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Variation of energy for capillary surfaces

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Abstract

Well known first order necessary conditions for a liquid mass to be in equilibrium in contact with a fixed solid surface declare that the free surface interface has mean curvature prescribed in terms of the bulk accelerations acting on the liquid and meets the solid surface in a materially dependent contact angle. We derive first order necessary conditions for capillary surfaces in equilibrium in contact with solid surfaces which may also be allowed to move. These conditions consist of the same prescribed mean curvature equation for the interface, the same prescribed contact angle condition on the boundary, and an additional integral condition which may be said to involve, somewhat surprisingly, only the wetted region.

An example of the kind of system under consideration is that of a floating ball in a fixed container of liquid. We apply our first order conditions to this particular problem.

A brief introduction to the notation we use for surfaces containing a few useful formulae is given as an appendix at the end.

1 Introduction: Expressions for Energy and Volume

In the calculus of variations, one is interested in minimizing an energy functional over some class of admissible functions. The energy is usually assumed to be an integral operator. For example, one often considers

$$\mathcal{F}[u] = \int_\Omega F(x, u, Du)dx + \int_{\partial\Omega} \bar{F}(x, u, Du)dx \tag{1}$$

where Ω is a fixed domain in $\mathbb{R}^n$ and the competing functions u are all defined on $\overline{\Omega}$.

Following Gauss, a standard approach to the derivation of equations governing equilibrium configurations for liquid masses in contact with solid bounding surfaces is via the calculus of variations. In this context, due to the geometric and in particular parametric nature of

the situation, it is more natural to consider energy functionals which are integral operators on a given Riemann surface M and which involve a parametric mapping $X : M \to \mathbb{R}^3$. (See the appendix to this paper for a brief discussion of these mappings and some formulae that will be used below.) Though the value of the energy involves universally the inclusion of surface area terms for the liquid *free surface interface* and the *wetted region* on the solid support surface, the precise identification in the literature of the Riemann surface M and the mapping X is somewhat more obscure.

In most, if not all, cases it is tacitly assumed that the entire configuration is determined by the free surface interface and, hence, that the functional $\mathcal{E} = \mathcal{E}[X]$ has as argument the mapping X which parameterizes the free surface interface. More precisely, it is assumed that the mapping X which determines the free surface interface also determines the contact line (the boundary of the free interface); the contact line determines a particular wetted region on the solid surface; the free surface interface together with the wetted region determine the enclosed volume or liquid mass, and this is enough information to compute the energy. Competing admissible mappings are then naturally considered as those mappings defined on the image Λ of X, or equivalently on its preimage M. Each such *free surface interface* leads to an admissible configuration defining a liquid mass as just described. From this point of view, the objective may be stated as finding a piecewise C^1 mapping X_0 determining a free surface interface for which $\mathcal{E}(X_0) \leq \mathcal{E}(X)$ for all piecewise C^1 competing or admissible mappings X. It should be noted that in this formulation, the energy is given implicitly at least to the extent that a particular integral describing, say, the wetted area, is never written down. Since the first variation formula in this context only involves integrals over Λ and $\partial\Lambda$, this point of view seems at first adequately justified, at least formally.

We give here an alternative approach which is somewhat more explicit at least to the extent that the initial energy is given in terms of a well defined integral operator. In situations in which portions of the solid boundary are free to move, or partially free to move, the image of the free surface does not always determine the wetted region or the volume; consider a cylindrical container $\{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 = 2, \ -3 \leq z \leq 1\} \cup \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 \leq 2, \ z = -3\}$ with the floating solid cylinder $\{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 \leq 1, \ -2 + h \leq z \leq 1 + h\}$ for h fixed with $-1 < h < 2$. An admissible interface for every value of h is given by the annulus $\{(x, y, 0) \in \mathbb{R}^3 : 1 < x^2 + y^2 < 2\}$; while this interface determines the contact line, the wetted region and enclosed volume both depend explicitly on h. In such situations, our alternative and more general approach is more natural, being further justified by the fact that additional integral terms, not expressible as integrals over the free surface interface or its boundary, appear in the first order necessary conditions. Furthermore, the explicit appearance of an independent parameter h in this type of problem suggests an analogous class of problems in the nonparametric case different from (1). Namely, one may consider

$$\mathcal{F}[u, h] = \int_\Omega F(x, u, Du, h)dx + \int_{\partial\Omega} \bar{F}(x, u, Du, h)dx \qquad (2)$$

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where the admissible functions u are defined on $\Omega \times \tilde{\Omega}$ for some set $\tilde{\Omega} \in \mathbb{R}^m$. We are unaware of any treatment of problems of this sort in the literature, but one can think of some interesting examples with little difficulty.

We now give formal mathematical assumptions used to model capillary systems in somewhat more detail than we have found in the literature. We model solid surfaces by the boundaries of initially prescribed closed sets in $\mathbb{R}^3$. The union of all such solid surfaces is denoted by $\Sigma = \partial\mathcal{C}$ where $\mathcal{C}$ denotes the union of the closed sets. We assume, for the moment, that all solid surfaces are fixed, and an *interior liquid mass*, modeled by an open set $\mathcal{M}$, occupies some portion of $\mathbb{R}^3 \backslash \mathcal{C}$. The boundary of $\mathcal{M}$ consists of a portion $\mathcal{W}$ in Σ, which we call the *wetted region*, and a portion $\Lambda = \partial\mathcal{M} \backslash \mathcal{W}$, the *free surface interface*.

Given a particular liquid mass $\mathcal{M}_0$, we assume $\mathcal{M}_0$ and the associated surfaces Λ_0 and $\mathcal{W}_0$ admit the structure of abstract Riemannian manifolds (of dimensions 3,2, and 2 respectively).

Relative to $\mathcal{M}_0$, we consider the admissible class of liquid mass configurations $\mathcal{M}$ obtained as parameterized images

$$X : \overline{\mathcal{M}}_0 \to \mathbb{R}^3.$$

More precisely, we consider liquid masses $\mathcal{M} = X(\mathcal{M}_0)$ with smooth parameterization $X\big|_{\mathcal{M}_0}$; it is assumed that $\partial\mathcal{M} = X(\Lambda_0) \cup X(\mathcal{W}_0)$ with free surface interface $\Lambda = X(\Lambda_0)$, parameterized by $X\big|_{\Lambda_0}$, and wetted region $\mathcal{W} = X(\mathcal{W}_0) = X(\overline{\mathcal{M}}_0) \cap \Sigma$ with parameterization $X\big|_{\mathcal{W}_0}$.

Given an admissible liquid mass configuration $\mathcal{M}$, we assume the energy of the system is given by

$$\mathcal{E} = A(\Lambda) - \beta A(\mathcal{W}) \tag{3}$$

where A denotes the area functional. The constant β is called the *adhesion coefficient*. See [7] for further details. For simplicity, we have not included in (3) the possibility that bulk accelerations are acting on the liquid. The applications given in this paper do not involve gravity or other accelerations, but for reference we will briefly indicate below what occurs in this more general setting.

We will assume that each global parameterization, such as $X\big|_{\Lambda_0} : \Lambda_0 \to \Lambda$, extends smoothly to an open neighborhood of the boundary. In particular, we can parameterize a small neighborhood of every point in the closure of Λ locally by some map defined in a neighborhood of $\mathbb{R}^2$. We will employ the usual abuse of notation in denoting this local parameterization also by X. More generally, we may emphasize the role of either parameter domain Λ_0 or $\mathcal{W}_0$ (or their union) as an abstract manifold by denoting it as M and referring to the appropriate restriction of $X : \overline{\mathcal{M}}_0 \to \mathbb{R}^3$ simply as $X : M \to \mathbb{R}^3$. The context is usually adequate to indicate which portion of $\partial\mathcal{M}$ is under consideration. If we wish to emphasize the distinction between the free surface Λ and wetted region $\mathcal{W}$, we will use a superscript "$\mathcal{W}$" in reference to the wetted region.
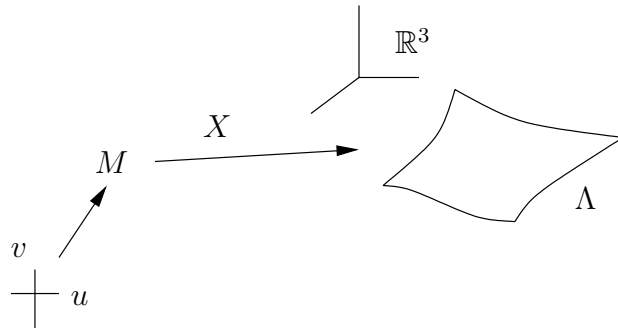
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Figure 1: A free surface interface parameterized on an abstract manifold

Under these assumptions, the area functional can be written explicitly as an integral over a fixed Riemann surface M of the Jacobian scaling factor $\sqrt{\det(dX^T dX)}$ where the superscript "T" denotes the transpose (or adjoint) of the linear transformation; see [6]. We can write, for example,

$$A(\Lambda) = \int_\Lambda 1 = \int_{\Lambda_0} \sqrt{\det(dX^T dX)} = \int_M \sqrt{\det(dX^T dX)}.$$

Using a similar expression for the wetting energy, we see that the energy of admissible configurations is well defined as a functional on pairs of restrictions $X : M = \Lambda_0 \to \mathbb{R}^3$, $X^{\mathcal{W}} : M^{\mathcal{W}} = \mathcal{W}_0 \to \mathbb{R}^3$ on a pair of fixed Riemann surfaces (or alternatively on admissible maps X on the union $M = \Lambda_0 \cup \mathcal{W}_0$):

$$\mathcal{E}(X) = \int_M \sqrt{\det(dX^T dX)} - \beta \int_{M^{\mathcal{W}}} \sqrt{\det(dX^{\mathcal{W}T} dX^{\mathcal{W}})} \tag{4}$$

The volume of the liquid mass can also be expressed in terms of integrals over the same parameter domains. In fact, using the divergence theorem

$$V = \int_{\mathcal{M}} 1 = \frac{1}{3} \int_{\mathcal{M}} \operatorname{div}_{\mathbb{R}^3} x = \frac{1}{3} \int_\Lambda X \cdot N + \frac{1}{3} \int_{\mathcal{W}} X_0^{\mathcal{W}} \cdot N_0^{\mathcal{W}}.$$

Thus, we may write

$$V(X) = \frac{1}{3} \int_M X \cdot N \sqrt{\det(dX^T dX)} + \frac{1}{3} \int_{M^{\mathcal{W}}} X_0^{\mathcal{W}} \cdot N_0^{\mathcal{W}} \sqrt{\det(dX^{\mathcal{W}T} dX^{\mathcal{W}})}. \tag{5}$$

The expressions (4) and (5) provide the basis for the main calculations in the next section.

When bulk accelerations are considered, one encounters in the energy an additional term of the form

$$\mathcal{G} = \int_{\mathcal{M}} g$$

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where g is a given scalar function of position. I am unaware of a means to express this term as an integral over $\partial\mathcal{M}_0$. Thus, we are apparently forced to use the full information of the mapping $X : \overline{\mathcal{M}}_0 \to \mathbb{R}^3$ associated with an admissible variation of the bulk liquid and admit the dependence of the energy on the third restriction $X\big|_{\mathcal{M}_0} : \mathcal{M}_0 \to \mathbb{R}^3$, which we also denote by X:

$$\mathcal{G}(X) = \int_{\mathcal{M}_0} g \circ X |\det DX|.$$

2 General Formulae

Given a liquid mass configuration as described above, we wish to consider a variation $X : \overline{\mathcal{M}}_0 \times (-\epsilon, \epsilon) \to \mathbb{R}^3$ such that $X(\cdot; 0) = \mathrm{id}_{\overline{\mathcal{M}}_0}(\cdot)$ and for each fixed $t \in (-\epsilon, \epsilon)$, the two restrictions

$$X\big|_{\Lambda_0 \times \{t\}} : \Lambda_0 = M \to \mathbb{R}^3 \quad \text{and} \quad X\big|_{\mathcal{W}_0 \times \{t\}} : \mathcal{W}_0 = M^{\mathcal{W}} \to \mathbb{R}^3,$$

which parameterize images Λ and $\mathcal{W}$ respectively, arise as restrictions to Λ_0 and $\mathcal{W}_0$ of the global map $X : \overline{\mathcal{M}}_0 \to \mathbb{R}^3$ associated to an admissible liquid mass $\mathcal{M} = \mathcal{M}_t$. The variations under consideration here are required to be smooth so that the functions

$$\mathcal{E}(t) = A(\Lambda) - \beta A(\mathcal{W}) \quad \text{and} \quad V(t) = \frac{1}{3} \int_\Lambda X \cdot N + \frac{1}{3} \int_{\mathcal{W}} X^{\mathcal{W}} \cdot N^{\mathcal{W}}$$

are differentiable on $(-\epsilon, \epsilon)$.

Finally, we note the compatibility of the restrictions to $\mathcal{W}_0$ and Λ_0 along their common boundary:

$$X^{\mathcal{W}}(q; t) = X(q; t) \quad \text{for every } q \in \partial M^{\mathcal{W}} = \partial M \tag{6}$$

This simple relation will play a key role in our derivation of a variational formula for floating bodies.

With no bulk accelerations, no condition on the interior restriction $X\big|_{\mathcal{M}_0 \times \{t\}}$ need be considered, and this part of the global map can be ignored.

We wish to compute $\dot{\mathcal{E}} = d\mathcal{E}/dt$. We begin with $A(\Lambda)$.

$$A(\Lambda) = \int_\Lambda 1 = \sum_U \int_U \sqrt{EG - F^2},$$

where we have written the abstract manifold M (up to a set of measure zero) as a disjoint union of images of coordinate neighborhoods U in the u, v-plane and E, F, and G are the coefficients of the first fundamental form on a given neighborhood U. That is, taking X to be locally defined on U,

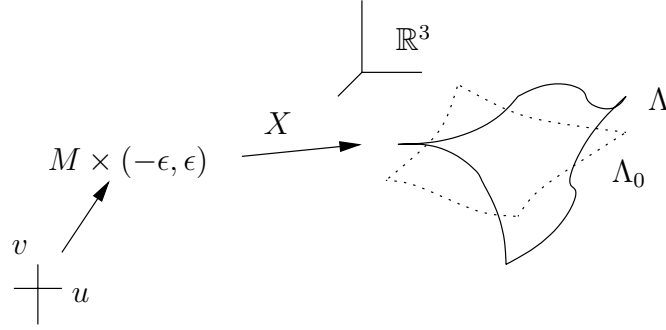$$E = |X_u|^2, \quad F = X_u \cdot X_v, \quad G = |X_v|^2,$$

Figure 2: Variation of Λ_0

and

$$\sqrt{EG - F^2} = |X_u \times X_v|;$$

see the appendix for details. We then have

$$\frac{d}{dt} \int_U \sqrt{EG - F^2}$$

$$= \int_U \frac{(X_u \times X_v) \cdot (\dot{X}_u \times X_v + X_u \times \dot{X}_v)}{\sqrt{EG - F^2}}$$

$$= \int_U \frac{(X_u \cdot \dot{X}_u)G - (X_v \cdot \dot{X}_u)F + (X_v \cdot \dot{X}_v)E - (X_u \cdot \dot{X}_v)F}{\sqrt{EG - F^2}}.$$

Now, if we write $\dot{X} = \phi N + \psi X_u + \eta X_v = \phi N + \dot{X}^{\mathrm{T}}$, where N is a unit normal to Λ then

$$\dot{X}_u = \phi_u N + \phi N_u + \psi_u X_u + \psi X_{uu} + \eta_u X_v + \eta X_{uv}$$
$$\dot{X}_v = \phi_v N + \phi N_v + \psi_v X_u + \psi X_{uv} + \eta_v X_v + \eta X_{vv},$$

so that

$$X_u \cdot \dot{X}_u = -\phi e + \psi_u E + \psi \frac{E_u}{2} + \eta_u F + \eta \frac{E_v}{2}$$

$$X_v \cdot \dot{X}_u = -\phi f + \psi_u F + \psi F_u - \psi \frac{E_v}{2} + \eta_u G + \eta \frac{G_u}{2}$$

$$X_u \cdot \dot{X}_v = -\phi f + \psi_v E + \psi \frac{E_v}{2} + \eta_v F + \eta F_v - \eta \frac{G_u}{2}$$

$$X_v \cdot \dot{X}_v = -\phi g + \psi_v F + \psi \frac{G_u}{2} + \eta_v G + \eta \frac{G_v}{2}$$

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where $e = X_{uu} \cdot N$, $f = X_{uv} \cdot N$, and $g = X_{vv} \cdot N$ are the coefficients of the second fundamental form. Multiplying the expressions above by the appropriate factors and adding we obtain

$$(X_u \cdot \dot{X}_u)G - (X_v \cdot \dot{X}_u)F + (X_v \cdot \dot{X}_v)E - (X_u \cdot \dot{X}_v)F =$$

$$-\phi eG + \psi_u EG + \psi \frac{E_u}{2}G + \eta_u FG + \eta \frac{E_v}{2}G$$

$$+\phi fF - \psi_u F^2 - \psi F F_u$$

$$+\psi \frac{E_v}{2}F - \eta_u FG - \eta \frac{G_u}{2}F$$

$$-\phi gE + \psi_v EF + \psi \frac{G_u}{2}E + \eta_v EG + \eta \frac{G_v}{2}E$$

$$+\phi fF - \psi_v EF - \psi \frac{E_v}{2}F - \eta_v F^2 - \eta F F_v + \eta \frac{G_u}{2}F$$

for the numerator in the integral above. Regrouping (in columns), this becomes

$$-\phi(eG - 2fF + gE)$$

$$+\psi_u(EG - F^2) + \psi \frac{(EG - F^2)_u}{2} + \eta_v(EG - F^2) + \eta \frac{(EG - F^2)_v}{2}.$$

Thus, we have

$$\frac{d}{dt}\int_U \sqrt{EG - F^2} = -\int_U \phi \frac{eG - 2fF + gE}{\sqrt{EG - F^2}}$$

$$+ \int_U \psi_u \sqrt{EG - F^2} + \psi \frac{(EG - F^2)_u}{2\sqrt{EG - F^2}}$$

$$+ \int_U \eta_v \sqrt{EG - F^2} + \eta \frac{(EG - F^2)_v}{2\sqrt{EG - F^2}}$$

$$= -\int_U \phi \frac{eG - 2fF + gE}{\sqrt{EG - F^2}}$$

$$+ \int_U (\psi \sqrt{EG - F^2})_u + (\eta \sqrt{EG - F^2})_v.$$

Finally, summing over the neighborhoods U we have

$$\dot{A}(\Lambda) = -\int_\Lambda 2H\phi$$

$$+ \sum_U \int_U \frac{(\psi \sqrt{EG - F^2})_u + (\eta \sqrt{EG - F^2})_v}{\sqrt{EG - F^2}} \sqrt{EG - F^2}$$

$$= -\int_\Lambda 2H\phi + \int_\Lambda \mathrm{div}_\Lambda \dot{X}^{\mathrm{T}}$$

$$= -\int_\Lambda 2HX \cdot N + \int_{\partial\Lambda} \dot{X}^{\mathrm{T}} \cdot \vec{n}$$

where H is the mean curvature of Λ, $\vec{n}$ is the outward conormal to $\partial\Lambda$ and the last term comes from the divergence theorem; see (41). We note finally, that $\dot{X}^{\mathrm{T}} \cdot \vec{n} = \dot{X} \cdot \vec{n}$ because, $N \cdot \vec{n} = 0$.

This formula should be recognized as the general formula for the first variation of area for an arbitrary surface in $\mathbb{R}^3$ under an arbitrary vector valued variation. While the derivation above is somewhat more general than that of Wente [15] or Finn [7], since we allow arbitrary variations, Spivak [13] derives an even more general version which allows arbitrary dimension and also immersion in an arbitrary submanifold. We felt it was worthwhile, however, to derive the generalization in the limited context of the present discussion.

We may apply the same calculation to the variation of $\mathcal{W}$ by $X^{\mathcal{W}}$ and obtain

$$\dot{A}(\mathcal{W}) = \int_{\partial\Lambda} \dot{X} \cdot \vec{\nu} \tag{7}$$

where ν is the outward conormal to $\partial\mathcal{W} = \partial\Lambda$ with respect to $\mathcal{W}$. Notice from (6) that

$$\dot{X}^{\mathcal{W}}(q;t) = \dot{X}(q;t)$$

for every $q \in \partial M^{\mathcal{W}}$, and in the case of a fixed rigid boundary surface Σ, we are requiring $\dot{X}^{\mathcal{W}} = \dot{X}$ to be tangent to Σ along $\partial\Lambda$. This is why the surface integral of $2H^{\mathcal{W}}\dot{X} \cdot N^{\mathcal{W}}$ does not appear in (7). Moreover, along $\partial\Lambda$ we may write $\dot{X} = (\dot{X} \cdot \vec{\nu})\vec{\nu} + \delta\vec{t}$ where $\vec{t}$ is a unit tangent to $\partial\Lambda$, so that

$$\dot{\mathcal{E}} = -\int_{\Lambda} 2H\dot{X} \cdot N + \int_{\partial\Lambda} \dot{X} \cdot \vec{\nu}(\vec{\nu} \cdot \vec{n} - \beta). \tag{8}$$

If γ is the angle measured inside $\mathcal{M}$ between Λ and Σ along $\partial\Lambda$, then we obtain the familiar form

$$\dot{\mathcal{E}} = -\int_{\Lambda} 2HX \cdot N + \int_{\partial\Lambda} (\cos\gamma - \beta)\dot{X} \cdot \vec{\nu}. \tag{9}$$

We next proceed to derive an expression for

$$\dot{V} = \frac{d}{dt}V(t)$$

where

$$V = V(t) = \int_{\mathcal{M}} 1 = \frac{1}{3}\int_{\mathcal{M}} \mathrm{div}_{\mathbb{R}^3}x = \frac{1}{3}\int_{\Lambda} X \cdot N + \frac{1}{3}\int_{\Sigma^{\mathcal{W}}} X^{\mathcal{W}} \cdot N^{\mathcal{W}}$$

by the divergence theorem.

Note: The expression used for the change in $V(\mathcal{M})$ in [2] and referenced by [10] apparently assumes that the variation mapping X (on $M \times [0, \epsilon]$) gives a parameterization of the volume under consideration. By following Wente [15] (who followed Blaschke [3]) we get an expression which is valid under this assumption and also without it. The computation, however, is a little more involved.

It is interesting to note that in their earlier paper [1], Barbosa and doCarmo use also Blaschke's expression for the volume. Presumably, the formulation in the latter paper [2] is a consequence of working in a curved ambient manifold. This suggests the following question: *Is there a generalization of the trivial formula* $\mathrm{div}_{\mathbb{R}^n} x = n$ *to situations in which* $\mathbb{R}^n$ *is replaced by an n-dimensional manifold?* The question in one lower dimension, also has relevance for the main problem under consideration in this paper, because if this were the case, it might be possible to replace the wetting energy term with an integral around $\partial\Lambda$ and, hence, simplify the overall construction. It should be noted, of course that in the case of a floating object, that vector field would be expected to depend on the position of the floating object.

Finally, in using Blaschke's expression for the volume, we have effectively replaced an integral over $\mathcal{M}$ by one over $\partial\mathcal{M}$. If we are willing to consider variation of the entire liquid mass (which we are apparently forced to do below to accommodate bulk accelerations), then an alternative approach is possible. This approach is presented in the context of the more complicated situation below. $\square$

Returning to the variation of Blaschke's volume expression, we begin as before, writing Λ as a union of parameter neighborhoods $X(U)$. Then

$$\frac{d}{dt}\int_\Lambda X \cdot N = \sum_U \frac{d}{dt}\int_U X \cdot (X_u \times X_v).$$

For each local domain U,

$$\frac{d}{dt}\int_U X \cdot (X_u \times X_v) = \int_U \dot{X} \cdot (X_u \times X_v) + X \cdot (\dot{X}_u \times X_v) + X \cdot (X_u \times \dot{X}_v).$$

Considering the last two terms in this expression,

$$
\begin{aligned}
X \cdot (\dot{X}_u \times X_v) + X \cdot (X_u \times \dot{X}_v) &= [X \cdot (\dot{X} \times X_v)]_u + [X \cdot (X_u \times \dot{X})]_v \\
&\quad -X_u \cdot (\dot{X} \times X_v) - X \cdot (X_{uv} \times \dot{X}) \\
&\quad -X_v \cdot (X_u \times \dot{X}) - X \cdot (\dot{X} \times X_{uv}) \\
&= [X \cdot (\dot{X} \times X_v)]_u + [X \cdot (X_u \times \dot{X})]_v \\
&\quad +2\dot{X} \cdot (X_u \times X_v).
\end{aligned}
$$

In the last line, we have used the identity $-X_v \cdot (X_u \times \dot{X}) = \mathrm{vol}(X_u, X_v, \dot{X}) = -X_u \cdot (\dot{X} \times X_v)$ where $\mathrm{vol}(X_u, X_v, \dot{X}) = \dot{X} \cdot (X_u \times X_v)$ is the volume of the parallelepiped spanned by $\dot{X}$, X_u and X_v (up to a sign). Substituting this into the integral under consideration, we find

$$
\begin{aligned}
\frac{d}{dt}\int_U X \cdot (X_u \times X_v) &= 3\int_U \dot{X} \cdot (X_u \times X_v) \\
&\quad + \int_U [X \cdot (\dot{X} \times X_v)]_u + [X \cdot (X_u \times \dot{X})]_v
\end{aligned}
$$

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We wish to express the last integrand as the local expression of the divergence of a globally defined vector field on Λ. Substituting the expression $\dot{X} = \phi N + \psi X_u + \eta X_v = \phi N + \dot{X}^{\mathrm{T}}$ for $\dot{X}$, we find that the integrand can be written as

$$
\begin{aligned}
[X \cdot (\dot{X} \times X_v)]_u + [X \cdot (X_u \times \dot{X})]_v &= \left[X \cdot \frac{\phi(X_u \times X_v) \times X_v}{\sqrt{EG - F^2}}\right]_u \\
&+ \left[\sqrt{EG - F^2}\,\psi X \cdot N\right]_u \\
&+ \left[X \cdot \frac{\phi X_u \times (X_u \times X_v)}{\sqrt{EG - F^2}}\right]_v \\
&+ \left[\sqrt{EG - F^2}\,\eta X \cdot N\right]_v.
\end{aligned}
$$

Notice that the second and fourth terms sum to precisely

$$
\sqrt{EG - F^2}\,\mathrm{div}_\Lambda[(X \cdot N)\dot{X}^{\mathrm{T}}].
$$

Applying the triple cross product rule, we find

$$
(X_u \times X_v) \times X_v = -GX_u + FX_v \quad \text{and} \quad X_u \times (X_u \times X_v) = FX_u - EX_v.
$$

Thus, the first and third terms sum to

$$
\left[\sqrt{EG - F^2}\,\phi\frac{-GX \cdot X_u + FX \cdot X_v}{EG - F^2}\right]_u \\
+ \left[\sqrt{EG - F^2}\,\phi\frac{-FX \cdot X_u - EX \cdot X_v}{EG - F^2}\right]_v.
$$

Comparing this with the expression of a general vector field in local coordinates (37), we see that this is

$$
-\sqrt{EG - F^2}\,\mathrm{div}_\Lambda[\phi X^{\mathrm{T}}].
$$

Going back to the original integral, we have

$$
\frac{d}{dt}\int_U X \cdot (X_u \times X_v) = 3\int_{X(U)} \dot{X} \cdot N + \int_{X(U)} \mathrm{div}_\Lambda[\phi\dot{X}^{\mathrm{T}} - (X \cdot N)X^{\mathrm{T}}].
$$

Properly speaking, the expressions X and $\dot{X}$ in the integrals on the right side of this equation represent respectively $X(X^{-1}(x;t);t)$ and $\dot{X}(X^{-1}(x;t);t)$ for $x \in X(U) \subset \Lambda$. The function $v(x;t) := \dot{X}(X^{-1}(\cdot;t);t)$ will play an important role in the case of bulk accelerations below.

Finally summing over all neighborhoods U and recalling that $\phi = \dot{X} \cdot N$, we have

$$
\begin{aligned}
\frac{1}{3}\frac{d}{dt}\int_\Lambda X \cdot N &= \int_\Lambda \dot{X} \cdot N + \frac{1}{3}\int_\Lambda \mathrm{div}_\Lambda[(X \cdot N)\dot{X}^{\mathrm{T}} - (\dot{X} \cdot N)X^{\mathrm{T}}] \\
&= \int_\Lambda \dot{X} \cdot N + \frac{1}{3}\int_{\partial\Lambda} [(X \cdot N)\dot{X}^{\mathrm{T}} - (\dot{X} \cdot N)X^{\mathrm{T}}] \cdot \vec{n} \\
&= \int_\Lambda \dot{X} \cdot N + \frac{1}{3}\int_{\partial\Lambda} [(X \cdot N)\dot{X} - (\dot{X} \cdot N)X] \cdot \vec{n}.
\end{aligned}
$$

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Again, we have used the divergence theorem. The last line results from the fact that the vectors $(X \cdot N)\dot{X}$ and $(\dot{X} \cdot N)X$ have the same component in the direction N.

The same calculation applies to the integral over $\mathcal{W}$ appearing in the expression for V so that

$$\dot{V} = \int_\Lambda \dot{X} \cdot N + \int_\mathcal{W} \dot{X}^\mathcal{W} \cdot N^\mathcal{W} + \mathrm{J}$$

where

$$3J = \int_{\partial\Lambda} [(X \cdot N)\dot{X} - (\dot{X} \cdot N)X] \cdot \vec{n} + [(X \cdot N^\mathcal{W})\dot{X} - (\dot{X} \cdot N^\mathcal{W})X] \cdot \vec{\nu}.$$

It would be usual at this point to assume $\dot{X} \cdot N^\mathcal{W} = 0$ along $\partial\Lambda$ so that one of the boundary integral terms vanishes. In keeping with the use of quite general vector valued variations, we will avoid such an assumption. We will substitute rather

$$\dot{X} = \phi N + \delta \vec{n} + \eta \vec{t},$$

$$\vec{\nu} = (\vec{\nu} \cdot N)N + (\vec{\nu} \cdot \vec{n})\vec{n},$$

and

$$N^\mathcal{W} = -(\vec{\nu} \cdot \vec{n})N + (\vec{\nu} \cdot N)\vec{n},$$

where $\vec{t}$ is a vector tangent to $\partial\Lambda$. In this way, we find

$$
\begin{aligned}
[(X \cdot N)\dot{X} &- (\dot{X} \cdot N)X] \cdot \vec{n} + [(X \cdot N^\mathcal{W})\dot{X} - (\dot{X} \cdot N^\mathcal{W})X] \cdot \vec{\nu} \\
&= \phi[-X \cdot \vec{n} + (X \cdot N^\mathcal{W})(\vec{\nu} \cdot N) + (\vec{\nu} \cdot \vec{n})(X \cdot \vec{\nu})] \\
&\quad + \delta[X \cdot N + (X \cdot N^\mathcal{W})(\vec{\nu} \cdot \vec{n}) - (\vec{\nu} \cdot N)(X \cdot \vec{\nu})] \\
&= \phi[-(X \cdot \vec{n})(1 - (\vec{\nu} \cdot \vec{n})^2 - (\vec{\nu} \cdot N)^2)] \\
&\quad + \delta[(X \cdot N)(1 - (\vec{\nu} \cdot \vec{n})^2 - (\vec{\nu} \cdot N)^2)] \\
&= 0.
\end{aligned}
$$

Thus, we have a general variational formula for volume with respect to an arbitrary admissible vector valued variation:

$$\dot{V} = \int_\Lambda \dot{X} \cdot N + \int_\mathcal{W} \dot{X}^\mathcal{W} \cdot N^\mathcal{W}. \tag{10}$$

In the situation presently under consideration, we also have that $\dot{X}^\mathcal{W}$ is in the tangent space of Σ, so that the second integral also vanishes, and we obtain the familiar

$$\dot{V} = \int_\Lambda \dot{X} \cdot N. \tag{11}$$

If the additional term $\mathcal{G} = \int_\mathcal{M} g$ is included in the energy, we require in addition that the third restriction of our variation $X\big|_{\mathcal{M}_0 \times \{t\}} : \mathcal{M}_0 \to \mathbb{R}^3$ parameterizes its image which is

required to be the admissible liquid mass $\mathcal{M} = \mathcal{M}_t$ bounded by Λ and $\mathcal{W}$. In this more complicated situation, we first write

$$\mathcal{G} = \int_{\mathcal{M}_0} g \circ X \det DX,$$

where we have omitted the absolute values on the Jacobian factor for small t under the assumption of smoothness in t on the third restriction and the observation that $\det DX(\cdot; 0) = 1 > 0$.

This puts us in a position to apply Euler's first kinematical theorem (identity (4.1) in [12]) to obtain

$$
\begin{aligned}
\dot{\mathcal{G}} &= \int_{\mathcal{M}} [Dg \cdot v + g \operatorname{div}_{\mathbb{R}^3} v] \\
&= \int_{\mathcal{M}} \operatorname{div}_{\mathbb{R}^3}[gv] \\
&= \int_{\Lambda} g \dot{X} \cdot N + \int_{\mathcal{W}} g \dot{X}^{\mathcal{W}} \cdot N^{\mathcal{W}}
\end{aligned}
$$

by the divergence theorem where $v(x; t) = \dot{X}(X^{-1}(x; t); t)$ is the *spatial velocity* of the flow induced by the variation. (In the last expression, we have returned to the use of $\dot{X}$ to represent $v(\cdot; t)$.) A detailed derivation of the kinematical identity in the special case under consideration may be found in the appendix.

If Σ is a fixed rigid surface, $\dot{X}^{\mathcal{W}} \cdot N^{\mathcal{W}} = 0$ as before, and we obtain, in complete generality, the formula

$$\dot{\mathcal{G}} = \int_{\Lambda} g \dot{X} \cdot N$$

which was obtained in special cases in [15] and [7] but not considered in [10] or [13].
Note: If one is willing to admit the full variation of the interior liquid mass, this approach gives also, in the special case $g \equiv 1$, formulae (10) and (11).

The next step in developing the variational theory of these surfaces would be to introduce a Lagrange parameter corresponding to the volume constraint of the problem. We will have the opportunity to consider this topic under more general circumstances below, and hence move directly to those considerations.

3 Movable Boundaries

We now wish to allow portions of the rigid bounding surface Σ to assume the character of free floating or partially free floating rigid obstacles. To this end, we assume Σ_0 (and correspondingly the relevant abstract manifold $M^{\mathcal{W}} = \mathcal{W}_0$) to be composed of disjoint components Σ_s and Σ_m. The former portions remain fixed, or stationary, as in the discussion above, while the latter are movable in the sense we now describe.

Let $w = w(x; h)$ be a family of rigid motions of $\mathbb{R}^3$ indexed by and smoothly depending on the parameter h which for simplicity we will assume is real valued. Correspondingly, we will generalize our notion of variation to that of smooth maps $X : \overline{\mathcal{M}}_0 \times (-\delta, \delta) \times (-\epsilon, \epsilon) \to \mathbb{R}^3$, which for each fixed $h \in (-\delta, \delta)$ satisfy the conditions set forth in the previous section, with one slight modification. The new parameter h lying between $-\delta$ and δ will index the position of the movable portion of Σ, so that for fixed h the image of $X_m^{\mathcal{W}}$ is required to lie in $w(\Sigma_m; h)$. Otherwise, the parameter t on $(-\epsilon, \epsilon)$ is the variation parameter as before. When convenient, we will continue to refer to the union $M_s^{\mathcal{W}} \cup M_m^{\mathcal{W}} = \mathcal{W}_{0s} \cup \mathcal{W}_{0m}$ simply as $M^{\mathcal{W}}$. As we have denoted derivatives with respect to t with the customary "˙", we will denote derivatives with respect to h with a "´". For each h and t, the energy of and volume associated with the configuration are given by the same expressions as before.

We will assume that $\mathcal{E}(X_0, X_0^{\mathcal{W}}) \leq \mathcal{E}(X, X^{\mathcal{W}})$ for all C^1 admissible variations $(X, X^{\mathcal{W}})$ in a C^0 neighborhood of $(X_0, X_0^{\mathcal{W}})$ whose associated volume is also constrained to match that of the original minimizing configuration.

We give now a brief justification for the existence of a Lagrange multiplier in this problem. The argument is slightly more general than that seen for integral operators of the form (1) because we avoid the use of "additive variations," i.e., the usual $u + \epsilon \phi$. This is essentially an avoidance of assuming a linear structure in the class of admissible variations. The argument is also somewhat different than that usually given for Fréchet differentiable functionals as in [9] or for Gâteaux differentiable functionals as in [11] or [5] for essentially the same reason.

We begin with a standard single parameter variation $Y : \overline{\mathcal{M}}_0 \times (-\epsilon, \epsilon) \to \mathbb{R}^3$ of $\mathcal{M}_0$ for which the volume has nonvanishing first variation. Examination of the variational formula for volume indicates that this is always possible; we can arrange for example that the first restriction (to Λ_0) satisfies $\dot{Y} \cdot N \geq 0$ with strict inequality on some open subset of Λ_0. We denote the parameter for this variation by τ so that $Y = Y(p; \tau)$. Corresponding to each fixed τ, we consider a further variation $X = X(p; h; t; \tau)$ with $X(p; 0; 0; \tau) = Y(p; \tau)$. This construction results in two smooth real valued functions of three variables,

$$f(h, t, \tau) = \mathcal{E}(X, X^{\mathcal{W}})$$

and

$$g(h, t, \tau) = V(X, X^{\mathcal{W}})$$

having the property that $f(0, 0, 0)$ is a minimum for $f(h, t, \tau)$ subject to the constraint $g(h, t, \tau) = V(X_0, X_0^{\mathcal{W}})$. The usual principle of Lagrange applied to these functions asserts the gradients of f and g must be parallel at the origin, i.e., for some constant λ

$$Df(0, 0, 0) = \lambda Dg(0, 0, 0).$$

This results in three equations, the first and second of which are

$$\acute{\mathcal{E}} - \lambda \acute{V} = 0 \quad \text{and} \quad \dot{\mathcal{E}} - \lambda \dot{V} = 0 \tag{12}$$

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where everything is evaluated at $h = t = 0$. We have computed expressions (9) and (11) for $\dot{\mathcal{E}}$ and $\dot{V}$ above, so that the second equation becomes

$$\dot{\mathcal{E}} - \lambda\dot{V} = -\int_{\Lambda_0} (2H + \lambda)\dot{X} \cdot N + \int_{\partial\Lambda_0} (\cos\gamma - \beta)\dot{X} \cdot \vec{n} = 0. \tag{13}$$

By taking first interior variations (for which $\dot{X} \equiv 0$ on $\partial\Lambda_0$ we find

$$2H = -\lambda = \text{constant} \quad \text{on } \Lambda_0.$$

Thus, the first integral vanishes, and we can take $\dot{X} \cdot \vec{n}$ arbitrary on $\partial\Lambda_0$, so that

$$\cos\gamma = \beta = \text{constant} \quad \text{on } \partial\Lambda_0.$$

These are the usual first order necessary conditions for a capillary surface in zero gravity. The inclusion of $\mathcal{G}$ evidently leads also to the usual prescribed mean curvature equation $2H = g - \lambda$ where g is a volumetric potential energy density appropriate to any accelerations present.

We now turn to the first equation in (12). Applying the reasoning that led to $\dot{\mathcal{E}}$ but differentiating instead with respect to h, we note that there is no reason to believe $\acute{X}^{\mathcal{W}}$ lies in the tangent space to Σ. Thus, we obtain a variational expression with all terms present:

$$\begin{aligned}
\acute{\mathcal{E}} - \lambda\acute{V} = &-\int_{\Lambda} 2H\acute{X} \cdot N + \int_{\partial\Lambda} \acute{X} \cdot \vec{n} + \beta\int_{\mathcal{W}} 2H^{\mathcal{W}}\acute{X}^{\mathcal{W}} \cdot N^{\mathcal{W}} - \beta\int_{\partial\Lambda} \acute{X} \cdot \vec{\nu} \\
&-\lambda\int_{\Lambda} \acute{X} \cdot N - \lambda\int_{\mathcal{W}} \acute{X}^{\mathcal{W}} \cdot N^{\mathcal{W}}
\end{aligned}$$

where $H^{\mathcal{W}}$ is the mean curvature on the wetted region $\mathcal{W}$. We note, that we could, of course eliminate some of the integrals over portions $\mathcal{W}_s$ of $\mathcal{W}$ which are stationary, but we will save on subscripts by including them for the moment. We can use the prescribed mean curvature equation to cancel the integrals over Λ, and we may also substitute the contact angle boundary condition so that for $\Lambda = \Lambda_0$ and $\Sigma = \Sigma_0$ we have

$$\int_{\partial\Lambda} \acute{X} \cdot (\vec{n} - \cos\gamma\vec{\nu}) + \cos\gamma\int_{\mathcal{W}} 2H^{\mathcal{W}}\acute{X}^{\mathcal{W}} \cdot N^{\mathcal{W}} + \int_{\mathcal{W}} 2H\acute{X}^{\mathcal{W}} \cdot N^{\mathcal{W}} = 0.$$

We have moved the constant $2H$ inside the integral; it may be easily checked that this same formula holds in the case of bulk accelerations where $2H = g - \lambda$ may be nonconstant. Noting that $\vec{n} - \cos\gamma\vec{\nu} = \vec{n} - (\vec{n} \cdot \vec{\nu})\vec{\nu}$ and that $N^{\mathcal{W}}$ and $\vec{\nu}$ form an orthonormal basis, we see that $\vec{n} - \cos\gamma\vec{\nu} = \sin\gamma N^{\mathcal{W}}$. Thus the condition becomes

$$\int_{\partial\Lambda} \sin\gamma\acute{X} \cdot N^{\mathcal{W}} + \cos\gamma\int_{\mathcal{W}} 2H^{\mathcal{W}}\acute{X}^{\mathcal{W}} \cdot N^{\mathcal{W}} + \int_{\mathcal{W}} 2H\acute{X}^{\mathcal{W}} \cdot N^{\mathcal{W}} = 0. \tag{14}$$

This is a first version of our new first order condition which applies to situations in which the solid support structures are deformable. We have not used, however, any special properties of the deformation $X^{\mathcal{W}}$ and of the smooth family of rigid motions $w : \mathbb{R}^3 \times (-\delta, \delta) \to \mathbb{R}^3$. Using these properties should allow us to express the formula in a manner that is essentially independent of $\acute{X}^{\mathcal{W}}$ in the interior of $\mathcal{W}$. The crucial observation is that because

$$\Sigma_{m;h} = \Sigma_h = w(\Sigma_m; h)$$

we have

$$w^{-1}(X^{\mathcal{W}}) \in \Sigma_0 \tag{15}$$

where we have suppressed the subscript "m," and $X^{\mathcal{W}}$ denotes $X_m^{\mathcal{W}}(x; h, t); h$. (We note that in the following discussion, we will also consistently suppress t dependence.)

The inclusion (15) implies

$$Dw^{-1}(X^{\mathcal{W}})\acute{X}^{\mathcal{W}} + \acute{w}^{-1}(X^{\mathcal{W}}) \in T_{w^{-1}(X^{\mathcal{W}})}\Sigma_0.$$

Recall, however, that for fixed h, the transformations w and w^{-1} are rigid motions. Therefore, we know that, for example, $w - w(0)$ is linear and has a corresponding matrix Dw which depends only on h. Since multiplication by this matrix also corresponds to the differential mapping $dw : T_{w^{-1}(X^{\mathcal{W}})}\Sigma_0 \to T_{X^{\mathcal{W}}}\Sigma$ of tangent spaces where $\Sigma = \Sigma_h$, we find that

$$\acute{X}^{\mathcal{W}} + Dw\acute{w}^{-1}(X^{\mathcal{W}}) \in T_{X^{\mathcal{W}}}\Sigma. \tag{16}$$

Differentiating the defining relation $w(w^{-1}(x; h); h) = x$ with respect to h, we find

$$Dw(w^{-1}(x; h); h)\acute{w}^{-1}(x; h) = -\acute{w}(w^{-1}(x; h); h).$$

Therefore, (16) can also be written as

$$\acute{X}^{\mathcal{W}} - \acute{w}(w^{-1}(X^{\mathcal{W}})) \in T_{X^{\mathcal{W}}}\Sigma. \tag{17}$$

From this it follows that each of the terms $\acute{X}^{\mathcal{W}} \cdot N^{\mathcal{W}}$ appearing in (14) may be replaced with $\acute{w} \cdot N^{\mathcal{W}} = \acute{w}(\cdot; 0) \cdot N^{\mathcal{W}}$.

$$\sin\gamma \int_{\partial\Lambda} \acute{w} \cdot N^{\mathcal{W}} + \cos\gamma \int_{\mathcal{W}} 2H^{\mathcal{W}}\acute{w} \cdot N^{\mathcal{W}} + \int_{\mathcal{W}} 2H\acute{w} \cdot N^{\mathcal{W}} = 0. \tag{18}$$

(We are still suppressing the subscript "m" on all the domains of integration; technically, $\partial\Lambda = \partial\mathcal{W}_s \cup \partial\mathcal{W}_m$ and the stationary portion of the first integral vanishes because $\acute{X}^{\mathcal{W}} \cdot N^{\mathcal{W}} \equiv 0$ there.)

Next, we temporarily fix h_0 and consider a specific motion of the wetted region $\mathcal{W}_{h_0}$ as follows. Let $\tilde{\mathcal{W}} = \tilde{\mathcal{W}}_h$ be parameterized on M_m by

$$p \mapsto w(w^{-1}(X^{\mathcal{W}}(p; h_0); h_0); h).$$

Notice that for $h = h_0$ the surface $\tilde{\mathcal{W}}$ is exactly $\mathcal{W}_{h_0}$ and for $h > h_0$ the surface $\tilde{\mathcal{W}}$ is just a rigid motion of the same wetted region and, therefore, has the same area. Thus, using the general formula for the variation of surface area in this special case, we have

$$0 = \frac{d}{dh} A(\tilde{\mathcal{W}}) = - \int_{\tilde{\mathcal{W}}} 2\tilde{H}^{\mathcal{W}} \acute{w} \cdot \tilde{N}^{\mathcal{W}} + \int_{\partial\tilde{\mathcal{W}}} \acute{w} \cdot \tilde{\nu},$$

where $\tilde{H}^{\mathcal{W}}$ is the mean curvature on $\tilde{\mathcal{W}}$, $\tilde{N}^{\mathcal{W}}$ is the normal on $\tilde{\mathcal{W}}$, $\tilde{\nu}$ is the conormal on $\partial\tilde{\mathcal{W}}$, and the vectors $\acute{w}$ appearing in the integrals are evaluated at $(w^{-1}(w(w^{-1}(x; h_0); h); h_0); h)$. We now change variables in the integrals above using the map $x \mapsto w(w^{-1}(x; h); h_0)$ which maps $\tilde{\mathcal{W}}_h$ back to $\tilde{\mathcal{W}}_{h_0} = \mathcal{W}_{h_0}$. We then have

$$- \int_{\mathcal{W}_{h_0}} 2H^{\mathcal{W}} Dw(w^{-1}(\cdot; h); h_0) \, Dw^{-1}(\cdot; h_0) \, \acute{w}(w^{-1}(\cdot; h_0); h) \cdot N^{\mathcal{W}} +$$

$$\int_{\partial\mathcal{W}_{h_0}} Dw(w^{-1}(\cdot; h); h_0) \, Dw^{-1}(\cdot; h_0) \, \acute{w}(w^{-1}(\cdot; h_0); h) \cdot \vec{\nu} = 0.$$

Setting $h = h_0$ and suppressing all the h's again so that $\acute{w}(w^{-1}) = \acute{w}(w^{-1}(\cdot; h); h)$, we get

$$- \int_{\mathcal{W}} 2H^{\mathcal{W}} \acute{w}(w^{-1}) \cdot N^{\mathcal{W}} + \int_{\partial\mathcal{W}} \acute{w}(w^{-1}) \cdot \vec{\nu} = 0.$$

Using this identity to replace the middle integral on the left in (18), we find at $h = 0$ the condition

$$\int_{\partial\mathcal{W}} \acute{w} \cdot (\sin\gamma N^{\mathcal{W}} + \cos\gamma \, \vec{\nu}) + \int_{\mathcal{W}} 2H \acute{w} \cdot N^{\mathcal{W}}.$$

Finally, we note that $\sin\gamma N^{\mathcal{W}} + \cos\gamma \vec{\nu} = \vec{n}$ and modify the notation to emphasize once again the significance of the movable portion Σ_m of the solid support surface to obtain the fundamental formula.

Theorem 1 *Assume X_0 parameterizes a capillary surface Λ_0 that meets solid support surfaces at a constant contact angle γ and defines a liquid mass $\mathcal{M}$ as described above. If moreover, Λ_0 provides a minimum among variations compatible with the motion of a portion Σ_m of the bounding surfaces allowed by the rigid motions $w = w(x; h)$, then*

$$\int_{\mathcal{W}_m} 2H\acute{w} \cdot N^{\mathcal{W}} + \int_{\partial\mathcal{W}_m} \acute{w} \cdot \vec{n} = 0,$$

where $\acute{w} = \partial w/\partial h(\cdot; 0)$. It should be noted in this formula that $\vec{n}$ is the conormal to $\partial\Lambda \supset \partial\mathcal{W}_m$ and H is the mean curvature of Λ. The formula holds both when H is constant and when $2H = g - \lambda$ is prescribed by bulk accelerations.

We now apply this result to a particular physical situation.

4 The Floating Ball

In [8] Finn considers the configuration shown in Figure 3 in which a planar interface $z \equiv 0$ for $\alpha = a \sin \gamma \leq r \leq R$ meets a cylindrical container of radius R at an angle $\pi/2$ and a sphere held rigidly on the axis of the container at an angle of contact γ. If the ball remains
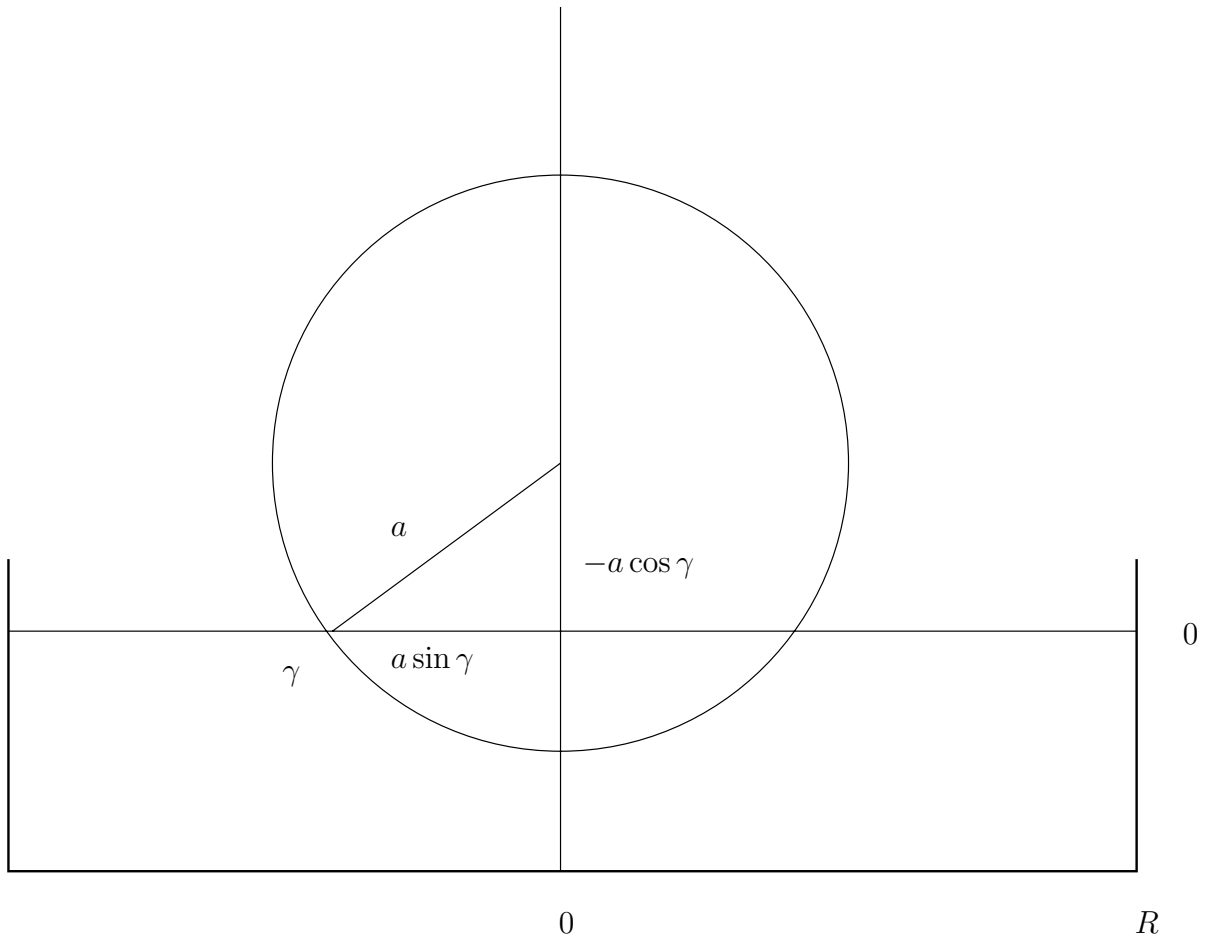

Figure 3: A planar interface determining a liquid mass of given volume and prescribed contact angle.

rigidly fixed with center at height $-a \cos \gamma$, then we may apply the stability theory of Vogel [14] to obtain the following result whose proof we give at the end of the section.

Theorem 2 *For the fixed ball problem, the planar interface is linearly stable and is a strong local minimum in the sense of Vogel.*

We conclude from this, that such an initial configuration is almost certainly experimentally reproducible, though with some external support to hold the ball in place. One would like

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to know if the ball will remain in this position and the planar interface will persist if the external support were to be removed. Will the ball float in this position as suggested by Finn?

A theory allowing such a decisive conclusion is apparently unavailable in the situation of moving boundaries. Our calculation leading to Theorem 1 however allows us to make an argument that stability with respect to vertical motion of the ball in this problem is also likely.

Finn shows in his paper that if the ball is moved (or suggestively pushed) downward by a small distance h, then there is a unique rotationally symmetric constant mean curvature surface that meets the ball still at the angle γ, the container still at angle $\pi/2$ and determines a liquid mass of the same volume. The same assertions are true if one raises the ball a small
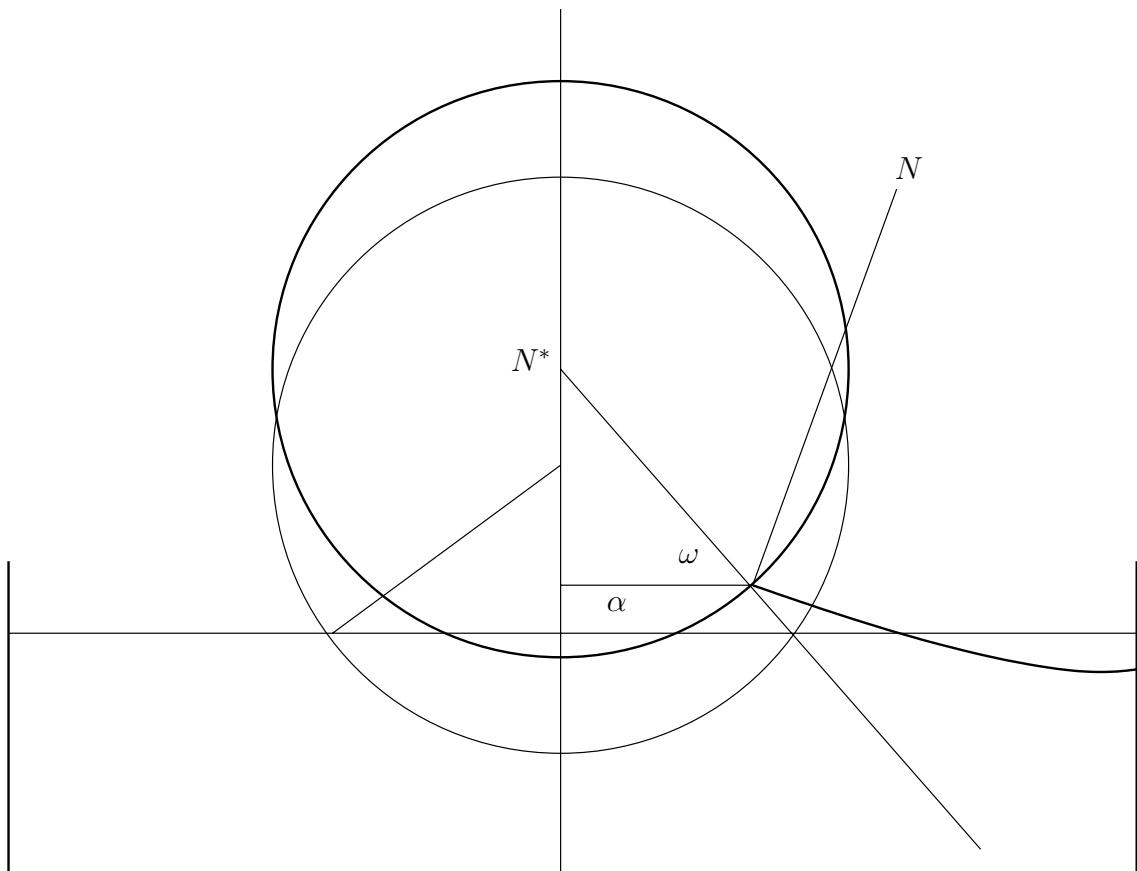

Figure 4: Nodoid interface.

distance h (see Figure 4), and the surfaces are, in each case, nodoids whose meridians are obtained as the path of the focal point of a hyperbola as it rolls along the vertical axis

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through the center of the ball. If for each $h \in (-\delta, \delta)$, these auxiliary configurations have free surface interface Λ_h and wetted area $\mathcal{W}_h$, where the center of the ball is located at $h - a \cos \gamma$, then presumably the overall stability of the configuration (at least with respect to vertical motion of the ball) should be characterized by the values

$$\mathcal{E}(h) = A(\Lambda_h) - \cos \gamma A(\mathcal{W}_h).$$

We wish to compute, in particular, $\acute{\mathcal{E}} = d\mathcal{E}/dh$; the volume term can be ignored, since all the volumes are the same. The quantity $\mathcal{E}(h)$ can be written explicitly in terms of integrals as

$$\mathcal{E} = 2\pi \int_\alpha^R r\sqrt{1 + u'(r)^2}dr - \cos \gamma \, 2\pi \int_0^\alpha \frac{ar}{\sqrt{a^2 - r^2}} \, dr \qquad (19)$$

where u and α are determined as follows. The function u solves the ordinary differential equation of constant mean curvature for rotationally symmetric surfaces. This equation is

$$\frac{1}{r}\left(\frac{ru'}{\sqrt{1 + u'^2}}\right)' = 2H_h. \qquad (20)$$

Denoting by $\alpha = \alpha(h)$ the radius at which the nodoid surface meets the ball, we can integrate to find

$$u(r) = u(\alpha) + \int_\alpha^r \frac{\alpha q + H_h(t^2 - \alpha^2)}{|H_h|\sqrt{(\rho_{\text{out}}^2 - t^2)(t^2 - \rho_{\text{in}}^2)}} \, dt, \qquad (21)$$

where

$$u(\alpha) = h - a \cos \gamma - \sqrt{a^2 - \alpha^2},$$

$$q = q(h) = \frac{u'(\alpha)}{\sqrt{1 + u'(\alpha)^2}} = -\left[\frac{\alpha}{a}\cos \gamma + \sqrt{1 - \left(\frac{\alpha}{a}\right)^2}\sin \gamma\right] = -\cos(\gamma - \omega)$$

where ω is the smaller angle made by a segment from the center of the ball to the contact line with the horizontal (see Figure 4),

$$H_h = -\frac{\alpha q}{R^2 - \alpha^2} = \frac{\alpha \cos(\gamma - \omega)}{R^2 - \alpha^2}$$

which follows from the fact that $u'(R) = 0$ and the first integral of equation (20), and $0 < \rho_{\text{in}} < \rho_{\text{out}}$ are the positive roots of $r^2 - [\alpha q + H_h(r^2 - \alpha^2)]^2 = 0$. Finally, it remains to describe $\alpha = \alpha(h)$ which is determined implicitly by the equation

$$\begin{aligned}
V &= \pi R^2 \ell + 2\pi \int_0^\alpha r(h - a \cos \gamma - \sqrt{a^2 - r^2})dr + 2\pi \int_\alpha^R ru(r)dr \\
&= \pi R^2(\ell + h - a \cos \gamma) - 2\pi \int_0^\alpha r\sqrt{a^2 - r^2} \, dr \\
&\qquad -2\pi\sqrt{a^2 - \alpha^2} \int_\alpha^R rdr + 2\pi \int_\alpha^R r \int_\alpha^r u'(t)dtdr \qquad (22)
\end{aligned}$$

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where ℓ denotes the initial depth of the liquid near the wall of the cylindrical container. In order to show that $\alpha = \alpha(h)$ is well defined for h in some neighborhood of zero, we think of the expression above as defining $V = V(\alpha, h)$ and show that

$$\frac{\partial V}{\partial \alpha}(a \sin \gamma, 0) \neq 0.$$

This calculation is contained in the proof of Lemma 2 below.

The expression for u may also be expressed in terms of standard elliptic integrals, as indicated in [8]. Then it will be noted that the volume above as well as $\mathcal{E}$ will depend on integrals of complicated expressions involving elliptic integrals. This should suffice to indicate the difficulty in calculating $\acute{\mathcal{E}}$ directly. Since we know that $\acute{V} = 0$ for this variation, however, we have direct recourse to the formula of Theorem 1 which is completely independent of the value of $u = u(r)$ for $\alpha < r < R$.

Lemma 1 *For the floating ball problem described above*

$$\acute{\mathcal{E}} = \frac{d\mathcal{E}}{dh} = \frac{2\pi\alpha R^2}{R^2 - \alpha^2} \cos(\gamma - \omega).$$

Proof: In this case, the rigid motion is simply translation in the vertical direction $w(x; h) = x + h\mathbf{e}_3$ and $\acute{w} = \mathbf{e}_3$. The wetted region (on the ball) $\mathcal{W}$ is the portion of lower hemisphere $\{(x, y, h - a\cos\gamma - \sqrt{a^2 - r^2}) : 0 \leq r^2 = x^2 + y^2 \leq \alpha^2\}$, and can be parameterized in spherical coordinates (θ, ϕ) for $0 \leq \theta \leq 2\pi$ and $0 \leq \phi \leq \pi/2 - \omega$. Thus from the formula in Theorem 1,

$$\begin{aligned}
\acute{\mathcal{E}} &= 2\pi\alpha\mathbf{e}_3 \cdot (\sin\gamma N^{\mathcal{W}} + \cos\gamma\vec{\nu}) + 2H_h 2\pi \int_0^{\pi/2-\omega} (\mathbf{e}_3 \cdot N^{\mathcal{W}})a^2 \sin\phi \, d\phi \\
&= 2\pi\alpha(\sin\gamma\sin\omega + \cos\gamma\cos\omega) + \frac{2\alpha\cos(\gamma - \omega)}{R^2 - \alpha^2} 2\pi a^2 \int_0^{\pi/2-\omega} \cos\phi\sin\phi \, d\phi \\
&= 2\pi\alpha\cos(\gamma - \omega)\left[1 + \frac{a^2}{R^2 - \alpha^2}\sin^2(\pi/2 - \omega)\right].
\end{aligned}$$

Since $a^2 \sin^2(\pi/2 - \omega) = \alpha^2$, this is the formula given above. $\square$

Notice that α and $R^2 - \alpha^2$ are both positive. Thus, the sign of $\acute{\mathcal{E}}$ in this case is determined by the factor $\cos(\gamma - \omega)$. We know that when $h = 0$ we have $\omega = \gamma - \pi/2$ so that $\acute{E}(0) = 0$. In order to determine the value of $\acute{\mathcal{E}}$ for other values of h we must examine more closely the volume equation for $\alpha = a\cos\omega$.

Lemma 2 *The function $\alpha = \alpha(h)$ defined in a neighborhood of $h = 0$ satisfies $\alpha'(0) < 0$. Consequently, given h in some neighborhood of zero, since $\cos\omega = \alpha/a$, if $h < 0$, then $\omega < \gamma - \pi/2$; if $h > 0$, then $\omega > \gamma - \pi/2$.*

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Proof: In this proof, we consider the expressions in the definition of the nodoid meridians (21) to be functions of both α and h as independent variables. From this point of view, we note that the expressions for q, H and u' have no explicit dependence on h. Furthermore, setting

$$v = v(\alpha; r) = \alpha q + H(r^2 - \alpha^2),$$

we obtain an alternative expression for $u' = u'(r; \alpha)$:

$$u' = \frac{v}{\sqrt{r^2 - v^2}}.$$

Differentiating the expression for the fixed volume (22) with respect to h, we find

$$0 = \pi R^2 + \left[\pi \left(\frac{\alpha}{\sqrt{a^2 - \alpha^2}} - u'(\alpha) \right) (R^2 - \alpha^2) + 2\pi \int_\alpha^R r \int_\alpha^r \frac{\partial u'}{\partial \alpha}(t) dt dr \right] \alpha'(h).$$

Thus, we wish to show

$$\lim_{h \to 0} \left[\left(\frac{\alpha}{\sqrt{a^2 - \alpha^2}} - u'(\alpha) \right) (R^2 - \alpha^2) + 2 \int_\alpha^R r \int_\alpha^r \frac{\partial u'}{\partial \alpha}(t) dt dr \right] > 0.$$

We consider each term individually; each term has a completely nonsingular limit.

$$\frac{\alpha}{\sqrt{a^2 - \alpha^2}} (R^2 - \alpha^2) \longrightarrow -\tan\gamma(R^2 - a^2 \sin^2\gamma) \quad \text{as } h \to 0.$$

Since $q, H \to 0$ as $h \to 0$, we see that v tends uniformly to zero for r in some neighborhood of $[a \sin\gamma, R]$. In particular, $\alpha u'(\alpha) \to 0$ as $h \to 0$, and the next term vanishes in the limit.

The last term requires some preliminary calculations and limits:

$$\begin{aligned}
\frac{\partial u'}{\partial \alpha} &= \frac{r^2}{(r^2 - v^2)^{3/2}} \frac{\partial v}{\partial \alpha} \\
&= \frac{r^2}{(r^2 - v^2)^{3/2}} \left(q + \alpha \frac{\partial q}{\partial \alpha} + \frac{\partial H}{\partial \alpha}(r^2 - \alpha^2) - 2\alpha H \right).
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\frac{\partial q}{\partial \alpha} &= -\frac{1}{a} \cos\gamma + \frac{\alpha}{a\sqrt{a^2 - \alpha^2}} \sin\gamma \\
&\to -\frac{1}{a}(\cos\gamma + \tan\gamma\sin\gamma) = -\frac{1}{a\cos\gamma} \quad \text{as } h \to 0,
\end{aligned}$$

and

$$\begin{aligned}
\frac{\partial H}{\partial \alpha} &= -\frac{\partial}{\partial \alpha}\left(\frac{\alpha}{R^2 - \alpha^2} \right) q - \left(\frac{\alpha}{R^2 - \alpha^2} \right) \frac{\partial q}{\partial \alpha} \\
&\to \frac{\tan\gamma}{R^2 - a^2 \sin^2\gamma} \quad \text{as } h \to 0.
\end{aligned}$$

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Therefore,

$$\frac{\partial u'}{\partial \alpha} \to \frac{1}{r}\left(-\tan\gamma + \tan\gamma \frac{r^2 - a^2 \sin^2\gamma}{R^2 - a^2 \sin^2\gamma}\right) = -\frac{\tan\gamma}{r(R^2 - a^2 \sin^2\gamma)}(R^2 - r^2) \quad \text{as } h \to 0.$$

Thus, since $-\tan\gamma > 0$, the first term has a positive limit; the second term vanishes in the limit, and the last term, having a positive integrand, has also a positive limit. $\square$

We recall that each of the nodoid interfaces is considered a critical interface for the fixed ball problem. That is, they satisfy the necessary conditions of having constant mean curvature and constant contact angle equal to γ on the contact line with the ball. With the addition of our third necessary condition in the floating ball problem, we can now state the following.

Theorem 3 *With respect to the variation of the planar interface through nodoid interfaces parameterized by the height of the ball h, the plane is the only equilibrium for h in some neighborhood of $h = 0$ and provides a strict minimum of energy among those interfaces.*

We were unable to analyze precisely the nodoid interfaces (even concerning existence and connectedness) for h far from zero, but since the formula in Lemma 1 results only from boundary considerations and we can say that $\cos(\gamma - \omega) \neq 0$ for every nodoid interface, we can make the following assertion:

Theorem 4 *The unique rotationally symmetric equilibrium for the floating ball problem is the planar interface; each nodoid interface is unstable to first order with respect to motion of the ball.*

It is, of course, unlikely that any of the other nodoid interfaces would have lower energy than the plane. Presumably, an analytic argument could be given to cover the component of those interfaces connected via h to the neighborhood of the planar interface considered in Theorem 3, and the global picture could be understood numerically if not analytically. For now, we have illustrated the use of Theorem 1.

Lastly, we return to the fixed ball problem and show that the planar interface is a strong local minimum in the sense of Vogel.

Proof of Theorem 2: This configuration satisfies the conditions stated in [14]. Therefore, the stability for the interface is determined by the eigenvalue problem

$$\begin{cases} \Delta\phi + \lambda\phi = 0, & r = \sqrt{x^2 + y^2} \in (\alpha, R) \\ \left.\frac{\partial\phi}{\partial r}\right|_{r=\alpha} = \frac{1}{\alpha}\phi\big|_{r=\alpha}; & \left.\frac{\partial\phi}{\partial r}\right|_{r=R} \equiv 0. \end{cases} \tag{23}$$

Vogel shows that if all eigenvalues for this problem are positive, the interface is strictly stable and a strong local minimum with respect to a large class of perturbations.

In polar coordinates, the equation becomes

$$\frac{1}{r}\left[\frac{\partial}{\partial r}\left(r\frac{\partial\phi}{\partial r}\right)+\frac{\partial}{\partial\theta}\left(\frac{1}{r}\frac{\partial\phi}{\partial\theta}\right)\right]+\lambda\phi=0. \tag{24}$$

Setting $\phi=\sum_{k=1}^{\infty}g_k(r)\sin(k\theta)+\sum_{k=0}^{\infty}f_k(r)\cos(k\theta)$, we are led to the Sturm-Liouville problems

$$\begin{cases}rf''+f'+\left(\lambda r-\frac{k^2}{r}\right)f=0, & \alpha<r<R\\ f'(\alpha)=\frac{1}{\alpha}f(\alpha); & f'(R)=0\end{cases} \tag{25}$$

(where f denotes f_k, $k=0,1,\dots$ or g_k, $k=1,2,\dots$).

We wish to show that there are no nontrivial solutions for $\lambda\le 0$. For $\lambda<0$ we set $\lambda=-\mu^2$ and change variables ($t=\mu r$) to obtain

$$\begin{cases}f''+\frac{1}{t}f'-\left(1+\frac{k^2}{t^2}\right)f=0, & \alpha\mu<t<R\mu\\ f'(\mu\alpha)=\frac{1}{\mu\alpha}f(\mu\alpha); & f'(\mu R)=0.\end{cases} \tag{26}$$

This is a regular problem which we can transform using the change of variables $\tau=\log(t)$. Then we get

$$f(t)=v(\log t);\qquad f'=\frac{v'}{t},\qquad f''=\frac{v''}{t^2}-\frac{v'}{t^2} \tag{27}$$

so that

$$\begin{cases}v''-(e^{2\tau}+k^2)v=0, & \ell_1=\log(\mu\alpha)<\tau<\log(\mu R)=\ell_2\\ v'(\ell_1)=v(\ell_1); & v'(\ell_2)=0.\end{cases} \tag{28}$$

We claim that this problem has no nontrivial solution. In fact, if $v(\ell_1)<0$, then $v'(\ell_1)=v(\ell_1)<0$ and $v''(\ell_1)=(e^{2\ell_1}+k^2)v(\ell_1)<0$. Since each of $v(\tau)$, $v'(\tau)$, and $v''(\tau)$ must remain negative, there can be no $\ell_2>\ell_1$ for which $v'(\ell_2)=0$. (Assume that one of $v,v'v''$ vanishes first at $\tau=\tau_1$. If $v(\tau_1)=0$, there is some $\tau_2<\tau_1$ for which $v'(\tau_2)>0$ (a contradiction). The same argument holds if $v'(\tau_1)=0$. If $v''(\tau_1)=0$, then $v(\tau_1)=\frac{1}{(e^{2\ell_1}+k^2)}v'(\tau_2)=0$, so we also get a contradiction.)

If $v(\ell_1)=0$, then uniqueness of solutions gives $v\equiv 0$.

If $v(\ell_1)>0$, then v, v', and v'' must remain forever positive, so we cannot have $v'(\ell_2)=0$.

If $\lambda=0$, the problem becomes

$$\begin{cases}r^2f''+rf'-k^2f=0, & \alpha<r<R\\ f'(\alpha)=\frac{1}{\alpha}f(\alpha); & f'(R)=0.\end{cases} \tag{29}$$

This is an Euler equation which transforms as above to

$$\begin{cases}v''=k^2v, & \ell_1=\log\alpha<\tau<\log R=\ell_2\\ v'(\ell_1)=v(\ell_1); & v'(\ell_2)=0.\end{cases} \tag{30}$$

An argument similar to that in the case $\lambda < 0$, shows that only the trivial solution is possible. In fact, more explicitly we have $v = ce^{kt} + de^{-kt}$ so that

$$\begin{cases} ce^{k\ell_1} - de^{-k\ell_1} = ce^{k\ell_1} + de^{-k\ell_1} \\ ce^{k\ell_2} - de^{-k\ell_2} = 0 \end{cases} \tag{31}$$

or

$$\begin{cases} (k-1)e^{k\ell_1}c - (k+1)e^{-k\ell_1}d = 0 \\ e^{k\ell_2}c - e^{-k\ell_2}d = 0 \end{cases} \tag{32}$$

which has only the trivial solution since

$$-(k-1)e^{k(\ell_1-\ell_2)} + (k+1)e^{k(\ell_2-\ell_1)} = k(e^{k(\ell_2-\ell_1)} - e^{-k(\ell_2-\ell_1)}) \tag{33}$$
$$+ e^{k(\ell_2-\ell_1)} + e^{-k(\ell_2-\ell-1)}$$
$$= 2k\sinh k(\ell_2-\ell_1) + 2\cosh k(\ell_2-\ell_1) \neq 0. \tag{34}$$

(If this were to vanish, we would have

$$\frac{\sinh k(\ell_2-\ell_1)}{\cosh k(\ell_2-\ell_1)} = -\frac{1}{k} < 0, \qquad k \neq 0 \tag{35}$$

or

$$\cosh k(\ell_2-\ell_1) = 0, \qquad k = 0, \tag{36}$$

both of which are impossible.)

Thus, the interface under consideration is stable and also a strong local minimum by the work of Vogel.

5 Appendix: Surfaces.

In this paper, in which we model liquid interfaces and rigid bounding structures by two dimensional surfaces embedded in $\mathbb{R}^3$, it is natural to adopt a notion of a surface more or less synonymous with that of a mapping $X : U \to \mathbb{R}^3$ defined on a domain U of $\mathbb{R}^2$. The surfaces we have in mind, of course, exist apart from the mapping; I can think about the sphere $\{(x, y, z) : x^2 + y^2 + z^2 = r^2\}$, which is a perfectly good surface, and I haven't said anything about a mapping. If we want to make any kind of quantitative calculations, however, then we usually want such a mapping. The problem arises that it is sometimes difficult (or impossible) to find a single domain U and a mapping X on U which gives a parameterization of the surface we have in mind. Even if we have such a mapping, furthermore, we very often wish to make calculations using a different parameterization; for this reason, it is useful to have at our disposal general formulae in terms of arbitrary coordinates.

It is also useful in this context to introduce the notion of a surface in $\mathbb{R}^3$ parameterized on an abstract Riemann surface on which the usual formalism of "overlapping charts" prevails. The Riemann surface, in this instance, is used primarily as a bookkeeping device to keep track of a surface globally. Technically, there is a distinction between the mapping $X : M \to \mathbb{R}^3$ which parameterizes such a surface and the composition of a coordinate chart $\xi : U \to M$ with such a map giving a local mapping $\Xi = X : U \to \mathbb{R}^3$ as described above. As in the main body of the paper, We will abuse notation, using the symbol X for both mappings and letting the context make the meaning clear.

Area and Integration with respect to Area

Having said that, let us temporarily distinguish between X (global) and $\Xi = X \circ \xi$ (local). Since $X : M \to \Lambda$ is a mapping of a two dimensional surface into $\mathbb{R}^3$, the formula for the area of Λ involves the usual Jacobian scaling factor:

$$A(\Lambda) = \int_M \sqrt{\det\left(dX^T dX\right)}$$

where $dX : T_P M \to T_{X(P)}\mathbb{R}^3$ is a linear map and the superscript T denotes the transpose or adjoint (so that $dX^T dX$ is a square matrix whose determinant makes sense). See [6, page 96] for further details.

On the other hand, $d\xi : \mathbb{R}^2 \to T_P M \sim \mathbb{R}^2$, so that changing variables gives

$$
\begin{aligned}
A(\Xi(U)) &= \int_{\xi(U)} \sqrt{\det\left(dX^T dX\right)} \\
&= \int_U \sqrt{\det\left(dX^T dX\right)} \left|\det d\xi\right| \\
&= \int_U \sqrt{\det\left(d\xi^T dX^T dX d\xi\right)} \\
&= \int_U \sqrt{\det\left(d\Xi^T d\Xi\right)}
\end{aligned}
$$

since $dX d\xi = d\Xi$.

Returning now to the abuse $X = \Xi$ and taking u, v-coordinates in $\mathbb{R}^2 \supset U$, we see that

$$dX = \begin{pmatrix} X_u^1 & X_v^1 \\ X_u^2 & X_v^2 \\ X_u^3 & X_v^3 \end{pmatrix} \quad \text{and} \quad \sqrt{\det\left(dX^T dX\right)} = \sqrt{EG - F^2}$$

where $E = |X_u|^2$, $F = X_u \cdot X_v$, and $G = |X_v|^2$ are the coefficients of the first fundamental form. Thus, we write

$$A(X(U)) = \int_U \sqrt{EG - F^2},$$

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and more generally if $f : \Lambda \to \mathbb{R}$ is a function on Λ, we define

$$\int_\Lambda f = \int_M f \circ X \sqrt{\det(dX^T dX)} \quad \text{(globally)}$$

or

$$\int_{X(U)} f = \int_U f \circ X \sqrt{EG - F^2} \quad \text{(locally)}.$$

Vector Fields and the Divergence Theorem

Like surfaces themselves, we can imagine vector fields that are defined globally on the surface, and these will have specific expressions in given coordinates. If T is a given (tangent) vector field on a surface Λ, then in local coordinates, T has the form $T = aX_u + bX_v$ where

$$a = \frac{GT \cdot X_u - FT \cdot X_v}{EG - F^2} \quad \text{and} \quad b = \frac{-FT \cdot X_u + ET \cdot X_v}{EG - F^2}. \tag{37}$$

We recall the Divergence Theorem for a vector field $\underline{v} = (v^1, v^2)$ on a domain U in $\mathbb{R}^2$ which states

$$\int_U \operatorname{div} \underline{v} = \int_{\partial U} \underline{v} \cdot \underline{n}$$

where

$$\operatorname{div} \underline{v} = \operatorname{div}_{\mathbb{R}^2} \underline{v} = \frac{\partial v^1}{\partial x} + \frac{\partial v^2}{\partial y},$$

and $\underline{n}$ is the outward pointing unit (co)normal along ∂U. There is a direct analogue of this theorem for a vector field T on a surface Λ with boundary. We begin consideration of this theorem locally, however, using a single parameter map $X : U \to \Lambda$ defined on a domain U in the u, v-plane. We assume that ∂U may be parameterized by a mapping $(u, v) = (u(s), v(s))$ in terms of arclength. Then a unit tangent vector to ∂U is given by (u_s, v_s) and the unit outward normal to ∂U satisfies

$$\underline{n} = \pm(-v_s, u_s) = \pm(-v_s \underline{e}_1 + u_s \underline{e}_2). \tag{38}$$

While we do not know the sign actually realized in this equation, compatibility of sign is forced, or equivalently a choice of this sign forces a compatible choice of rotation along $\partial X(U)$, as follows.

We can always define a local orthonormal frame by

$$\mathbf{e}_1 = \frac{X_u}{\sqrt{E}}; \quad \mathbf{e}_2 = \frac{\sqrt{E} X_v - F X_u / \sqrt{E}}{\sqrt{EG - F^2}},$$

and there is a choice ($\pm$) of orthogonal rotation in each tangent plane to $X(U)$ determined by the condition

$$\mathbf{e}_1{}^\perp = (\pm)\mathbf{e}_2. \tag{39}$$

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Applying the rotation twice to any vector results in reversal of sign so that

$$\mathbf{e_2}^\perp = (\mp)\mathbf{e_1}.$$

We may assume that the choice $(\pm)$ is compatible with the condition

$$\vec{n} = \frac{X_s^\perp}{|X_s|} \tag{40}$$

where $\vec{n}$ is the outward pointing conormal and $X_s = u_s X_u + v_s X_v$ is a tangent vector along $\partial X(U)$. Equivalently, we may note that (40) determines a choice of rotation and then take (39) as a consequence of that choice. In any case, we do not presently know the sign $(\pm)$ or its relation to the sign $\pm$ appearing in (38).

We may observe, however, that

$$X_v = (F\mathbf{e_1} + \sqrt{EG - F^2}\,\mathbf{e_2})/\sqrt{E}$$

and

$$X_v^\perp = (\pm)(F\mathbf{e_2} - \sqrt{EG - F^2}\,\mathbf{e_1})/\sqrt{E}.$$

Compatibility follows from the fact that $dX(\underline{n})$ must point out of $X(U)$. That is,

$$
\begin{aligned}
0 \quad &\leq \quad dX(\underline{n}) \cdot \vec{n} \\
&= \quad \pm[-v_s dX(\underline{e_1}) + u_s dX(\underline{e_2})] \cdot (u_s X_u^\perp + v_s X_v^\perp)/|X_s| \\
&= \quad \pm(-v_s X_u + u_s X_v) \cdot (u_s X_u^\perp + v_s X_v^\perp)/|X_s| \\
&= \quad \pm(-v_s^2 X_u \cdot X_v^\perp + u_s^2 X_u^\perp \cdot X_v)/|X_s| \\
&= \quad \pm(\pm)(u_s^2 + v_s^2)\sqrt{EG - F^2}/|X_s|.
\end{aligned}
$$

It follows that $\pm = (\pm)$.

Using the expression $T = aX_u + bX_v$ in local coordinates, we may now compute

$$
\begin{aligned}
\int_{\partial X(U)} T \cdot \vec{n} &= \int_{\partial U} (aX_u + bX_v) \cdot X_s^\perp \\
&= \int_{\partial U} (aX_u + bX_v) \cdot (u_s X_u^\perp + v_s X_v^\perp) \\
&= \int_{\partial U} av_s X_u \cdot X_v^\perp + bu_s X_u^\perp \cdot X_v \\
&= \pm \int_{\partial U} (a, b) \cdot (-v_s, u_s)\sqrt{EG - F^2}.
\end{aligned}
$$

Using (38) and the divergence theorem in $\mathbb{R}^2$, we have

$$\int_{\partial X(U)} T \cdot \vec{n} = \int_{\partial U} \sqrt{EG - F^2}\,(a, b) \cdot \underline{n} = \int_U (\sqrt{EG - F^2}\,a)_u + (\sqrt{EG - F^2}\,b)_v.$$

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Finally, expressing things back in terms of the surface, we can write

$$\int_{\partial X(U)} T \cdot \vec{n} = \int_{X(U)} \frac{1}{\sqrt{EG - F^2}} [(\sqrt{EG - F^2}\, a)_u + (\sqrt{EG - F^2}\, b)_v].$$

This last expression suggests the definition (at least locally)

$$\operatorname{div}_\Lambda T = \frac{1}{\sqrt{EG - F^2}} [(\sqrt{EG - F^2}\, a)_u + (\sqrt{EG - F^2}\, b)_v], \tag{41}$$

so that our first version of the divergence theorem on a surface reads simply

$$\int_{\partial X(U)} T \cdot \vec{n} = \int_{X(U)} \operatorname{div}_\Lambda T.$$

Next, using this local theorem, we find

$$\lim_{A(\Lambda^*) \to 0} \frac{1}{A(\Lambda^*)} \int_{\partial \Lambda^*} T \cdot \vec{n} = \lim_{A(\Lambda^*) \to 0} \frac{1}{A(\Lambda^*)} \int_{\Lambda^*} \operatorname{div}_\Lambda T = \operatorname{div}_\Lambda T(X^*)$$

where the subdomains $\Lambda^* \subset \Lambda$ are well behaved enough for the conormal to be defined and converge as sets to the point $X^* \in \Lambda$. This shows that the definition of the divergence on Λ does not depend on the local coordinates. (Using geodesic balls on the surface for the Λ^* would be sufficient for this reasoning.)

Finally, we claim that the same theorem holds for any surface Λ. That is,

$$\int_{\partial \Lambda} T \cdot \vec{n} = \int_\Lambda \operatorname{div}_\Lambda T.$$

In fact, we can take a sequence of partitions of Λ involving disjoint pieces Λ_j^* whose maximum diameter tends to zero. Then taking points $X_j^* \in \Lambda_j^*$, we can write

$$\int_{\partial \Lambda} T \cdot \vec{n} = \sum_j \int_{\partial \Lambda_j^*} T \cdot \vec{n}$$

$$= \sum_j \left[\frac{1}{A(\Lambda_j^*)} \int_{\partial \Lambda_j^*} T \cdot \vec{n} - \operatorname{div}_\Lambda T(X_j^*) \right] A(\Lambda_j^*) + \sum_j \operatorname{div}_\Lambda T(X_j^*) A(\Lambda_j^*).$$

The first term (involving a sum) tends to zero and the second term tends to

$$\int_\Lambda \operatorname{div}_\Lambda T$$

as the maximum diameter of the partition pieces tends to zero.

Notice there is no requirement on orientability here as is often suggested in treatments of this theorem based on the formalism of differential forms. We avoid this restriction by only integrating real valued functions with respect to area.

A Kinematical Identity

We conclude this appendix with the derivation of a "kinematical identity" adapted from formula (4.1) in [12]. Given a region (of liquid) which moves through space under a flow, the identity describes how an integral over the region of a spatially dependent function changes in time with respect to the flow.

Let $\mathcal{M}_0$ be an open set in $\mathbb{R}^3$. We introduce a *smooth variation* $X : \mathcal{M}_0 \times (-\epsilon, \epsilon) \to \mathbb{R}^3$ such that $X(\cdot\,; 0) = \mathrm{id}_{\mathcal{M}_0}$ and set $\mathcal{M} = X(\mathcal{M}_0; t)$. Note that for $|t|$ small $\det DX = \det\left(\frac{\partial x^i}{\partial x^j}\right) \approx 1 > 0$, where X denotes here the restriction map $X\big|_{\mathcal{M}_0 x \{t\}} : \mathcal{M}_0 \to \mathcal{M}$. Let $g : U \to \mathbb{R}^3$ be a smooth real valued function on an open set $U \subseteq \mathbb{R}^3$ containing $\bigcup_{-\epsilon < t < \epsilon} X(\mathcal{M}_0; t)$ and consider

$$\mathcal{G} = \mathcal{G}(t) = \int_{\mathcal{M}} g = \int_{\mathcal{M}_0} g \circ X \det DX.$$

Evidently,

$$\dot{\mathcal{G}} = \int_{\mathcal{M}_0} Dg(X) \cdot \dot{X} + \int_{\mathcal{M}_0} g \circ X \frac{\partial}{\partial t} [\det DX]. \tag{42}$$

Following Serrin's derivation, we now show that

$$\frac{\partial}{\partial t} [\det DX] = \mathrm{div}_{\mathbb{R}^3} \, v(X; t) \det DX. \tag{43}$$

where $v(x; t) := \dot{X}(X^{-1}(x; t); t)$.

Lemma 3 (A Matrix Identity) *If $M = (\mu_{ij})$ is any matrix, then*

$$\delta_{ij} \det M = \sum_{\ell} \mu_{i\ell} \, \mathrm{cof}(M; j, \ell) \tag{44}$$

where $\delta_{ij} = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases}$ is the usual Kronecker δ and $\mathrm{cof}(M; i, j)$ is the determinant of the matrix obtained from M by deleting the i-th row and j-th column—the i, j cofactor determinant.

Proof: For each fixed i, we have a row expansion

$$\det M = \sum_{\ell} \mu_{i\ell} \, \mathrm{cof}(M; i, \ell).$$

This implies

$$\delta_{ij} \det M = \delta_{ij} \sum_{\ell} \mu_{i\ell} \, \mathrm{cof}(M; i, \ell) = \delta_{ij} \sum_{\ell} \mu_{i\ell} \, \mathrm{cof}(M; j, \ell). \tag{45}$$

(We can exchange the i for a j in $\mathrm{cof}(M; i, \ell)$ since $\delta_{ij} = 0$ unless $i = j$.) But, in fact, $\sum_\ell \mu_{i\ell} \, \mathrm{cof}(M; j, \ell) = 0$ already if $i \neq j$, because this is the determinant of the matrix $\tilde{M}$ obtained from M by replacing the jth row with the ith row. (If $i \neq j$, the ith row appears twice in $\tilde{M}$, so $\det \tilde{M} = 0$.) Thus, the δ_{ij} may be erased in the last expression in (45) on the right. $\qquad\square$

Lemma 4 (A Second Matrix Identity) *If $M = (\mu_{ij})$ has entries differentiable in t, then*

$$\frac{\partial}{\partial t} \det M = \sum_{i, \ell} \dot{\mu}_{i\ell} \, \mathrm{cof}(M; i, \ell) \tag{46}$$

Proof: We begin with the permutation expression

$$\det M = \sum_\sigma \epsilon(\sigma) \mu_{1\sigma(1)} \cdots \mu_{n\sigma(n)}$$

where $\epsilon(\sigma) = \begin{cases} -1 & \text{if } \sigma \text{ is an odd permutation} \\ 1 & \text{if } \sigma \text{ is an even permutation} \end{cases}$; see [4, pg. 160]. Thus,

$$\frac{d}{dt} \det M = \sum_{\sigma, i} \epsilon(\sigma) \mu_{1\sigma(1)} \cdots \dot{\mu}_{i\sigma(i)} \cdots \mu_{n\sigma(n)} = \sum_i \det \tilde{M}^i$$

where $\tilde{M}^i$ is the matrix obtained from M by replacing the ith row by $\dot{\mu}_{i1}, \dot{\mu}_{i2}, \ldots, \dot{\mu}_{in}$ (i.e., by differentiating the ith row). Expressing $\det \tilde{M}^i$ via a column expansion, one obtains (46). $\square$

Proof of (43): From the second identity,

$$\frac{\partial}{\partial t} \left[\det DX \right] = \sum_{i, \ell} \frac{\partial}{\partial t} \left[\frac{\partial X^i}{\partial x^\ell} \right] \mathrm{cof}(DX; i, \ell)$$

$$= \sum_{i, \ell} \frac{\partial \dot{X}^i}{\partial x^\ell} \, \mathrm{cof}(DX; i, \ell)$$

Now, we consider the composition

$$v(x; t) = \dot{X}(X^{-1}(x; t), t),$$

which gives the spatial velocity of the flow generated by the variation $X : \mathcal{M}_0 \times (-\epsilon, \epsilon) \to \mathbb{R}^3$. It is evident that $\dot{X}(x, t) = v(X(x; t); t)$. Therefore, by the chain rule

$$\frac{\partial \dot{X}}{\partial x^\ell} = \sum_j \frac{\partial v}{\partial x^j} (X(x; t); t) \frac{\partial X^j}{\partial x^\ell} (x, t).$$

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Therefore,

$$\sum_{i,\ell} \frac{\partial \dot{X}^i}{\partial x^\ell} \operatorname{cof}(DX; i, \ell) = \sum_{i,j,\ell} \frac{\partial v^i}{\partial x^j}(X(x,t);t)\frac{\partial X^j}{\partial x^\ell}(x;t)\operatorname{cof}(DX; i, \ell)$$

$$= \sum_{i,j} \frac{\partial v^i}{\partial x^i}(x;t) \sum_\ell \frac{\partial X^j}{\partial x^\ell}(x;t)\operatorname{cof}(DX; i, \ell)$$

$$= \sum_{i,j} \frac{\partial v^i}{\partial x^j}(X;t)\delta_{ij} \det DX$$

$$= \operatorname{div}_{\mathbb{R}^3} v(X(x;t);t) \det DX.$$

This establishes (43). $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \square$

Returning to (42), we have

$$\dot{\mathcal{G}} = \int_{\mathcal{M}} Dg \cdot v + \int_{\mathcal{M}} g \operatorname{div}_{\mathbb{R}^3} v,$$

which is the form taken by Serrin's formula (4.1) in our context.

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