and $\widehat{A'B'C'} = \widehat{ABC}$. Then:

$$BC \geqslant B'C'.$$

Similarly, if $\Delta A''B''C''$ is a comparison triangle in M_{-K} such that $A''B'' = AB$, $A''C'' = AC$ and $\widehat{A''B''C''} = \widehat{ABC}$, then:

$$BC \leqslant B''C''.$$

Topogonov's second comparison theorem may be derived as a corollary to this result:

Theorem 3.8 [Topogonov II]

Let ΔABC be a triangle in M and let $\Delta A'B'C'$ be the comparison triangle in M_{-k} such that $A'B' = AB$, $A'C' = AC$ and $B'C' = BC$. Then:

$$\widehat{BAC} \leqslant \widehat{B'A'C'}, \qquad \widehat{ACB} \leqslant \widehat{A'C'B'}, \qquad \widehat{CBA} \leqslant \widehat{C'B'A'}.$$

Similarly, if $\Delta A''B''C''$ is the comparison triangle in M_{-K} such that $A''B'' = AB$, $A''C'' = AC$ and $B''C'' = BC$, then:

$$\widehat{BAC} \geqslant \widehat{B''A''C''}, \qquad \widehat{ACB} \geqslant \widehat{A''C''B''}, \qquad \widehat{CBA} \geqslant \widehat{C''B''A''}.$$

If ΔABC is a triangle in M and if $\Delta A'B'C'$ is the comparison triangle in M_λ such that $A'B' = AB$, $A'C' = AC$ and $B'C' = BC$, then we define the mapping φ_λ sending the vertices of $\Delta A'B'C'$ to those of ΔABC by:

$$\varphi_\lambda(A') = A, \qquad \varphi_\lambda(B') = B, \qquad \varphi_\lambda(C') = C.$$

We then extend φ_λ to a function over the whole of $\Delta A'B'C'$ by linear interpolation, and we obtain the **canonical homeomorphism** from $\Delta A'B'C'$ to ΔABC. Viewing triangles in general as metric subspaces of the corresponding ambiant manifolds, we obtain the following result:

Theorem 3.9

Let ΔABC be a triangle in M and let $\Delta A'B'C'$ be the comparison triangle in M_{-k} such that $A'B' = AB$, $A'C' = AC$ and $B'C' = BC$. Let $\varphi_{-k} : \Delta A'B'C' \to \Delta ABC$ be the canonical homeomorphism. Then φ_{-k} is contracting. In otherwords, if d and d_{-k} denote the distance functions over M and M_{-k} respectively, then, for all $P, Q \in \Delta A'B'C'$:

$$d(\varphi_{-k}(P), \varphi_{-k}(Q)) \leqslant d_{-k}(P, Q).$$

Similarly, if $\Delta A''B''C''$ is the comparison triangle in M_{-K} such that $A''B'' = AB$, $A''C'' = AC$ and $B''C'' = BC$ and if $\varphi_{-K} : \Delta A''B''C'' \to \Delta ABC$ is the canonical homeomorphism, then φ_{-K} is dilating. In otherwords, if d and d_{-K} denote the distance functions over M and M_{-K} respectively, then, for all $P, Q \in \Delta A'B'C'$:

$$d(\varphi_{-K}(P), \varphi_{-K}(Q)) \geqslant d_{-K}(P, Q).$$

3.4 Degenerate Triangles Having One Point at Infinity.

We now wish to consider degenerate triangles having one point at infinity in a Hadamard manifold of pinched sectional curvature. For $\lambda > 0$, let ΔABC be a triangle in M_λ such that the angle between BC and BA is equal to $\pi/2$ and A is in $\partial_\infty M_\lambda$ (see figure 3.2).

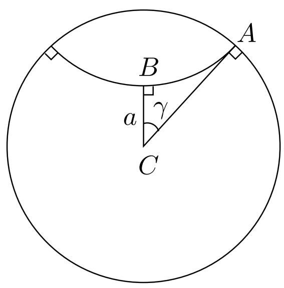

Figure 3.2

Under these conditions, the angle γ is uniquely defined by the length $a = BC$. We thus obtain, for all λ, a function $\Gamma_\lambda : a \mapsto \gamma$. By studying this triangle, for example, in the Poincaré disc model for $\mathbb{H}^2$, and by placing C at the origin, we find that Γ_λ is strictly decreasing, $\Gamma_\lambda(0) = \pi/2$ and $\Gamma_\lambda(t)$ tends to zero as t tends to infinity.

Let $K \geqslant k > 0$ be positive real numbers and let M be a Hadamard manifold of sectional curvature pinched between $-K$ and $-k$. Let ABC be a triangle in M such that the angle $\widehat{ABC}$ is equal to $\pi/2$ and such that A is a point in $\partial_\infty M$. We obtain the following result:

Lemma 3.10

Let a be the length of the geodesic arc segment BC. The angle $\widehat{BCA}$ is less than or equal to $\Gamma_k(a)$. Let D be a point in BA.

Proof: We construct the comparison triangle $\Delta D'B'C'$ in M_{-k} such that $D'B' = DB$, $B'C' = BC = a$ and that $\widehat{D'B'C'} = \widehat{DBC} = \pi/2$. Let b and b' be the lengths of CD and $C'D'$ respectively (see figure 3.3). By the comparison principal, we know that $b' \leqslant b$. Let D'' be such that $D''B' = D'B'$ and $D''C' = b$.

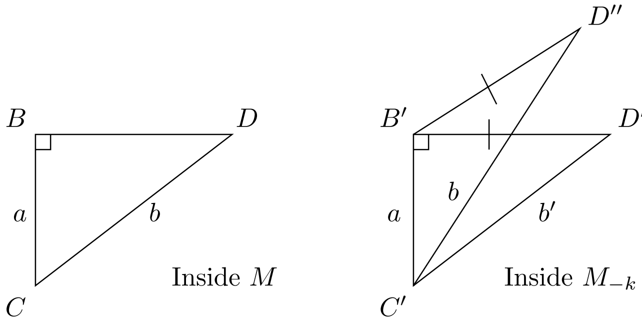

Figure 3.3

By lemma 3.5, the angle $\widehat{D''B'C'}$ is greater than or equal to $\widehat{D'B'C'}$, and thus, by lemma 3.6, the angle $\widehat{B'C'D''}$ is less than or equal to $\widehat{B'C'D'}$. Finally, by the comparison principal, we obtain:

$$\widehat{BCD} \leqslant \widehat{B'C'D''} \leqslant \widehat{B'C'D'} \leqslant \Gamma_k(a).$$

The result now follows by letting D tend towards A. $\square$

We now bound the angle $\widehat{BCA}$ from below:

Lemma 3.11

Let a be the length of the geodesic arc segment BC. The angle $\widehat{BCA}$ is greater than or equal to $\Gamma_K(a)$. Let $D \in BA$ be such that $BD \geqslant BC$.

Proof: We construct the comparison triangle $\Delta D'B'C'$ in M_K such that $D'B' = DB$, $B'C' = BC = a$ and $\widehat{D'B'C'} = \widehat{DBC} = \pi/2$. Let b and b' be the lengths of CD and $C'D'$ respectively (see figure 3.4). By the comparison principal, $b' \geqslant b$. Let D'' be such that $D''B' = D'B'$ and $D''C' = b$.

By lemma 3.5, the angle $\widehat{D''B'C'}$ is less than or equal to $\widehat{D'B'C'}$. Consequently, by lemma 3.6, the angle $\widehat{B'C'D''}$ is greater than or equal to $\widehat{B'C'D'}$. However, by the comparison principal:

$$\widehat{BCD} \geqslant \widehat{B'C'D''} \geqslant \widehat{B'C'D'}.$$

The result now follows by letting D tend towards A. $\square$

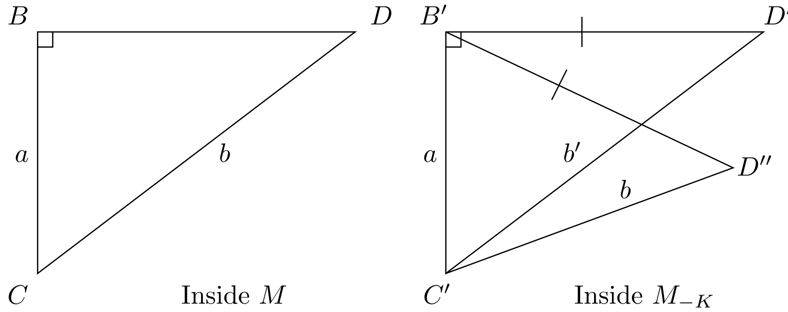

Figure 3.4

3.5 Degenerate Triangles Having Two Points at Infinity.

We now study degenerate triangles having two points at infinity. Let $\lambda > 0$ be a positive real number and let ΔABC be a triangle in M_λ such that the points B and C are in $\partial_\infty M_\lambda$. Let BC be the geodesic joining B to C. The distance $d(A, BC)$ from A to BC is uniquely determined by the angle $\alpha = \widehat{BAC}$ (see figure 3.5). We thus obtain a function $D_\lambda : \alpha \mapsto d(A, BC)$.

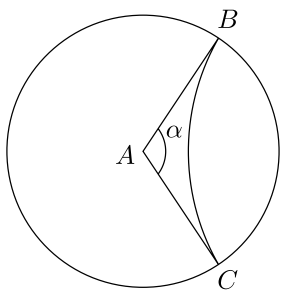

Figure 3.5

By dilating the Poincare disk model of $\mathbb{H}^2$ according to our needs, we see that D_λ is continuous and strictly decreasing. Moreover, we find that $D_\lambda(\alpha)$ tends towards infinity and zero as α tends to zero and π respectively. In fact, we find that, for $t \geqslant 0$ and for all $\alpha \in [0, \pi]$:

$$\begin{aligned} D_\lambda(2\Gamma_\lambda(t)) &= t, \\ 2\Gamma_\lambda(D_\lambda(\alpha)) &= \alpha. \end{aligned}$$

Let M be a Hadamard manifold with sectional curvature pinched between $-K$ and $-k$ and let ΔABC be a triangle in M such that the points B and C lie in $\partial_\infty M$. We have the following result:

Lemma 3.12

Let α denote the angle $\widehat{BAC}$. The distance $d(A,BC)$ satisfies the following double inequality:

$$D_K(\alpha) \leqslant d(A,BC) \leqslant D_k(\alpha).$$

Proof: Let η be the unique geodesic joining B to C:

$$\eta(-\infty) = B, \qquad \eta(+\infty) = C.$$

Now let D be the point in η that minimises the distance from A. We may suppose that η is normalised such that $\eta(0) = D$. For all $|t| > AD$, let us define B_t and C_t by $B_t = \eta(-t)$ and $C_t = \eta(t)$ (see figure 3.6).

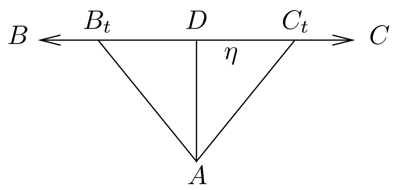

Figure 3.6

As in the proof of lemma 3.10, we have:

$$\begin{array}{lll} & \widehat{B_tAD}, \widehat{C_tAD} & \leqslant \Gamma_k(AD), \\ \Rightarrow & \widehat{B_tAC_t} & \leqslant 2\Gamma_k(AD). \end{array}$$

Since D_k is monotone decreasing, we obtain:

$$D_k(\widehat{B_tAC_t}) \geqslant D_k(2\Gamma_k(AD)) = AD.$$

The inequality result now follows. We cannot use the triangle inequality to bound AD from below, and we are thus obliged to reason differently. We know that $\widehat{B_tAC_t} \to \widehat{BAC}$ as t tends to infinity. Let $\Delta B_t'A'C_t'$ be the comparison triangle in M_{-K} such that $B_t'A' = B_tA$, $C_t'A' = C_tA$ and $\widehat{B_t'A'C_t'} = \widehat{B_tAC_t}$. By the comparison principal, we have:

$$B_t'C_t' \geqslant B_tC_t.$$

We thus define B_t'' such that $B_t''C_t' = B_tC_t$ and $B_t''A' = B_t'A' = B_tA$ (see figure 3.7):

By monotonicity (lemma 3.5), we have:

$$\widehat{B_t''A'C_t'} \leqslant \widehat{B_t'A'C_t'} = \widehat{B_tAC_t}.$$

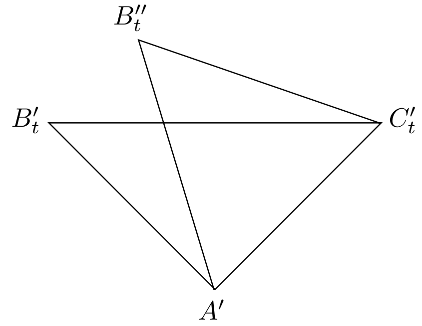

Figure 3.7

Let $(t_n)_{n\in\mathbb{N}}$ be a sequence which tends monotonically to infinity. Since $(A'B''_{t_n})_{n\in\mathbb{N}}$ and $(A'C'_{t_n})_{n\in\mathbb{N}}$ tend towards infinity, and since $M_{-K}\cup\partial_\infty M_{-K}$ is compact, we may suppose that there exists $B'', C' \in \partial_\infty M_{-K}$ such that $(B''_{t_n})_{n\in\mathbb{N}}$ and $(C'_{t_n})_{n\in\mathbb{N}}$ tend respectively towards B'' and C'. Now:

$$\begin{aligned}\widehat{B''AC'} &= \mathrm{Lim}_{n\to\infty}\,\widehat{B''_{t_n}A'C'_{t_n}}\\ &= \mathrm{LimSup}_{n\to\infty}\,\widehat{B''_{t_n}A'C'_{t_n}}\\ &\leqslant \mathrm{LimSup}_{n\to\infty}\,\widehat{B_{t_n}AC_{t_n}}\\ &= \widehat{BAC}.\end{aligned}$$

Since D_K is monotone decreasing, we now have:

$$d(A', B''C') = D_K(\widehat{B''A'C'}) \geqslant D_K(\widehat{BAC}).$$

Now, by theorem 3.9, the canonical homeomorphism between $\Delta A'B''_tC'_t$ and ΔAB_tC_t is dilating, and thus:

$$d(A, B_tC_t) \geqslant d(A', B''_tC'_t).$$

Consequently:

$$\begin{aligned}d(A', B''C') &\leqslant \mathrm{LimInf}_{n\to\infty}\, d(A', B''_{t_n}C'_{t_n})\\ &\leqslant \mathrm{LimInf}_{n\to\infty}\, d(A, B_{t_n}C_{t_n})\\ &= d(A, BC).\end{aligned}$$

The second result now follows. □

3.6 A Final Bound.

Now, let γ be a straight line (geodesic) in $M_0=\mathbb{R}^2$ and let us denote:

$$A^- = \gamma(-\infty), \qquad A^+ = \gamma(+\infty).$$

Let P be a point in γ. For $\rho > 0$ and $\varphi \in]0,\pi]$, let us choose the point R such that $d(R,P)=\rho$ and that the angle $\widehat{RPA^+}$ is equal to φ (see figure 3.8). We now define the function Δ_0 such that:

$$\Delta_0(\rho,\varphi) = d(R,\gamma).$$

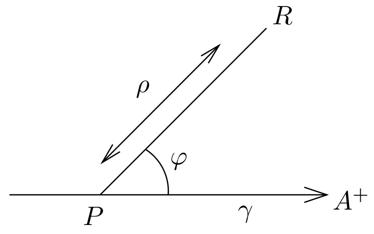

Figure 3.8

For a fixed value of φ, we have:

$$\begin{array}{lll} \Delta_0(\rho,\varphi) & \to 0 & \text{as } \rho \to 0, \\ \Delta_0(\rho,\varphi) & \to +\infty & \text{as } \rho \to +\infty. \end{array}$$

Moreover, Δ_0 is monotone decreasing in ρ. At the same time, for a fixed value of ρ, Δ_0 is monotone decreasing in φ. Finally:

$$\begin{array}{ll} \Delta_0(\rho,\pi/2) & = \rho, \\ \Delta_0(\rho,0) & = 0. \end{array}$$

For M a Hadamard manifold, we have the following result:

Lemma 3.13

Let γ be a geodesic in M and let P be a point in γ. For $\rho > 0$ and for $\varphi \in]0,\pi/2]$, let R be such that $d(P,R) = \rho$ and that the angle between PR and γ is equal to φ. We have:

$$d(R,\gamma) \geqslant \Delta_0(R,\varphi)$$

Proof: Let $Q \in \gamma$ be the point in γ minimising the distance from R. We have $\widehat{PQR} = \pi/2$. Let $\Delta P'Q'R'$ be the comparison triangle in $M_0 = \mathbb{R}^2$ such that $R'Q' = RQ$, $R'P' = RP$ and $P'Q' = PQ$ (see figure 3.9).

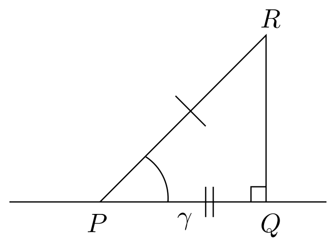

Figure 3.9

By Topogonov's second result (theorem 3.8), we have:

$$\widehat{R'P'Q'} \geqslant \widehat{RPQ}.$$

If we denote by γ' the straight line (geodesic) obtained by extending $P'Q'$ in both directions, we obtain:

$$RQ = R'Q' \geqslant d(R', \gamma') = \Delta_0(R'P', \widehat{R'P'Q'}).$$

Since Δ_0 is strictly increasing in φ, we have:

$$\Delta_0(R'P', \widehat{R'P'Q'}) \geqslant \Delta_0(R'P', \widehat{RPQ}) = \Delta_0(RP, \widehat{RPQ}).$$

The result now follows. □

3.7 Controlling Geodesics.

We have the following result:

Lemma 3.14

Let M be a Hadamard manifold with curvature pinched between $-K$ and $-k$. Let p be a point in the ideal boundary $\partial_\infty M$ of M. Let U be a neighbourhood of p in $M \cup \partial_\infty M$. There exists a neighbourhood Ω of p in $\partial_\infty M$ such that if γ is a geodesic in M such that:

$$\gamma(-\infty), \gamma(+\infty) \in \Omega,$$

then the whole of γ is contained in U.

Proof: Let q be a point in M. For $r \in M \cup \partial_\infty M$, we define $\widehat{pqr}$ to be the angle at q between the geodesic arcs pq and rq, and we define qr to be the distance from q to r. For $\theta \in]0, \pi]$ and for $\rho > 0$, we define the neighbourhood $\Omega_{\theta,\rho}$ of p by:

$$\Omega_{\theta,\rho} = \left\{ r \in M \cup \partial_\infty M | \widehat{pqr} < \theta, qr > \rho \right\}.$$

By definition of the topology of $M \cup \partial_\infty M$, for θ_0 sufficiently small, and for ρ_0 sufficiently large:

$$\Omega_{\theta_0,\rho_0} \subseteq U.$$

Since $\Gamma_k(t)$ tends towards zero as t tends to infinity, there exists $\rho_1 > \rho_0$ such that $\Gamma_k(\rho_1) < \theta_0/3$. Next, since $D_K(\theta)$ tends towards infinity as θ tends towards zero, there exists $\theta_1 < \theta_0/3$ such that $D_K(2\theta_1) > \rho_1$. Let r, r' be points in $\partial_\infty M$ such that $\widehat{pqr}, \widehat{pqr'} < \theta_1$ and let us denote the geodesic joining r to r' by rr'. By the triangle inequality, $\widehat{rqr'} < 2\theta_1$ and thus, by lemma 3.12, we have:

$$d(q, rr') \geqslant D_K(\widehat{rqr'}) \geqslant D_K(2\theta_1) > \rho_1 > \rho_0.$$

Let r_0 be the point in rr' that minimises the distance from q. The geodesic qr_0 makes a right angle with the geodesic rr' at r_0. For all $s \in M$, let us denote by $\widehat{sqr_0}$ the angle at

q between the geodesics sq and qr_0. As in the proof of lemma 3.10, for all $s \in rr_0$ and for all $s' \in r_0r'$:

$$\widehat{sqr_0}, \widehat{s'qr_0} \leqslant \Gamma_k(d(q, rr')) < \Gamma_k(\rho_1) < \theta_0/3.$$

In particular, by taking limits, we obtain:

$$\begin{array}{rll} & \widehat{rqr_0} & < \frac{\theta_0}{3} \\ \Rightarrow & \widehat{pqr_0} & < \frac{2\theta_0}{3}. \end{array}$$

where the second inequality follows from the definition of r and the triangle inequality. By applying the triangle inequality a second time, we obtain:

$$\widehat{sqp}, \widehat{s'qp} < \theta_0.$$

It thus follows that every point in the geodesic rr' is contained in Ω_{θ_0,ρ_0} and the result now follows. $\square$

Remark: We could equally well have used the stronger result [1] of Anderson which states that for all $p \in \partial_\infty M$ and for every neighbourhood U of p in $M \cup \partial_\infty M$, there exists a convex neighbourhood V of p in $M \cup \partial_\infty M$ which is contained in U.

3.8 Proofs of Main Results of This Section.

We may now prove lemma 3.2:

Lemma 3.2

Let $K \geqslant k > 0$ be real numbers and let M be a Hadamard manifold with curvature pinched between $-K$ and $-k$. Let α be a hyperbolic isometry of M having p_+ and p_- as attractive and repulsive fixed points respectively. Let p_0 be a point distinct from $p_\pm$. For every $B > 0$, there exists a neighbourhood Ω of p_0 in $\partial_\infty M$ such that if $q_\pm \in \Omega$ and if β is a hyperbolic isometry of M having $q_\pm$ as fixed points such that $\|\beta\| > B$, then the subgroup $\langle\alpha, \beta\rangle$ of $\mathrm{Isom}(M)$ generated by α and β is a Schottky group.

Proof: Let us also denote the geodesic joining p_- to p_+ by α:

$$\alpha(-\infty) = p_-, \qquad \alpha(+\infty) = p_+.$$

Let N_α be the normal vector bundle over α and let Σ_α be the normal unit sphere bundle over α. Let $\overrightarrow{n} : \Sigma_\alpha \to \partial_\infty M$ be the Gauss-Minkowski mapping. By lemma 2.5, there exists a unique vector $X \in \Sigma_\alpha$ such that $\overrightarrow{n}(X) = p_0$. We suppose that α is normalised such that X lies in the fibre over $\alpha(0)$. We define α_0 by $\alpha_0 = \alpha(0)$. The angles $\widehat{p_0\alpha_0p_-}$ and $\widehat{p_0\alpha_0p_+}$ are right angles. We define $N_0 = N_{\alpha(0)}\alpha$. Let $i : \mathbb{R} \times N_0 \to N_\alpha$ be the isometry of vector bundles generated by parallel transport. We define the isometry $\tilde{\alpha}$ of $\mathbb{R} \times N_0$ such that the following diagram commutes:

$$\begin{tikzcd}
\mathbb{R} \times N_0 \arrow[r, "i"] \arrow[d, "\tilde{\alpha}"] & N_\alpha \arrow[r, "\mathrm{Exp}"] \arrow[d, "T\alpha"] & M \arrow[d, "\alpha"] \\
\mathbb{R} \times N_0 \arrow[r, "i"] & N_\alpha \arrow[r, "\mathrm{Exp}"] & M
\end{tikzcd}$$

Since α commutes with parallel transport along the geodesic α, there exists $T_0 \in \mathbb{R}$ and a rotation R_0 of N_0 such that, for all $(t,v) \in \mathbb{R} \times N_0$:

$$\tilde{\alpha}(t,v) = (t + T_0, R_0 v).$$

Since α goes from p_- to p_+, $T_0 > 0$. We define $\tilde{U}_+, \tilde{U}_- \subseteq \mathbb{R} \times N_0$ by:

$$\begin{array}{ll} \tilde{U}_+ & = \left\{(t,v) | t > \frac{T_0}{2}\right\}, \\ \tilde{U}_- & = \left\{(t,v) | t < -\frac{T_0}{2}\right\}. \end{array}$$

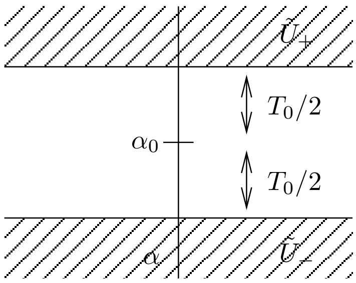

Figure 3.10

We define $\tilde{D}_+ = \partial \tilde{U}_+$ and $\tilde{D}_- = \partial \tilde{U}_-$ and we orient them such that:

$$\begin{array}{ll} \tilde{U}_+ & = \mathrm{Int}(\tilde{D}_+), \\ \tilde{U}_- & = \mathrm{Int}(\tilde{D}_-). \end{array}$$

This construction is illustrated in figure 3.10. We define $U_+ = (\mathrm{Exp} \circ i)(\tilde{U}_+)$ and $U_- = (\mathrm{Exp} \circ i)(\tilde{U}_-)$ and we define D_+ and D_- in a similar fashion. We now have:

$$\tilde{\alpha}(\mathrm{Ext}(\tilde{D}_-)) = \mathrm{Int}(\tilde{D}_+).$$

Consequently:

$$\alpha(\mathrm{Ext}(D_-)) = \mathrm{Int}(D_+).$$

For $r \in M \cup \partial_\infty M$, let $\widehat{r\alpha_0 p_0}$ be the angle at α_0 between the geodesics $r\alpha_0$ and $p_0\alpha_0$. For all $\theta \in]0,\pi]$ and for all $\rho > 0$, we define $\Omega_{\theta,\rho}$ by:

$$\Omega_{\theta,\rho} = \left\{r \in M \cup \partial_\infty M | \widehat{r\alpha_0 p_0} < \theta, d(r,\alpha_0) > \rho\right\}.$$

For all θ and for all ρ, $\Omega_{\theta,\rho}$ is a neighbourhood of p_0. Since p_0 lies in the exteriors of the closures of $\partial_\infty U_\pm$, there exists θ_1 and ρ such that:

$$\Omega_{\theta_1,\rho} \cap U_\pm = \emptyset.$$

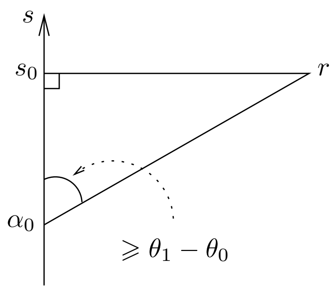

Figure 3.11

Next, let $\theta_0 < \theta_1$ be another angle. Let r be a point in $\Omega_{\theta_0,\rho}$. Let s be a point in $U_\pm$. If $\widehat{r\alpha_0 s}$ denotes the angle between the geodesic arcs $\alpha_0 r$ and $\alpha_0 s$ at α_0, then:

$$\widehat{r\alpha_0 s} \leqslant \theta_1 - \theta_0.$$

Let us note denote by $\alpha_0 s$ the complete geodesic which passes by α_0 and s (see figure 3.11)). By lemma 3.13:

$$d(r, \alpha_0 s) \geqslant \Delta_0(\rho, \theta_1 - \theta_0).$$

Let $s_0 \in \alpha_0 s$ be the point which minimises the distance from r. Let $\widehat{s' r s_0}$ be the angle at r_0 between the geodesics $s'r$ and $s_0 r$. As in the proof of lemma 3.10, for all $s' \in \alpha_0 s$, we have:

$$\widehat{s' r s_0} \leqslant \Gamma_k(d(r, \alpha_0 s)) \leqslant \Gamma_k(\Delta_0(\rho, \theta_1 - \theta_0)).$$

In particular, taking limits, and using the triangle inequality, we obtain:

$$\widehat{s r \alpha_0} \leqslant 2\Gamma_k(\Delta_0(\rho, \theta_1 - \theta_0)).$$

Since, for a given fixed φ_0, $\Delta_0(\rho, \varphi)$ tends to infinity as ρ tends to infinity, and since $\Gamma_k(t)$ tends to zero as t tends to infinity, we may choose ρ such that:

$$2\Gamma_k(\Delta_0(\rho, \theta_1 - \theta_0)) < \frac{\pi}{2} - \Gamma_K(B/2).$$

By lemma 3.14 there exists a neighbourhood Ω of p_0 such that every geodesic γ whose extremities lie in Ω is contained entirely in $\Omega_{\theta_0,\rho}$. We suppose that γ is normalised such that 0 minimises the distance from α_0. Let us define $M_0 = N_{\gamma(0)}\gamma$ and let $j : \mathbb{R} \times M_0 \to N_\gamma$ be the isomorphism of vector bundles generated by parallel transport. As before, we define $\tilde{V}_\pm \in \mathbb{R} \times M_0$ by:

$$\begin{aligned} \tilde{V}_+ &= \left\{(t, v) | t > \tfrac{B}{2}\right\}, \\ \tilde{V}_- &= \left\{(t, v) | t < -\tfrac{B}{2}\right\}. \end{aligned}$$

We define $E_\pm \in \mathbb{R} \times M_0$ by $E_\pm = \partial V_\pm$, and we orient them such that:

$$V_\pm = \mathrm{Int}(E_\pm).$$

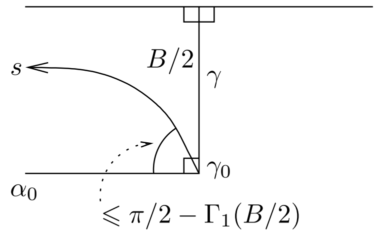

Figure 3.12

If $\tilde{\gamma}$ is a hyperbolic isometry which preserves the geodesic γ such that $\|\tilde{\gamma}\| > K$, then:

$$\tilde{\gamma}(\mathrm{Ext}(E_-)) \subseteq \mathrm{Int}(E_+).$$

However, for all $s \in U_\pm$:

$$\widehat{s\gamma(0)\alpha_0} \leqslant 2\Gamma_k(\Delta_0(\rho, \theta_1 - \theta_0)) < \pi/2 - \Gamma_K(B/2).$$

Consequently (see figure 3.12):

$$D_\pm \subseteq \mathrm{Ext}(E_+) \cap \mathrm{Ext}(E_-).$$

It thus follows that the subgroup of $\mathrm{Isom}(M)$ generated by α and by $\tilde{\gamma}$ is a Schottky group, and the result now follows. $\square$

The proof of lemma 3.4 is much simpler:

Lemma 3.4

Let $k > 0$ be a real number. Let M be a Hadamard manifold of sectional curvature bounded above by $-k$. Let $p_{1,\pm}, p_{2,\pm}$ be distinct points in $\partial_\infty M$. There exists $B > 0$ such that, if α_1 and α_2 are hyperbolic isometries of M such that $p_{k,\pm}$ are fixed points of α_k for each k and if $\|\alpha_1\|, \|\alpha_2\| > B$, then the subgroup of $\mathrm{Isom}(M)$ generated by α_1 and α_2 is a Schottky group.

Proof: For each k, let $U_{k,\pm}$ be a neighbourhood in $M \cup \partial_\infty M$ of $p_{k,\pm}$. We may assume that $U_{1,\pm}$ and $U_{2,\pm}$ are all disjoint. Next, let α_k be the geodesic joining $p_{k,-}$ to $p_{k,+}$. For each k:

$$\alpha_k(-\infty) = p_{k,-}, \qquad \alpha_k(+\infty) = p_{k,+}.$$

We define $N_k = N_{\alpha_k(0)}\alpha_k$. Let $i_k : \mathbb{R} \times N_0 \to N_\alpha$ be the isometry of vector bundles generated by parallel transport. Let $\pi_k : \mathbb{R} \times N_0 \to \mathbb{R}$ be the projection onto the first component. For $T > 0$, we define $\tilde{\Omega}_{k,\pm}(T) \subseteq \mathbb{R} \times N_k$ by:

$$\begin{aligned} \tilde{\Omega}_{k,+}(T) &= \{(t,v) | t > T/2\}, \\ \tilde{\Omega}_{k,-}(T) &= \{(t,v) | t < -T/2\}. \end{aligned}$$

We now define $\Omega_{k,\pm}(T) = (\mathrm{Exp} \circ i_k)(\tilde{\Omega}_k)$. We find that for sufficiently large values of T:

$$\Omega_{k,+}(T) \subseteq U_{k,+}, \Omega_{k,-}(T) \subseteq U_{k,*}.$$

We now see that if $\tilde{\alpha}_1$ and $\tilde{\alpha}_2$ are hyperbolic applications which preserve α_1 and α_2 respectively and if:

$$\|\alpha_1\|, \|\alpha_2\| > T,$$

then the subgroup of $\mathrm{Isom}(M)$ generated by α_1 and α_2 is a Schottky group, and the result now follows. $\square$

◇

4 - The Trouser Decomposition.

4.1 Introduction.

Throughout this section (M, Q) denotes a pointed, compact, Riemannian manifold of strictly negative sectional curvature, $(\tilde{M}, \tilde{Q})$ denotes its universal cover, (Σ, P) denotes a pointed, compact surface of genus at least 2 and $\theta : \pi_1(\Sigma, P) \to \pi_1(M, Q)$ denotes a representation of $\pi_1(\Sigma, P)$ in $\pi_1(M, Q)$.

In [4], Gallo, Kapovich and Marden obtain a trouser decomposition of Σ such that the θ-image of the fundamental group of each trouser is a Schottky group. In this section, we will prove the following lemma, which is the generalisation to our framework of this part of Gallo, Kapovich and Marden's result:

Lemma 4.1

Let (M, Q) be a pointed, compact, three dimensional Riemannian manifold of strictly negative sectional curvature. Let (Σ, θ) be a pointed, orientable, compact surface of hyperbolic type (i.e. of genus greater than or equal to 2) with holonomy in $\pi_1(M, Q)$. Then there exist bound, marked trousers $(T_i, \theta_i, \beta_i)_{1 \leqslant i \leqslant 2g-2}$ with holonomy in $\pi_1(M, Q)$ such that:

(i) for all i, the image of θ_i is a Schottky group, and

(ii)

$$(\Sigma, \theta) \cong \bigcup_{i=1}^{g} (\bigcirc(T_i, \theta_i, \beta_i)) \cup (\bigcup_{i=g+1}^{2g-2} (T_i, \theta_i, \beta_i)).$$

The terms used in the statement of this lemma will be explained in the third and fourth parts of this section. The proof of this lemma is an immediate consequence of propositions 4.9, 4.13 and 4.16.

In the second part of this section, we review certain algebraic properties of elements of $\pi_1(M, Q)$, showing, in particular, that, if such an element is different to the identity, then it is hyperbolic. In the third part, we define the notion of a marked surface with holonomy in a group G, and in the fourth part, we define the notion of a binding, which is required as a bookkeeping measure in order to recover correctly the fundamental group after the surface has been cut up into trousers. In the fifth part, we prove various technical algebraic results which are necessary for our construction. In the remaining sections, we successively decompose the surface Σ, proving propositions 4.9, 4.13 and 4.16.

4.2 Hyperbolicity of Elements of $\pi_1(M)$.

We review the properties of the fundamental group of a compact pointed manifold of strictly negative sectional curvature. We recall the following result:

Lemma 4.2

Let $\gamma \in \pi_1(M,Q)$ be different from the identity. Then there exists a unique closed geodesic $\hat{\gamma}$ in M which is freely homotopic to γ.

Remark: We see that two curves $\gamma, \gamma' \in \pi_1(M,Q)$ are freely homotopic if and only if they are conjugate in $\pi_1(M,Q)$. We thus obtain a bijection $[\gamma] \mapsto \hat{\gamma}$ between the conjugacy classes of $\pi_1(M,Q)$ and the closed geodesics in M.

We now obtain information concerning the action of elements of $\pi_1(M,Q)$ on $(\tilde{M},\tilde{Q})$:

Lemma 4.3

Let $\gamma \in \pi_1(M,Q)$ be different from the identity. Let us denote equally by γ the action of γ on $(\tilde{M},\tilde{Q})$. Then, γ is hyperbolic, and:

$$\|\gamma\| = \mathrm{Long}(\gamma).$$

where $\mathrm{Long}(\gamma)$ is the length of the unique closed geodesic in M which is freely homotopic to γ.

Consequently, every element of $\pi_1(M,Q)\setminus\{\mathrm{Id}\}$ has a hyperbolic action over $(\tilde{M},\tilde{Q})$. Thus, in the sequel, when we wish to show that an element of Γ is hyperbolic, it will be sufficient for us to show that it is different to the identity. Moreover, since M is compact, its injectivity radius is bounded below by ρ, say, and so the minimal displacement of every hyperbolic element of $\pi_1(M,Q)$ is also bounded below by ρ. Next, if a and b are two points in $\partial_\infty\tilde{M}$, then γ cannot send a to b and b to a. Indeed, let us denote by η the geodesic in $\tilde{M}$ going from a to b. If γ exchanges a and b, then it preserves η whilst reversing its orientation. It follows that γ has a fixed point in η and thus in the interior of $\tilde{M}$, which is impossible for hyperbolic elements. Finally, we recall the following result concerning the fixed points of the actions of elements of $\pi_1(M,Q)$ over $(\tilde{M},\tilde{Q})$:

Lemma 4.4

Let γ and γ' be elements of $\pi_1(M,Q)\setminus\{\mathrm{Id}\}$ and let us denote also by γ and γ' the actions of these elements over $(\tilde{M},\tilde{Q})$. If they have a fixed point in common, then there exists $m,n \in \mathbb{Z}$ such that:

$$\gamma^m = \gamma'^n.$$

In particular, γ and γ' have the same fixed points.

4.3 Marked Surfaces and Handles.

Let G be a group. We define a **marked surface** with **holonomy** in G to be a triplet of:

(i) a pointed compact surface with boundary $(\Sigma, \partial\Sigma, P_0)$, with $\partial\Sigma$ oriented such that the interior of Σ lies to its left,

(ii) for every connected component C_α of $\partial\Sigma$, a point $Q_\alpha \in C_\alpha$, which is the base point of C_α, and

(iii) a homomorphism $\theta : \pi_1(\Sigma, P_0) \to G$.

By slight abuse of notation, we will refer to the marked surface with holonomy simply by (Σ, θ). Every closed curve in Σ defines a conjugacy class in $\pi_1(\Sigma, P_0)$. Indeed, let a be a closed curve in Σ and let Q be the base point of a. Let x be a curve joining P_0 to Q, and let us define a_x by:

$$a_x = x^{-1}ax.$$

Now let y be another curve joining P_0 to Q. Then $(y^{-1}x) \in \pi_1(\Sigma, P_0)$ and:

$$a_y = (y^{-1}x)a_x(y^{-1}x)^{-1}.$$

Moreover, every element in the conjugacy class of a_x may be constructed in this manner. We will denote the conjugacy class defined by a by $[a]$.

The θ-image of $[a]$ defines a conjugacy class in G which we denote by $\theta[a]$. If G is a subgroup of the group of isometries of a three dimensional Hadamard manifold, and if one element in a given conjugacy class of G is hyperbolic, then all elements in that conjugacy class are hyperbolic. We refer to a conjugacy class in G containing only hyperbolic elements as a **hyperbolic class**. A closed curve a in (Σ, θ) is said to be **hyperbolic** if and only if $\theta[a]$ is a hyperbolic class.

Similarly, every pair (a, b) of simple closed curves in Σ with the same base point generates a conjugacy class of pairs in $\pi_1(\Sigma, P_0)$. Indeed, with x defined as before, we define:

$$(a, b)_x = (a_x, b_x).$$

As before, every element of the conjugacy class of $(a, b)_x$ may be constructed in this manner. We denote the conjugacy class defined by (a, b) by $[a, b]$.

The θ-image of the class $[a, b]$ defines a conjugacy class of pairs in G which we denote by $\theta[a, b]$. Every element in the conjugacy class $\theta[a, b]$ generates a subgroup of G. Since all these subgroups are in the same conjugacy class, it follows that $\theta[a, b]$ defines a conjugacy class of subgroups of G which we denote by $\langle\theta[a, b]\rangle$. If G is a subgroup of the group of isometries of a three dimensional Hadamard manifold, and if a subgroup in a given conjugacy class in G is a Schottky group, then so is every group in that class. We refer to a conjugacy class of subgroups of G consisting only of Schottky groups as a **Schottky class**. We say that the pair (a, b) in (Σ, θ) is **Schottky** if and only if $\langle\theta[a, b]\rangle$ is a Schottky class. Similarly, if a subgroup in a given conjugacy class in G is non-elementary, then so is every subgroup in that class. We refer to a conjugacy class of subgroups of G consisting only of non-elementary groups as a **non-elementary class**. We say that a pair (a, b) in (Σ, θ) is **non-elementary** if and only if $\langle\theta[a, b]\rangle$ is a non-elementary class.

Let (Σ, θ) be a marked surface with holonomy in G. A **marked handle** in (Σ, θ) is a pair (a, b) of simple closed curves in (Σ, θ) corresponding to a topological handle of θ such that:

(i) a intersects b at only one point, which we take to be the common base point of these two curves, and

(ii) the pair (a, b) is oriented positively. In otherwords, b crosses a from right to left.

When G is a subgroup of the group of isometries of a Hadamard manifold, we say that (a, b) is a **hyperbolic handle** if and only if both a and b are hyperbolic, (a, b) is a **non-elementary handle** if and only if (a, b) is non-elementary, and (a, b) is a **Schottky handle** if and only if the pair (a, b) is Schottky.

Finally, for (Σ, θ) a marked surface with boundary with holonomy in G, and for a given family, $(Q_i)_{i\in F}$ of points in (Σ, θ), we define a **binding sash** to be a set of simple curves $(\gamma_i)_{i\in F}$ in (Σ, θ) indexed by the same set as $(Q_i)_{i\in F}$ such that:

(i) for every i, γ_i goes from P_0 to Q_i,

(ii) for every $i \neq j$, γ_i only intersects γ_j at P_0,

(iii) for every i, γ_i can only intersect $\partial\Sigma$ at Q_i.

Binding sashes permit us to identify objects in Σ (curves, marked handles, etc.) explicitly with elements or pairs of elements of $\pi_1(\Sigma, P_0)$.

When we define many such geometric structures over (Σ, θ), we will assume them to be disjoint, except, possibly, at their end or base points.

4.4 Bindings.

Let G be a group. Let (Σ, θ) be a marked surface with holonomy in G. Suppose that Σ has n boundary components, and let $(C_i)_{1\leqslant i\leqslant n}$ be the oriented boundary components of Σ. For every i, let Q_i denote the base point of C_i, and let us denote the set of all homotopy classes of curves in Σ going from P_0 to Q_i by $\pi_1(\Sigma, P_0, Q_i)$. A **binding** of (Σ, θ) is a set of mappings $(\beta_i)_{1\leqslant i\leqslant n}$ such that:

(i) for every i, β_i maps $\pi_1(\Sigma, P_0, Q_i)$ into G, and

(ii) for every i, β_i is equivariant under θ with respect to the canonical right action of $\pi_1(\Sigma, P_0)$ on $\pi_1(\Sigma, P_0, Q_i)$. In otherwords, for all $\xi \in \pi_1(\Sigma, P_0, Q_i)$, and for all $\eta \in \pi_1(\Sigma, P_0)$:

$$\beta_i(\xi\eta) = \beta_i(\xi)\theta(\eta).$$

Bindings are indispensable for recovering the holonomy of a surface obtained by joining two surfaces together along the boundary components. They encode the algebraic information in the holonomy that is lost when we cut a surface open along a simple closed curve, and allow us to identify boundary components of Σ with specific elements of G. We encourage the reader not to lose to much time over this concept which is essentially only a conveniant bookkeeping measure. We will denote the binding by β, and, for every i, we will write β instead of β_i when there is no danger of ambiguity.

When a marked surface (Σ, θ) with holonomy in G carries a given binding, β, then we will say that the surface is **bound**. We will denote the bound marked surface with holonomy in G by (Σ, θ, β). For any oriented, pointed boundary component (C, Q) of Σ, we define the element $\beta(C) \in G$ such that, for any (and thus, equivalently, for all) $\xi \in \pi_1(\Sigma, P_0, Q)$:

$$\beta(C) = \beta(\xi)\theta(\xi^{-1}C\xi)\beta(\xi)^{-1}.$$

Let (Σ, θ, β) and $(\Sigma', \theta', \beta')$ be bound marked surfaces with holonomy in G. Let (C, Q) and (C', Q') be oriented, pointed boundary components of Σ and Σ' respectively. Let $\varphi : (C, Q) \to (C'^{-1}, Q')$ be an orientation reversing homeomorphism. We define $\Sigma \cup_\varphi \Sigma'$, the **joined union** of Σ and Σ' **along** φ by:

$$\Sigma \cup_\varphi \Sigma' = \Sigma \sqcup \Sigma' / \varphi.$$

Let $\pi : \Sigma \sqcup \Sigma' \to \Sigma \cup_\varphi \Sigma'$ be the canonical projection. We identify all objects (points, curves, etc.) in Σ and Σ' with their images under π and vice versa. Let P_0 be the base point of Σ. We take P_0 to be the base point of $\Sigma \cup_\varphi \Sigma'$.

We say that (θ, β) and (θ', β') may be **joined** along φ if and only if:

$$\beta(C)^{-1} = \beta'(C').$$

If (θ, β) and (θ', β') may be joined along φ, we define $\theta \cup_\varphi \theta'$, the **joined union** of θ and θ' **along** φ, such that:

(i) for all $x \in \pi_1(\Sigma, P_0)$:

$$(\theta \cup_\varphi \theta')(x) = \theta(x),$$

and

(ii) for all $x \in \pi_1(\Sigma', P'_0)$, $\xi \in \pi_1(\Sigma, P_0, Q)$ and $\eta \in \pi_1(\Sigma', P'_0, Q')$:

$$(\theta \cup_\varphi \theta')(\xi^{-1}\eta x \eta^{-1}\xi) = \beta(\xi)^{-1}\beta'(\eta)\theta'(x)\beta'(\eta)^{-1}\beta(\xi).$$

The join condition on (θ, β) and (θ', β') ensures that $\theta \cup_\varphi \theta'$ is a well defined homomorphism from $\pi_1(\Sigma \cup_\varphi \Sigma', P_0)$ into G. Similarly, if (θ, β) and (θ', β') may be joined along φ, we define $\beta \cup_\varphi \beta'$, the **joined union** of β and β' **along** φ, such that:

(i) for every pointed boundary component (C_1, Q_1) of Σ different to (C, Q), and for every $\gamma \in \pi_1(\Sigma, P_0, Q_1)$:

$$(\beta \cup_\varphi \beta')(\gamma) = \beta(\gamma),$$

and,

(ii) for every pointed boundary component (C_1', Q_1') of Σ' different to (C', Q'), for every $\gamma \in \pi_1(\Sigma', P_0', Q_1')$, for every $\xi \in \pi_1(\Sigma, P_0, Q)$ and for every $\eta \in \pi_1(\Sigma', P_0', Q')$:

$$(\beta \cup_\varphi \beta')(\gamma\eta^{-1}\xi) = \beta'(\gamma)\beta'(\eta)^{-1}\beta(\xi).$$

As before, the join condition on (θ, β) and (θ', β') ensures that $\beta \cup_\varphi \beta'$ is a well defined binding over $\Sigma \cup_\varphi \Sigma'$. We now define $(\Sigma, \theta, \beta) \cup_\varphi (\Sigma', \theta', \beta')$, the **joined union** of (Σ, θ, β) and $(\Sigma', \theta', \beta')$ along φ, by:

$$(\Sigma, \theta, \beta) \cup_\varphi (\Sigma', \theta', \beta') = (\Sigma \cup_\varphi \Sigma', \theta \cup_\varphi \theta', \beta \cup_\varphi \beta').$$

When we discuss the join of two marked surfaces with holonomy, the actual homeomorphism that we use is of little importance. In the sequel, we will thus omit the subscript φ when discussing joined unions.

In a similar manner, if (C, Q) and (C', Q') are distinct boundary components of the same bound marked surface with holonomy in G, (Σ, θ, β), then we can consider joining (Σ, θ, β) to itself along pointed homeomorphisms from (C, Q) to $({C'}^{-1}, Q')$. Let $\varphi : (C, Q) \to ({C'}^{-1}, q')$ be an orientation reversing homeomorphism. We say that (Σ, θ, β) may be **joined to itself** along φ if and only if:

$$\beta(C)^{-1} = \beta(C').$$

In this case, we define $\bigcirc_\varphi \Sigma$, the **join** of Σ **to itself** along φ by:

$$\bigcirc_\varphi \Sigma = \Sigma/\varphi.$$

As before, we will often identify all objects (points, curves, etc.) in Σ with their images under the canonical projection from Σ to $\bigcirc_\varphi \Sigma$, and vice versa. We define $\bigcirc_\varphi \theta$ and $\bigcirc_\varphi \beta$ in a similar manner as before, and we define $\bigcirc_\varphi(\Sigma, \theta, \beta)$ by:

$$\bigcirc_\varphi(\Sigma, \theta, \beta) = (\bigcirc_\varphi \Sigma, \bigcirc_\varphi \theta, \bigcirc_\varphi \beta).$$

As for joined unions, the pointed homeomorphism φ that we use is of little importance. In the sequel, we will thus refer to the join of (Σ, θ, β) by:

$$\bigcirc(\Sigma, \theta, \beta).$$

For surfaces with many boundary components, this process may be iterated, and we will refer to the n-fold join of (Σ, θ, β) by:

$$\bigcirc^n(\Sigma, \theta, \beta).$$

4.5 Technical Algebraic Results.

Various algebraic properties of elements of $\pi_1(M, Q)$ may be established by carefully following the sets of fixed points of hyperbolic elements. The following technical propositions correspond to the Dehn twists that will be carried out in the sequel:

Proposition 4.5

Let M be a Hadamard manifold. Let Γ be a subgroup of the group of isometries of M consisting only of hyperbolic elements (and the identity). Let α,β,ξ and η be elements of Γ such that the subgroup $\langle\alpha, \beta\rangle$ of Γ is non-elementary. There exists $K \in \mathbb{N}$ and a function $N : \{|n| \geqslant K\} \to \mathbb{N}$ such that, for all $|k| \geqslant K$ and for all $|n| \geqslant N(k)$:

(1) $\beta\alpha^k$ is hyperbolic,

(2) $\delta_k = \eta\beta\alpha^k$ is hyperbolic,

(3) $\delta_k^n\alpha$ is hyperbolic and has no fixed point in common with $\beta\alpha^k$, and

(4) $\delta_k^n\xi$ is hyperbolic.

Proof:

(1) Since $\langle\alpha, \beta\rangle$ is non-elementary, for all k, the element $\beta\alpha^k$ is different to the identity. Indeed, otherwise, β and α would have the same fixed points, which is absurd. In particular, it follows that $\beta\alpha^k$ is hyperbolic for all k.

(2) Let us assume that there exists $k \neq k'$ such that δ_k and $\delta_{k'}$ are not hyperbolic. We have:

$$\alpha^{k-k'} = (\eta\beta\alpha^{k'})^{-1}(\eta\beta\alpha^k) = \delta_{k'}^{-1}\delta_k = \mathrm{Id}.$$

In particular, we obtain $\alpha = \mathrm{Id}$, which is absurd. It follows that there exists at most one value of k for which $\delta_k = \mathrm{Id}$, and consequently that there exists K such that, for $|k| \geqslant K$, the application δ_k is hyperbolic.

(3) We suppose that there exists $n \neq n'$ and a point P such that:

$$P \cdot (\delta_k^n\alpha) = P \cdot (\delta_k^{n'}\alpha) = P \cdot (\beta\alpha^k) = P.$$

Then, firstly:

$$\begin{array}{rlll} & P \cdot \delta_k^{n-n'} & = P \cdot (\delta_k^n\alpha) \cdot (\delta_k^{n'}\alpha) & = P \\ \Rightarrow & P \cdot \delta_k & = P. & \end{array}$$

Consequently:

$$P \cdot \alpha = P \cdot \delta_k^{-n} \cdot (\delta_k^n\alpha) = P.$$

Finally, we have:

$$P \cdot \beta = P \cdot (\beta\alpha^k) \cdot \alpha^{-k} = P.$$

It thus follows that α and β share P as a common fixed point. This is absurd, since $\langle\alpha, \beta\rangle$ is non-elementary. It thus follows that there exists at most one value of n for which $\delta_k^n\alpha$ has a fixed point in common with $\beta\alpha^k$. Consequently, there exists $N_1(k)$, which only depends on k, such that, for all $|n| \geqslant N_1(k)$, the application $\delta_k^n\alpha$ has no fixed point in common with $\beta\alpha^k$. In particular, for all such n, the mapping $\delta_k^n\alpha$ is not equal to the identity, and is thus hyperbolic.

(4) As in (2), we assume that there exists $n \neq n'$ such that:

$$\delta_k^n\xi = \delta_k^{n'}\xi = \mathrm{Id}.$$

We obtain:

$$\delta_k^{n-n'} = (\delta_k^n\xi)(\delta_k^{n'}\xi)^{-1} = \mathrm{Id}.$$

This is absurd. It thus follows that there exists at most one value of n for which $\delta_k^n\xi$ is not hyperbolic. Consequently, there exists $N_2(k) \geqslant N_1(k)$ such that, for $|n| \geqslant N_2(k)$, the application $\delta_k^n\xi$ is hyperbolic. $\square$

Next, we have:

Proposition 4.6

Let M be a Hadamard manifold. Let Γ be a subgroup of the group of isometries of M consisting only of hyperbolic elements (and the identity). Let α,β,γ and η be elements of Γ such that the subgroup $\langle\alpha, \beta\rangle$ generated by α and β is non-elementary. Suppose moreover that ξ and η are both hyperbolic.

For $k \in \mathbb{Z}$, we define $\delta_k = \eta\beta\alpha^k$. After exchanging ξ and η if necessary, there exists $K \in \mathbb{N}$ and a function $N : \{|n| \geqslant K\} \to \mathbb{N}$ such that, for all $|k| \geqslant K$ and for all $|n| \geqslant N(k)$, $\langle\delta_k^n\xi\delta_k^{-n}, \eta\rangle$ is a Schottky group.

Remark: The hypotheses of proposition 4.6 are stronger than those of propositon 4.5. Consequently, the conclusions of proposition 4.5 are also valid in this case.

Proof:

(1) We begin by showing that, after exchanging ξ and η if necessary, there exists $K \in \mathbb{N}$ such that for $|k| \geqslant K$, the mapping $\delta_k = \eta\beta\alpha^k$ has no fixed point in common with ξ. We suppose the contrary in order to obtain a contradiction. There thus exists $k \neq k'$ such that, for both $P \in \mathrm{Fix}(\xi)$:

$$P \cdot (\eta\beta\alpha^k) = P, \qquad P \cdot (\eta\beta\alpha^{k'}) = P.$$

Then:

$$\begin{array}{rlll} & P \cdot \alpha^{k-k'} & = P \cdot (\eta\beta\alpha^{k'})^{-1} \cdot (\eta\beta\alpha^k) & = P \\ \Rightarrow & P \cdot \alpha & = P. & \end{array}$$

Consequently:

$$\mathrm{Fix}(\alpha) = \mathrm{Fix}(\xi).$$

Next, we suppose that there exists $\hat{k} \neq \hat{k}'$ such that, for both $P \in \mathrm{Fix}(\eta)$:

$$P \cdot (\xi\beta\alpha^{\hat{k}}) = P, P \cdot (\xi\beta\alpha^{\hat{k}'}) = P.$$

Then, in a similar fashion, we obtain:

$$\mathrm{Fix}(\alpha) = \mathrm{Fix}(\eta).$$

Thus, in particular, $\mathrm{Fix}(\eta) = \mathrm{Fix}(\xi)$. Consequently, for both $P \in \mathrm{Fix}(\xi) = \mathrm{Fix}(\eta)$:

$$P \cdot \beta = P \cdot \eta^{-1} \cdot (\eta\beta\alpha^k) \cdot \alpha^{-k} = P.$$

Thus:

$$\mathrm{Fix}(\beta) = \mathrm{Fix}(\xi) = \mathrm{Fix}(\alpha).$$

This is absurd. Consequently, either there exists at most one k such that $\eta\beta\alpha^k$ has the same fixed points as ξ, or there exists at most one k such that $\xi\beta\alpha^k$ has the same fixed points as η. Thus, by exchanging ξ and η if necessary, we find that there exists K such that, for $|k| \geqslant K_1$, the application $\delta_k = \eta\beta\alpha^k$ does not have both fixed points in common with ξ. Since $\xi, \delta_k \in \Gamma$, the sets $\mathrm{Fix}(\delta_k)$ and $\mathrm{Fix}(\xi)$ are either disjoint or identical, and the result now follows.

(2) We suppose that there exists $k \neq k'$ and P such that:

$$P \cdot (\eta\beta\alpha^k) = P \cdot (\eta\beta\alpha^{k'}) = P \cdot \eta = P.$$

Then:

$$\begin{array}{rlll} & P \cdot \alpha^{k-k'} & = P \cdot (\eta\beta\alpha^{k'})^{-1} \cdot (\eta\beta\alpha^k) & = P \\ \Rightarrow & P \cdot \alpha & = P. & \end{array}$$

Consequently:

$$P \cdot \beta = P \cdot \eta^{-1} \cdot (\eta\beta\alpha^k) \cdot \alpha^{-k} = P.$$

It follows that P is a fixed point of α and β. Since $\langle \alpha, \beta \rangle$ is non-elementary, this is absurd. Consequently, there exists at most one value of k for which $\delta_k = \eta\beta\alpha^k$ has a fixed point in common with η. It thus follows that there exists K such that for $|k| \geqslant K$:

$$\mathrm{Fix}(\delta_k) \cap \mathrm{Fix}(\eta) = \emptyset$$

In particular, $\delta_k \neq \mathrm{Id}$, and is thus hyperbolic. Moreover, by (1), we may assume that, for all $|k| \geqslant K$:

$$\mathrm{Fix}(\delta_k) \cap \mathrm{Fix}(\xi) = \emptyset.$$

It follows by corollary 3.3 that there exists $N_1(k)$ such that, for $|n| \geqslant N_1(k)$ the group generated by $\langle \delta_k^n \xi \delta_k^{-n}, \eta \rangle$ is a Schottky group. The result now follows. $\square$

The following result will also be required:

Lemma 4.7

Let M be a Hadamard manifold. Let Γ be a subgroup of the group of isometries of M consisting only of hyperbolic elements (and the identity). Let α,β be elements of Γ such that the subgroup $\langle \alpha, \beta \rangle$ is non-elementary.

For all $\sigma \in \Gamma$, there exists $K \in \mathbb{N}$ such that, for all $|k| \geqslant K$, $J_k = \sigma^k \alpha \sigma^{-k} \beta$ is hyperbolic.

Proof: If $\sigma = \mathrm{Id}$, then $J_k = \alpha\beta$ for all k. This is necessarily hyperbolic, since $\langle\alpha,\beta\rangle$ is non-elementary, and the result immediately follows in this case. We now suppose that $\sigma \neq \mathrm{Id}$ and consequently that σ is hyperbolic. Suppose that there exists $k \neq k'$ such that $J_k = J_{k'} = \mathrm{Id}$. Then:

$$\begin{array}{rll} & \mathrm{Id} & = J_k J_{k'}^{-1} \\ & & = \sigma^k\alpha\sigma^{-k}\beta\beta^{-1}\sigma^{k'}\alpha^{-1}\sigma^{-k'} \\ & & = \sigma^k\alpha\sigma^{k'-k}\alpha^{-1}\sigma^{-k'} \\ \Rightarrow & \sigma^{k-k'}\alpha\sigma^{k'-k}\alpha^{-1} & = \mathrm{Id}. \end{array}$$

We denote $n = k - k'$, and we obtain:

$$\begin{array}{rll} & \sigma^n\alpha\sigma^{-n}\alpha^{-1} & = \mathrm{Id} \\ \Rightarrow & \sigma^n\alpha\sigma^{-n} & = \alpha \\ \Rightarrow & \mathrm{Fix}(\sigma^n\alpha\sigma^{-n}) & = \mathrm{Fix}(\alpha) \\ \Rightarrow & \mathrm{Fix}(\alpha)\cdot\sigma^{-n} & = \mathrm{Fix}(\alpha) \end{array}$$

Since $\sigma \in \Gamma$ is hyperbolic, it cannot exchange the two elements of $\mathrm{Fix}(\alpha)$. Consequently:

$$\mathrm{Fix}(\alpha) = \mathrm{Fix}(\sigma).$$

By a similar reasoning, we obtain:

$$\mathrm{Fix}(\beta) = \mathrm{Fix}(\sigma).$$

Consequently, α and β have the same fixed points, and this is absurd since $\langle\alpha,\beta\rangle$ is non-elementary. Consequently, there is at most one value of k for which $J_k = \mathrm{Id}$, and the result now follows. □

The final technical lemma will be used in order to cut a surface of genus one having two boundary components into two trousers such that the images of their fundamental groups are both Schottky groups:

Lemma 4.8

Let M be a Hadamard manifold. Let Γ be a subgroup of the group of isometries of M consisting only of hyperbolic elements (and the identity). Let α,β,ξ and η be elements of Γ such that:

(i) the subgroup $\langle\alpha,\beta\rangle$ is non-elementary,

(ii) $\alpha\beta\alpha^{-1}\beta^{-1} = \eta\xi$,

(iii) the subgroup $\langle\xi,\eta\rangle$ is a Schottky group,

(iv) $\xi\beta$ is not equal to the identity.

Then, there exists k and n such that, if we define γ_k and δ_k by:

$$\gamma_k = \beta\alpha^k, \qquad \delta_k = \xi\gamma_k,$$

then, $\langle\gamma_k, (\delta_k^n\alpha)^{-1}\gamma_k^{-1}(\delta_k^n\alpha)\rangle$ and $\langle\xi, \delta_k^{-n}\eta\delta_k^n\rangle$ are Schottky groups:

Proof:

(1) Since $\langle\xi,\eta\rangle$ is a Schottky group:

$$\begin{array}{rll} & \alpha\beta\alpha^{-1}\beta^{-1} & =\eta\xi \quad \neq\xi \\ \Rightarrow & \alpha\beta\alpha^{-1}\beta^{-1}\xi^{-1} & \neq \mathrm{Id}. \end{array}$$

It thus follows also by conjugation that:

$$\beta\alpha^{-1}\beta^{-1}\xi^{-1}\alpha\neq\mathrm{Id}.$$

(2) We suppose that there exists $k\neq k'$ and P such that:

$$P\cdot(\beta\alpha^k)=P\cdot(\beta\alpha^{k'})=P.$$

This yields:

$$P\cdot\alpha^{k-k'}=P\cdot(\beta\alpha^{k'})^{-1}\cdot(\beta\alpha^k)=P.$$

Thus:

$$P\cdot\alpha=P.$$

However:

$$P\cdot\beta=P\cdot(\beta\alpha^k)\cdot\alpha^{-k}=P.$$

Consequently:

$$P\in\mathrm{Fix}(\beta)\cap\mathrm{Fix}(\alpha)\neq\emptyset.$$

This is absurd, since $\langle\alpha,\beta\rangle$ is non-elementary. Thus, for any given point, P, there exists at most one value of k for which $\gamma_k=\beta\alpha^k$ fixes P. Consequently, there exists K_1 such that, for $|k|\geqslant K_1$, the application γ_k does not have any fixed point in common with ξ, η or $\beta\alpha^{-1}\beta^{-1}\xi^{-1}\alpha$. By conjugating the last mapping with α, we see that, for $|k|\geqslant K_1$, $\alpha\gamma_k\alpha^{-1}$ has no fixed point in common with $\alpha\beta\alpha^{-1}\beta^{-1}\xi^{-1}$. Moreover, for $|k|\geqslant K_1$, the application $\delta_k=\xi\gamma_k$ does not have any fixed points in common with ξ or γ_k.

(3) If $\mathrm{Fix}(\delta_k)=\mathrm{Fix}(\eta)$, then:

$$\begin{array}{rll} & [\delta_k\eta] & =\mathrm{Id} \\ \Rightarrow & \delta_k^{-n}\eta\delta_k^n & =\eta. \end{array}$$

Thus, in particular $\langle\xi,\delta_k^{-n}\eta\delta_k^n\rangle=\langle\xi,\eta\rangle$ is a Schottky group for all n. Otherwise, $\mathrm{Fix}(\delta_k)\cap\mathrm{Fix}(\eta)=\emptyset$. Let us denote the repulsive and attractive fixed points of δ_k by $\delta_{k,-}$ and $\delta_{k,+}$ respectively. We see that:

$$\begin{array}{ll} \mathrm{Fix}(\delta_k^{-n}\eta\delta_k^n) & \rightarrow\delta_{k,+}\text{ as }n\rightarrow+\infty, \\ \mathrm{Fix}(\delta_k^{-n}\eta\delta_k^n) & \rightarrow\delta_{k,-}\text{ as }n\rightarrow-\infty. \end{array}$$

Moreover, $\|\delta_k^{-n}\eta\delta_k^n\|=\|\eta\|>0$. Thus, since $\mathrm{Fix}(\delta_k)\cap\mathrm{Fix}(\xi)\neq\emptyset$, it follows by corollary 3.3 that there exists $N_1(k)$ such that, for $|n|\geqslant N_1(k)$, the group $\langle\xi,\delta_k^{-n}\eta\delta_k^n\rangle$ is a Schottky group.

(4) Since $\mathrm{Fix}(\gamma_k)\cap\mathrm{Fix}(\delta_k)=\emptyset$, we have:

$$\begin{aligned}\mathrm{Fix}((\delta_k^n\alpha)^{-1}\gamma_k(\delta_k^n\alpha)) &\rightarrow \delta_{k,+}\cdot\alpha \text{ as } n\rightarrow+\infty,\\ \mathrm{Fix}((\delta_k^n\alpha)^{-1}\gamma_k(\delta_k^n\alpha)) &\rightarrow \delta_{k,-}\cdot\alpha \text{ as } n\rightarrow-\infty.\end{aligned}$$

Suppose that there exists P such that:

$$P\cdot(\alpha\gamma_k)=P\cdot\alpha,\qquad P\cdot\delta_k=P\cdot(\xi\gamma_k)=P.$$

This yields:

$$P\cdot(\alpha\gamma_k\alpha^{-1})=P.$$

Consequently:

$$\begin{aligned}P &= P\cdot(\alpha\gamma_k\alpha^{-1})\cdot(\xi\gamma_k)^{-1}\\ &= P\cdot(\alpha\beta\alpha^{k-1}\alpha^{-k}\beta\xi^{-1})\\ &= P\cdot(\alpha\beta\alpha^{-1}\beta^{-1}\xi^{-1}).\end{aligned}$$

Thus $\alpha\gamma_k\alpha^{-1}$ has a fixed point in common with $\alpha\beta\alpha^{-1}\beta^{-1}\xi^{-1}$, which is absurd by (2). We thus see that:

$$\mathrm{Fix}(\delta_k)\cdot\alpha\cap\mathrm{Fix}(\gamma_k)=\emptyset.$$

Since $\|(\delta_k^n\alpha)^{-1}\gamma_k(\delta_k^n\alpha)\|=\|\gamma_k\|>0$ it follows by corollary 3.3 that there exists $N_2(k)\geqslant N_1(k)$ such that, for $|n|\geqslant N_2(k)$, the group $\langle\gamma_k,(\delta_k^n\alpha)^{-1}\gamma_k(\delta_k^n\alpha)\rangle$ is a Schottky group. $\square$

4.6 Slicing Open The Surface.

In this subsection, we will prove the following result:

Proposition 4.9

Let (M,Q_0) be a pointed, compact, three dimensional manifold of strictly negative sectional curvature. let (Σ,θ) be a compact surface without boundary of genus $g\geqslant 2$ with holonomy in $\pi_1(M,Q_0)$.

If the image of θ is non-elementary, then there exists $(\Sigma_{2g-2},\theta_{2g-2},\beta_{2g-2})$, a bound marked surface with holonomy in $\pi_1(M,Q_0)$ of genus 1 with a non-elementary marked handle (a,b) and having $2g-2$ boundary components, such that:

(i) every connected component of $\partial\Sigma_{2g-2}$ is hyperbolic, and

(ii) (Σ,θ) is homeomorphic to $\bigcirc^{g-1}(\Sigma_{2g-2},\theta_{2g-2},\beta_{2g-2})$.

If Σ is a closed surface of genus $g\geqslant 2$, then we define a **canonical basis** of Σ to be an n-tuple $[(a_1,b_1),...,(a_n,b_n)]$ of pairs of simple closed curves in Σ which correspond to handles in Σ (see figure 4.13), such that:

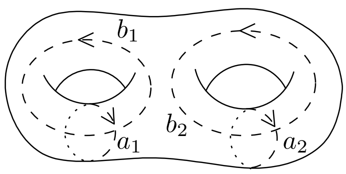

Figure 4.13

(i) for all $i \neq j$ the pairs (a_i, b_i) and (a_j, b_j) are disjoint,

(ii) for all i, the curve a_i intersects the curve b_i at only one point. We may suppose that the shared point of a_i and b_i is the base point of both these curves, and we will denote it by Q_i, and

(iii) for all i, the pair (a_i, b_i) is oriented positively. In otherwords b_i crosses a_i from right to left.

We have the following result:

Proposition 4.10

Let (M, Q_0) be a pointed compact three dimensional Hadamard manifold of strictly negative sectional curvature. Let (Σ, θ) be a compact surface without boundary of genus $g \geqslant 2$ with holonomy in $\pi_1(M, Q_0)$. Let $[(a_1, b_1), ..., (a_n, b_n)]$ be a canonical basis of Σ.

If θ is non-elementary, then there exists a homeomorphism Φ of Σ such that $\Phi_*(a_1, b_1)$ is non-elementary.

Proof: Since the image of θ is non-elementary, after composing with a homeomorphism of Σ which permutes the generators of $\pi_1(\Sigma)$, we may suppose that the class $\theta(a_1)$ is hyperbolic. For all i, let Q_i be the common base point of a_i and b_i. Let $(c_i)_{1 \leqslant i \leqslant g}$ be a binding sash of $(Q_i)_{1 \leqslant i \leqslant g}$. Let us define $a = c_1^{-1} a_1 c_1$ and $b = c_1^{-1} b_1 c_1$. Let us next define $\alpha = \theta(a)$ and $\beta = \theta(b)$. There are two cases to study:

(1) β does not have any fixed points in common with α. In this case, we take $\Phi = \mathrm{Id}$ and we obtain the desired result.

(2) β has two fixed points in common with α (in particular, β could be the identity). Since Γ is non-elementary, there exists $x \in \left\{ c_i^{-1} a_i c_i, c_i^{-1} b_i c_i | 2 \leqslant i \leqslant n \right\}$ such that $\xi = \theta(x)$ does not have both fixed points in common with α, and thus, by lemma 4.4, ξ and α have no fixed points in common.

We may orient x such that $C = ax$ is freely homotopic to a simple closed curve. Let T be the Dehn twist about C. In order to simplify the discussion, we identify Q_1 with P_0. Figure 4.14 shows the handle of Σ defined by (a, b) opened up by cutting along a and b.

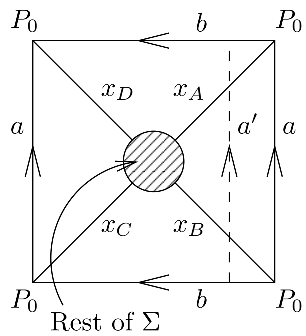

Figure 4.14

The curve x terminates at P_0 without crossing either a or b. This corresponds to one of the four topologically distinct configurations x_A,x_B,x_C or x_D shown in figure 4.14. By defining C using a slightly displaced copy of a, we may ensure that C never crosses a. If x is in one of the configurations A or B, this would correspond to taking a copy of a displaced to its left, and if x is in one of the configurations C or D, then this would correspond to taking a copy of a displaced to its right, as shown in figure 4.14. If we choose x in configuration A, then the Dehn twist, T, about C satisfies:

$$T_*a = a, \qquad T_*b = bax.$$

Now:

$$\mathrm{Fix}(\beta\alpha\xi) = \mathrm{Fix}(\alpha) \Leftrightarrow \mathrm{Fix}(\xi) = \mathrm{Fix}(\alpha).$$

It thus follows that $\mathrm{Fix}(\beta\alpha\xi) \cap \mathrm{Fix}(\alpha) = \emptyset$. Defining $\Psi = T$, we obtain the desired homeomorphism. $\square$

We next have:

Proposition 4.11

Let (M, Q_0) be a pointed compact three dimensional Hadamard manifold of strictly negative sectional curvature. Let (Σ, θ) be a compact surface without boundary of genus $g \geqslant 2$ with holonomy in $\pi_1(M, Q_0)$. Let $[(a_1, b_1), ..., (a_g, b_g)]$ be a canonical basis of Σ.

If (a_1, b_1) is non-elementary, then, for all i different to 1, there exists a homeomorphism Ψ of Σ such that:

(1) $\Psi_*(a_1, b_1)$ is non-elementary,

(2) Ψ_*a_i and Ψ_*b_i are hyperbolic, and

(3) for all j different from 1 and i, the application Ψ_* acts on a_j and b_j by conjugation.

Using induction, we thus obtain the following result:

Corollary 4.12

With the same hypotheses as in proposition 4.11, there exists a homeomorphism Ψ such that:

(1) $\Psi_*(a_1, b_1)\rangle$ is non-elementary, and

(2) for all i different from 1, Ψ_*a_i and Ψ_*b_i are hyperbolic.

Proof of proposition 4.11: For all i, let Q_i be the common base point of a_i and b_i. Let $(c_i)_{1\leqslant i\leqslant g}$ be a binding sash of $(Q_i)_{1\leqslant i\leqslant g}$. In order to simplify our reasoning, we identify Q_1 with P_0. As before, we denote $a = c_1^{-1}a_1c_1$ and $b = c_1^{-1}b_1c_1$. We now define $\alpha = \theta(a)$, and $\beta = \theta(b)$. For a given i different to 1, we denote $x = c_i^{-1}a_ic_i$, and $y = c_i^{-1}b_ic_i$, and we define $\xi = \theta(x)$ and $\eta = \theta(y)$.

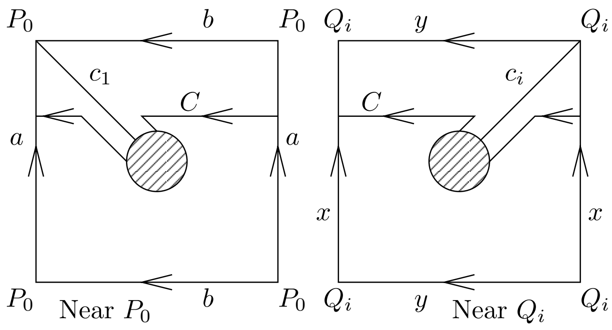

Figure 4.15

We orient y such that $C = yb$ is freely homotopic to a simple closed curve. We denote $d_0 = yb$. Let T_0 be a Dehn twist about C. If we choose the configurations of c_1 and c_i as in figure 4.15, then we obtain:

$$(T_0)_*a = d_0a, \quad (T_0)_*b = b,$$
$$(T_0)_*x = d_0x, \quad (T_0)_*y = y.$$

Since $(T_0)_*$ is a homomorphism leaving d_0 invariant, we obtain, for all n:

$$(T_0^n)_*a = d_0^n a, \quad (T_0^n)_*b = b,$$
$$(T_0^n)_*x = d_0^n x, \quad (T_0^n)_*y = y.$$

Moreover, for all j different from 1 and i, since d_0 only intersects c_j near P_0, it follows that $(T_0)_*$ acts on (a_j, b_j) by conjugation. Let T_a be the Dehn twist about a such that, for all k:

$$(T_a^k)_*a = a, \quad (T_a^k)_*b = ba^k,$$
$$(T_a^k)_*x = x, \quad (T_a^k)_*y = y.$$

For all j different to 1, since a stays away from (a_j, b_j), $(T_a)_*$ has no effect on this pair. We define $\Psi_{k,n} : \Sigma \to \Sigma$ by:

$$\Psi_{k,n} = T_a^k T_0^n T_a^{-k}.$$

If, for all k, we define d_k by $d_k = yba^k$, then we obtain:

$$(\Psi_{k,n})_*a = d_k^n a, \quad (\Psi_{k,n})_*ba^k = ba^k,$$
$$(\Psi_{k,n})_*x = d_k^n x, \quad (\Psi_{k,n})_*y = y.$$

Where $d_k = yba^k$. Moreover, for all j different to 1 and i, the mapping $\Psi_{k,n}$ acts on (a_j, b_j) by conjugation. We now choose k and n as in lemma 4.5. If η is hyperbolic, then, taking $\Psi = \Psi_{k,n}$, we obtain the desired result. Otherwise, $\eta = \mathrm{Id}$, and we may take a Dehn twist T about x such that $Ty = xy$ and which leaves all the other curves invariant. We see that:

$$\theta(\Psi_{k,n})_*T_*y = \theta\Psi_{k,n*}xy = \theta\delta_k^n xy = \delta_k^n\xi,$$

and this is hyperbolic. We thus denote $\Psi = \Psi_{k,n} \circ T$ and we once again obtain the desired result. $\square$

The proof of proposition 4.9 is now elementary:

Proof of proposition 4.9: Let $[(a_1, b_1), ..., (a_n, b_n)]$ be a canonical basis of Σ. By proposition 4.10 and corollary 4.12, we may suppose that (a_1, b_1) is a non-elementary marked handle and that, for all $2 \leqslant i \leqslant n$, a_i and b_i are hyperbolic.

We obtain Σ_{2g-2} by cutting Σ along each a_i for $2 \leqslant i \leqslant n$. We identify objects (points, curves etc.) in Σ with the corresponding objects in Σ_{2g-2}, and vice-versa. For each i, we denote by $C_{i,-}$ the copy of a_i in Σ_{2g-2} that b_i leaves from and by $C_{i,+}$ the copy of a_i^{-1} in Σ_{2g-2} that b_i arrives at (bearing in mind that b_i crosses a_i from right to left, these orientations ensure that the interior of Σ_{2g-2} lies to the left of $C_{i,-}$ and $C_{i,+}$). For all $2 \leqslant i \leqslant n$, we define $Q_{i,\pm}$ to be the copy of Q_i lying in $C_{i,\pm}$. For $2 \leqslant i \leqslant n$, let γ_i be any curve in Σ_{2g-2} running from P_0 to $Q_{i,-}$. We define the the binding β_{2g-2} over Σ_{2g-2} such that, for all i:

$$\begin{aligned} \beta_{2g-2}(\gamma_i) &= \mathrm{Id}, \\ \beta_{2g-2}(b_i\gamma_i) &= \theta(\gamma_i^{-1}b_i\gamma_i). \end{aligned}$$

We may verify that this binding satisfies the appropriate join conditions and that:

$$(\Sigma, \theta) \cong \bigcirc^{g-1}(\Sigma_{2g-2}, \theta_{2g-2}, \beta_{2g-2}).$$

Finally, we define the marked handle (a, b) in Σ_{2g-2} by:

$$(a, b) = (a_1, b_1).$$

Since (a_1, b_1) is hyperbolic, so is (a, b), and the result now follows. $\square$

4.7 Pruning.

Let G be a group. Let (Σ, θ, β) be a bound marked surface with holonomy in G. Let $\partial\Sigma/\sim$ denote the set of connected components of $\partial\Sigma$. If $\partial\Sigma$ has an even number, $2n$, of connected components, then we define a **pairing** on $\partial\Sigma$ to be a bijective map ι from $\partial\Sigma/\sim$ to $\{1, ..., n\} \times \{\pm\}$. For all i, we define $C_{i,\pm}$ to be the oriented connected component of $\partial\Sigma$ such that $\iota(C_{i,\pm}) = (i, \pm)$. We say that the pairing ι is **compatible** with the binding β if and only if, for all $1 \leqslant i \leqslant n$:

$$\beta(C_{i,-})^{-1} = \beta(C_{i,+}).$$

In proposition 4.9, we constructed a bound marked surface $(\Sigma_{2g-2}, \theta_{2g-2}, \beta_{2g-2})$ with holonomy if $\pi_1(M, Q_0)$ and a pairing on $\partial\Sigma_{2g-2}$ compatible with β_{2g-2}. In this subsection, we aim to prove the following result:

Proposition 4.13

Let (M, Q) be a pointed, compact, three dimensional manifold of strictly negative sectional curvature. Let $(\Sigma_{2g-2}, \theta_{2g-2}, \beta_{2g-2})$ be a bound marked surface with holonomy in $\pi_1(M, Q)$ of genus 1 having $2g-2$ boundary components, all of which are hyperbolic, and a non-elementary marked handle (a, b). Let ι be a pairing on $\partial\Sigma$ compatible with β_{2g-2}. There exists:

(i) a bound marked surface $(\Sigma_2, \theta_2, \beta_2)$ *with holonomy in* $\pi_1(M, Q)$*, and*

(ii) $2g-4$ *bound, marked trousers* $(T_i, \varphi_i, \beta_i)_{1 \leqslant i \leqslant 2g-4}$ *with holonomy in* $\pi_1(M, Q)$*,*

such that:

$$(\Sigma_{2g-2}, \theta_{2g-2}, \beta_{2g-2}) \cong (\Sigma_2, \varphi_2, \beta_2) \bigcup_{i=1}^{2g-4} (T_i, \varphi_i, \beta_i),$$

and:

(i) $(\Sigma_2, \theta_2, \beta_2)$ *has genus* 1*, two hyperbolic boundary components and a non-elementary marked handle* (a', b')*,*

(ii) for all i*, the image of* φ_i *in* $\pi_1(M, Q)$ *is a Schottky group, and*

(iii) after identifying objects (points, curves, etc.) in Σ_{2g-2} *with the corresponding objects in* $\Sigma_2 \cup_{i=1}^{2g-4} T_i$*, and vice-versa, for all* $1 \leqslant i \leqslant g-1$*, the curves* $C_{i,+}$ *and* $C_{i,-}$ *are boundary components of* T_i*.*

Remark: We could also add in the statement of this proposition that this decomposition takes the form of a tree, but this is a combinatorial consequence of the facts that Σ_2 has genus 1 and that the handles are formed by glueing the first $g-1$ trousers to themselves.

We begin with the following proposition:

Proposition 4.14

Let (M, Q) *be a pointed compact three dimensional manifold of strictly negative sectional curvature. Let* $(\Sigma_n, \theta_n, \beta_n)$ *be a bound marked surface with holonomy in* $\pi_1(M, Q)$ *of genus* 1 *having* n *hyperbolic boundary components and a non-elementary marked handle* (a, b)*.*

Let x_1 *and* y_1 *be two boundary components of* Σ_n*. Let* c *be a simple curve joining* x *to* y *which is disjoint from* x_1*,* y_1*,* a_1 *and* b_1 *except possibly at its end points (see figure 4.16).*

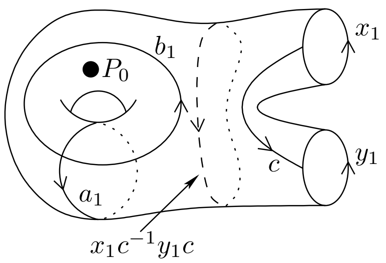

Figure 4.16

There exists a homeomorphism $\Psi : (R, \partial R, P_0) \to (R, \partial R, P_0)$ *of* R *such that:*

(1) $\Psi_*(a_1, b_1)$ *is non-elementary,*

(2) if z *is a boundary component of* Σ *different to* x_1 *and* y_1*, then* Φ_* *acts on* z *by conjugation (thus, if* $\theta(z)$ *is hyperbolic, then* $\theta\Phi_* z$ *is hyperbolic), and*

(3) $\Psi_*(x_1, c^{-1}y_1c)$ *is Schottky (and thus, in particular,* $x_1c^{-1}y_1c$ *is hyperbolic).*

Proof: Let Q_0 be the common base point of a_1 and b_1. Let Q_x and Q_y be the base points of x and y resp. Let (c_0, c_x) be a binding fan of (Q_0, Q_x). We define:

$$\begin{array}{ll} a = c_0^{-1} a_1 c_0, & b = c_0^{-1} b_1 c_0, \\ x = c_x^{-1} x_1 c_x, & y = (c c_x)^{-1} y_1 (c c_x). \end{array}$$

We denote $\alpha = \theta_{2n-2}(a)$, $\beta = \theta_{2n-2}(b)$, $\xi = \theta_{2n-2}(x)$ and $\eta = \theta_{2n-2}(y)$.

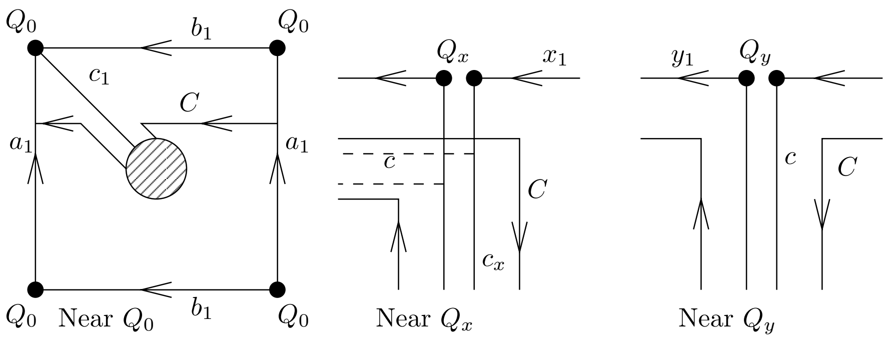

Figure 4.17

We may choose a copy y' of y oriented such that $C = y'b$ is homotopic to a simple closed curve. we denote $d_0 = y'b$. Let T_0 be a Dehn Twist about C. If we choose the configurations of c_a and c_x as in figure 4.17, then we obtain:

$$\begin{array}{ll} (T_0)_* a = d_0 a, & (T_0)_* b = b, \\ (T_0)_* x = d_0 x d_0^{-1}, & (T_0)_* y = y. \end{array}$$

Since $(T_0)_*$ is a homomorphism which leaves d_0 invariant, we obtain, for all n:

$$\begin{array}{ll} (T_0^n)_* a = d_0^n a, & (T_0^n)_* b = b, \\ (T_0^n)_* x = d_0^n x d_0^{-n}, & (T_0^n)_* y = y. \end{array}$$

As before, we choose a Dehn twist T_a about a such that, for all k:

$$\begin{array}{ll} (T_a^k)_* a = a, & (T_a^k)_* b = b a^k, \\ (T_a^k)_* x = x, & (T_a^k)_* y = y. \end{array}$$

We define $\Psi_{k,n} : \Sigma_{2n-2} \to \Sigma_{2n-2}$ by:

$$\Psi_{k,n} = T_a^k T_0^n T_a^{-k}$$

Denoting $d_k = y b a^k$, We have:

$$\begin{array}{ll} (\Psi_{k,n})_* a = d_k^n a, & (\Psi_{k,n})_* b a^k = b a^k, \\ (\Psi_{k,n})_* x = d_k^n x d_k^{-n}, & (\Psi_{k,n})_* y = y. \end{array}$$

We choose k and n as in lemmata 4.5 and 4.6, and we denote $\Psi = \Psi_{k,n}$. The first result follows from the first and third conclusions of lemma 4.5. Since Ψ is a product of Dehn twists, it acts on paths corresponding to boundary components by conjugation. The second result now follows. The third result follows from lemma 4.6. $\square$

This yields:

Corollary 4.15

With the same hypotheses as in proposition 4.14, there exists:

(i) a bound marked trouser (T, θ, β) with holonomy in $\pi_1(M, Q)$, and

(ii) a bound marked surface $(\Sigma_{n-1}, \theta_{n-1}, \beta_{n-1})$ with holonomy in $\pi_1(M, Q)$,

such that:

$$(\Sigma_n, \theta_n, \beta_n) \cong (\Sigma_{n-1}, \theta_{n-1}, \beta_{n-1}) \cup (T, \theta, \beta),$$

and,

(i) $(\Sigma_{n-1}, \theta_{n-1}, \beta_{n-1})$ has genus 1, $n-1$ hyperbolic boundary components, and a non-elementary marked handle (a', b'),

(ii) the image of θ in $\pi_1(M, Q)$ is a Schottky group, and

(iii) after identifying objects (points, curves, etc.) in Σ_n with the corresponding objects in $\Sigma_{n-1} \cup T$, and vice-versa, the curves x and y correspond to boundary components of T.

Proof: We may suppose that (Σ, θ) satisfies the conclusions of proposition 4.14. Cutting Σ_n along a curve freely homotopic to $x_1 c^{-1} y_1 c$ which does not intersect either a or b, we obtain the desired result as in the proof of proposition 4.9. $\square$

We may now prove proposition 4.13:

Proof of proposition 4.13: This follows directly from induction and corollary 4.15, taking care to pair off the correct boundary components of Σ_{2g-2} in order to obtain the first $(g-1)$ trousers. $\square$

4.8 Untying The Root.

In this subsection, we aim to prove the following result:

Proposition 4.16

Let (M, Q) be a pointed compact three dimensional manifold of strictly negative sectional curvature. Let $(\Sigma_2, \theta_2, \beta_2)$ be a bound marked surface with holonomy in $\pi_1(M, Q)$ of genus 1 having 2 hyperbolic boundary components and a non-elementary marked handle (a, b).

There exist bound marked trousers $(T_i, \theta_i, \beta_i)_{1 \leqslant i \leqslant 2}$ with holonomy in $\pi_1(M)$ such that:

(i) $(\Sigma_2, \theta_2, \beta_2) \cong [\bigcirc(T_1, \theta_1, \beta_1)] \cup (T_2, \theta_2, \beta_2)$, and

(ii) the images of θ_1 and θ_2 are Schottky groups.

We begin by proving the following proposition:

Proposition 4.17

Let (M, Q) be a pointed compact three dimensional manifold of strictly negative sectional curvature. Let $(\Sigma_2, \theta_2, \beta_2)$ be a bound marked surface with holonomy in $\pi_1(M, Q)$ of genus 1 having 2 boundary components and a non-elementary marked handle (a, b).

Let x_1 and y_1 be the two oriented boundary components of Σ_2, and let c be a curve in Σ_2 joining the base point of y to that of x, but otherwise disjoint from a_1,b_1,x_1 and y_1. There exists a homeomorphism Ψ of Σ_2 such that $\Psi_*(c^{-1}x_1c, y_1)$ and $\Psi_*(a_1^{-1}b_1a_1, b_1)$ are Schottky.

Proof: Let Q_0 be the common base point of a and b. Let Q_x and Q_y be the base points of x and y respectively. Let (c_0, c_x, c_y) be a binding sash of (Q_0, Q_x, Q_y). We denote:

$$\begin{aligned} a = c_a^{-1}a_1c_a, \quad b = c_a^{-1}b_1c_a, \\ x = c_x^{-1}x_1c_x, \quad y = c_y^{-1}y_1c_y. \end{aligned}$$

We may assume that $c = a^{-1}b^{-1}ab$ is freely homotopic to $(yx)^{-1}$ as in figure 4.18.

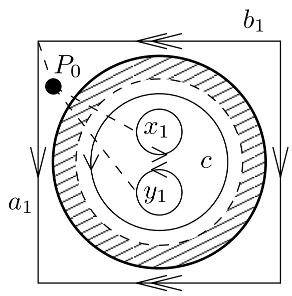

Figure 4.18

We denote:

$$\alpha = \theta_2(a), \beta = \theta_2(b), \xi = \theta_2(x), \eta = \theta_2(y), \sigma = \theta_2(c).$$

We may suppose that $\langle \xi, \eta \rangle$ is a Schottky group and that $\langle \alpha, \beta \rangle$ is non-elementary. Let T_c be the Dehn twist about c such that, for all m:

$$\begin{aligned} T_c^m a = a, \qquad & T_c^m b = b, \\ T_c^m x = c^m x c^{-m}, \quad & T_c^m y = c^m y c^{-m}. \end{aligned}$$

By lemma 4.7, after replacing θ with $\theta \circ (T_c)_*$ if necessary, we may suppose that $J = \xi\beta$ is not equal to the identity.

For $k \in \mathbb{N}$, we define $d_k = xba^k$. We may assume that xb is freely homotopic to a simple closed curve, and, as in the proofs of lemmata 4.9 and 4.13, using Dehn twists, for all k and for all n, we may construct a homeomorphism $\Psi_{k,n}$ of Σ_2 such that:

$$\begin{aligned} (\Psi_{k,n})_* a &= d_k^n a, \quad (\Psi_{k,n})_* ba^k &= ba^k, \\ (\Psi_{k,n})_* x &= x, \qquad (\Psi_{k,n})_* y &= d_k^{-n} y d_k^n. \end{aligned}$$

Where $d_k = xba^k$. Choosing k and n as in lemma 4.8, we obtain the desired result. $\square$

We now obtain proposition 4.16:

Proof of proposition 4.16: This follows directly by applying proposition 4.17 and then cutting along curves freely homotopic to yx and to b. $\square$

◇

5 - Invariant Domains of Schottky Groups.

5.1 Introduction.

Throughout this section, (M, Q) will be a pointed three dimensional Hadamard manifold and $\Gamma = \langle\alpha, \beta\rangle$ will be a Schottky subgroup of $\mathrm{Isom}(M, Q)$. In this section, we study the algebraic properties of Schottky groups.

We define an **invariant domain** of Γ to be a Jordan domain Ω contained in $\partial_\infty M$ which is invariant under the action of Γ. Invariant domains are easy to construct. Indeed, let Γ' be any Schottky subgroup of $\mathbb{P}SL(2, \mathbb{C})$ which preserves the real line. Let $\phi : \Gamma \to \Gamma'$ be an isomorphism, and let $\Phi : \partial_\infty M \to \hat{\mathbb{C}}$ be a homeomorphism which intertwines with ϕ (which we constructed in appendix A). $\Phi^{-1}(\hat{\mathbb{R}})$ is a Jordan curve in $\partial_\infty M$, and both connected components of its complement are invariant domains.

Trivially, for any invariant domain, Ω:

$$\mathrm{Fix}(\Gamma) \subseteq \partial\Omega.$$

We define $\hat{\Omega}$ by:

$$\hat{\Omega} = \overline{\Omega} \setminus \mathrm{Fix}(\Gamma).$$

For every element γ of Γ we define γ_- and γ_+ to be the repulsive and attractive fixed points of γ respectively. We say that γ is simple when there exists no other element η in Γ and no $n \geqslant 2$ such that $\gamma = \eta^n$. For γ a simple element, we say that a connected component I of $\partial\Omega \setminus \mathrm{Fix}(\Gamma)$ is **adapted to** γ if and only if:

(i) if I is viewed as an open subset of $\partial\Omega$, then:

$$\partial I = \gamma_\pm,$$

and

(ii) if I is viewed as an oriented subarc of $\partial\Omega$, then it runs from γ_- to γ_+.

Such a connected component, when it exists, is unique. We say that Ω is **adapted** to the generators (α, β) when both of α and β have adapted components in $\partial\Omega$. Trivially, the invariant domain Ω, constructed as above is adapted to the generators (α, β).

We show (proposition 5.2) that Γ acts properly discontinuously over $\hat{\Omega}$ and that its quotient is a compact topological surface with boundary. Since its fundamental group is the free group on two generators, and since it is oriented, it is either a trouser (three boundary components) or a punctured torus (one boundary component). When Ω is adapted to the generators (α, β), the adapted components of α and β project down to boundary components of $\hat{\Omega}/\Gamma$, and since α and β cannot be conjugate in Γ, these components are distinct. Consequently, $\hat{\Omega}/\Gamma$ is a trouser.

This would already allow us to construct equivariant Plateau problems over a given bound, marked trouser (T, θ, β) with holonomy in $\pi_1(M, Q)$ and such that θ is a Schottky group. The solution is by no means unique, and in order to solve the more general problem of finding an equivariant Plateau problem for an arbitrary bound, marked surface, we are required to study in more depth the algebraic properties of adapted invariant domains.

If γ is a hyperbolic element of Γ, we may define the torus $\mathbb{T}_\gamma$ by:

$$\mathbb{T}_\gamma = (\partial_\infty \tilde{M} \setminus \{\gamma_\pm\})/\langle\gamma\rangle.$$

For any hyperbolic element γ of Γ, we may associate to any invariant domain Ω of Γ a unique element $[\Omega]_\gamma$ in $H_1(\mathbb{T}_\gamma)$ (see lemma 5.7). Moreover, for all such γ, we may also canonically define a preferred one dimensional subspace L_γ of $H_1(\mathbb{T}_\gamma)$ and we may show that, for any invariant domain, Ω, of Γ, the element $[\Omega]_\gamma$ lies in L_γ (see lemma 5.8).

Let $\mathrm{Homeo}_0(\partial_\infty M)$ denote the connected component of the space of homeomorphisms of $\partial_\infty M$ which contains the identity. There exists a canonical homomorphism of $\pi_1(M, Q)$ into $\mathrm{Homeo}_0(\partial_\infty M)$. We may show (see corollary 5.5) that if $\widetilde{\mathrm{Homeo}}_0(\partial_\infty M)$ denotes the universal cover of $\mathrm{Homeo}_0(\partial_\infty M)$, then it is a two-fold covering. For every hyperbolic element γ of Γ, we may canonically define a mapping $\mathrm{Lift}_\gamma : L_\gamma \to \widetilde{\mathrm{Homeo}}_0(\partial_\infty M)$.

The main result of this section, which allows us to establish the obstruction to constructing a $\pi_1(M, Q)$ structure over a closed surface Σ with holonomy in $\pi_1(M, Q)$ may now be expressed as follows:

Lemma 5.1

Let $\Gamma = \langle\alpha, \beta\rangle$ be a Schottky group. Let us denote $\gamma = \alpha\beta$. Let $([a]_\alpha, [b]_\beta, [c]_\gamma)$ be a triplet in $L_\alpha \times L_\beta \times L_\gamma$. There exists an invariant domain, Ω, of Γ in $\partial_\infty M$ adapted to the generators (α, β) such that:

$$[\Omega]_\alpha = [a]_\alpha, \qquad [\Omega]_\beta = [b]_\beta, \qquad [\Omega]_\gamma = [c]_\gamma,$$

if and only if:

$$\mathrm{Lift}_\gamma([c]_\gamma)^{-1}\mathrm{Lift}_\alpha([a]_\alpha)\mathrm{Lift}_\beta([b]_\beta) \neq \mathrm{Id}.$$

The proof of this result follows immediately from propositions 5.10 and 5.11.

In the second part of this section, we review the geometric properties of invariant domains, justifying the assertions made in this introduction. In the third part, we review the topology of the group of homeomorphisms of the sphere and the properties of tress groups of order three in the sphere. In the fourth part, we show how to define the element $[\Omega]_\gamma$, for any hyperbolic element, γ, of Γ and any invariant domain, Ω, of Γ, and we prove that it must lie in L_γ. Finally, in the fifth part, we prove propositions 5.10 and 5.11.

5.2 Invariant Domains.

In this section, we study the geometric properties of invariant domains. First, we have:

Propositon 5.2

Γ acts properly discontinuously on $\hat{\Omega}$ and $\hat{\Omega}/\Gamma$ is compact.

Remark: Since the fundamental group of $\hat{\Omega}/\Gamma$ is isomorphic to the free group on two generators, $\hat{\Omega}/\Gamma$ is either a trouser or a punctured torus.

Proof: Let $(C_\alpha^\pm, C_\beta^\pm)$ be generating circles of Γ with respect to the generators (α,β). We define $X \subseteq \partial_\infty M$ by:

$$X = \partial_\infty M \setminus \left(\mathrm{Int}(C_\alpha^+) \cup \mathrm{Int}(C_\alpha^-) \cup \mathrm{Int}(C_\beta^+) \cup \mathrm{Int}(C_\beta^-)\right).$$

By lemma A.11:

$$\begin{array}{lll} & \partial_\infty M & = (\cup_{\gamma\in\Gamma}\gamma X) \cup \mathrm{Fix}(\Gamma) \\ \Rightarrow & \hat{\Omega} & = \cup_{\gamma\in\Gamma}\gamma(X \cap \hat{\Omega}). \end{array}$$

Thus, if $\pi : \hat{\Omega} \to \hat{\Omega}/\Gamma$ denotes the canonical projection, then its restriction to $X \cap \hat{\Omega}$ is surjective. However, since $X \cap \mathrm{Fix}(\Gamma) = \emptyset$, it follows that $X \cap \hat{\Omega} = X \cap \overline{\Omega}$ is compact, and thus so is $\hat{\Omega}/\Gamma$.

Suppose that $p \in \partial_\infty M$ is a concentration point of Γ. There exists $p_0 \in \partial_\infty M$ and distinct elements $(\gamma_n)_{n\in\mathbb{N}}$ in Γ such that $(p_0 \cdot \gamma_n)_{n\in\mathbb{N}}$ tends to p_0. If $p_0 \in \mathrm{Fix}(\Gamma)$, then so is $p_0 \cdot \gamma_n$ for all n, and thus, by compactness, so is p. Consequently, we only need to study the case where $p_0 \in X$. Let $(\boldsymbol{\gamma}_n)_{n\in\mathbb{N}}$ in $W_{\alpha,\beta}$ be such that, for all n:

$$\mathrm{Eval}(\boldsymbol{\gamma}_n) = \gamma_n.$$

By compactness, we may suppose that there exists $\boldsymbol{\gamma} \in W_{\alpha,\beta}^\infty$ such that $(\boldsymbol{\gamma}_n)_{n\in\mathbb{N}}$ converges to $\boldsymbol{\gamma}$. Consequently, by proposition A.12, $(p_0 \cdot \gamma_n)_{n\in\mathbb{N}}$ converges to $\mathcal{P}(\boldsymbol{\gamma}) \in \mathrm{Fix}(\Gamma)$. Thus the only concentration points of Γ lie in $\mathrm{Fix}(\Gamma)$.

Likewise, the only fixed points of elements of Γ lie in $\mathrm{Fix}(\Gamma)$. Thus, since $\hat{\Omega} \cap \mathrm{Fix}(\Gamma) = \emptyset$, Γ acts properly discontinuously on $\hat{\Omega}$. □

However, the generators (α,β) of Γ do not necessarily correspond to boundary components of $\hat{\Omega}/\Gamma$. We consequently restrict our attention to invariant domains which are adapted to these generators, since, in this case, as we observed in the introduction, the adapted components of α and β project down to distinct boundary components of $\hat{\Omega}/\Gamma$ and $\hat{\Omega}/\Gamma$ is consequently a trouser.

In the sequel, we will require the following technical result concerning adapted invariant domains:

Proposition 5.3

Let Ω be an invariant domain of Γ adapted to the generators (α,β). Let γ and δ be the elements of Γ defined by:

$$\gamma = \alpha\beta, \qquad \delta = \beta\alpha.$$

If $\alpha_\pm$,$\beta_\pm$,$\gamma_\pm$ and $\delta_\pm$ are the fixed points of these four elements, then they are distributed round $\partial\Omega$ in the following order:

$$\alpha_-, \alpha_+, \delta_+, \delta_-, \beta_-, \beta_+, \gamma_+, \gamma_-.$$

Proof: Let p be one of the fixed points of γ. We have:

$$(p \cdot \alpha) \cdot \delta = p \cdot (\alpha\beta\alpha) = (p \cdot \gamma) \cdot \alpha = p \cdot \alpha.$$

Consequently, α sends the fixed points of γ to those of δ, and so $\gamma_\pm \cdot \alpha = \delta_\pm$. Likewise $\delta_\pm \cdot \beta = \gamma_\pm$. Let I_α and I_β be the adapted components of α and β respectively.

Since α preserves orientation, it also preserves I_α and I_α^C, Moreover, it shifts all points in I_α and I_α^C towards α^+. Consequently, the attractive and repulsive fixed points of α,γ and δ appear in the following order as one moves around $\partial\Omega$ in the positive direction:

$$\alpha_-, \alpha_+, \delta_\pm, \gamma_\pm.$$

Similarly, the mapping β shifts all points in I_β and I_β^C towards β^+. Thus, the attractive and repulsive fixed points of β,γ and δ appear the the following order as one moves around $\partial\Omega$ in the positive direction:

$$\beta_-, \beta_+, \gamma_\pm, \delta_\pm.$$

Since the mapping α, which sends the fixed points of γ to those of δ, is an orientation preserving mapping, it follows that, as one moves round $\partial\Omega$ in the positive direction, γ_- and γ_+ appear in the same order as δ_- and δ_+. Moreover, since I_α and I_β are adapted components, they have no fixed points of Γ in their interiors, and so, combining all this information, we find that these eight fixed points appear round $\partial\Omega$ in the following order:

$$\alpha_-, \alpha_+, \delta_\pm, \delta_\mp, \beta_-, \beta_+, \gamma_\pm, \gamma_\mp.$$

Let p be the fixed point of γ lying closest to β_+ in I_β^C. Let q be the fixed point of δ lying closest to α_+ in I_α^C. The point p is also the fixed point of γ lying closest to α_+ in I_α^C. Since α preserves orientation, we have $p \cdot \alpha = q$. The points $\alpha_\pm$ lie between q and β_+ in I_β^C. Consequently, the points $\alpha_\pm \cdot \gamma = \alpha_\pm \cdot \beta$ lie between $p = p \cdot \gamma = q \cdot \beta$ and β_+ in I_β^C.

Let I_1 and I_2 be the two connected components of $\partial\Omega \setminus \{\gamma_\pm\}$. The mapping γ shifts all points in the intervals I_1 and I_2 towards γ_+. We may suppose that I_1 is the component containing $\alpha_\pm$. Since $\alpha_+ \cdot \gamma$ lies between p and β_+ in I_β^C, it follows that it also lies between α_+ and p in I_1. The point p is consequently the attractive fixed point of γ and the result now follows. $\square$

5.3 Braids and the Topology of $\mathrm{Homeo}_0(\partial_\infty M)$.

For any topological surface Σ, let $\mathrm{Homeo}_0(\Sigma)$ be the connected component of $\mathrm{Homeo}(\Sigma)$ that contains the identity. There exists a canical embedding of the group $\pi_1(M,Q)$ into $\mathrm{Homeo}_0(\partial_\infty M)$. We may thus consider the mapping θ as a homomorphism of $\pi_1(\Sigma,P_0)$ taking values in the group $\mathrm{Homeo}_0(\partial_\infty M)$.

Let $X\subseteq Y$ be topological spaces. We define a **strong deformation retraction** of Y onto X to be a mapping $\psi : I\times Y\to Y$ such that:

(i) $\psi_0 : Y\to Y$ is the identity,

(ii) $\psi_1(Y)\subseteq X$, and

(iii) for all $t\in I$, the restriction of ψ_t to X is the identity.

Let us denote by $S^2\subseteq\mathbb{R}^3$ the sphere of radius 1 in $\mathbb{R}^3$. We recall the following result concerning the homotopy type of $\mathrm{Homeo}_0(S^2)$ (see [3], [6]):

Theorem 5.4 [Friberg, 1973]

The space $\mathrm{Homeo}_0(S^2)$ *retracts by strong deformations onto* $\mathrm{SO}(3,\mathbb{R})$.

In particular, we obtain:

Corollary 5.5

$$\pi_1(\mathrm{Homeo}_0(\partial_\infty M),\mathrm{Id})\cong\pi_1(\mathrm{SO}(3,\mathbb{R}),\mathrm{Id})\cong\mathbb{Z}_2.$$

A **tress of order** 3 in $\partial_\infty M$ is a triple $\boldsymbol{\gamma}=(\gamma_1,\gamma_2,\gamma_3)$ where $\gamma_1,\gamma_2,\gamma_3 : I\to S^2$ are curves such that, for all $t\in I$, the points $\gamma_1(t)$, $\gamma_2(t)$ and $\gamma_3(t)$ are all distinct. The interested reader may find a more detailed treatment of braids in general in [9] or in appendix D of [8]. For all $t\in I$, we denote:

$$\boldsymbol{\gamma}(t)=(\gamma_1(t),\gamma_2(t),\gamma_3(t)).$$

We call the point $\boldsymbol{\gamma}(0)$ the **base point** of the tress $\boldsymbol{\gamma}$. We say that the tress is closed if and only if:

$$\boldsymbol{\gamma}(0)=\boldsymbol{\gamma}(1).$$

Let $\boldsymbol{\gamma}_0$ and $\boldsymbol{\gamma}_1$ be two braids having the same endpoints. A **homotopy** between $\boldsymbol{\gamma}_0$ and $\boldsymbol{\gamma}_1$ is a continuous family $(\boldsymbol{\eta}_t)_{t\in I}$ of braids having the same extremities as $\boldsymbol{\gamma}_0$ and $\boldsymbol{\gamma}_1$ such that:

$$\boldsymbol{\eta}_0=\boldsymbol{\gamma}_0,\boldsymbol{\eta}_1=\boldsymbol{\gamma}_1$$

For $\mathbf{p}=(p_1,p_2,p_3)$ a triplet of distinct points in $\partial_\infty M$, we denote by $T_{\mathbf{p}}$ the family of braids of order 3 in $\partial_\infty M$ having $\mathbf{p}$ as a base point. Likewise, we denote by $T^0_{\mathbf{p}}$ the subfamily of $T_{\mathbf{p}}$ consisting of all the closed braids in $T_{\mathbf{p}}$. Let $\sim$ be the homotopy equivalence relation over $T_{\mathbf{p}}$. The law of composition of curves yields the law of composition of braids, and the set $T_{\mathbf{p}}/\sim$ thus forms a semigroup. Likewise, $T^0_{\mathbf{p}}/\sim$ forms a group. For $\boldsymbol{\gamma}=(\gamma_1,\gamma_2,\gamma_3)$ a tress in $T_{\mathbf{p}}$, we denote by $[\boldsymbol{\gamma}]=[\gamma_1,\gamma_2,\gamma_3]$ its projection in $T_{\mathbf{p}}/\sim$.

Let $C_0(I, \mathrm{Homeo}_0(\partial_\infty M))$ denote the family of continuous curves in $\mathrm{Homeo}_0(\partial_\infty M)$ leaving from the identity. For any triplet $\mathbf{p} = (p_1, p_2, p_3)$ of distinct points in $\partial_\infty M$, we define the mapping $\mathcal{T}_{\mathbf{p}}$ from $C_0(I, \mathrm{Homeo}_0(\partial_\infty M))$ to $T_{\mathbf{p}}$ such that, for all $c \in C_0(I, \mathrm{Homeo}_0(\partial_\infty M))$, and for all $t \in I$:

$$(\mathcal{T}_{\mathbf{p}} c)(t) = (p_1 \cdot c(t), p_2 \cdot c(t), p_3 \cdot c(t)).$$

In the case where $\partial_\infty M = \hat{\mathbb{C}}$, for all $\mathbf{p}$, the mapping $\mathcal{T}_{\mathbf{p}}$ restricts to a homeomorphism from $C_0(I, \mathbb{P}SL(2, \mathbb{C}))$ onto $T_{\mathbf{p}}$. Consequently, in the general case, the mapping $\mathcal{T}_{\mathbf{p}}$ is surjective. Moreover, for all $\mathbf{p}$, the mapping $\mathcal{T}_{\mathbf{p}}$ quotients down to a surjective mapping from $\widetilde{\mathrm{Homeo}}_0(\partial_\infty M)$ to $T_{\mathbf{p}}/\sim$ which we also denote by $\mathcal{T}_{\mathbf{p}}$. Theorem 5.4 and the fact that $\mathbb{P}SL(2, \mathbb{C})$ also retracts by strong deformation retraction onto $\mathrm{SO}(3, \mathbb{R})$ permit us to show that this mapping is bijective.

Let $\boldsymbol{\gamma}$ be a tress in $T_{\mathbf{p}}(p_1, p_2, p_3)$. We may suppose that there exists a point p_∞ in $\partial_\infty M$ which does not lie in the image of $\boldsymbol{\gamma}$. Let $\alpha : \partial_\infty M \setminus \{p_\infty\} \to \mathbb{R}^2$ be a homeomorphism. For any closed curve η in $\mathbb{R}^2 \setminus \{0\}$, let $\mathrm{Wind}(\eta)$ be the winding number of η about 0. The quantity:

$$\sum_{i<j} \mathrm{Wind}(\alpha \circ \gamma_i - \alpha \circ \gamma_j) \ \mathrm{Mod}\ 2$$

is well defined and independant of α and p_∞ (see, for example, [9], or appendix D of [8]). We thus define $\mathrm{Wind}_r(\boldsymbol{\gamma})$, the **relative winding number** of $\boldsymbol{\gamma}$ to be equal to this quantity.

Wind_r defines an isomorphism from $T^0_{\mathbf{p}}/\sim$ to $\mathbb{Z}_2$. We define the mapping $W_{r,\boldsymbol{p}}$ which sends $\pi_1(\mathrm{Homeo}_0(\partial_\infty M))$ into $\mathbb{Z}_2$ by:

$$W_{r,\boldsymbol{p}} = \mathrm{Wind}_r \circ \mathcal{T}_{\mathbf{p}}.$$

We have the following result:

Lemma 5.6

$W_{r,\boldsymbol{p}}$ defines an isomorphism from $\pi_1(\mathrm{Homeo}_0(\partial_\infty M))$ to $\mathbb{Z}_2$, and does not depend on the choice of $\mathbf{p}$.

We thus denote $W_r = W_{r,\boldsymbol{p}}$.

5.4 Homological Classes.

For $\gamma \in \Gamma$, we define the torus $\mathbb{T}_\gamma$ by:

$$\mathbb{T}_\gamma = (\partial_\infty M \setminus \{\gamma_-, \gamma_+\})/\langle\gamma\rangle.$$

Let us denote by π_γ the canonical projection from $\partial_\infty M \setminus \{\gamma_-, \gamma_+\}$ onto $\mathbb{T}_\gamma$. We obtain the following elementary result:

Lemma 5.7

Let Ω be an invariant domain of Γ in $\partial_\infty M$. For all $\gamma \in \Gamma$, there exists a unique homology class $[\Omega]_\gamma \in H_1(\mathbb{T}_\gamma)$ such that, for all $p_0 \in \Omega$ and for all $c : I \to T$ such that:

$$c(0) = p_0, \qquad c(1) = p_0 \cdot \gamma.$$

we have:

$$[\Omega]_\gamma = [\pi \circ c].$$

Proof: Indeed, let p_0 and p_1 be two points in Ω. Let c_0 and c_1 be curves such that:

$$\begin{aligned} c_0(0) &= p_0, & c_0(1) &= p_1 \cdot \gamma, \\ c_1(0) &= p_1, & c_1(1) &= p_1 \cdot \gamma. \end{aligned}$$

Since Ω is connected, simply connected and invariant under the action of γ, there exists a homotopy $(c_t)_{t\in[0,1]}$ between c_0 and c_1 such that, for all t:

$$c_t(1) = \gamma(c_t(0)).$$

Consequently:

$$[\pi \circ c_0] = [\pi \circ c_1].$$

The result now follows. □

Let C be a closed curve which turns once around the cylinder $\partial_\infty M \setminus \{\gamma_\pm\}$ in such a manner that γ_+ lies to its left (in otherwords, in its interior). C defines a unique homology class $[C]_\gamma$ in $H_1(\mathbb{T}_\gamma)$. Using Poincare duality, we obtain a scalar product $\langle \cdot, \cdot \rangle$ over $H_1(\mathbb{T}_\gamma)$, and we define $L_\gamma \subseteq H_1(\mathbb{T}_\gamma)$ by:

$$L_\gamma = \{[a] \in H_1(\mathbb{T}_\gamma) \mid \langle [C_\gamma], [a] \rangle = 1\}.$$

Heuristically, L_γ is a straight line which contains all the curves in $\mathbb{T}_\gamma$ which cross C_γ exactly once, going from right to left. For $[a], [b] \in L_\gamma$, we define $[a] - [b]$ to be the unique integer n such that:

$$[a] = [b] + n[C]_\gamma.$$

Similarly, for $[a] \in L_\gamma$ and $n \in \mathbb{Z}$, we define $[a] + n$ by:

$$[a] + n = [a] + n[C]_\gamma.$$

We now obtain the following result:

Lemma 5.8

Let Ω be an invariant domain of Γ in $\partial_\infty M$. For all $\gamma \in \Gamma$, $[\Omega]_\gamma \in L_\gamma$.

Proof: Let p_0 be a point in Ω and let c be a curve in Ω joining p_0 to $\gamma(p_0)$. We may suppose that c only intersects C at its end point. By definition of C, the curve $\pi \circ c$ crosses $\pi \circ C$ from the right to the left, and the result now follows. □

5.5 Liftings of Applications.

Let γ be an element in Γ. Let $\gamma_\pm$ be the fixed points of γ. Let $c : I \to \partial_\infty M \setminus \{\gamma_\pm\}$ be a curve such that:

$$c(0) \cdot \gamma = c(1).$$

The curve c projects down to an element of L_γ. We consider $\gamma_\pm$ as being constant curves, and we define $\hat{\gamma}_c$ to be the unique lifting of γ in $\widetilde{\mathrm{Homeo}}_0(\partial_\infty M)$ such that:

$$\mathcal{T}_{(\gamma_-, c(0), \gamma_+)} \hat{\gamma}_c = [\gamma_-, c, \gamma_+].$$

Trivially, $\hat{\gamma}_c$ only depends on the homotopy class of $[\pi \circ c]$ in $L_\gamma \subseteq H_1(\mathbb{T}_\gamma)$. We thus obtain a mapping $\mathrm{Lift}_\gamma : L_\gamma \rightarrow \widetilde{\mathrm{Homeo}}_0(\partial_\infty M)$ which is defined such that, for all c such that $c(0) \cdot \gamma = c(1)$:

$$\mathrm{Lift}_\gamma([\pi_\gamma \circ c]) = \hat{\gamma}_c.$$

We have:

Proposition 5.9

Let $c, c' : I \rightarrow M \setminus \{\gamma_\pm\}$ be curves such that $c(0) \cdot \gamma = c(1)$ and $c'(0) \cdot \gamma = c'(1)$. Then:

$$\mathrm{Lift}_\gamma([\pi_\gamma \circ c]) = \mathrm{Lift}_\gamma([\pi_\gamma \circ c']) \Leftrightarrow [\pi_\gamma \circ c] - [\pi_\gamma \circ c'] = 0 \ \mathit{Mod}\ 2.$$

Proof: Let us define:

$$\alpha = \mathrm{Lift}_\gamma([\pi_\gamma \circ c])^{-1} \cdot \mathrm{Lift}_\gamma([\pi_\gamma \circ c']).$$

The element α is a lifting of the identity in $\widetilde{\mathrm{Homeo}}_0(\partial_\infty M)$, and:

$$\mathcal{T}_{(\gamma_-, c'(0), \gamma_+)} = [\gamma_-, c^{-1}c', \gamma_+].$$

By lemma 5.6, the element α is equal to the identity if and only if:

$$W_r[\gamma_-, c^{-1}c', \gamma_+] = 0.$$

However, since $\gamma_\pm$ are trivial paths, we obtain:

$$W_r[\gamma_-, c^{-1}c', \gamma_+] = [\pi_\gamma \circ c'] - [\pi_\gamma \circ c] \text{ Mod } 2.$$

The result now follows. □

If Ω is an invariant domain of Γ in $\partial_\infty M$, then, for every element $\gamma \in \Gamma$, we define $\hat{\gamma}_\Omega$ by:

$$\hat{\gamma}_\Omega = \mathrm{Lift}_\gamma[\Omega]_\gamma.$$

We obtain the following result:

Proposition 5.10

Let $\Gamma = \langle \alpha, \beta \rangle$ be a Schottky group. Let Ω be an invariant domain of Γ in $\partial_\infty M$ which is adapted to the generators (a, b). Let us denote $\gamma = \alpha\beta$, and let us define Δ by:

$$\Delta = \hat{\gamma}_\Omega^{-1} \hat{\alpha}_\Omega \hat{\beta}_\Omega.$$

Then Δ, which is a closed curve in $\mathrm{Homeo}_0(\partial_\infty M)$, is homotopically non-trivial.

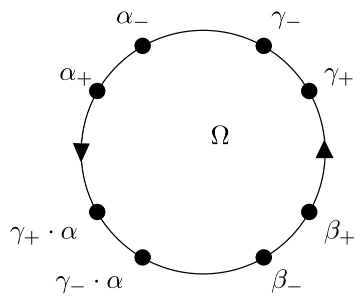

Figure 5.19

Proof: By lemma A.2, it suffices to prove this result when $\partial_\infty \tilde{M} = \hat{\mathbb{C}}$ and Γ is a subgroup of $\mathbb{P}SL(2, \mathbb{C})$. In this case, for any triplet $\mathbf{p} = (p_1, p_2, p_3)$ of distinct points in $\hat{\mathbb{C}}$, the restriction of the mapping $\mathcal{T}_{\mathbf{p}}$ to $C_0(I, \mathbb{P}SL(2, \mathbb{C}))$ is bijective. Let $\mathcal{S}_{\mathbf{p}}$ denote its inverse. Since, for any tress, $\boldsymbol{\eta}$ in $T_{\mathbf{p}}$, the point $\mathbf{p}$ is the base point of $\boldsymbol{\eta}$, and is thus determined by $\boldsymbol{\eta}$, we may write $\mathcal{S}$ instead of $\mathcal{S}_{\mathbf{p}}$.

$\partial\Omega$ is homeomorphic to the circle S^1 and is invariant under the action of Γ on $\mathbb{C}$. By proposition 5.3, since Ω is adapted to the generators (α, β), the 8 points $\alpha_\pm$,$\beta_\pm$,$\gamma_\pm$ and $\gamma_\pm \cdot \alpha$ are distributed around $\partial\Omega$ according to the diagram in figure 5.19.

For every point, p in $\partial_\infty M$, let us denote also by p the constant curve which sends the unit interval onto p. Let p_0 be a point in Ω. Let a be a curve in Ω joining p_0 to $p_0 \cdot \alpha$. Let b be a curve in Ω joining $p_0 \cdot \alpha$ to $p_0 \cdot (\alpha\beta)$. By definition:

$$\begin{aligned} \hat{\alpha}_\Omega &= [\mathcal{S}(\alpha_-, a, \alpha_+)], \\ \hat{\beta}_\Omega &= [\mathcal{S}(\beta_-, b, \beta_+)], \\ \hat{\gamma}_\Omega^{-1} &= [\mathcal{S}(\gamma_-, (a \cdot b)^{-1}, \gamma_+)]. \end{aligned}$$

Let $\xi_\pm : I \to \partial\Omega$ be such that $\xi_\pm$ avoids $\alpha_\mp$ and that:

$$\xi_\pm(0) = \alpha_\pm, \qquad \xi_\pm(1) = \gamma_\pm.$$

Since $\partial\Omega$ is invariant under the action of α, the curves $\xi_\pm \cdot \alpha$ lie in $\partial\Omega$ and join $\alpha_\pm$ to $\gamma_\pm \cdot \alpha$ whilst avoiding $\alpha_\mp$. Let $(\aleph_t^\pm)_{t \in I}$ be a continuous family of curves in $\partial\Omega$ such that:

(i) for all s:

$$\aleph_0^\pm(s) = \alpha_\pm,$$

and,

(ii) for all t:

$$\begin{aligned} \aleph_t^\pm(0) &= \xi_\pm(t), \\ \aleph_t^\pm(1) &= \xi_\pm(t) \cdot \alpha. \end{aligned}$$

To be more precise, we deform $\aleph^-$ slightly towards the interior of $\partial\Omega$ and $\aleph^+$ slightly towards the exterior of $\partial\Omega$ so that they do not intersect each other. Heuristically, $\aleph_1^\pm$ is a curve which goes from $\gamma_\pm$ to $\gamma_\pm \cdot \alpha$ in the clockwise direction. For all t and for all s, we may suppose that the three points $(\aleph_t^-(0), a(0), \aleph_t^+(0))$ are distinct. Moreover, for all t:

$$(\aleph_t^-(0), a(0), \aleph_t^+(0)) \cdot \alpha = (\aleph_t^-(1), a(1), \aleph_t^+(1)).$$

Consequently, the family $(\aleph_t^-, a, \aleph_t^+)_{t\in I}$ defines a homotopy between the braids (α_-, a, α_+) and $(\aleph_1^-, a, \aleph_1^+)$, respecting the fact that α sends the start point of each braid onto its end point. Since the set of liftings of α in $\widetilde{\mathbb{P}SL}(2,\mathbb{C})$ is discrete, it follows by continuity that:

$$\mathcal{S}(\aleph_1^-, a, \aleph_1^+) = \mathcal{S}(\alpha_-, a, \alpha_+).$$

We define $(\beth_t^\pm)_{t\in I}$ in a similar fashion using β instead of α, and we obtain:

$$\begin{aligned}\hat{\gamma}_T^{-1}\hat{\alpha}_T\hat{\beta}_T &= [\mathcal{S}(\gamma_-, (a\cdot b)^{-1}, \gamma_+)\mathcal{S}(\alpha_-, a, \alpha_+)\mathcal{S}(\beta_-, b, \beta_+)]\\ &= [\mathcal{S}(\gamma_-, (a\cdot b)^{-1}, \gamma_+)\mathcal{S}(\aleph_1^-, a, \aleph_1^+)\mathcal{S}(\beth_1^-, b, \beth_1^+)]\\ &= [\mathcal{S}(\gamma_-\cdot\aleph_1^-\cdot\beth_1^-, (a\cdot b)^{-1}\cdot a\cdot b, \gamma_+\cdot\aleph_1^+\cdot\beth_1^+)].\end{aligned}$$

We thus define the closed tress $\mathcal{T}$ by:

$$\mathcal{T} = (\gamma_-\cdot\aleph_1^-\cdot\beth_1^-, (a\cdot b)^{-1}\cdot a\cdot b, \gamma_+\cdot\aleph_1^+\cdot\beth_1^+).$$

Since $\aleph_1^\pm$ and $\beth_1^\pm$ stay close to $\partial\Omega$ and $(a\cdot b)^{-1}\cdot a\cdot b$ lies in the interior of Ω, there exists a homotopy between $(a\cdot b)^{-1}\cdot a\cdot b$ and the constant curve p_0 which stays away from $\aleph_1^\pm$ and $\beth_1^\pm$. Consequently:

$$\begin{aligned}\mathcal{T} &\sim (\gamma_-\cdot\aleph_1^-\cdot\beth_1^-, p_0, \gamma_+\cdot\aleph_1^+\cdot\beth_1^+)\\ &\sim (\aleph_1^-\cdot\beth_1^-, p_0, \aleph_1^+\cdot\beth_1^+).\end{aligned}$$

Heuristically, $\aleph_1^+\cdot\beth_1^+$ is a curve which turns once about $\partial\Omega$ in the clockwise direction. Moreover, this curve lies to the exterior of $\partial\Omega$. There thus exists a homotopy which stays away from p_0 and $\aleph_1^-\cdot\beth_1^-$ between this curve and the constant curve q_+. Thus:

$$\mathcal{T} \sim (\aleph_1^-\cdot\beth_1^-, p_0, q_+).$$

Finally, heuristically, $\aleph_1^-\cdot\beth_1^-$ is a simple closed curve which separates p_0 from q_+. Consequently:

$$\begin{aligned} & \mathrm{Wind}_r(\mathcal{T}) &= 1\\ \Rightarrow\ & W_r(\hat{\gamma}_\Omega^{-1}\hat{\alpha}_\Omega\hat{\beta}_\Omega) &= 1.\end{aligned}$$

The result now follows. □

We also obtain the converse to this result:

Proposition 5.11

Let $\Gamma = \langle\alpha,\beta\rangle$ be a Schottky group. If $[a]_\alpha \in L_\alpha$, $[b]_\beta \in L_\beta$ and $[c]_\gamma \in L_\gamma$ are such that:

$$(\mathrm{Lift}_\gamma[c]_\gamma)^{-1}(\mathrm{Lift}_\alpha[a]_\alpha)(\mathrm{Lift}_\beta[b]_\beta) \neq \mathrm{Id},$$

then there exists an invariant domain Ω of $\partial_\infty M$ adapted to the generators (α,β) such that:

$$[\Omega]_\alpha = [a]_\alpha, \qquad [\Omega]_\beta = [b]_\beta, \qquad [\Omega]_\gamma = [c]_\gamma.$$

Proof: By lemma A.2, it suffices to prove this result in the case where $\partial_\infty M = \hat{\mathbb{C}}$ and Γ is a Schottky subgroup of $\mathbb{P}SL(2,\mathbb{C})$.

Let $(C^\pm_\alpha, C^\pm_\beta)$ be generating circles of Γ with respect to the generators (α, β). We define the set $X \subseteq \hat{\mathbb{C}}$ by:

$$X = \hat{\mathbb{C}} \setminus \left(\mathrm{Int}(C^+_\alpha) \cup \mathrm{Int}(C^-_\alpha) \cup \mathrm{Int}(C^+_\beta) \cup \mathrm{Int}(C^-_\beta)\right).$$

We now define the curves (a, b, c_1, c_2) to be non-intersecting, simple curves, lying in the interior of X except at their end points, such that:

$$\begin{aligned} a(1) &= a(0)\cdot\alpha, & \qquad b(1) &= b(0)\cdot\beta, \\ c_1(0) &= c_2(1)\cdot\alpha, & \qquad c_2(0) &= c_2(1)\cdot\beta. \end{aligned}$$

We refer to the quadruplet (a, b, c_1, c_2) as **generating curves** for Γ with respect to the generating circles $(C^\pm_\alpha, C^\pm_\beta)$. These curves are geometrically distributed in $\hat{\mathbb{C}}$ is in figure 5.20.

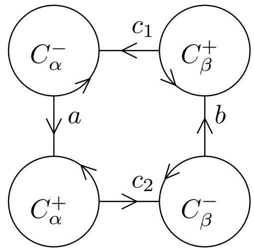

Figure 5.20

By taking the images of these four curves under the actions of elements of γ, and by then adding $\mathrm{Fix}(\gamma)$, we obtain uniquely from these four curves a Jordan curve which is invariant under the action of Γ and which we denote by $\Gamma(a, b, c_1, c_2)$. The interior of $\Gamma(a, b, c_1, c_2)$ is an invariant domain of Γ which is adapted to the generators (α, β). let us denote this domain by $\Omega(a, b, c_1, c_2)$. By definition:

$$[a]_\alpha = [\Omega(a, b, c_1, c_2)]_\alpha, \qquad [b]_\beta = [\Omega(a, b, c_1, c_2)]_\beta.$$

We define the curve c by $c = c_2^{-1}(c_1 \cdot \beta^{-1})^{-1}$, and we observe that:

$$c(1) = c(0)\cdot\gamma.$$

Consequently:

$$[c]_\gamma = [\Omega(a, b, c_1, c_2)]_\gamma.$$

By proposition 5.10:

$$\mathrm{Lift}_\gamma([c]_\gamma)^{-1}\mathrm{Lift}_\beta([b]_\beta)\mathrm{Lift}_\alpha([a]_\alpha) \neq \mathrm{Id}.$$

Let T_1 be the Dehn twist about C_α^+ such that:

$$T_1[a]_\alpha = [a]_\alpha + 1, \qquad T_1[b]_\beta = [b]_\beta, \qquad T_1[c]_\gamma = [c]_\gamma + 1.$$

Let T_2 be the Dehn twist about C_β^+ such that:

$$T_2[a]_\alpha = [a]_\alpha, \qquad T_2[b]_\beta = [b]_\beta + 1, \qquad T_2[c]_\gamma = [c]_\gamma + 1.$$

Finally, let T_3 be a Dehn twist about a curve that separates $C_\alpha^\pm$ from $C_\beta^\pm$. We may choose T_3 such that:

$$T_3[a]_\alpha = [a]_\alpha, \qquad T_3[b]_\beta = [b]_\beta, \qquad T_3[c]_\gamma = [c]_\gamma + 2.$$

Using combinations of these three Dehn twists, for any triple $(x, y, z) \in L_\alpha \times L_\beta \times L_\gamma$ satisfying:

$$([a]_\alpha - x) + ([b]_\beta - y) + ([c]_\gamma - z) = 0 \text{ Mod } 2,$$

we can construct a quadruplet (a', b', c_1', c_2') of generating curves of Γ such that:

$$[\Omega(a', b', c_1', c_2')]_\alpha = x, \qquad [\Omega(a', b', c_1', c_2')]_\beta = y, \qquad [\Omega(a', b', c_1', c_2')]_\gamma = z.$$

By proposition 5.9, the triple (x, y, z) satisfies this condition precisely when:

$$\mathrm{Lift}_\gamma(z)^{-1}\mathrm{Lift}_\alpha(x)\mathrm{Lift}_\beta(y) \neq \mathrm{Id}.$$

The result now follows. □

◇

6 - Constructing The Local Homeomorphism.

6.1 Introduction.

Throughout this section, (M, Q) will be a pointed, compact, three dimensional manifold of strictly negative sectional curvature, $(\tilde{M}, \tilde{Q})$ its universal cover, and $\Gamma = \langle \alpha, \beta \rangle$ a Schottky subgroup of $\pi_1(M, Q)$.

In this section, we will prove the main results of this paper. First we have:

Theorem 1.5

Suppose that (M, Q) is a pointed, compact manifold of strictly negative sectional curvature. Let (Σ, P) be a pointed, compact surface of hyperbolic type (i.e. of genus at least two). Let $\theta : \pi_1(\Sigma, P) \to \pi_1(M, Q)$ be a homomorphism. Suppose that θ is non-elementary and may be lifted to a homomorphism $\hat{\theta}$ of $\pi_1(\Sigma, P)$ into the group $\widetilde{\mathrm{Homeo}}_0(\partial_\infty \tilde{M})$. Then there exists an equivariant Plateau problem for θ.

We then prove:

Theorem 1.6

If θ is non-elementary and lifts, then there exists a convex immersion $i : \Sigma \rightarrow M$ such that:

$$\theta = i_*.$$

In the second part of this section, we define the notion of a $\pi_1(M, Q)$ structure, and prove an existence result for such structures over bound, marked trousers. In the third part of this section, we state results which summarise the glueing technique using by Gallo, Kapovich and Marden in section 8 of [4]. In the fourth section we provide a proof of theorem 1.5. Finally, in the fifth section we show how to obtain convex solutions to the Plateau problem in Hadamard manifolds, and this permits us to prove theorem 1.6.

6.2 Constructing the Solution Over Trousers.

Let (Σ, θ, β) be a bound marked surface. Let $\tilde{\Sigma}$ be the universal cover of Σ. We define a $\pi_1(M, Q)$ **structure** over (Σ, θ, β) to be a local homeomorphism $\varphi : \tilde{\Sigma} \rightarrow \partial_\infty \tilde{M}$ such that:

(i) φ is equivariant under the action of θ, and

(ii) for every boundary component C of $\tilde{\Sigma}$, the restriction of φ to C is a homeomorphism onto its image. In otherwords $\varphi(C)$ is a non self-intersecting curve.

Let P_0 be the base point of Σ, and let $\tilde{P}_0$ be the corresponding base point in the universal cover $\tilde{\Sigma}$ of Σ. Let (C, P) be a pointed boundary component of Σ. Let η be an element of $\pi_1(\Sigma, P_0, P)$, and let $\tilde{\eta}$ be the lift of η such that:

$$\eta(0) = \tilde{P}_0.$$

Let $\hat{P}_\eta$ be the endpoint of $\tilde{\eta}$. $\hat{P}_\eta$ is thus a lift of P. Viewing C as a parametrised simple closed curve in Σ leaving P, we define $\hat{C}_\eta$ to be the lift of C starting from $\hat{P}_\eta$. The curve C_η is thus a segment of one of the boundary components of $\tilde{\Sigma}$. Moreover, by definition:

$$\hat{C}_\eta(0) \cdot (\eta^{-1} C \eta) = \hat{C}_\eta(1).$$

Since $\varphi \circ \hat{C}_\eta$ is non self-intersecting, it must avoid the fixed points of $\theta(\eta^{-1} C \eta)$ in $\partial_\infty \tilde{M}$. Consequently $\varphi \circ \hat{C}_\eta$ projects down to a closed curve in $\mathbb{T}_{\theta(\eta^{-1} C \eta)}$. We will denote this element by $[C]_{\eta, \varphi}$.

Let γ be an element of $\pi_1(M, Q_0)$. Let ξ be another element. The mapping ξ sends $\gamma_\pm$ onto the fixed points $\delta_\pm$ of $\delta = \xi^{-1} \gamma \xi$. It follows that ξ defines a homeomorphism from $\mathbb{T}_\gamma$ to $\mathbb{T}_\delta$, which we will also denote by ξ. Moreover, we observe that, for any $\alpha \in \pi_1(\Sigma)$:

$$\hat{C}_\eta \cdot \theta(\alpha) = \hat{C}_{\eta\alpha}.$$

Consequently:

$$[C]_{\eta,\varphi} \cdot \theta(\alpha) = [C]_{\eta\alpha,\varphi}.$$

We may thus define the element $\beta_\varphi(C)$ lying in $\mathbb{T}_{\beta(C)}$ such that, for any (and thus, equivalently, for all) $\eta \in \pi_1(\Sigma, P_0, P)$:

$$\beta_\varphi(C) = [C]_{\eta,\varphi} \cdot \beta(\eta)^{-1}.$$

We are now in a position to construct $\pi_1(M, Q_0)$ structures over bound, marked trousers:

Proposition 6.1

Let (M, Q_0) be a pointed three dimensional Hadamard manifold of strictly negative sectional curvature. Let (T, θ, β) be a bound marked trouser with holonomy in $\pi_1(M, Q_0)$. Suppose that the image of θ is a Schottky group.

Let P_0 be the base point of T and let $(C_j, Q_j)_{1\leqslant j\leqslant 3}$ be the three oriented boundary components of T. Let $(\xi_j)_{1\leqslant j\leqslant 3}$ be a binding sash of T with respect to the points $(Q_j)_{1\leqslant j\leqslant 3}$ such that:

$$\xi_3^{-1}C_3^{-1}\xi_3 = (\xi_1^{-1}C_1^{-1}\xi_1)(\xi_2^{-1}C_2^{-1}\xi_2).$$

Let $(x_1, x_2, x_3) \in L_{\beta(C_1)} \times L_{\beta(C_2)} \times L_{\beta(C_3)}$ be a triplet such that:

$$\mathrm{Lift}_{\theta(\xi_3^{-1}C_3\xi_3)}(x_3 \cdot \beta(\xi_3))^{-1}\mathrm{Lift}_{\theta(\xi_1^{-1}C_1\xi_1)}(x_1 \cdot \beta(\xi_1))\mathrm{Lift}_{\theta(\xi_2^{-1}C_2\xi_2)}(x_2 \cdot \beta(\xi_2)) \neq \mathrm{Id}.$$

Then, there exists a $\pi_1(M, Q_0)$ structure, φ, over (T, θ, β) such that, for each i:

$$\beta_\varphi(C_i) = x_i.$$

Proof: Let P_T be the base point of T. Let us denote $a = \xi_1^{-1}C_1\xi_1$ and $b = \xi_2^{-1}C_2\xi_2$. Let $\Gamma = \mathrm{Im}(\theta)$. Denote $\alpha = \theta(a)$ and $\beta = \theta(b)$. By lemma 5.1, there exists an invariant domain Ω of Γ adapted to the generators (α, β) such that:

$$[\Omega]_\alpha = x_1 \cdot \beta(\xi_1), \qquad [\Omega]_\beta = x_2 \cdot \beta(\xi_2), \qquad [\Omega]_\gamma = x_3 \cdot \beta(\xi_3).$$

Let P_0 be a point in $\hat{\Omega}$. Let $\pi : \hat{\Omega} \to \hat{\Omega}/\Gamma$ be the canonical projection. Let $\phi : \pi_1(\hat{\Omega}/\Gamma, \pi(P_0)) \to \Gamma$ be the unique isomorphism defined such that, for all $\gamma \in \Gamma$ and for any curve c joining P_0 to $\gamma(P_0)$:

$$\phi([\pi \circ c]) = \gamma.$$

In particular, since Ω is adapted to the generators (α, β), we see that there exist boundary components a' and b' of $\hat{\Omega}/\Gamma$ and a binding sash $(c_{a'}, c_{b'})$ of $(\hat{\Omega}/\Gamma, \pi(P_0))$ with respect to the base points of a' and b' respectively such that:

$$\phi(c_{a'}^{-1}a'c_{a'}) = \alpha, \qquad \phi(c_{b'}^{-1}b'c_{b'}) = \beta.$$

There exists a homeomorphism, $\varphi : (T, P_T) \to (\hat{\Omega}/\Gamma, \pi(P_0))$ such that:

$$\varphi_* a \sim c_{a'}^{-1} a' c_{a'}, \qquad \varphi_* b \sim c_{b'}^{-1} b' c_{b'}.$$

Consequently:

$$\theta = \phi \circ \varphi_*.$$

Let $(\tilde{T}, \tilde{P}_T)$ denote the pointed universal cover of (T, P_T). The mapping φ lifts to a unique pointed homeomorphism, $\tilde{\varphi}$, from $(\tilde{T}, \tilde{P}_T)$ to $(\hat{\Omega}, P_0)$ which is equivariant under the action of θ. The curve $\tilde{\varphi} \circ C_1$ lies entirely in $\overline{\Omega}$. Thus, by lemma 5.7:

$$\begin{array}{rlll} & [C_1]_{\xi_1, \tilde{\varphi}} & = [\Omega]_\alpha & = x_1 \cdot \beta(\xi_1) \\ \Rightarrow & \beta_{\tilde{\varphi}}(C_1) & = [C_1]_{\xi_1, \tilde{\varphi}} \cdot \beta(\xi_1)^{-1} & = x_1. \end{array}$$

Similarly, $\beta_{\tilde{\varphi}}(C_2) = x_2$ and $\beta_{\tilde{\varphi}}(C_3) = x_3$. The mapping $\tilde{\varphi}$ is consequently the desired $\pi_1(M, Q_0)$ structure, and the result now follows. □

6.3 Joining the Trousers.

Let $(\Sigma_i, \theta_i, \beta_i)_{i \in \{1,2\}}$ be two bound marked surfaces with holonomy in $\pi_1(M, Q)$. For each i, let (C_i, Q_i) be a pointed boundary component of Σ_i. Let $\psi : (C_1, Q_1) \to (C_2^{-1}, Q_2)$ be a homeomorphism and suppose that $(\Sigma_1, \theta_1, \beta_1)$ may be joined to $(\Sigma_2, \theta_2, \beta_2)$ along ψ. We recall that this means that $\beta_1(C_1)^{-1} = \beta_2(C_2)$ and consequently that:

$$\mathbb{T}_{\beta_1(C_1)^{-1}} = \mathbb{T}_{\beta_2(C_2)}.$$

We observe that, for any hyperbolic element γ of Γ, we may identify $\mathbb{T}_\gamma$ and $\mathbb{T}_{\gamma^{-1}}$, and:

$$L_{\gamma^{-1}} = -L_\gamma.$$

For each i, let φ_i be a $\pi_1(M, Q)$ structure over $(\Sigma_i, \theta_i, \beta_i)$. We say that φ_1 may be **joined** to φ_2 along ψ if and only if:

$$-\beta_{1,\varphi_1}(C_1) = \beta_{2,\varphi_2}(C_2).$$

The glueing procedure described by Gallo, Kapovich and Marden in section 8 of [4] yields the following result:

Proposition 6.2

Let (M, Q) be a compact, pointed, three dimensional manifold of strictly negative sectional curvature.

Let $(\Sigma_i, \theta_i, \beta_i)_{i \in \{1,2\}}$ be two bound marked surfaces with holonomy in $\pi_1(M, Q)$. For each i, let (C_i, Q_i) be pointed boundary components of Σ_i and let $\psi : (C_1, Q_1) \to (C_2^{-1}, Q_2)$ be a homeomorphism such that $(\Sigma_1, \theta_1, \beta_1)$ may be joined to $(\Sigma_2, \theta_2, \beta_2)$ along ψ.

Suppose that there exists, for each i, a $\pi_1(M, Q)$ structure φ_i over $(\Sigma_i, \theta_i, \beta_i)$. If moreover, φ_1 may be joined to φ_2 along ψ, then there exists a $\pi_1(M, Q)$ structure φ over $(\Sigma_1, \theta_1, \beta_1) \cup_\varphi$

$(\Sigma_2, \theta_2, \beta_2)$ *such that, after identifying objects in each of* Σ_1 *and* Σ_2 *with the corresponding objects in* $\Sigma_1 \cup_\varphi \Sigma_2$*:*

(i) for every pointed boundary component (C', Q') *of* Σ_1*:*

$$(\beta_1 \cup \beta_2)_\varphi(C') = \beta_{1,\varphi_1}(C').$$

and,

(ii) for every pointed boundary component (C', Q') *of* Σ_2*:*

$$(\beta_1 \cup \beta_2)_\varphi(C') = \beta_{2,\varphi_2}(C').$$

Let (Σ, θ, β) be a bound marked surface with holonomy in $\pi_1(M, Q)$. Let $(C_i, Q_i)_{i\in\{1,2\}}$ be pointed boundary components of Σ. Let $\psi : (C_1, Q_1) \to (C_2^{-1}, Q_2)$ be a homeomorphism and suppose that (Σ, θ, β) may be joined to itself along ψ. We recall that this means that $\beta(C_1)^{-1} = \beta(C_2)$ and consequently that:

$$\mathbb{T}_{\beta(C_1)^{-1}} = \mathbb{T}_{\beta(C_2)}.$$

Let φ be a $\pi_1(M, Q)$ structure over (Σ, θ, β). We say that φ may be **joined to itself** along ψ if and only if, for any (and thus, equivalently, for all) $\xi \in \pi_1(\Sigma, P, Q_1)$ and $\eta \in \pi_1(\Sigma, P, Q_2)$:

$$-\beta_\varphi(C_1) = \beta_\varphi(C_2).$$

Once again, the glueing procedure described by Gallo, Kapovich and Marden in section 8 of [4] permits us to obtain the following analogue of lemma 6.2:

Proposition 6.3

Let (M, Q) *be a compact, pointed, three dimensional manifold of strictly negative sectional curvature.*

Let (Σ, θ, β) *be a bound, marked surface with holonomy in* $\pi_1(M, Q)$*. Let* $(C_i, Q_i)_{i\in\{1,2\}}$ *be pointed boundary components of* Σ*. Let* $\psi : (C_1, Q_1) \to (C_2^{-1}, Q_2)$ *be a homeomorphism such that* (Σ, θ, β) *may be joined to itself along* ψ*.*

Suppose that there exists, for each i*, a* $\pi_1(M, Q)$ *structure,* φ*, over* (Σ, θ, β)*. If, moreover,* φ *may be joined to itself along* ψ*, then there exists a* $\pi_1(M, Q)$ *structure* φ' *over* $\bigcirc_\psi(\Sigma, \theta, \beta)$ *such that, after identifying objects in* Σ *with the corresponding objects in* $\bigcirc_\varphi \Sigma$*, for every pointed boundary component* (C', Q') *of* Σ*:*

$$(\bigcirc \beta)_{\varphi'}(C') = (\bigcirc \beta)_\varphi(C).$$

6.4 The Construction of a Local Homeomorphism.

We are now in a position to prove theorem 1.5:

Proof of theorem 1.5: By lemma 4.1, there exist $2g-2$ bound, marked trousers $(T_i, \theta_i, \beta_i)_{1\leqslant i\leqslant 2g-2}$ with holonomy in $\pi_1(M,Q)$ such that the image of every θ_i is a Schottky group, and:

$$(\Sigma,\theta) \cong \bigcup_{i=1}^{g}(\bigcirc(T_i,\theta_i,\beta_i))\cup(\bigcup_{i=g+1}^{2g-2}(T_i,\theta_i,\beta_i)).$$

Let $\pi : \widetilde{\mathrm{Homeo}}_0(\partial_\infty\tilde{M}) \to \mathrm{Homeo}_0(\partial_\infty\tilde{M})$ be the canonical projection. For every i, using the lifting, $\hat{\theta}$, of θ, we may construct liftings $\hat{\theta}_i$ and $\hat{\beta}_i$ of θ_i and β_i such that:

(i) for all i, $\pi\circ\hat{\theta}_i = \theta_i$ and $\pi\circ\hat{\beta}_i = \beta_i$, and

(ii) the $(T_i, \hat{\theta}_i, \hat{\beta}_i)_{1\leqslant i\leqslant 2g-2}$ may be joined along the same edges as the $(T_i,\theta_i,\beta_1)_{1\leqslant i\leqslant 2g-2}$.

Choose $1\leqslant i\leqslant 2g-2$. Let (C,Q) be a boundary component of T_i. We define the element E_C such that:

(i) $E_C \in L_{\beta(C)}$, and

(ii) $\mathrm{Lift}_{\beta(C)}(E_C) \neq \hat{\beta}(C)$.

We may choose the (E_C) such that, if T_i and T_j are distinct trousers and if (C,Q) and (C',Q') are oriented, pointed boundary components of T_i and T_j respectively along which these two trousers are joined, then:

$$E_C = -E_{C'}.$$

Likewise, we may suppose that if T_i is a trouser and if (C,Q) and (C',Q') are distinct oriented, pointed boundary components of the same trouser T_i along which this trouser is joined to itself, then:

$$E_C = -E_{C'}.$$

Choose $1\leqslant i\leqslant 2g-2$. Let $\mathcal{P}_i$ be the base point of T_i and let $(C_j,Q_j)_{1\leqslant j\leqslant 3}$ be the three oriented, pointed boudary components of T_i. Let $(\xi_j)_{1\leqslant j\leqslant 3}$ be a binding sash of T_i with respect to the points $(Q_j)_{1\leqslant j\leqslant 3}$ such that:

$$\xi_3^{-1}C_3\xi_3 = (\xi_1^{-1}C_1\xi_1)(\xi_2^{-1}C_2\xi_2).$$

For each j, let us denote $\gamma_j = \xi_j^{-1}C_j\xi_j$ and $a_j = E_{C_j}\cdot\beta(\xi_j)$. Let $\Delta\in\widetilde{\mathrm{Homeo}}_0(\partial_\infty\tilde{M})$ be the element of $\pi^{-1}(\mathrm{Id})$ that is different to the identity. We recall that Δ commutes with every element of $\widetilde{\mathrm{Homeo}}(\partial_\infty\tilde{M})$. For each j, we have:

$$\begin{aligned}\mathrm{Lift}_{\theta(\gamma_j)}(a_j) &= \beta(\xi_j)^{-1}[\mathrm{Lift}_{\beta(\xi_j)\theta(\gamma_j)\beta(\xi_j)^{-1}}(E_{C_j})]\beta(\xi_j)\\ &= \hat{\beta}(\xi_j)^{-1}\mathrm{Lift}_{\beta(C_j)}(E_{C_j})\hat{\beta}(\xi_j)\\ &= \hat{\beta}(\xi_j)^{-1}\hat{\beta}(C_j)\Delta\hat{\beta}(\xi_j)\\ &= \hat{\theta}(\gamma_j)\Delta.\end{aligned}$$

Thus:

$$\begin{aligned}\mathrm{Lift}_{\theta(\gamma_3)}(a_3)^{-1}\mathrm{Lift}_{\theta(\gamma_1)}(a_1)\mathrm{Lift}_{\theta(\gamma_2)}(a_2) &= \hat{\theta}(\gamma_3)^{-1}\hat{\theta}(\gamma_1)\hat{\theta}(\gamma_2)\Delta^3\\ &= \hat{\theta}(\gamma_3^{-1}\gamma_1\gamma_2)\Delta^3\\ &= \Delta^3\\ &= \Delta.\end{aligned}$$

Thus, by proposition 6.1, there exists a $\pi_1(M,Q)$ structure φ_i over (T_i,θ_i,β_i) such that, for each j:

$$\beta_{i,\varphi_i}(C_j) = E_{C_j}.$$

These $\pi_1(M,Q)$ structures may all be joined to each other, and the existence of a $\pi_1(M,Q)$ structure over (Σ,θ) now follows using a process of induction and propositions 6.2 and 6.3. The result now follows. $\square$

6.5 Convex Subsets of Hadamard Manifolds.

We now require the following results concerning convex sets:

Proposition 6.4

Let p be a point in $\partial_\infty\tilde{M}$ and let Ω be a neighbourhood of p in $\partial_\infty\tilde{M}$. There exists a complete convex subset X of $\tilde{M}$ such that:

$$p \notin \partial_\infty X, \qquad \Omega^c \subseteq \partial_\infty X.$$

Proof: Let $K \geqslant k > 0$ be such that the sectional curvature of $\tilde{M}$ is contained in $[-K,-k]$. For all q in $\tilde{M}$, and for any two points r and r' in $\partial_\infty\tilde{M}$, let $\widehat{rqr'}$ be the angle at q between the two geodesics joining q to r and q to r'. We define the angle metric at q, θ_q, such that, for all $r,r' \in \partial_\infty\tilde{M}$:

$$\theta_q(r,r') = \widehat{rqr'}.$$

For q a point in $\tilde{M}$, and for r a point in $\partial_\infty\tilde{M}$, and for $\theta \in [0,2\pi]$ an angle, we define the neighbourhood $\Omega_{q,r,\theta}$ of r in $\partial_\infty\tilde{M}$ by:

$$r' \in \Omega_{q,r,\theta} \Leftrightarrow \theta_q(r,r') < \theta.$$

By theorem 3.1 of [1], there exists $\pi/2 > \varphi > \psi > 0$ such that, for all $q \in \tilde{M}$ and for all $r \in \partial_\infty\tilde{M}$, there exists a complete convex subset X of $\tilde{M}$ such that:

$$\Omega_{q,r,\psi} \subseteq \partial_\infty X \subseteq \Omega_{q,r,\varphi}.$$

Let q_0 be any point in $\tilde{M}$. Let γ be the unique geodesic running from q_0 to p, normalised such that $\gamma(0) = q_0$. Let $\delta \in \mathbb{R}^+$ be such that:

$$\Omega_{q_0,p,\delta} \subseteq \Omega.$$

Let us define $B \in \mathbb{R}^+$ such that:

$$B > \Gamma_k^{-1}(\psi) + \Gamma_k^{-1}(\delta).$$

Let $q_1 = \gamma(B)$. Let X be a complete convex subset of $\tilde{M}$ such that:

$$\Omega_{q_1,\gamma(-\infty),\psi} \subseteq \partial_\infty X.$$

By lemma 3.10:

$$\Omega^c_{q_0,\gamma(+\infty),\delta} \subseteq \partial_\infty X.$$

The result now follows. □

This result now permits us to show:

Lemma 6.5

Let D be a closed subset of $\partial_\infty \tilde{M}$. There exists a complete strictly convex subset X with C^1 boundary in $\tilde{M}$ such that:

$$D = \partial_\infty X.$$

Proof: For all $p \in D^c$, by proposition 6.4, there exists a complete convex subset X_p of $\partial_\infty \tilde{M}$ such that $p \notin \partial_\infty X_p$ and $D \subseteq \partial_\infty X_p$. We define $\hat{X}$ by:

$$\hat{X} = \bigcap_{p \in C^c} X_p.$$

$\hat{X}$ is a complete convex subset of $\tilde{M}$ and $\partial_\infty \hat{X} = D$. In particular, $\hat{X}$ is non-empty. For $\epsilon \in \mathbb{R}^+$, let Σ_ϵ be the surface obtained by moving $\partial \hat{X}$ normally along geodesics. Σ_ϵ is strictly convex, is of type C^1 and bounds a complete convex subset X of $\tilde{M}$. X is thus the desired subset, and the result now follows. □

We are now in a position to prove theorem 1.6:

Proof of theorem 1.6: Since Σ is compact, and since φ is θ-equivariant, there exists a finite open cover $\mathcal{A} = (\Omega_i)_{1 \leqslant i \leqslant k}$ of Σ such that, if $\tilde{\mathcal{A}} = (\Omega_{i,\gamma})_{1 \leqslant i \leqslant k, \gamma \in \pi_1(\Sigma,P)}$ is the lifting of $\mathcal{A}$ to $\tilde{\Sigma}$, then, for every $\Omega \in \tilde{\mathcal{A}}$:

(i) the closure of Ω in $\tilde{\Sigma}$ is homeomorphic to a closed disc, and

(ii) for every $\Omega \in \tilde{\mathcal{A}}$, the restriction of φ to $\overline{\Omega}$ is a homeomorphism onto its image.

By lemma 6.5, for every i, we may find a family $(X_{i,\gamma})_{\gamma \in \pi_1(\Sigma,P)}$ of complete strictly convex subsets with C^1 boundary in $\tilde{M}$ such that:

(i) $(X_{i,\gamma})_{\gamma \in \pi_1(\Sigma,P)}$ is θ-equivariant. In other words, for all $\gamma, \delta \in \pi_1(\Sigma, P)$:

$$X_{i,\gamma\delta} = X_{i,\gamma} \cdot \theta(\delta),$$

and

(ii) for all $\gamma \in \pi_1(\Sigma, P)$:

$$\partial_\infty X_{1,\gamma} = \varphi(\Omega_{i,\gamma})^c.$$

We may assume moreover that the boundaries of these sets intersect transversally in the region that will be of interest to us. Using these convex sets, we construct a family

$(P_{i,\gamma})_{1\leqslant i\leqslant k,\gamma\in\pi_1(\Sigma,P)}$ of polygonal subsets of $\tilde{\Sigma}$ and a family $(\psi_{i,\gamma})_{1\leqslant i\leqslant k,\gamma\in\pi_1(\Sigma,P)}$ of immersions such that:

(i) $(P_{i,\gamma})_{1\leqslant i\leqslant k,\gamma\in\pi_1(\Sigma,P)}$ provides a polygonal decomposition of $\tilde{\Sigma}$. In otherwords:

$$\tilde{\Sigma} = \mathop{\cup}_{\substack{1\leqslant i\leqslant k\\ \gamma\in\pi_(\Sigma,P)}} P_{i,\gamma},$$

and two polygons in this family only intersect if they share a common boundary component;

(ii) the family $(P_{i,\gamma})_{1\leqslant i\leqslant k,\gamma\in\pi_1(\Sigma,P)}$ is equivariant. In otherwords, for each i, and for all $\gamma,\delta\in\pi_1(\Sigma,P)$:

$$P_{i,\gamma\delta} = P_{i,\gamma}\cdot\delta;$$

(iii) for each i and for all γ, $\psi_{i,\gamma}$ is a homeomorphism from $P_{i,\gamma}$ onto a subset of the boundary of $X_{i,\gamma}$ in $\tilde{M}$, and

(iv) the family $(\psi_{i,\gamma})_{1\leqslant i\leqslant k,\gamma\in\pi_1(\Sigma,P)}$ is θ-equivariant. In otherwords, for each i, and for all $\gamma,\delta\in\pi_1(\Sigma,P)$, and for all $p\in P_{i,\gamma}$:

$$\psi_{i,\gamma\delta}(p\cdot\delta) = \psi_{i,\gamma}(p)\cdot\theta(\delta).$$

Joining together the elements of $(\psi_{i,\gamma})_{1\leqslant i\leqslant k,\gamma\in\pi_1(\Sigma,P)}$, we obtain a θ-equivariant locally strictly convex immersion $\hat{\psi}$ of $\tilde{\Sigma}$ into $\tilde{M}$ such that if $\overrightarrow{n}$ is the Gauss-Minkowski mapping sending $U\tilde{M}$ into $\partial_\infty\tilde{M}$, then:

$$\overrightarrow{n}\circ\hat{\psi} = \varphi.$$

Taking quotients, we obtain a locally strictly convex immersion ψ of Σ into M which realises θ. Finally, since Σ is compact, by deforming ψ slightly, we may suppose that it is also smooth and the result now follows. □

◇

A - Homeomorphism Equivalence of Schottky Groups.

A.1 Introduction.

Throughout this appendix, (M,Q) will be a pointed three dimensional Hadamard manifold and $\Gamma=\langle\alpha,\beta\rangle$ a Schottky subgroup of $\mathrm{Isom}(M,Q)$.

In this appendix, we provide a proof of the equivalence up to homeomorphisms of the Schottky groups that we will be using.

We define $\mathrm{Fix}(\Gamma)$, the **fixed point set** of Γ by:

$$\mathrm{Fix}(\Gamma) = \overline{\mathop{\cup}_{\gamma\in\Gamma\setminus\{\mathrm{Id}\}}\mathrm{Fix}(\gamma)}.$$

We define a **reduced word** over (α,β) to be a sequence $\boldsymbol{\gamma}=(\gamma_k)_{1\leqslant k\leqslant n}$ of elements of $\left\{\alpha^{\pm 1},\beta^{\pm 1}\right\}$ which does not contain any of the subwords $\alpha\alpha^{-1}$, $\alpha^{-1}\alpha$, $\beta\beta^{-1}$ or $\beta^{-1}\beta$. In otherwords, a reduced word is the shortest length word expressing the corresponding element of Γ. Let $W_{\alpha,\beta}$ denote the set of all reduced words over (α,β) of finite length. Let $W^n_{\alpha,\beta}$ denote the subset of $W_{\alpha,\beta}$ consisting of words of length n. Let $W^\infty_{\alpha,\beta}$ denote the set of all reduced words over (α,β) of infinite length. For all $m\geqslant n$, we define the truncation map $T_n:W^m_{\alpha,\beta}\rightarrow W^n_{\alpha,\beta}$ to be the map which sends a word of length m to the word given by its first n letters. This definition is trivially also valid when $m=\infty$. We give $W_{\alpha,\beta}$ the discrete topology, and we then give $W^\infty_{\alpha,\beta}$ the coarsest topology with respect to which every T_n is continuous. In otherwords, a basis of open sets of $W^\infty_{\alpha,\beta}$ is given by:

$$\mathcal{B}=\underset{n\in\mathbb{N}}{\cup}\left\{T_n^{-1}(\{\boldsymbol{\gamma}\})\text{ s.t. }\boldsymbol{\gamma}\text{ is of length }n\right\}.$$

This topology is trivially Hausdorff. Moreover, since the alphabet is finite, it is not difficult to show that $W^\infty_{\alpha,\beta}$ is compact with respect to this topology.

The first result of this appendix is the construction of a canonical homeomorphism from $W^\infty_{\alpha,\beta}$ to $\mathrm{Fix}(\Gamma)$. In order to explicitely describe the homeomorphism, we are required to construct a few more objects. Let $(D^\pm_\alpha,D^\pm_\beta)$ be generating disks for Γ in $\tilde{M}\cup\partial_\infty\tilde{M}$ with respect to the generators (α,β). Let $(C^\pm_\alpha,C^\pm_\beta)$ be the corresponding generating circles in $\partial_\infty\tilde{M}$. In otherwords:

$$C^\pm_\alpha=\partial_\infty D^\pm_\alpha,\qquad C^\pm_\beta=\partial_\infty D^\pm_\beta.$$

We define the set S_1 by $S_1=\left\{C^\pm_\alpha,C^\pm_\beta\right\}$ and we define S_n for $n\geqslant 2$ inductively by:

$$S_n=\left(\underset{\gamma\in\{\alpha^{\pm1},\beta^{\pm1}\}}{\cup}\gamma S_{n-1}\right)\setminus\overset{n-1}{\underset{k=0}{\cup}}S_{n-1}.$$

For all $n\geqslant 2$, and for all $C\in S_n$, there exists a unique $C'\in S_1$ such that $C\subseteq\mathrm{Int}(C')$. We thus orient C such that $\mathrm{Int}(C)\subseteq\mathrm{Int}(C')$. We observe that, for all n, and for any distinct $C,C'\in S_n$, the interiors of C and C' are also distinct.

We define the mapping $D_1:W^1_{\alpha,\beta}\rightarrow S_1$ by:

$$D_1(\alpha^{\pm1})=C^\pm_\alpha,\qquad D_1(\beta^{\pm1})=C^\pm_\beta.$$

We define D_n for $n\geqslant 2$ such that, for all $\boldsymbol{\gamma}=(\gamma_k)_{1\leqslant k\leqslant n}$ in $W^n_{\alpha,\beta}$:

$$D_n(\boldsymbol{\gamma})=D_1(\gamma_n)\cdot\gamma_{n-1}\cdot\ldots\cdot\gamma_1.$$

We then define D over $W_{\alpha,\beta}$ such that it restricts to D_n over each $W^n_{\alpha,\beta}$. We observe that, for all n, D_n defines a bijection between $W^n_{\alpha,\beta}$ and S_n. We now obtain the following result:

Lemma A.1

Suppose that the sectional curvature of M is bounded above by $-k<0$. For all $\boldsymbol{\gamma}=(\gamma_n)_{n\in\mathbb{N}}$ in $W^\infty_{\alpha,\beta}$, there exists a unique point $\mathcal{P}(\boldsymbol{\gamma})\in\mathrm{Fix}(\Gamma)$ such that the sequence $(\mathrm{Int}(D(T_n(\boldsymbol{\gamma}))))_{n\in\mathbb{N}}$ converges towards $\{\mathcal{P}(\boldsymbol{\gamma})\}$ in the Hausdorff topology. Moreover, $\mathcal{P}(\boldsymbol{\gamma})$ is contained in $\mathrm{Int}(D(T_n(\boldsymbol{\gamma})))$ for every n, and $\mathcal{P}$ defines a homeomorphism between $W^\infty_{\alpha,\beta}$ and $\mathrm{Fix}(\Gamma)$.

Extending this result to a homeomorphism between $\partial_\infty M$ and an abstract space over which Γ acts in a trivial manner, we obtain the following result which tells us that all the Schottky groups that we will be studying are essentially equivalent to Schottky subgroups of $\mathbb{P}SL(2,\mathbb{C})$:

Lemma A.2

Suppose that the sectional curvature of M is bounded above by $-k < 0$. Let $\Gamma \subseteq \mathrm{Isom}_0(M)$ and $\Gamma' \subseteq \mathrm{Isom}_0(M')$ be Schottky subgroups. There exists an isomorphism $\phi : \Gamma \to \Gamma'$ and a homeomorphism $\Phi : \partial_\infty M \to \partial_\infty M'$ such that, for all $\gamma \in \Gamma$:

$$\Phi \circ \gamma = \varphi(\gamma) \circ \Phi.$$

In the second part of this appendix, we prove lemma A.1, and then in the third part we show how this result may be used to prove lemma A.2.

A.2 The Fixed Point Set of a Schottky Group.

In this subsection, we prove lemma A.1. We begin with the following more elementary result concerning Hadamard manifolds of strictly negative sectional curvature:

Proposition A.3

Let M be a Hadamard manifold of sectional curvature bounded above by $-k < 0$. Let UM be the unitary bundle over M and let $\pi : UM \to M$ be the canonical projection. Let $(v_n)_{n\in\mathbb{N}}$ be a sequence of vectors in UM such that $(\pi \circ v_n)_{n\in\mathbb{N}}$ converges to a point p_0 in $\partial_\infty M$. For all n, let γ_n be the geodesic in M leaving $\pi \circ v_n$ with velocity v_n. After extraction of a subsequence, at least one of $(\gamma_n(-\infty))_{n\in\mathbb{N}}$ and $(\gamma_n(+\infty))_{n\in\mathbb{N}}$ converges also to p_0.

Proof: Let q_0 be a point in M. For all n, let us denote $p_n = \pi \circ v_n$ and $D_n = d(q_0, p_n)$. Since $(p_n)_{n\in\mathbb{N}}$ converges to a point in $\partial_\infty M$, the sequence $(D_n)_{n\in\mathbb{N}}$ tends to infinity. For all n, let η_n be the unique geodesic such that:

$$\eta_n(0) = q_0, \eta_n(D_n) = p_n.$$

For all n, let Σ_n be the normal circle bundle over η_n, and let $\Sigma_{n,t}$ be the fibre over $\eta_n(t)$. Let $\overrightarrow{n} : UM \to \partial_\infty M$ be the Gauss-Minkowski mapping and, for all n, let $\Omega_{n,t}$ be the connected component of $\partial_\infty M \setminus \overrightarrow{n}(\Sigma_{n,t})$ containing $\eta_n(+\infty)$. Without loss of generality, we may suppose that, for all n, the angle that v_n makes with the positive direction of η_n at p_n is not greater than $\pi/2$. Thus, by considering the restriction of $\overrightarrow{n}$ to the fibre of UM over p_n, we obtain, for all n:

$$\gamma_n(+\infty) \in \overline{\Omega_{n,D_n}}.$$

By lemma 3.10, for all n, the angle that $\gamma_n(+\infty)$ makes with $\eta_n(+\infty)$ at q_0 is bounded above by $\Gamma_k(D_n)$. Since $(D_n)_{n\in\mathbb{N}}$ tends to infinity, $(\Gamma_k(D_n))_{n\in\mathbb{N}}$ tends to zero. Finally, since $(p_n)_{n\in\mathbb{N}}$ tends to p_0, the angle that $(\eta_n(+\infty))_{n\in\mathbb{N}}$ makes with p_0 at q_0 also tends to zero. Thus, by the triangle inequality for angles in the sphere, the angle that $(\gamma_n(+\infty))_{n\in\mathbb{N}}$ makes with p_0 at q_0 also tends to zero and the result now follows. $\square$

We now define the subset X of M by:

$$X = M \setminus \underset{\substack{\gamma\in\{\alpha,\beta\}\\ \mathrm{sgn}\in\{\pm\}}}{\cup} \mathrm{Int}(D_\gamma^{\mathrm{sgn}}).$$

Trivially:

$$\partial_\infty X = \partial_\infty M \setminus \underset{\substack{\gamma\in\{\alpha,\beta\}\\ \mathrm{sgn}\in\{\pm\}}}{\cup} \mathrm{Int}(C_\gamma^{\mathrm{sgn}}).$$

For all $\gamma \in \Gamma \setminus \{\mathrm{Id}\}$, let γ^- and γ^+ be the repulsive and attractive fixed points of γ respectively, and let g_γ be the unique geodesic joining γ^- to γ^+. We define G_Γ by:

$$G_\Gamma = \underset{\gamma\in\Gamma}{\cup} g_\gamma(\mathbb{R}).$$

We obtain the following result:

Proposition A.4

The intersection $X \cap G_\Gamma$ is bounded in M.

Proof: Suppose the contrary. There exists a sequence $(p_n)_{n\in\mathbb{N}} \in X \cap G_\Gamma$ and $p_0 \in \partial_\infty M$ such that $(p_n)_{n\in\mathbb{N}}$ converges to p_0. For all n, let $\gamma_n \in \Gamma$ be such that that p_n lies in g_{γ_n}. For all n, let $\gamma_{n,\pm}$ be the fixed points of γ_n. By lemma A.3, we may assume that $(\gamma_{n,+})_{n\in\mathbb{N}}$ tends to p_0, in which case $p_0 \in \mathrm{Fix}(\Gamma)$. However, $p_0 \in \partial_\infty X$, which is absurd, since $\mathrm{Fix}(\Gamma)$ and $\partial_\infty X$ are disjoint. The result now follows. $\square$

We define the evaluation map $\mathrm{Eval} : W_{\alpha,\beta} \to \Gamma$ such that, for all $\boldsymbol{\gamma} = (\gamma_k)_{1\leqslant k\leqslant n}$:

$$\mathrm{Eval}(\boldsymbol{\gamma}) = \gamma_n \cdot \ldots \cdot \gamma_1.$$

We have the following elementary result:

Lemma A.5

For every $\boldsymbol{\gamma} = (\gamma_k)_{1\leqslant k\leqslant n}$ in $W_{\alpha,\beta}$, we have:

$$\mathrm{Ext}(D_1(\gamma_n^{-1})) \cdot \mathrm{Eval}(\boldsymbol{\gamma}) \subseteq \mathrm{Int}(D_1(\gamma_1)).$$

Proof: We prove this result by induction on the length of $\boldsymbol{\gamma}$. The result follows immediately from the definition of generating circles when $\boldsymbol{\gamma}$ is of length 1. We suppose now that the result is true when $\boldsymbol{\gamma}$ is of length n. Let $\boldsymbol{\gamma} = (\gamma_k)_{1\leqslant k\leqslant n}$ be a reduced word of length $n+1$ over (α,β). By the induction hypothesis, we have:

$$\mathrm{Ext}(D_1(\gamma_{n+1}^{-1})) \cdot \mathrm{Eval}(\boldsymbol{\gamma}) \subseteq \mathrm{Int}(D_1(\gamma_2)) \cdot \gamma_1.$$

Since $\boldsymbol{\gamma}$ is a reduced word, $\gamma_2 \neq \gamma_1^{-1}$. Consequently:

$$\begin{array}{rll} & \mathrm{Int}(D_1(\gamma_2)) & \subseteq \mathrm{Ext}(D_1(\gamma_1^{-1})) \\ \Rightarrow & \mathrm{Int}(D_1(\gamma_2)) \cdot \gamma_1 & \subseteq \mathrm{Int}(D_1(\gamma_1)). \end{array}$$

The result now follows. □

This yields the following two corollaries concerning $D(\boldsymbol{\gamma})$ for $\boldsymbol{\gamma}$ in $W_{\alpha,\beta}$. First, we have:

Corollary A.6

For every $\boldsymbol{\gamma} = (\gamma_k)_{1\leqslant k\leqslant n}$ in $W_{\alpha,\beta}$:

$$\mathrm{Int}(D(\boldsymbol{\gamma})) = \mathrm{Ext}(D_1(\gamma_n^{-1})) \cdot \mathrm{Eval}(\boldsymbol{\gamma}).$$

Proof: This follows from lemma A.5 and the fact that $D(\boldsymbol{\gamma})$ is oriented such that its interior is contained in the interior of the unique circle in S_1 in which it lies. □

Next we have:

Corollary A.7

For every $\boldsymbol{\gamma} = (\gamma_k)_{k\in\mathbb{N}}$ in $W^\infty_{\alpha,\beta}$ and for all $n \in \mathbb{N}$:

$$\mathrm{Int}(D(T_{n+1}(\boldsymbol{\gamma}))) \subseteq \mathrm{Int}(D(T_n(\boldsymbol{\gamma}))).$$

Proof: Indeed:

$$\begin{array}{rl} \mathrm{Int}(D(T_{n+1}(\boldsymbol{\gamma}))) & = \mathrm{Ext}(D_1(\gamma_{n+1}^{-1})) \cdot \mathrm{Eval}(T_{n+1}(\boldsymbol{\gamma})) \\ & = \mathrm{Ext}(D_1(\gamma_{n+1}^{-1})) \cdot \gamma_{n+1} \cdot \mathrm{Eval}(T_n(\boldsymbol{\gamma})) \\ & = \mathrm{Int}(D_1(\gamma_{n+1})) \cdot \mathrm{Eval}(T_n(\boldsymbol{\gamma})). \end{array}$$

Since $\boldsymbol{\gamma}$ is a reduced word, $\gamma_{n+1} \neq \gamma_n^{-1}$, and so:

$$\begin{array}{rll} & \mathrm{Int}(D_1(\gamma_{n+1})) & \subseteq \mathrm{Ext}(D_1(\gamma_n)) \\ \Rightarrow & \mathrm{Int}(D_1(\gamma_{n+1})) \cdot \mathrm{Eval}(T_n(\boldsymbol{\gamma})) & \subseteq \mathrm{Ext}(D_1(\gamma_n)) \cdot \mathrm{Eval}(T_n(\boldsymbol{\gamma})) \\ \Rightarrow & \mathrm{Int}(D(T_{n+1}(\boldsymbol{\gamma}))) & \subseteq \mathrm{Int}(D(T_n(\boldsymbol{\gamma}))). \end{array}$$

The result follows. □

We define $W^0_{\alpha,\beta}$ to be the set of all reduced words $\boldsymbol{\gamma} = (\gamma_k)_{1\leqslant k\leqslant n}$ in $W_{\alpha,\beta}$ such that $\gamma_n \neq \gamma_1^{-1}$. We have the following result:

Propositon A.8

$\|\mathrm{Eval}(\boldsymbol{\gamma})\|$ tends to infinity as the length of $\boldsymbol{\gamma}$ tends to infinity in $W^0_{\alpha,\beta}$.

Proof: Suppose the contrary. Without loss of generality, there exists $K > 0$ and infinitely many distinct elements $(\gamma_n)_{n\in\mathbb{N}} = (\mathrm{Eval}(\boldsymbol{\gamma}_n))_{n\in\mathbb{N}}$ in Γ such that, for all n:

$$\|\gamma_n\| \leqslant K.$$

Since, for all n, $\gamma_n \in W^0_{\alpha,\beta}$, by lemma A.5, we may suppose that there exist two distinct circles C_1 and C_2 in S_1 such that, for all n:

$$\mathrm{Ext}(C_1)\cdot\gamma_n \subseteq \mathrm{Int}(C_2).$$

For every n, let $p_{n,-}$ and $p_{n,+}$ be the repulsive and attractive fixed points respectively of γ_n. We have:

$$p_{n,-} \subseteq \mathrm{Int}(C_1), \qquad p_{n,+} \subseteq \mathrm{Int}(C_2).$$

Consequently, if η_n is the geodesic fixed by γ_n, then η_n intersects X non-trivially. Thus, for all n, there exists $p_n \in X \cap G_\Gamma$ such that:

$$d(p_n, p_n\cdot\gamma_n) \leqslant K.$$

Let p be an arbitrary point in M. By compactness, there exists $B > 0$ such that, for all $q \in X \cap G_\Gamma$:

$$d(p,q) \leqslant B.$$

Thus, for all n:

$$\begin{aligned} d(p, p\cdot\gamma_n)) &\leqslant d(p,p_n) + d(p_n, p_n\cdot\gamma_n) + d(p_n\cdot\gamma_n), p\cdot\gamma_n) \\ &= 2d(p,p_n) + d(p_n,p_n\cdot\gamma_n) \\ &\leqslant 2B + K. \end{aligned}$$

Consequently, by compactness, the set $\{p\}\cdot\Gamma$ has a concentration point in $\tilde{M}$, which is absurd, since $\Gamma \subseteq \pi_1(M,Q)$ acts properly discontinuously on $\tilde{M}$. The result now follows. $\square$

For $\boldsymbol{\gamma}$ an element of $W^\infty_{\alpha,\beta}$, we define the sequence $(l_n(\boldsymbol{\gamma}))_{n\in\mathbb{N}}$ such that, for all n:

$$l_n(\boldsymbol{\gamma}) = \mathrm{Sup}\left\{k \text{ s.t. } 1 \leqslant k \leqslant n \ \&\ \gamma_k \neq \gamma_1^{-1}\right\}.$$

We now obtain the following partial proof of lemma A.1:

Proposition A.9

Suppose that the sectional curvature of M is bounded above by $-k < 0$. For all $\boldsymbol{\gamma} = (\gamma_n)_{n\in\mathbb{N}}$ in $W^\infty_{\alpha,\beta}$ such that $(l_n(\boldsymbol{\gamma}))_{n\in\mathbb{N}}$ tends to infinity, there exists a unique point $\mathcal{P}(\boldsymbol{\gamma}) \in \mathrm{Fix}(\Gamma)$ such that the sequence $(\mathrm{Int}(D(T_n(\boldsymbol{\gamma}))))_{n\in\mathbb{N}}$ converges towards $\{\mathcal{P}(\boldsymbol{\gamma})\}$ in the Hausdorff topology.

Proof: For all n, we define $\boldsymbol{\mu}_n \in W_{\alpha,\beta}^{l_n(\boldsymbol{\gamma})}$ and $\boldsymbol{\nu}_n \in W_{\alpha,\beta}^{n-l_n(\boldsymbol{\gamma})}$ such that:

$$\boldsymbol{\mu}_n = T_{l_n(\boldsymbol{\gamma})}(\boldsymbol{\gamma}), \qquad T_n(\boldsymbol{\gamma}) = \boldsymbol{\nu}_n\boldsymbol{\mu}_n.$$

For all n, and for all $1 \leqslant k \leqslant l_n(\boldsymbol{\gamma})$, we denote by $\mu_{n,k}$ the k'th letter of $\boldsymbol{\mu}_n$. By corollary A.7, for all n:

$$\begin{aligned}\mathrm{Int}(D(T_n(\boldsymbol{\gamma}))) &\subseteq \mathrm{Int}(D(T_{l_n(\boldsymbol{\gamma})}(\boldsymbol{\gamma})))\\ &= \mathrm{Int}(D(\boldsymbol{\mu}_n)).\end{aligned}$$

Likewise, for all n:

$$\mathrm{Int}(D(\boldsymbol{\mu}_{n+1})) \subseteq \mathrm{Int}(D(\boldsymbol{\mu}_n)).$$

For $p \in M$, let θ_p denote the angle metric over $\partial_\infty M$ with respect to p. Thus, if $x, y \in \partial_\infty M$, if px and py are the geodesic arcs linking p to x and to y respectively, and if $\widehat{xpy}$ is the angle between px and py at p, then:

$$\theta_p(x, y) = \widehat{xpy}.$$

Let p be a fixed point in M. We recall that all the circles in S_2 are contained in the union of the interiors of the circles in S_1. Moreover, $\mathrm{Fix}(\Gamma)$ is contained in the union of the interiors of the circles in S_2. It follows by compactness that there exists $\delta > 0$ such that, for all $q \in \mathrm{Fix}(\Gamma)$, and for all $C \in S_1$:

$$\theta_p(q, C) > \delta.$$

By compactness and uniform continuity, there exists $\epsilon > 0$ such that, for all $p' \in X \cap G_\Gamma$, for all $q \in \mathrm{Fix}(\Gamma)$ and for all $C \in S_1$:

$$\theta_{p'}(q, C) > \epsilon.$$

Let $\theta > 0$ be an angle. Again, by compactness and uniform continuity, there exists $\varphi > 0$ such that, for all $p' \in X \cap G_\Gamma$ and for all $q, q' \in \partial_\infty \tilde{M}$:

$$\theta_{p'}(q, q') < \varphi \Rightarrow \theta_p(q, q') < \theta.$$

We define B by:

$$B = \Gamma_k^{-1}(2\epsilon) + \Gamma_k^{-1}(\varphi/2).$$

Since $\boldsymbol{\mu}_n$ is a member of $W_{\alpha,\beta}^0$ for all n, by proposition A.8, there exists $N \in \mathbb{N}$ such that, for $n \geqslant N$:

$$\|\mathrm{Eval}(\boldsymbol{\mu}_n)\| > B.$$

By lemma A.5, for all n:

$$\mathrm{Ext}(D_1(\mu_{n,l_n(\boldsymbol{\gamma})}^{-1})) \cdot \mathrm{Eval}(\boldsymbol{\mu}_n) \subseteq \mathrm{Int}(D_1(\mu_{n,1})).$$

Let us denote by η_n the unique geodesique preserved by $\mathrm{Eval}(\boldsymbol{\mu}_n)$. It follows that η_n intersects X non-trivially. Let p_n be any point in $X \cap \eta_n$. Let p_n^- and p_n^+ be the repulsive and attractive fixed points respectively of $\mathrm{Eval}(\boldsymbol{\mu}_n)$. We have:

$$\mathrm{Ext}(D_1(\mu_{n,l_n(\boldsymbol{\gamma})}^{-1})) \subseteq \left\{q \in \partial_\infty M \text{ s.t. } \theta_{p_n}(q, p_n^-) > \epsilon\right\}.$$

Thus, by lemma 3.10:

$$\begin{aligned}\mathrm{Int}(D(\boldsymbol{\mu}_n)) \quad &= \mathrm{Ext}(D_1(\mu_{n,l_n(\boldsymbol{\gamma})}^{-1})) \cdot \mathrm{Eval}(\boldsymbol{\mu})\\ &\subseteq \{q \in \partial_\infty M \text{ s.t. } \theta_{p_n}(q, p_n^+) < \varphi/2\}.\end{aligned}$$

Consequently, the diameter of $\mathrm{Int}(D(\boldsymbol{\mu}_n))$ with respect to the metric θ_{p_n} is less than φ, and so its diameter with respect to the metric θ_p is less than θ. Since we may chose θ as small as we like, it follows that the diameter of $(\mathrm{Int}(D(\boldsymbol{\mu}_n)))_{n\in\mathbb{N}}$ with respect to the metric θ_p tends to zero.

Since $(\overline{\mathrm{Int}(D(\boldsymbol{\mu}_n))})_{n\in\mathbb{N}}$ is a nested sequence of compact sets, its intersection is non-empty. Since the diameter of the intersection is zero, it contains at most 1 point. Finally, since, for all n, $\mathrm{Int}(D(T_n(\boldsymbol{\gamma}_n))) \subseteq \mathrm{Int}(D(\boldsymbol{\mu}_n))$, this sequence of sets converges towards this point in the Hausdorff topology, and the result follows. $\square$

We may then use this result to obtain:

Proposition A.10

Suppose that the sectional curvature of M is bounded above by $-k < 0$. For all $\boldsymbol{\gamma}$ in $W_{\alpha,\beta}^\infty$, there exists a unique point $\mathcal{P}(\boldsymbol{\gamma}) \in \mathrm{Fix}(\Gamma)$ such that the sequence $(\mathrm{Int}(D(T_n(\boldsymbol{\gamma}))))_{n\in\mathbb{N}}$ converges towards $\{\mathcal{P}(\boldsymbol{\gamma})\}$ in the Hausdorff topology.

Proof: By proposition A.9, it suffices to prove the result when $(l_n(\boldsymbol{\gamma}))_{n\in\mathbb{N}}$ is bounded. Let k be such that, for all n:

$$l_n(\boldsymbol{\gamma}) \leqslant k.$$

By definition of $(l_n(\boldsymbol{\gamma}))_{n\in\mathbb{N}}$, for all $m \geqslant k$:

$$\gamma_m = \gamma_1^{-1}.$$

We thus define $\boldsymbol{\mu} = T_k(\boldsymbol{\gamma})$, and we define $\boldsymbol{\gamma}' \in W_{\alpha,\beta}^\infty$ to be the word obtained by removing the first k letters from $\boldsymbol{\gamma}$. Thus:

$$\boldsymbol{\gamma} = \boldsymbol{\gamma}'\boldsymbol{\mu}.$$

Trivially, for all n, $l_n(\boldsymbol{\gamma}') = n$. Thus, by proposition A.9, there exists a unique point $\mathcal{P}(\boldsymbol{\gamma}') \in \mathrm{Fix}(\Gamma)$ such that $(\mathrm{Int}(D(T_n(\boldsymbol{\gamma}'))))_{n\in\mathbb{N}}$ converges to $\{\mathcal{P}(\boldsymbol{\gamma}')\}$ in the Hausdorff topology. However, for all n:

$$\begin{aligned}\mathrm{Int}(D(T_n(\boldsymbol{\gamma}'))) \cdot \mathrm{Eval}(\boldsymbol{\mu}) \quad &= \mathrm{Ext}(D_1({\gamma'}_n^{-1})) \cdot \mathrm{Eval}(T_n(\boldsymbol{\gamma}')) \cdot \mathrm{Eval}(\boldsymbol{\mu})\\ &= \mathrm{Ext}(D_1(\gamma_{n+k}^{-1})) \cdot \mathrm{Eval}(T_{n+k}(\boldsymbol{\gamma}))\\ &= \mathrm{Int}(D(T_{n+k}(\boldsymbol{\gamma}))).\end{aligned}$$

Since $\mathcal{P}(\boldsymbol{\gamma}) = \mathcal{P}(\boldsymbol{\gamma}') \cdot \mathrm{Eval}(\boldsymbol{\mu})$ is also a member of $\mathrm{Fix}(\Gamma)$, existence now follows, and uniqueness then follows by uniqueness of $\mathcal{P}(\boldsymbol{\gamma}')$. $\square$

We now obtain a proof of lemma A.1:

Lemma A.1

Suppose that the sectional curvature of M is bounded above by $-k < 0$. For all $\boldsymbol{\gamma} = (\gamma_n)_{n\in\mathbb{N}}$ in $W^\infty_{\alpha,\beta}$, there exists a unique point $\mathcal{P}(\boldsymbol{\gamma}) \in \mathrm{Fix}(\Gamma)$ such that the sequence $(\mathrm{Int}(D(T_n(\boldsymbol{\gamma}))))_{n\in\mathbb{N}}$ converges towards $\{\mathcal{P}(\boldsymbol{\gamma})\}$ in the Hausdorff topology. Moreover, $\mathcal{P}(\boldsymbol{\gamma})$ is contained in $\mathrm{Int}(D(T_n(\boldsymbol{\gamma})))$ for every n, and $\mathcal{P}$ defines a homeomorphism between $W^\infty_{\alpha,\beta}$ and $\mathrm{Fix}(\Gamma)$.

Proof: Existence and uniqueness of $\mathcal{P}$ follow from propositions A.9 and A.10. Moreover, by corollary A.7, $(\mathrm{Int}(D(T_n(\boldsymbol{\gamma}))))_{n\in\mathbb{N}}$ is a nested sequence of sets, and so, for all n:

$$\mathcal{P}(\boldsymbol{\gamma}) \in \mathrm{Int}(D(T_n(\boldsymbol{\gamma}))).$$

Let $\boldsymbol{\gamma}$ be a point in $W^\infty_{\alpha,\beta}$. Let $(\boldsymbol{\gamma}_n)_{n\in\mathbb{N}}$ be a sequence of points in $W^\infty_{\alpha,\beta}$ converging to $\boldsymbol{\gamma}$. Let p be a point in M. Let $\theta > 0$ be an angle. There exists $N \in \mathbb{N}$ such that, for $n \geqslant N$:

$$\mathrm{Int}(D(T_n(\boldsymbol{\gamma}))) \subseteq \{q \in \partial_\infty M \text{ s.t. } \theta_p(q, \mathcal{P}(\boldsymbol{\gamma})) < \theta/2\}\,.$$

However, since $(\boldsymbol{\gamma}_n)_{n\in\mathbb{N}}$ converges to γ, there exists $M \in \mathbb{N}$ such that, for $n \geqslant M$:

$$T_N(\boldsymbol{\gamma}_n) = T_N(\boldsymbol{\gamma}).$$

Thus, for $n \geqslant M$:

$$\begin{array}{lll} & \mathcal{P}(\boldsymbol{\gamma}_n) & \subseteq \mathrm{Int}(D(T_N(\boldsymbol{\gamma}_n))) \quad = \mathrm{Int}(D(T_N(\boldsymbol{\gamma}))) \\ \Rightarrow & \theta_p(\mathcal{P}(\boldsymbol{\gamma}_n), \mathcal{P}(\boldsymbol{\gamma})) & < \theta. \end{array}$$

The continuity of $\mathcal{P}$ now follows.

For all n, the mapping D defines a bijection between $W^n_{\alpha,\beta}$ and S_n. We recall that, for all n, the interiors of any two circles in S_n are disjoint, and for all $m > n$, and for all $C \in S_m$, there exists a unique circle, C', in S_n such that C lies in the interior of C'. Moreover, by the preceeding reasoning, for all $m > n$ and for all $\boldsymbol{\gamma}$ in $W^m_{\alpha,\beta}$:

$$\mathrm{Int}(D(\boldsymbol{\gamma})) \subseteq \mathrm{Int}(D(T_n(\boldsymbol{\gamma}))).$$

Let p be a point in $\mathrm{Fix}(\Gamma)$. For all n, p lies in the union of the interiors of the circles in S_n. Consequently, there exists $\boldsymbol{\gamma}_n$ such that:

$$p \in \mathrm{Int}(D(\boldsymbol{\gamma}_n)).$$

Moreover, since, for $m > n$, the intersection of $\mathrm{Int}(D(\boldsymbol{\gamma}_m))$ with $\mathrm{Int}(D(\boldsymbol{\gamma}_n))$ contains p and is thus non-empty, it follows that:

$$\begin{array}{lll} & D(T_n(\boldsymbol{\gamma}_m)) & = D(\boldsymbol{\gamma}_n) \\ \Rightarrow & T_n(\boldsymbol{\gamma}_m) & = \boldsymbol{\gamma}_n. \end{array}$$

We may thus define $\boldsymbol{\gamma} \in W^{\infty}_{\alpha,\beta}$ such that, for all n:

$$T_n(\boldsymbol{\gamma}) = \boldsymbol{\gamma}_n$$

Since p lies in $\mathrm{Int}(D(T_n(\boldsymbol{\gamma})))$ for all n, it follows that $\mathcal{P}(\boldsymbol{\gamma}) = p$, and the surjectivity of $\mathcal{P}$ now follows.

Let $\boldsymbol{\gamma}$ and $\boldsymbol{\gamma}'$ be two points in $W^{\infty}_{\alpha,\beta}$ such that $\mathcal{P}(\boldsymbol{\gamma}) = \mathcal{P}(\boldsymbol{\gamma}) = p$. For all n, p is contained in the interiors of both $D(T_n(\boldsymbol{\gamma}))$ and $D(T_n(\boldsymbol{\gamma}'))$. Consequently, for all n:

$$\begin{array}{lll} & D(T_n(\boldsymbol{\gamma})) & = D(T_n(\boldsymbol{\gamma}')) \\ \Rightarrow & T_n(\boldsymbol{\gamma}) & = T_n(\boldsymbol{\gamma}'). \end{array}$$

Consequently $\boldsymbol{\gamma} = \boldsymbol{\gamma}'$, and it follows that $\mathcal{P}$ is injective.

It follows that $\mathcal{P}$ is a bijective continuous mapping between two compact sets and is thus a homeomorphism. The result now follows. □

A.3 Homeomorphism Equivalence of Schottky Groups.

Let $(C^{\pm}_{\alpha}, C^{\pm}_{\beta})$ be generating circles of Γ with respect to the generators (α, β). We define $\Omega \subseteq \partial_\infty M$ by:

$$\Omega = \partial_\infty M \setminus \mathop{\cup}_{\substack{\gamma\in\{\alpha,\beta\} \\ \mathrm{sgn}\in\{\pm\}}} \mathrm{Int}(C^{\mathrm{sgn}}_{\gamma}).$$

We define the continuous mapping $\Phi : \Omega \times \Gamma \to \partial_\infty M$ by:

$$\Phi(x, \gamma) = x \cdot \gamma.$$

We define the equivalence relation $\sim$ over $\Omega \times \Gamma$ such that:

$$(x, \gamma) \sim (y, \eta) \Leftrightarrow \Phi(x, \gamma) = \Phi(y, \eta).$$

Since Ω is a fundamental domain for the action of Γ, we find that $(x, \gamma) \sim (y, \eta)$ if and only if either $x = y$ and $\gamma = \eta$ or:

(i) $x, y \in \partial\Omega$, and

(ii) there exists $\mu \in \left\{\alpha^{\pm 1}, \beta^{\pm 1}\right\}$ such that:

$$(x, \gamma) = (y \cdot \mu, \mu^{-1} \cdot \eta).$$

We see that $\Omega \times \Gamma / \sim$ is a Hausdorff space and that Φ quotients down onto a homeomorphism of $\Omega \times \Gamma / \sim$ onto its image.

We have the following result:

Proposition A.11

$$\partial_\infty M = \mathrm{Im}(\Phi) \cup \mathrm{Fix}(\Gamma).$$

Proof: Let p be a point in $\partial_\infty M \setminus \mathrm{Im}(\Phi)$. For $\gamma \in \Gamma$, let $l(\gamma)$ denote its length in the word metric. For all n, we have:

$$\begin{array}{rll} & p & \notin \cup_{l(\gamma)<n} \Phi(\Omega \times \{\gamma\}) \\ \Rightarrow & p & \in \cup_{C \in S_n} \mathrm{Int}(C). \end{array}$$

Thus, for all n, there exists $\boldsymbol{\gamma}_n \in W^n_{\alpha,\beta}$ such that:

$$p \in \mathrm{Int}(D(\boldsymbol{\gamma}_n)).$$

For $m > n$, the intersection of $\mathrm{Int}(D(\boldsymbol{\gamma}_m))$ with $\mathrm{Int}(D(\boldsymbol{\gamma}_n))$ contains p and is thus non-empty. Consequently, the intersection of $\mathrm{Int}(D(T_n(\boldsymbol{\gamma}_m)))$ with $\mathrm{Int}(D(\boldsymbol{\gamma}_n))$ is also non-empty, and so:

$$\begin{array}{rll} & D(T_n(\boldsymbol{\gamma}_m)) & = D(\boldsymbol{\gamma}_n) \\ \Rightarrow & T_n(\boldsymbol{\gamma}_m) & = \boldsymbol{\gamma}_n. \end{array}$$

We may thus define $\boldsymbol{\gamma} \in W^\infty_{\alpha,\beta}$ such that, for all n:

$$T_n(\boldsymbol{\gamma}) = \boldsymbol{\gamma}_n.$$

Since $p \in \mathrm{Int}(D(T_n(\boldsymbol{\gamma})))$ for all n, it follows that $p = \mathcal{P}(\boldsymbol{\gamma})$ and is thus an element of $\mathrm{Fix}(\Gamma)$. The result now follows. $\square$

We now define the space Σ_Γ by:

$$\Sigma_\Gamma = (\Omega \times \Gamma/\sim) \cup W^\infty_{\alpha,\beta}.$$

For all n, and for all $\boldsymbol{\gamma} \in W^n_{\alpha,\beta}$, we define the subset $U_{\boldsymbol{\gamma}}$ of Σ_Γ by:

$$U_{\boldsymbol{\gamma}} = \mathrm{Int}\left(\bigcup_{\substack{\boldsymbol{\eta} \in W_{\alpha,\beta} \text{ s.t} \\ l(\boldsymbol{\eta}) \geqslant n, T_n(\boldsymbol{\eta}) = \boldsymbol{\gamma}}} \Omega \times \{\mathrm{Eval}(\boldsymbol{\eta})\}/\sim\right) \cup \left\{\boldsymbol{\gamma}' \in W^\infty_{\alpha,\beta} \text{ s.t. } T_n(\boldsymbol{\gamma}') = \boldsymbol{\gamma}\right\}.$$

For $m \geqslant n$, for $\boldsymbol{\gamma} \in W^m_{\alpha,\beta}$ and $\boldsymbol{\eta} \in W^n_{\alpha,\beta}$, either $T_n(\boldsymbol{\gamma}) \neq \boldsymbol{\eta}$, in which case $U_{\boldsymbol{\gamma}} \cap U_{\boldsymbol{\eta}} = \emptyset$, or $T_n(\boldsymbol{\gamma}) = \boldsymbol{\eta}$, in which case $U_{\boldsymbol{\gamma}} \subseteq U_{\boldsymbol{\eta}}$. Consequently, if τ denotes the topology of $\Omega \times \Gamma/\sim$, then $\tau \cup \{U_{\boldsymbol{\gamma}} \text{ s.t. } \boldsymbol{\gamma} \in W_{\alpha,\beta}\}$ defines a base for a topology over Σ_Γ. This topology is Hausdorff and restricts to the initial topologies over $\Omega \times \Gamma/\sim$ and $W^\infty_{\alpha,\beta}$. We now have the following result:

Proposition A.12

The space Σ_Γ is compact. Moreover, if $\boldsymbol{\gamma} \in W^\infty_{\alpha,\beta}$ and if $(p_n, \gamma_n)_{n \in \mathbb{N}}$ is a sequence in $\Omega \times \Gamma/\sim$ converging to $\boldsymbol{\gamma}$, then, for all $k \in \mathbb{N}$, there exists $N \in \mathbb{N}$ such that:

$$n \geqslant N \Rightarrow T_k(\boldsymbol{\gamma}_n) = T_k(\boldsymbol{\gamma}).$$

Proof: Let $(q_n)_{n\in\mathbb{N}}$ be a sequence of points in Σ_Γ. Since $W^\infty_{\alpha,\beta}$ is compact, we only need to treat the case where $(q_n)_{n\in\mathbb{N}}$ is contained in $\Omega\times\Gamma/\sim$. For all n, let $(p_n,\gamma_n)\in\Omega\times\Gamma$ be such that $q_n=(p_n,\gamma_n)$. For all n, let $l(\gamma_n)$ denote the length of γ_n in the word metric. If $(l(\gamma_n))_{n\in\mathbb{N}}$ is bounded, then, after taking a subsequence, we may assume that $(\gamma_n)_{n\in\mathbb{N}}$ is constant, and the result now follows by the compactness of Ω.

It thus remains to study the case where $(l(\gamma)_n)_{n\in\mathbb{N}}$ tends to infinity. For all n, let $\boldsymbol{\gamma}_n\in W_{\alpha,\beta}$ be such that:

$$\gamma_n=\mathrm{Eval}(\boldsymbol{\gamma}_n).$$

For all n, let k_n denote the length of $\boldsymbol{\gamma}_n$. We may suppose that $(k_n)_{n\in\mathbb{N}}$ is a strictly increasing sequence. Since the alphabet $\{\alpha^{\pm1},\beta^{\pm1}\}$ is finite, we may suppose that there exist sequences $(\boldsymbol{\eta}_n)_{n\in\mathbb{N}}$ and $(\boldsymbol{\mu}_n)_{n\in\mathbb{N}}$ in $W_{\alpha,\beta}$ such that:

(i) for all n, $\boldsymbol{\eta}_n$ is of length n,

(ii) for all $m>n$, $T_n(\boldsymbol{\eta}_m)=\boldsymbol{\eta}_n$, and

(iii) for all n, $\boldsymbol{\gamma}_n=\boldsymbol{\mu}_n\boldsymbol{\eta}_n$.

We define $\boldsymbol{\eta}\in W^\infty_{\alpha,\beta}$ such that, for all n:

$$T_n(\boldsymbol{\eta})=\boldsymbol{\eta}_n.$$

Let $\boldsymbol{\delta}\in W_{\alpha,\beta}$ be such that $\boldsymbol{\eta}\in U_{\boldsymbol{\delta}}$. Let k be the length of $\boldsymbol{\delta}$. We have:

$$\boldsymbol{\delta}=T_k(\boldsymbol{\eta}).$$

Thus, for all $n\geqslant k$:

$$\boldsymbol{\delta}=T_k(\boldsymbol{\eta}_n).$$

Consequently, for all $n\geqslant k$, $(p_n,\gamma_n)\in U_{\boldsymbol{\delta}}$. Consequently, for every neighbourhood, V, of $\boldsymbol{\eta}$, there exists $N\in\mathbb{N}$ such that:

$$n\geqslant N\Rightarrow(p_n,\gamma_n)\in V.$$

It follows that $(q_n)_{n\in\mathbb{N}}$ converges to $\boldsymbol{\eta}$, and the result now follows. $\square$

We now define the mapping $\Psi:\Sigma_\Gamma\to\partial_\infty\tilde{M}$ by:

$$\Psi(p)=\begin{cases}\mathcal{P}(p) & \text{if } p\in W^\infty_{\alpha,\beta},\\ \Phi(p) & \text{if } p\in\Omega\times\Gamma/\sim.\end{cases}$$

By lemma A.1 and proposition A.11, the mapping Ψ is bijective and restricts to a homeomorphism onto its image over both $W^\infty_{\alpha,\beta}$ and $\Omega\times\Gamma/\sim$. We have the following result:

Lemma A.13

The mapping Ψ is a homeomorphism.

Proof: Since Σ_Γ is compact, and since Ψ is bijective, it suffices to prove that Ψ is continuous. Since the restrictions of Ψ to both $W^\infty_{\alpha,\beta}$ and $\Omega\times\Gamma/\sim$ are continuous, it thus suffices to show that if $\boldsymbol{\gamma}\in W^\infty_{\alpha,\beta}$ and if $(p_n,\gamma_n)_{n\in\mathbb{N}}$ is a sequence in $\Omega\times\Gamma/\sim$ which converges to $\boldsymbol{\gamma}$, then $(\Phi(p_n,\gamma_n))_{n\in\mathbb{N}}$ converges to $\mathcal{P}(\boldsymbol{\gamma})$. We will show that every subsequence of $(\Phi(p_n,\gamma_n))_{n\in\mathbb{N}}$ has a subsubsequence which converges to $\mathcal{P}(\boldsymbol{\gamma})$. The result then follows.

For all n, let $\boldsymbol{\gamma}_n\in W_{\alpha,\beta}$ be such that $\mathrm{Eval}(\boldsymbol{\gamma}_n)=\gamma_n$. Since the alphabet $\left\{\alpha^{\pm1},\beta^{\pm1}\right\}$ is finite, we may suppose that there exist sequences $(\boldsymbol{\eta}_n)_{n\in\mathbb{N}}$ and $(\boldsymbol{\mu}_n)_{n\in\mathbb{N}}$ in $W_{\alpha,\beta}$ such that:

(i) for all n, $\boldsymbol{\eta}_n$ is of length n,

(ii) for all $m>n$, $T_n(\boldsymbol{\eta}_m)=\boldsymbol{\eta}_n$, and

(iii) for all n, $\boldsymbol{\gamma}_n=\boldsymbol{\mu}_n\boldsymbol{\eta}_n$.

Since $(p_n,\gamma_n)_{n\in\mathbb{N}}$ converges to $\boldsymbol{\gamma}$ for all n, we have:

$$\boldsymbol{\eta}_n=T_n(\boldsymbol{\gamma}).$$

For all $k\leqslant n$, let $\eta_{n,k}$ be the k'th letter of $\boldsymbol{\eta}_n$. For all n, let l_n be the length of $\boldsymbol{\mu}_n$, and, for all $k\leqslant l_n$, let $\mu_{n,k}$ be the k'th letter of $\boldsymbol{\mu}_n$. Since the interior of Ω is a subset of $\mathrm{Ext}(D_1(\mu_{n,l_n}^{-1}))$, lemma A.5 yields:

$$\Omega\cdot\mathrm{Eval}(\boldsymbol{\mu}_n)\subseteq\overline{\mathrm{Int}(D_1(\mu_{n,1}))}.$$

Since $\boldsymbol{\gamma}_n$ is a reduced word, $\mu_{n,1}\neq\eta_{n,n}^{-1}$. Consequently, for all n:

$$\begin{array}{rll} & \overline{\mathrm{Int}(D_1(\mu_{n,1}))} & \subseteq \mathrm{Ext}(D_1(\eta_{n,n}^{-1}))\\ \Rightarrow & \Omega\cdot\mathrm{Eval}(\boldsymbol{\mu}_n) & \subseteq \mathrm{Ext}(D_1(\eta_{n,n}^{-1}))\\ \Rightarrow & \Omega\cdot\mathrm{Eval}(\boldsymbol{\gamma}_n) & \subseteq \mathrm{Ext}(D_1(\eta_{n,n}^{-1}))\cdot\mathrm{Eval}(\boldsymbol{\eta}_n).\end{array}$$

However, by corolloary A.6, for all n:

$$\mathrm{Int}(D(\boldsymbol{\eta}_n))=\mathrm{Ext}(D_1(\eta_{n,n}^{-1}))\cdot\mathrm{Eval}(\boldsymbol{\eta}_n).$$

Thus, for all n:

$$\begin{array}{rll} & p_n\cdot\mathrm{Eval}(\boldsymbol{\gamma}_n) & \in\mathrm{Int}(D(\boldsymbol{\eta}_n))\\ \Rightarrow & \Phi(p_n,\gamma_n) & \in\mathrm{Int}(D(\boldsymbol{\eta}_n)).\end{array}$$

The result now follows by lemma A.1. $\square$

We have constructed a topological space homeomorphic to the sphere. Moreover, there exists a canonical action of Γ over this space with which Ψ intertwines. We are now able to prove lemma A.2:

Lemma A.2

Suppose that the sectional curvature of M is bounded above by $-k<0$. Let $\Gamma\subseteq\mathrm{Isom}_0(M)$ and $\Gamma'\subseteq\mathrm{Isom}_0(M')$ be Schottky subgroups. There exists an isomorphism $\phi:\Gamma\rightarrow\Gamma'$ and a homeomorphism $\Phi:\partial_\infty M\rightarrow\partial_\infty M'$ such that, for all $\gamma\in\Gamma$:

$$\Phi\circ\gamma=\varphi(\gamma)\circ\Phi.$$

Proof: Let (α, β) and (α', β') be generators of Γ and Γ' respectively. Let $(C_\alpha^\pm, C_\beta^\pm)$ and $({C'_\alpha}^\pm, {C'_\beta}^\pm)$ be generating circles of Γ and Γ' respectively with respect to these generators. We define $\Omega \subseteq \partial_\infty M$ and $\Omega' \subseteq \partial_\infty M'$ by:

$$\begin{aligned} \Omega &= \partial_\infty M \setminus \left(\mathrm{Int}(C_\alpha^+) \cup \mathrm{Int}(C_\alpha^-) \cup \mathrm{Int}(C_\beta^+) \cup \mathrm{Int}(C_\beta^-)\right), \\ \Omega' &= \partial_\infty M' \setminus \left(\mathrm{Int}({C'_\alpha}^+) \cup \mathrm{Int}({C'_\alpha}^-) \cup \mathrm{Int}(C'_\beta) \cup \mathrm{Int}({C'_\beta}^-)\right). \end{aligned}$$

We define $\phi : \Gamma \rightarrow \Gamma'$ such that:

$$\phi(\alpha) = \alpha', \qquad \phi(\beta) = \beta'.$$

Let $\hat{\Phi} : \Omega \rightarrow \Omega'$ be a homeomorphism such that, for all $p \in C_\alpha^-$ and for all $q \in C_\beta^-$:

$$\begin{aligned} \hat{\Phi}(p) \cdot \alpha' &= \hat{\Phi}(p \cdot \alpha), \\ \hat{\Phi}(q) \cdot \beta' &= \hat{\Phi}(q \cdot \alpha). \end{aligned}$$

There exists a unique extension of $\hat{\Phi}$ to a mapping from Σ_Γ to $\Sigma_{\Gamma'}$ which intertwines with ϕ. We define $\Psi_1 : \Sigma_\Gamma \rightarrow \partial_\infty M$ and $\Psi_2 : \Sigma_\Gamma \rightarrow \partial_\infty M'$ as before. For all $\gamma \in \Gamma$, the following diagram trivially commutes:

$$\begin{tikzcd} \partial_\infty M \arrow[r, "\Psi_1^{-1}"] \arrow[d, "\gamma"] & \Sigma_\Gamma \arrow[r, "\hat{\Phi}"] \arrow[d, "\gamma"] & \Sigma_{\gamma'} \arrow[r, "\Psi_1'"] \arrow[d, "\phi(\gamma)"] & \partial_\infty M' \arrow[d, "\phi(\gamma)"] \\ \partial_\infty M \arrow[r, "\Psi_1^{-1}"] & \Sigma_\Gamma \arrow[r, "\hat{\Phi}"] & \Sigma_{\gamma'} \arrow[r, "\Psi_1'"] & \partial_\infty M' \end{tikzcd}$$

The mapping $\Phi = \Psi_1' \circ \hat{\Phi} \circ \Psi_1^{-1}$ is thus the desired homeomorphism, and the result now follows. $\square$

◇

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