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by

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By investigating the convex property of the function R, appeared in computing the entanglement of formation for isotropic states in Phys. Rev. Lett. **85**, 2625 (2000), and a tight lower bound of entanglement of formation for arbitrary bipartite mixed states in Phys. Rev. Lett. **95**, 210501 (2005), we show analytically that the very nice results in these papers are valid not only for dimensions 2 and 3 but any dimensions.

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Quantum entangled states are playing fundamental roles in quantum information processing [1]. The entanglement of formation (EOF) is a most important measure in quantifying the degree of entanglement [2, 3]. Considerable efforts have been spent on deriving EOF or its lower bound through analytical and numerical approaches ([4, 5] and references therein). In [4] the EOF for isotropic states are presented. It is shown that for $F \geq 1/d$, the EOF for isotropic states ρ_F is $E(\rho_F) = co\big(R(F)\big)$, where "co($R$)" stands for the convex hull, the largest convex function that is bounded above by the function R. An explicit expression of $co\big(R(F)\big)$ has been derived for dimensions $d = 2, 3$, and its general form is conjectured for arbitrary d. While in [5] a tight lower bound of EOF for arbitrary bipartite mixed states is given, by using the property of R function derived in [4]. For any bipartite $m \otimes n$ ($m \leq n$) mixed quantum state ρ, the EOF $E(\rho)$ satisfies $E(\rho) \geq co\big(R(\Lambda)\big)$. In both cases the results depend on the convex property of the function R. These results are correct when the second derivative of R has one zero point, which has been shown to be true if the dimension is 2 or 3.

The $R(\Lambda)$-function (Λ corresponds to $d\,F$ in $R(F)$ [4]) has the form

$$R(\Lambda) = H_2\big(\gamma(\Lambda)\big) + \big(1 - \gamma(\Lambda)\big)\log_2(m-1), \tag{1}$$

where

$$\gamma(\Lambda) = \frac{1}{m^2}\left(\sqrt{\Lambda} + \sqrt{(m-1)(m-\Lambda)}\right)^2 \tag{2}$$

and $H_2(.)$ is the standard binary entropy function, $\Lambda \in [1, m]$. For $m \geq 4$, it is conjectured that the second derivative of R w.r.t. Λ has still only one zero point, by numerically calculating the function R for a given m. In deed for $m = 4$, one can easily see this is true by plotting $R(\Lambda)$. In the following we show analytically that the results in both papers valid for arbitrary dimensions $m \geq 5$.

For simplicity we replace $\log_2$ in (1) by the natural log. Without confusion we still use the notion $R(\Lambda)$ below, which, in fact, differs a positive factor $\log_2 e$ from the $R(\Lambda)$ above. We first prove that there is one and only one point Λ_0 between 1 and $m-1$ such that $R''(\Lambda_0) = 0$ for $m \geq 5$. Then we further show that there is no more zero points for $R''(\Lambda)$ between $m-1$ and m. By direct calculation we have the second derivative of R w.r.t Λ

$$R''(\Lambda) = \gamma''(\Lambda)\log\frac{1-\gamma(\Lambda)}{(m-1)\gamma(\Lambda)} - \frac{1}{\Lambda(m-\Lambda)}, \tag{3}$$

where

$$\gamma''(\Lambda) = -\frac{\sqrt{m-1}}{2}(\Lambda(m-\Lambda))^{-3/2}. \tag{4}$$

From which we get $R''(1) = \mathrm{Lim}_{\epsilon\to 0} R''(1+\epsilon) = +\infty$. On the other hand,

$$R''(m-1) = -\frac{1}{m-1}\left(\log\frac{m-2}{2(m-1)} + 1\right),$$

which is less than 0 for $m \geq 5$. Therefore for $m \geq 5$ there exits $\Lambda_0 \in (1, m-1)$ such that $R''(\Lambda_0) = 0$. From (3) and (4) Λ_0 is the solution of $g(\Lambda) = f(\Lambda)$, where

$$g(\Lambda) = \log\frac{1-\gamma(\Lambda)}{(m-1)\gamma(\Lambda)}, \quad f(\Lambda) = -2\sqrt{\frac{\Lambda(m-\Lambda)}{m-1}}.$$

As $g'(\Lambda) > 0$, $g(\Lambda)$ is a monotonically increasing function taking values from $g(1) \to -\infty$ to

$$g(m-1) = 2\log\frac{m-2}{2(m-1)} > -2.$$

While $f(1) = f(m-1) = -2$, $f''(\Lambda) > 0$, i.e. f is convex. Therefore there is one and only one solution Λ_0 to the equation $g(\Lambda) = f(\Lambda)$ for $\Lambda \in (1, m-1)$.

We now show that there are no more solutions to $R''(\Lambda) = 0$ for $\Lambda \in (m-1, m)$, i.e. $R''(m-1+\delta) \neq 0$, $\forall\, \delta \in (0,1)$. From (2), (3) and (4) this is equivalent to show $F(\delta) \equiv \frac{1}{2}B(\delta)\log A(\delta) \neq -1$, where,

$$B(\delta) = \sqrt{\frac{m-1}{(m-1+\delta)(1-\delta)}}, \; A(\delta) = \frac{(mC(\delta))^2 - 1}{m-1},$$

and $C(\delta)=\left(\sqrt{m-1+\delta}+\sqrt{(m-1)(1-\delta)}\right)^{-1}$. It is straight forward to verify that $A(0)>0$. As the derivative of $C(\delta)$ w.r.t δ, $C'(\delta)>0$, we have $A'(\delta)>0$. Hence $\log A(\delta)$ increases as δ increases. Similarly, as the derivative of $(m-1)/\big((m-1+\delta)(1-\delta)\big)$ w.r.t δ is positive, $B(\delta)$ also increases as δ increases. Therefore $F(\delta)$ is an increasing function of δ. Moreover $F(0)=\log(m-2)/\big(2(m-1)\big)\geq\log 3/8>-1$. We have $F(\delta)\geq F(0)>-1$, $\forall\,\delta\in(0,1)$ and $m\geq 5$. Thus $R''(\Lambda)=0$ has no solutions for $\Lambda\in(m-1,m)$.

We have shown that the R-function has only one reflection point for any dimensions. Therefore the construction of the largest convex function that is bounded above by the R-function in [4] is correct. Either the EOF for isotropic states in [4] and the tight lower bound of EOF in [5] are valid for arbitrary dimensions.

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