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Generalized Tractability for Multivariate
Problems

by

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Part I: Linear Tensor Product Problems and Linear Information

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Abstract

There are many papers studying *polynomial* tractability for multivariate problems. Polynomial tractability means that the minimal number $n(\varepsilon, d)$ of information evaluations needed to reduce the initial error by a factor of ε for a multivariate problem defined on a space of d -variate functions may be bounded by a polynomial in ε^{-1} and d and this holds for all $(\varepsilon^{-1}, d) \in [1, \infty) \times \mathbb{N}$.

We propose to study *generalized* tractability by verifying when $n(\varepsilon, d)$ can be bounded by a power of $T(\varepsilon^{-1}, d)$ for all $(\varepsilon^{-1}, d) \in \Omega$, where Ω can be a proper subset of $[1, \infty) \times \mathbb{N}$. Here T is a *tractability* function which is non-decreasing in both variables and grows slower than exponentially to infinity. In this paper we study the set $\Omega = [1, \infty) \times \{1, 2, \dots, d^*\} \cup [1, \varepsilon_0^{-1}) \times \mathbb{N}$ for some $d^* \geq 1$ and $\varepsilon_0 \in (0, 1)$. We study linear tensor product problems for which we can compute arbitrary linear functionals as information evaluations. We present necessary and sufficient conditions on T such that generalized tractability holds for linear tensor product problems. We show a number of examples for which polynomial tractability does *not* hold but generalized tractability *does* hold.

1 Introduction

Tractability of multivariate problems has become a popular research problem in information-based complexity, see [9] where this concept was defined, and [3] and papers cited there for a survey of results. Tractability studies approximation of operators defined on spaces of d -variate functions and the emphasis is on large d . Problems with d in the hundreds and thousands occur in numerous applications such as in financial mathematics, physics and chemistry, see [6] for the discussion of this point. Tractability can be studied in various settings. In this paper we study the worst case setting in which the error of an algorithm is defined by its worst performance over a given class of functions.

Let $n(\varepsilon, d)$ denote the minimal number of information evaluations needed to reduce the initial error by a factor of $\varepsilon \in (0, 1]$. By one information evaluation we mean the evaluation of one linear functional or one function value at some point. The initial error is defined as the minimal error which can be achieved without sampling the function. For linear operators, the initial error is just the norm of the linear operator.

The essence of tractability is to find necessary and sufficient conditions on spaces of functions and operators for which $n(\varepsilon, d)$ does *not* depend exponentially on ε^{-1} and on d . There are various ways of measuring the lack of exponential dependence. So far, the tractability study was done for the *polynomial* case in which we insist that $n(\varepsilon, d)$ can be bounded by a polynomial in ε^{-1} and d for all $\varepsilon \in (0, 1]$ and all $d \in \mathbb{N}$. Many results were obtained for polynomial tractability. A typical result is that for classical spaces in which all variables play the same role, we do *not* have tractability since $n(\varepsilon, d)$ grows faster than any polynomial in ε^{-1} or d . This holds, in particular, for linear tensor product problems. We note in passing that these negative results motivated the study of *weighted* spaces in which variables or groups of variables play different roles and are moderated by weights. For weighted spaces, a typical result is that polynomial tractability holds for sufficiently small weights. Furthermore, we may even have *strong* tractability, i.e., $n(\varepsilon, d)$ is bounded by a polynomial in ε^{-1} independently of d , see [4], which was probably the first paper on tractability of weighted spaces, and [3] for a survey of results.

This is the first paper in a series where we study *generalized* tractability for multivariate problems. Generalized tractability may differ from polynomial tractability in two points. The first point is the domain of (ε, d) . For polynomial tractability, ε and d are independent, and $(\varepsilon^{-1}, d) \in [1, \infty) \times \mathbb{N}$. For some applications, like in mathematical finance, when d is huge, we may be interested only in a rough approximation and then ε is not too small. There may be also problems for which d is never too large although for a relatively small d we may be interested in a very accurate approximation which corresponds to a very small ε . For generalized tractability, we assume that $(\varepsilon^{-1}, d) \in \Omega$, where

$$[1, \infty) \times \{1, 2, \dots, d^*\} \cup [1, \varepsilon_0^{-1}) \times \mathbb{N} \subset \Omega \subset [1, \infty) \times \mathbb{N} \quad (1)$$

for some non-negative integer d^* and $\varepsilon_0 \in (0, 1]$ such that $d^* + (1 - \varepsilon_0) > 0$.

The essence of (1) is that for all such Ω we know that at least one of the parameters (ε^{-1}, d) may go to infinity but not necessarily two of them. Hence, for generalized tractability we assume that $(\varepsilon^{-1}, d) \in \Omega$ and we may choose an arbitrary Ω satisfying (1) for some d^* and ε_0 .

The second point in which generalized tractability may differ from polynomial tractability is the way we measure the lack of exponential dependence.

We define a *tractability* function

$$T : [1, \infty) \times [1, \infty) \rightarrow [1, \infty)$$

which is non-decreasing in both variables and which grows to infinity slower than any exponential function a^x for x tending to infinity with $a > 1$. More precisely, for a given Ω satisfying (1) we assume that $T(x, y)/a^{x+y}$ tends to zero as $(x, y) \in \Omega$ and $x + y$ approaches infinity. This is equivalent to assuming that

$$\lim_{(x, y) \in \Omega, x+y \rightarrow \infty} \frac{\ln T(x, y)}{x + y} = 0. \quad (2)$$

With Ω satisfying (1) and T satisfying (2), we study generalized tractability by insisting that there are positive numbers C and t such that

$$n(\varepsilon, d) \leq C T(\varepsilon^{-1}, d)^t \quad \text{for all } (\varepsilon^{-1}, d) \in \Omega.$$

We also have generalized strong tractability if we replace $T(\varepsilon^{-1}, d)$ above by $T(\varepsilon^{-1}, 1)$. In both cases, we are interested in the smallest exponents t which are called the exponents of (generalized) tractability and strong tractability. The precise definitions are given in Section 2. Note that generalized tractability coincides with polynomial tractability if we take $\Omega = [1, \infty) \times \mathbb{N}$ and $T(x, y) = xy$.

We are mainly interested in how the choice of Ω and T affects the class of tractable problems. Some promising results were already obtained in [10] with $\Omega = [1, \infty) \times \mathbb{N}$ and $T(x, y) = f_1(x)f_2(y)$ for $f_i(t) = \exp(\ln^{1+\alpha_i} t)$ and non-negative α_i . Namely, it was proved that linear tensor product problems with polynomially decaying eigenvalues are tractable iff $\alpha_1\alpha_2 \geq 1$. Hence, these problems are *not* polynomial tractable since this corresponds to $\alpha_1 = \alpha_2 = 0$ but are tractable if, for example, $\alpha_1 = \alpha_2 = 1$.

In this first paper on generalized tractability, we study linear tensor product problems for which we can use arbitrary bounded linear functionals. This type of information is called *linear information* which explains the subtitle of our paper. Linear tensor product problems are fully characterized by eigenvalues $\lambda = \{\lambda_j\}$ for $d = 1$ which are ordered and normalized so that $1 = \lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_j \geq 0$ and $\lim_{j \rightarrow \infty} \lambda_j = 0$. In particular, we study eigenvalues with exponential, polynomial and logarithmic rates of convergence.

We also choose the “smallest” set Ω , called the restricted tractability domain,

$$\Omega = \Omega^{\text{res}} = [1, \infty) \times \{1, 2, \dots, d^*\} \cup [1, \varepsilon_0^{-1}) \times \mathbb{N}.$$

We provide necessary and sufficient conditions on the tractability function T for which generalized tractability holds. These conditions depend on the parameters d^* , ε_0 and the sequence λ . In particular, the following results hold. Assume that $d^* \geq 1$ and $\varepsilon_0 < 1$. Then if the largest eigenvalue has multiplicity at least two, i.e., when $\lambda_2 = 1$, then generalized tractability does *not* hold no matter how we choose the tractability function T .

Assume then that $\lambda_2 < 1$ and that we have a polynomial rate of convergence, $\lambda_j = \Theta(j^{-\beta})$ for a positive β . This case is typical and corresponds to many classical Sobolev or Korobov tensor product spaces of smooth functions whose smoothness is measured by the parameter β .

Assume first that $\varepsilon_0^2 < \lambda_2$. Then generalized strong tractability does *not* hold no matter how we choose T . Generalized tractability holds iff $\liminf_{x \rightarrow \infty} (\ln T(x, 1)) / \ln x \in (0, \infty]$ and

$$\liminf_{d \rightarrow \infty} \inf_{\varepsilon \in [\varepsilon_0, \sqrt{\lambda_2})} \frac{\ln T(\varepsilon^{-1}, d)}{\alpha(\varepsilon) \ln d} \in (0, \infty],$$

where $\alpha(\varepsilon) = \lceil 2 \ln(1/\varepsilon) / \ln(1/\lambda_2) \rceil - 1$. In particular, if we take $T(x, y) = xy$ then generalized tractability *holds* with the exponent

$$t^{\text{tra}} = \max \left\{ \frac{2}{\beta}, \alpha(\varepsilon_0) \right\}.$$

Note that t^{tra} goes to $\max\{2/\beta, 1\}$ as $\lambda_2 - \varepsilon_0^2$ tends to zero, and t^{tra} goes to infinity as ε_0 tends to zero.

Assume now that $\lambda_2 \leq \varepsilon_0^2$. Then generalized strong tractability holds iff $\liminf_{x \rightarrow \infty} (\ln T(x, 1)) / \ln x \in (0, \infty]$. For $T(x, y) = xy$, this holds and the exponent of generalized strong tractability is $t^{\text{str}} = 2/\beta$.

We end this introduction by a note on future research. We plan to study linear tensor product problems still with linear information for more general Ω including $\Omega = [1, \infty) \times \mathbb{N}$. Next we want to study a standard class of information for which we can only compute function values. We want to verify which results on generalized tractability for linear information also hold for standard information. Finally we plan to study weighted spaces and to verify how conditions on weights may be relaxed for generalized tractability.

2 Preliminaries

In this section we define a multivariate problem and generalized tractability. Let m be a given positive integer. The multivariate problem will be given as a sequence of (dm) -variate problems defined on spaces of functions f of dm variables. Here $d = 1, 2, \dots$, and our main emphasis will be on large d . Usually, $m = 1$ but there are natural multivariate problems for which $m \geq 2$, see [1, 2, 7]. We wish to compute an ε -approximation for each d , and measure the difficulty of the computation by the minimal number of information evaluations needed for such an approximation. One information evaluation may be given by one function value of f or, more generally, by one linear functional value of f . The essence of tractability is to assure that the minimal number of information evaluations is *not* exponentially dependent either on d or on ε^{-1} . In the tractability study it has been so far assumed that we want to guarantee that the minimal number of information evaluations is polynomial in d and ε^{-1} , see [9]. The concept of generalized tractability is to study more general non-exponential functions, and to verify how tractability of a multivariate problem depends on the choice of this non-exponential function.

2.1 Multivariate Problems

For $m, d \in \mathbb{N}$, let F_d be a normed linear space of functions

$$f : D_d \subseteq \mathbb{R}^{dm} \rightarrow \mathbb{R}$$

and let G_d be a normed linear space. We consider a sequence $S = \{S_d\}$ of operators $S_d : F_d \rightarrow G_d$. We stress that S_d can be a non-linear operator, although technical results in this paper will be obtained for bounded linear operators. We call S a *multivariate problem*.

Let $\Lambda_d \subseteq F_d^*$ be a class of admissible continuous linear functionals. Two examples of Λ_d are mainly studied. The first one is denoted by Λ_d^{all} and is called *linear information*. It is equal to $\Lambda_d^{\text{all}} = F_d^*$ and is the set of all continuous linear functionals defined on F_d . The second class is denoted by Λ_d^{std} and is called *standard information*. It is equal to the set of specific continuous linear functionals given by function values. More precisely, for this class, we assume that $L_x(f) = f(x)$, for all $f \in F_d$, is a continuous linear functional for all $x \in D_d$. Then $\Lambda_d^{\text{std}} = \{L_x \mid x \in D_d\}$.

We consider algorithms which use finitely many admissible information evaluations. An algorithm $A_{n,d}$ has the form

$$A_{n,d}(f) = \phi(L_1(f), L_2(f), \dots, L_n(f)) \quad (3)$$

for some $L_i \in \Lambda_d$ and some mapping $\phi : \mathbb{R}^n \rightarrow G_d$. Adaptive choice of the functionals L_i is also allowed as explained in e.g., [5].

In this paper we restrict ourselves to the worst case setting although settings such as the average case, randomized and probabilistic may be also studied. The *worst case error* of the algorithm $A_{n,d}$ is defined as

$$e^{\text{wor}}(A_{n,d}) = \sup_{f \in F_d, \|f\|_{F_d} \leq 1} \|S_d(f) - A_{n,d}(f)\|_{G_d}. \quad (4)$$

For $n = 0$ we do not sample the function f , and $A_{0,d}(f) = g$ is a constant mapping with $g \in G_d$. By the initial error we mean the minimal error of constant algorithms,

$$e^{\text{init}}(S_d) = \inf_{g \in G_d} \sup_{f \in F_d, \|f\|_{F_d} \leq 1} \|S_d(f) - g\|_{G_d}.$$

It is clear that if S_d is a bounded linear operator then the initial error is achieved for $g = 0$, and thus

$$e^{\text{init}}(S_d) = \|S_d\| = e^{\text{wor}}(A_{0,d}^*),$$

where $A_{0,d}^* = 0$ is the zero algorithm. Let

$$n(\varepsilon, S_d, \Lambda_d) = \min\{n \mid \exists A_{n,d} : e^{\text{wor}}(A_{n,d}) \leq \varepsilon e^{\text{init}}(S_d)\} \quad (5)$$

denote the minimal number of admissible information evaluations from Λ_d needed to reduce the initial error by a factor ε . Without loss of generality we may assume that $\varepsilon \in (0, 1]$ since for $\varepsilon > 1$, we obviously have $n(\varepsilon, S_d, \Lambda_d) = 0$.

We say that the algorithm $A_{n,d}$ computes an ε -approximation of S_d if its worst case error is at most $\varepsilon e^{\text{init}}(S_d)$. This can only happen if $n \geq n(\varepsilon, S_d, \Lambda_d)$.

The number $n(\varepsilon, S_d, \Lambda_d)$ is the so-called information complexity of the problem S_d . It is also of interest to study the total complexity of the problem S_d which is defined as the minimal cost of computing an ε -approximation. For linear problems, it follows from general results that the total complexity

is roughly proportional to the information complexity $n(\varepsilon, S_d, \Lambda_d)$. For non-linear problems, it is also true if there exists an algorithm which computes an ε -approximation and whose combinatory cost is of order of the information complexity, see [5] for more details. For simplicity, we restrict ourselves in this paper only to the information complexity. More specifically, we study when $n(\varepsilon, S_d, \Lambda_d)$ does not depend exponentially on ε^{-1} and d . Obviously, this will lead to necessary conditions that the total complexity is also not exponentially dependent on ε^{-1} and d .

2.2 Generalized Tractability

The essence of tractability is to guarantee that $n(\varepsilon, S_d, \Lambda_d)$ does *not* depend exponentially on ε^{-1} and d . There are various ways to measure the lack of exponential dependence. First of all, we must agree how the parameters ε and d vary. In the tractability study, it has been so far assumed that ε and d are independent, and $\varepsilon \in (0, 1]$, $d \in \mathbb{N}$. In particular, it was assumed that both ε^{-1} and d may go to infinity. For some applications, e.g., in finance, we are interested in huge d and relatively small ε^{-1} . For instance, d may be in the hundreds or thousands, and $\varepsilon \geq 0.01$ since it is enough to know the solution within at most one percent. In such cases, the assumption that both ε^{-1} and d may go to infinity is too demanding.

That is why we assume that (ε^{-1}, d) belongs to Ω , where the domain Ω is, in general, a proper subset of $[1, \infty) \times \mathbb{N}$. Obviously, the domain Ω cannot be “too small” to model properly the essence of multivariate problems.

Let us use the notation $[n] := \{1, \dots, n\}$ for any integer n . In particular, $[n] = \emptyset$ if $n \leq 0$. We assume that

$$[1, \infty) \times [d^*] \cup [1, \varepsilon_0^{-1}) \times \mathbb{N} \subseteq \Omega \quad (6)$$

for some $d^* \in \mathbb{N}_0$ and some $\varepsilon_0 \in (0, 1]$ such that $d^* + (1 - \varepsilon_0) > 0$. Condition (6) is the only restriction we impose on Ω . The constraint $d^* + (1 - \varepsilon_0) > 0$ excludes the case $d^* = 0$ and $\varepsilon_0 = 1$ corresponding to no restriction on Ω .

Tractability for multivariate problems has been so far defined by demanding that $n(\varepsilon, S_d, \Lambda_d)$ can be bounded by a multiple of powers of ε^{-1} and d . Obviously there are different ways of guaranteeing that $n(\varepsilon, S_d, \Lambda_d)$ does not depend exponentially on ε^{-1} and d .

For instance, in theoretical computer science, tractability for discrete problems is usually understood by demanding that the cost bound of an

algorithm is a multiple of a power of $k = \lceil \log_2(1 + \varepsilon^{-1}) \rceil$. That is, we can compute k correct bits of the solution in a polynomial time in k . We note in passing that if one adopts this definition of tractability for multivariate problems then most of them become intractable since even for the univariate case, $d = 1$, a typical behavior of $n(\varepsilon, S_1, \Lambda_1)$ is a polynomial in ε^{-1} .

One may also take an opposite point of view to the theoretical computer science approach presented above, and study tractability when $n(\varepsilon, S_d, \Lambda_d)$ can be bounded by a multiple of powers of functions depending on ε^{-1} and d which grow faster than polynomials. This has been partially done in [10] by demanding that $n(\varepsilon, S_d, \Lambda_d)$ can be bounded by a multiple of powers of $f_1(\varepsilon^{-1})$ and $f_2(d)$ with functions f_i such as $f_i(x) = \exp(\ln^{1+\alpha_i}(x))$ with $\alpha_i > 0$. Indeed, such functions grow faster than any polynomial as x tends to infinity, but slower than any exponential function a^x with $a > 1$. It was shown in [10] that the class of tractable multivariate problems is larger for such functions f_i than the tractability class studied before.

The approach of [10] is not fully general since it studies tractability with the separate dependence on the parameters ε^{-1} and d . Indeed, this is so since functions f_i depend only on one of these parameters. For some multivariate problems, such as tensor product problems studied also here, this restriction is essential. It is therefore better not to insist on separate dependence on ε^{-1} and d , and study tractability without assuming this property.

That is why we study tractability in this paper in terms of a function T of two variables and use a multiple of a power of $T(\varepsilon^{-1}, d)$ in the definition of generalized tractability. Obviously, we need to assume a couple of natural properties of T . First of all, if ε decreases the problem of computing an ε -approximation usually becomes harder. Furthermore, with a proper definition of the operators S_d , the increase of d should also make the problem harder. That is why we assume that the function T is non-decreasing in both arguments. To rule out the exponential behavior of T , we assume that $T(x, y)/a^{x+y}$ tends to zero as $x + y$ tends to infinity for any $a > 1$. This is equivalent to assuming that $\ln T(x, y)/(x + y)$ tends to zero as $x + y$ approaches infinity. As we shall see in a moment, it will be convenient to define the domain of T as the set $[1, \infty) \times [1, \infty)$. In particular, this domain allows us to say that T is non-decreasing, and will be useful for the concept of generalized strong tractability. This discussion motivates the following definitions.

A function $T : [1, \infty) \times [1, \infty) \rightarrow [1, \infty)$ is a *tractability function* if T is

non-decreasing in x and y and

$$\lim_{(x,y) \in \Omega, x+y \rightarrow \infty} \frac{\ln T(x,y)}{x+y} = 0. \quad (7)$$

The multivariate problem $S = \{S_d\}$ is (T, Ω) -tractable in the class $\Lambda = \{\Lambda_d\}$ if there exist non-negative numbers C and t such that

$$n(\varepsilon, S_d, \Lambda_d) \leq C T(\varepsilon^{-1}, d)^t \quad \text{for all } (\varepsilon^{-1}, d) \in \Omega. \quad (8)$$

The *exponent t^{tra}* of (T, Ω) -tractability in the class Λ is defined as the infimum of all non-negative t for which there exists a $C = C(t)$ such that (8) holds.

Let $\varepsilon_0 < 1$. Then it is easy to see that if

$$n(\varepsilon_0, S_d, \Lambda_d) \geq \kappa^d \quad \text{for almost all } d \in \mathbb{N} \text{ with } \kappa > 1, \quad (9)$$

then S is *not* (T, Ω) -tractable in the class Λ for an arbitrary tractability function T and an arbitrary domain Ω satisfying (6). Indeed, suppose on the contrary that S is (T, Ω) -tractable in the class Λ . Then

$$\frac{\ln C + t \ln T(\varepsilon_0^{-1}, d)}{\varepsilon_0^{-1} + d} \geq \frac{d \ln \kappa}{\varepsilon_0^{-1} + d}$$

implies $\liminf_{d \rightarrow \infty} \ln T(\varepsilon_0^{-1}, d)/(\varepsilon_0^{-1} + d) \geq t^{-1} \ln \kappa > 0$ which contradicts (7).

Similarly, if $d^* \geq 1$ and there exist $d \in [d^*]$ and $\kappa > 1$ such that

$$n(\varepsilon, S_d, \Lambda_d) \geq \kappa^{1/\varepsilon} \quad \text{for sufficiently small } \varepsilon, \quad (10)$$

then S is *not* (T, Ω) -tractable in the class Λ for an arbitrary tractability function T and an arbitrary domain Ω satisfying (6). As before, this follows from the fact that $\liminf_{\varepsilon \rightarrow 0} \ln T(\varepsilon^{-1}, d)/(\varepsilon^{-1} + d) \geq t^{-1} \ln \kappa > 0$ which contradicts (7).

For some multivariate problems, it has been shown that $n(\varepsilon, S_d, \Lambda_d)$ is bounded by a multiple of some power of ε^{-1} which does not depend on d . This property is called *strong tractability*. In our case, we can define generalized strong tractability by insisting that the bound in (8) is independent of d . Formally, we replace $T(\varepsilon^{-1}, d)$ by $T(\varepsilon^{-1}, 1)$. We stress that (ε^{-1}, d) from Ω does not necessarily imply that $(\varepsilon^{-1}, 1)$ is in Ω . Nevertheless, due to the more general domain of T , the value $T(\varepsilon^{-1}, 1)$ is well defined, and due to monotonicity of T we have $T(\varepsilon^{-1}, 1) \leq T(\varepsilon^{-1}, d)$.

The multivariate problem S is *strongly* (T, Ω) -tractable in the class $\Lambda = \{\Lambda_d\}$ if there exist non-negative numbers C and t such that

$$n(\varepsilon, S_d, \Lambda_d) \leq C T(\varepsilon^{-1}, 1)^t \quad \text{for all } (\varepsilon^{-1}, d) \in \Omega. \quad (11)$$

The *exponent* t^{str} of *strong* (T, Ω) -tractability in the class Λ is defined as the infimum of all non-negative t for which there exists a $C = C(t)$ such that (11) holds.

Clearly, strong (T, Ω) -tractability in the class Λ implies (T, Ω) -tractability in the class Λ . Furthermore, $t^{\text{tra}} \leq t^{\text{str}}$. For some multivariate problems the exponents t^{tra} and t^{str} are the same, and for some they are different. We shall see such examples also in this paper.

We will be also using simplified notations. If Ω and Λ are clear from the context, we say that S is T -tractable or strongly T -tractable. If also T is clear from the context, we say that S is tractable or strongly tractable. Finally, we talk about generalized tractability or generalized strong tractability if we consider various T , Ω and Λ .

Suppose we have two tractability functions T_1 and T_2 such that $T_1 = T_2^\alpha$ for some positive α . It is clear that the concepts of T_i -tractability are then essentially the same with the obvious changes of their exponents. Therefore we can obtain substantially different tractability results for T_1 and T_2 only if they are *not* polynomially related.

We now introduce a couple of specific cases of generalized tractability depending on the domain Ω and the form of the function T . We begin with two examples of Ω which seem especially interesting.

- **Restricted tractability domain.** Let

$$\Omega^{\text{res}} = [1, \infty) \times [d^*] \cup [1, \varepsilon_0^{-1}) \times \mathbb{N}$$

for some $d^* \in \mathbb{N}_0$ and $\varepsilon_0 \in (0, 1]$ with $d^* + (1 - \varepsilon_0) > 0$. This corresponds to the smallest set Ω used for tractability study. This case is called the *restricted tractability domain* independently of the function T .

We may consider the special subcases where $d^* = 0$ or $\varepsilon_0 = 1$. If $d^* = 0$ then $\varepsilon_0 < 1$ and we have $\Omega^{\text{res}} = [1, \varepsilon_0^{-1}) \times \mathbb{N}$. Hence, we now want to compute an ε -approximation for only $\varepsilon \in (\varepsilon_0, 1]$ and for all d . We call this subcase *restricted tractability in ε* .

If $\varepsilon_0 = 1$ then $d^* \geq 1$ and we have $\Omega^{\text{res}} = [1, \infty) \times [d^*]$. Hence, we now want to compute an ε -approximation for all $\varepsilon \in (0, 1]$ and only for $d \leq d^*$. We call this subcase *restricted tractability in d* .

- **Unrestricted tractability domain.** Let

$$\Omega^{\text{unr}} = [1, \infty) \times \mathbb{N}.$$

This corresponds to the largest set Ω used for tractability study. This case is called the *unrestricted tractability domain* independently of the function T .

We now present a couple of examples of generalized tractability in terms of specific functions T which we think are of a particular interest.

- **Polynomial tractability.** Let

$$T(x, y) = xy.$$

In this case (T, Ω^{unr}) -tractability coincides with the previously studied tractability. For this T , independently of Ω , tractability means that $n(\varepsilon, S_d, \Lambda_d)$ is bounded by a polynomial in ε^{-1} and d ; hence the name.

- **Separable tractability.** Let

$$T(x, y) = f_1(x)f_2(y)$$

with non-decreasing functions $f_i : [1, \infty) \rightarrow [1, \infty)$ for $i = 1, 2$. To guarantee (7) we assume that

$$\lim_{x \rightarrow \infty} \frac{\ln f_i(x)}{x} = 0 \quad \text{for } i = 1, 2.$$

In this case (T, Ω^{unr}) -tractability coincides with the notion of (f_1, f_2) -tractability studied in [10]. For this T , independently of Ω , the roles of ε^{-1} and d are separated; hence the name. Observe that polynomial tractability is a special case of separable tractability for $f_i(x) = x$.

For separable tractability, we can modify the condition (8) by taking possibly different exponents of ε^{-1} and d . That is, the problem S is (T, Ω) -tractable in the class Λ if there are non-negative numbers C, p and q such that

$$n(\varepsilon, S_d, \Lambda_d) \leq C f_1(\varepsilon^{-1})^p f_2(d)^q \quad \text{for all } (\varepsilon^{-1}, d) \in \Omega. \quad (12)$$

The exponents p and q are called the ε -exponent and the d -exponent. We stress that, in general, they do not need to be uniquely defined. Note that we obtain (8) from (12) by taking $t = \max\{p, q\}$. Similarly, the notion of strong (T, Ω) -tractability in the class Λ is obtained if $q = 0$ in the bound above, and the exponent t^{str} is the infimum of p satisfying the bound above with $q = 0$. Again for $f_i(x) = x$ these notions coincide with the already studied notions for polynomial tractability.

- **Separable restricted tractability.** Let

$$T(x, y) := \begin{cases} f_1(x) & \text{if } (x, y) \in [1, \infty) \times [1, d^*], \\ f_2(y) & \text{if } (x, y) \in [1, \varepsilon_0^{-1}) \times \mathbb{N} \setminus [1, d^*], \\ \max\{f_1(x), f_2(y)\} & \text{otherwise,} \end{cases}$$

where f_1, f_2 are as above with $f_2(d^*) \geq f_1(\varepsilon_0^{-1})$.

It is easy to check that T is indeed a generalized tractability function. Suppose that the function T is considered on the restricted tractability domain Ω^{res} . Then (T, Ω^{res}) -tractability corresponds to the smallest set Ω and we have a separate dependence on ε and d ; hence the name. As already discussed, such generalized tractability seems especially relevant for the case when for huge d we are only interested in a rough approximation to the solution.

- **Non-separable symmetric tractability.** Let

$$T(x, y) = \exp(f(x)f(y)) \tag{13}$$

with a non-decreasing function $f : [1, \infty) \rightarrow \mathbb{R}_+$. To guarantee (7) we need to assume that $\lim_{x+y \rightarrow \infty} f(x)f(y)/(x+y) = 0$. This holds, for example, if $f(x) = x^\alpha$ with $\alpha < 1/2$ or if $f(x) = \ln^{1+\alpha}(x+1)$ with a positive α . This choice of the tractability function with $f(x) = \ln^{1+\alpha}(x+1)$ will be useful in the study of linear tensor product problems.

It is easy to see that this tractability function is *not* separable if f is not a constant function. Indeed, assume by contrary that $T(x, y) = f_1(x)f_2(y)$ for some f_i . For $x = 1$ we get $f_2(y) = f_1(1)^{-1} \exp(f(1)f(y))$, and similarly by taking $y = 1$, we obtain $f_1(x) = f_2(1)^{-1} \exp(f(1)f(x))$. Hence, $\exp(f(x)f(y)) = [f_1(1)f_2(1)]^{-1} \exp(f(1)(f(x) + f(y)))$. From $f_1(1)f_2(1) = \exp(f^2(1))$, and by taking $x = y$ we obtain $f^2(x) =$

$2f(1)f(x) - f^2(1)$ which leads to $f(x) = f(1)$ for all x . This contradicts that f is not a constant function. Thus, T is not separable and since the roles of ε^{-1} and d are the same, this motivates the name of this generalized tractability.

We finish this subsection by an example of a function T that is *not* a tractability function. Consider $T(x, y) = \exp(y^{1-1/x})$. This function is bounded in x for fixed y and increases sub-exponentially in y for fixed x . Nevertheless,

$$\limsup_{x+y \rightarrow \infty} \frac{\ln T(x, y)}{x + y} \geq \lim_{x=y \rightarrow \infty} \frac{x^{1-1/x}}{2x} = \frac{1}{2},$$

proving that T is not a tractability function. This example shows that the notion of tractability functions does not admit functions that increase asymptotically as fast as an exponential function in some direction.

3 Linear Tensor Product Problems

In this section we consider multivariate problems defined as linear tensor product problems and study generalized tractability.

Let F_1 be a separable Hilbert space of real valued functions defined on $D_1 \subset \mathbb{R}^m$, and let G_1 be an arbitrary separable Hilbert space. Let $S_1 : F_1 \rightarrow G_1$ be a compact linear operator. Then the non-negative self-adjoint operator

$$W_1 := S_1^* S_1 : F_1 \rightarrow F_1$$

is also compact. Let $\{\lambda_i\}$ denote the sequence of non-increasing eigenvalues of W_1 , or equivalently, the sequence of squares of singular values of S_1 . If $k = \dim(F_1)$ is finite then W_1 has just finitely many eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_k$. Then we formally put $\lambda_j = 0$ for $j > k$. In any case, the eigenvalues λ_j converge to zero.

There exist orthonormal bases $\{\zeta_i\}, \{\eta_i\}$ of F_1 and G_1 respectively such that

$$S_1 f = \sum_{i \in \mathbb{N}} \sqrt{\lambda_i} \langle f, \zeta_i \rangle_{F_1} \eta_i \quad \text{for all } f \in F_1.$$

Without loss of generality, we assume that S_1 is not a zero operator, and normalize the problem by assuming that $\lambda_1 = 1$. Hence,

$$1 = \lambda_1 \geq \lambda_2 \geq \dots \geq 0.$$

This implies that $\|S_1\| = 1$ and the initial error is also one.

For $d \geq 2$, let

$$F_d = F_1 \otimes \cdots \otimes F_1$$

be the complete d -fold tensor product Hilbert space of F_1 of real valued functions defined on $D_d = D_1 \times \cdots \times D_1 \subseteq \mathbb{R}^{dm}$. Similarly, let $G_d = G_1 \otimes \cdots \otimes G_1$, d times.

The linear operator S_d is defined as tensor product operator

$$S_d = S_1 \otimes \cdots \otimes S_1 : F_d \rightarrow G_d.$$

We have $\|S_d\| = \|S_1\|^d = 1$, thus the initial error is one for all d . We call the multivariate problem $S = \{S_d\}$ a *linear tensor product problem*.

In this paper we analyze the problem S for the class of linear information $\Lambda^{\text{all}} = \{\Lambda_d^{\text{all}}\}$ leaving out the case of standard information for future study. For linear information, we can compute arbitrary inner products. In particular, we can compute $\langle f, \zeta_{i_1} \otimes \zeta_{i_2} \otimes \cdots \otimes \zeta_{i_d} \rangle_{F_d}$. It is known, see e.g., [5], that the algorithm

$$\sum_{(i_1, i_2, \dots, i_d) : \lambda_{i_1} \lambda_{i_2} \cdots \lambda_{i_d} > \varepsilon^2} \sqrt{\lambda_{i_1} \cdots \lambda_{i_d}} \langle f, \zeta_{i_1} \otimes \cdots \otimes \zeta_{i_d} \rangle_{F_d} \eta_{i_1} \otimes \cdots \otimes \eta_{i_d}$$

computes an ε -approximation of S_d and

$$n(\varepsilon, S_d, \Lambda_d^{\text{all}}) = |\{(i_1, \dots, i_d) \in \mathbb{N}^d \mid \lambda_{i_1} \cdots \lambda_{i_d} > \varepsilon^2\}|, \quad (14)$$

with the convention that the cardinality of the empty set is zero. Note that $n(\varepsilon, S_d, \Lambda_d^{\text{all}})$ is finite for all $\varepsilon \in (0, 1]$ and all d since $\lim_{j \rightarrow \infty} \lambda_j = 0$. For $d \geq 2$, we have

$$\begin{aligned} n(\varepsilon, S_d, \Lambda_d^{\text{all}}) &= \sum_{j=1}^{\max\{i : \lambda_i > \varepsilon^2\}} n(\varepsilon/\sqrt{\lambda_j}, S_{d-1}, \Lambda_{d-1}^{\text{all}}) \\ &= \sum_{j=1}^{n(\varepsilon, S_1, \Lambda_1^{\text{all}})} n(\varepsilon/\sqrt{\lambda_j}, S_{d-1}, \Lambda_{d-1}^{\text{all}}). \end{aligned}$$

Since $n(\varepsilon/\sqrt{\lambda_j}, S_{d-1}, \Lambda_{d-1}^{\text{all}}) \leq n(\varepsilon, S_{d-1}, \Lambda_{d-1}^{\text{all}})$ we obtain for all $d \geq 2$,

$$n(\varepsilon, S_1, \Lambda_1^{\text{all}}) \leq n(\varepsilon, S_{d-1}, \Lambda_{d-1}^{\text{all}}) \leq n(\varepsilon, S_d, \Lambda_d^{\text{all}}) \leq n(\varepsilon, S_1, \Lambda_1^{\text{all}})^d. \quad (15)$$

We now show a simple lemma relating generalized tractability to the eigenvalues λ_i .

Lemma 3.1. *Let T be an arbitrary tractability function, Ω a domain satisfying (6) with $\varepsilon_0 < 1$, $S = \{S_d\}$ a linear tensor product problem defined as above, and $\Lambda = \{\Lambda_d\}$ an arbitrary class of information evaluations.*

- *Let $\lambda_2 = 1$. Then S is not (T, Ω) -tractable in the class Λ .*
- *Let $\varepsilon_0^2 < \lambda_2 < 1$. Then S is not strongly (T, Ω) -tractable in the class Λ .*
- *Let $\lambda_2 = 0$. Then S is strongly (T, Ω) -tractable in the class Λ^{all} since $n(\varepsilon, S_d, \Lambda_d^{\text{all}}) = 1$ for all $(\varepsilon^{-1}, d) \in \Omega$ with $\varepsilon < 1$, and $t^{\text{str}} = 0$.*

Proof. Since $\Lambda_d \subset \Lambda_d^{\text{all}}$ we have $n(\varepsilon, S_d, \Lambda_d) \geq n(\varepsilon, S_d, \Lambda_d^{\text{all}})$. If $\lambda_2 = 1$, then we can take $i_j \in \{1, 2\}$ to conclude from (14) that

$$n(\varepsilon_0, S_d, \Lambda_d^{\text{all}}) \geq 2^d \quad \text{for all } d.$$

Hence (9) holds with $\kappa = 2$ and S is not (T, Ω) -tractable in the class Λ .

If $\varepsilon_0^2 < \lambda_2 < 1$, then we take $d-1$ values of $i_j = 1$ and one value of $i_j = 2$. Since we have at least d products of eigenvalues λ_{i_j} equal to λ_2 , we get

$$n(\varepsilon_0, S_d, \Lambda_d^{\text{all}}) \geq d \quad \text{for all } d.$$

This contradicts strong (T, Ω) -tractability in the class Λ since $n(\varepsilon_0, S_d, \Lambda_d^{\text{all}})$ cannot be bounded by $CT(\varepsilon_0^{-1}, 1)^t$ for all d .

Finally, if $\lambda_2 = 0$ then S_1 as well as S_d is a bounded linear functional which can be computed exactly using only one information evaluation. This completes the proof. \square

In what follows we will need a simple bound for $n(\varepsilon, S_d, \Lambda_d^{\text{all}})$ which was proved in [8, Remark 3.1]. For the sake of completeness we restate here the short argument.

Lemma 3.2.

- *For $\varepsilon \in (0, 1)$ and $\lambda_2 \in (0, 1)$ let*

$$\alpha(\varepsilon) := \lceil 2 \ln(1/\varepsilon) / \ln(1/\lambda_2) \rceil - 1,$$

$\beta(\varepsilon) := n(\varepsilon, S_1, \Lambda_1^{\text{all}})$, and $a := \min\{\alpha(\varepsilon), d\}$. Then

$$\binom{d}{a} \leq n(\varepsilon, S_d, \Lambda_d^{\text{all}}) \leq \binom{d}{a} \beta(\varepsilon)^a. \quad (16)$$

- If $\lambda_2 \leq \varepsilon_0^2 < 1$ then

$$n(\varepsilon, S_d, \Lambda_d^{\text{all}}) = 1 \quad \text{for all } \varepsilon \in [\varepsilon_0, 1) \text{ and for all } d \in \mathbb{N}. \quad (17)$$

Proof. Let us consider a product $\lambda_{i_1} \dots \lambda_{i_d}$ of eigenvalues of W_1 such that $\lambda_{i_1} \dots \lambda_{i_d} > \varepsilon^2$. Let k denote the number of indices i_j , $j \in [d]$, with $i_j \geq 2$. Then necessarily $\lambda_2^k > \varepsilon^2$, which implies $k \leq \alpha(\varepsilon)$. Consequently we have at most a indices that are not one. From (14), it follows that $\beta(\varepsilon) = |\{j \mid \lambda_j > \varepsilon^2\}|$, which implies that $i_j \leq \beta(\varepsilon)$ for all $j \in [d]$. This leads to (16). For $\lambda_2 \leq \varepsilon_0^2$, we may assume without loss of generality that $\lambda_2 > 0$, and we have $\alpha(\varepsilon) = a = 0$ for all $\varepsilon \in [\varepsilon_0, 1)$ and for all d . Then (16) implies (17). \square

4 Restricted Tractability Domain

In this section we study generalized tractability for the linear tensor product problem S and the restricted tractability domain

$$\Omega^{\text{res}} = [1, \infty) \times [d^*] \cup [1, \varepsilon_0^{-1}) \times \mathbb{N}$$

for $d^* \in \mathbb{N}_0$ and $\varepsilon_0 \in (0, 1]$ with $d^* + (1 - \varepsilon_0) > 0$.

As before, $\lambda = \{\lambda_j\}$ denotes the sequence of non-increasing eigenvalues of the compact operator W_1 with $\lambda_1 = 1$. We first treat the two subcases of restricted tractability in ε and in d . We will see that in the first case, when $d^* = 0$, the second largest eigenvalue λ_2 is the only eigenvalue which effects tractability, while in the second case, when $\varepsilon_0 = 1$, the convergence rate of the sequence λ is the important criterion for tractability. Then we consider the case of restricted tractability domain with $d^* \geq 1$ and $\varepsilon_0 < 1$.

4.1 Restricted Tractability in ε

We now provide necessary and sufficient conditions for restricted tractability in ε , and then illustrate them for a couple of tractability functions. In this subsection $\varepsilon_0 < 1$, and due to Lemma 3.1 we restrict our attention to the case when $\lambda_2 < 1$.

Theorem 4.1. *Let $\varepsilon_0 < 1$ and $d^* = 0$, so that*

$$\Omega^{\text{res}} = [1, \varepsilon_0^{-1}) \times \mathbb{N}.$$

Let S be a linear tensor product problem with $\lambda_2 < \lambda_1 = 1$.

- S is strongly (T, Ω^{res}) -tractable in the class of linear information iff $\lambda_2 \leq \varepsilon_0^2$. If this holds then $n(\varepsilon, S_d, \Lambda_d^{\text{all}}) = 1$ for all $(\varepsilon, d) \in [\varepsilon_0, 1) \times \mathbb{N}$, and the exponent of strong restricted tractability is $t^{\text{str}} = 0$.
- Let $\lambda_2 > \varepsilon_0^2$. Then S is (T, Ω^{res}) -tractable in the class of linear information iff

$$B := \liminf_{d \rightarrow \infty} \inf_{\varepsilon \in [\varepsilon_0, \sqrt{\lambda_2})} \frac{\ln T(\varepsilon^{-1}, d)}{\alpha(\varepsilon) \ln d} \in (0, \infty],$$

where, as in Lemma 3.2, $\alpha(\varepsilon) = \lceil 2 \ln(1/\varepsilon) / \ln(1/\lambda_2) \rceil - 1$. If this holds then the exponent of restricted tractability is $t^{\text{tra}} = 1/B$.

Proof. The first part of the lemma follows directly from Lemma 3.1 and 3.2. Before we verify the second part, we present an estimate of $n(\varepsilon, S_d, \Lambda_d^{\text{all}})$. Let $\varepsilon \in [\varepsilon_0, 1)$. For $d \geq \alpha(\varepsilon)$, we get from (16) of Lemma 3.2,

$$n(\varepsilon, S_d, \Lambda_d^{\text{all}}) \leq \frac{\beta(\varepsilon)^{\alpha(\varepsilon)}}{\alpha(\varepsilon)!} d(d-1) \dots (d-\alpha(\varepsilon)+1) \leq C_1 d^{\alpha(\varepsilon)}, \quad (18)$$

where C_1 depends only on ε_0 and S_1 .

Let now $B \in (0, \infty]$. We want to show the existence of some positive C and t such that

$$n(\varepsilon, S_d, \Lambda_d^{\text{all}}) \leq C T(\varepsilon^{-1}, d)^t \quad \text{for all } (\varepsilon, d) \in [\varepsilon_0, 1) \times \mathbb{N}. \quad (19)$$

Let $\{B_n\}$ be a sequence in $(0, B)$ that converges to B . Then we find for each $n \in \mathbb{N}$ a number $d_n \in \mathbb{N}$ such that

$$\inf_{\varepsilon \in [\varepsilon_0, \sqrt{\lambda_2})} \frac{\ln T(\varepsilon^{-1}, d)}{\alpha(\varepsilon) \ln d} \geq B_n \quad \text{for all } d \geq d_n.$$

Due to (18), to prove (19) it is sufficient to show $C_1 d^{\alpha(\varepsilon)} \leq C T(\varepsilon^{-1}, d)^t$, which is equivalent to

$$\frac{\ln(C_1/C)}{t \ln d} + \frac{\alpha(\varepsilon)}{t} \leq \frac{\ln T(\varepsilon^{-1}, d)}{\ln d}.$$

If $C \geq C_1$ and $1/t = B_n$, then for all $d \geq d_n$ and all $\varepsilon \in [\varepsilon_0, 1]$

$$n(\varepsilon, S_d, \Lambda_d^{\text{all}}) \leq C T(\varepsilon^{-1}, d)^t.$$

To make the last estimate hold for every $(\varepsilon, d) \in [\varepsilon_0, 1] \times \mathbb{N}$, we only have to increase the number C if necessary. Taking n to infinity, we see that $t^{\text{tra}} \leq 1/B$.

Now let (19) hold for some positive C and t . To prove that $B \in (0, \infty]$, we apply (16) of Lemma 3.2 for $\varepsilon \in [\varepsilon_0, \sqrt{\lambda_2})$ and $d \geq \alpha(\varepsilon)$. Then

$$n(\varepsilon, S_d, \Lambda_d^{\text{all}}) \geq \binom{d}{\alpha(\varepsilon)} \geq \left(\frac{d}{\alpha(\varepsilon)}\right)^{\alpha(\varepsilon)} \geq C_2 d^{\alpha(\varepsilon)}$$

with¹ $C_2 = \alpha(\varepsilon_0)^{-\alpha(\varepsilon_0)}$. Thus for $\varepsilon \in [\varepsilon_0, \sqrt{\lambda_2})$ we have

$$C_2 d^{\alpha(\varepsilon)} \leq C T(\varepsilon^{-1}, d)^t \quad \forall d \geq \alpha(\varepsilon),$$

which is equivalent to

$$\frac{\ln T(\varepsilon^{-1}, d)}{\ln d} \geq \frac{\alpha(\varepsilon)}{t} + \frac{\ln(C_2/C)}{t \ln d}.$$

The condition $\varepsilon^2 < \lambda_2$ implies $\alpha(\varepsilon) \geq 1$, and we get

$$\liminf_{d \rightarrow \infty} \inf_{\varepsilon \in [\varepsilon_0, \sqrt{\lambda_2})} \frac{\ln T(\varepsilon^{-1}, d)}{\alpha(\varepsilon) \ln d} \geq \frac{1}{t}.$$

This proves that $B > 0$ and $t^{\text{tra}} \geq 1/B$, and completes the proof. \square

We illustrate Theorem 4.1 for a number of tractability functions T assuming that $\lambda_2 \in (\varepsilon_0^2, 1)$. In this case we do *not* have strong tractability whereas tractability depends on T .

- Polynomial tractability, $T(x, y) = xy$. Then (T, Ω^{res}) -tractability in the class of linear information holds with the exponent $t^{\text{tra}} = 1/B$ with

$$B = \frac{1}{\alpha(\varepsilon_0)} = \frac{1}{\lceil 2 \ln(1/\varepsilon_0) / \ln(1/\lambda_2) \rceil - 1}.$$

- Separable restrictive tractability, $T(x, y) = f_2(y)$ for $x, y \in [1, \varepsilon_0^{-1}] \times \mathbb{N}$, and with a non-decreasing function $f_2 : [1, \infty) \rightarrow [1, \infty)$ such that

¹Here we use the inequality $\binom{d}{k} \geq (d/k)^k$ for $d \geq k$ which can be easily checked by induction on d .

$\lim_{y \rightarrow \infty} (\ln f_2(y))/y = 0$. Then (T, Ω^{res}) -tractability in the class of linear information holds iff

$$B_1 := \liminf_{d \rightarrow \infty} \frac{\ln f_2(d)}{\ln d} \in (0, \infty];$$

in this case we get $t^{\text{tra}} = 1/B$, where

$$B = \frac{B_1}{\alpha(\varepsilon_0)} = \frac{B_1}{\lceil 2 \ln(1/\varepsilon_0) / \ln(1/\lambda_2) \rceil - 1}.$$

Note that $B_1 > 0$ iff $f_2(d)$ is at least of order d^β for some positive β . Hence, if we take $f_2(d) = \lceil \ln(d+1) \rceil$ then we do not have tractability. On the other hand, if $f_2(d) = d^\beta$ for a positive β then $B_1 = \beta$. For $f_2(d) = \exp(\ln^{1+\beta}(d))$ with $\beta > 0$ we obtain $B_1 = \infty$ and $t^{\text{tra}} = 0$. This means that in this case for an arbitrarily small positive t we have

$$n(\varepsilon, S_d, \Lambda_d^{\text{all}}) = o(T(\varepsilon^{-1}, d)^t) \quad \text{for all } \varepsilon \in [\varepsilon_0, 1], d \in \mathbb{N}.$$

- Non-separable symmetric tractability, $T(x, y) = \exp(f(x)f(y))$ with f as in (13). Then (T, Ω^{res}) -tractability in the class of linear information holds iff

$$B_2 = \liminf_{d \rightarrow \infty} \frac{f(d)}{\ln d} \in (0, \infty],$$

and the exponent $t^{\text{tra}} = 1/B$ with

$$B = B_2 \inf_{\varepsilon \in [\varepsilon_0, \sqrt{\lambda_2})} \frac{f(x)}{\alpha(x)}.$$

Note that $B_2 > 0$ iff $f(d)$ is at least of order $\ln(d)$. For example, if $f(x) = \beta \ln d$ for a positive β then $B_2 = \beta$, whereas $f(d) = d^\alpha$ with $\alpha > 0$ yields $B_2 = \infty$ and $t^{\text{tra}} = 0$.

4.2 Restricted Tractability in d .

We now assume that $d^* \geq 1$ and $\varepsilon_0 = 1$ such that

$$\Omega^{\text{res}} = [1, \infty) \times [d^*].$$

We provide necessary and sufficient conditions for restricted tractability in d in terms of the sequence of eigenvalues $\lambda = \{\lambda_j\}$ of the compact operator

$W_1 = S_1^* S_1$. Assume first that W_1 has a finite number of positive eigenvalues λ_j . Then

$$\lim_{\varepsilon \rightarrow 0} n(\varepsilon, S_1, \Lambda_1^{\text{all}}) < \infty$$

and (15) yields for all d ,

$$\lim_{\varepsilon \rightarrow 0} n(\varepsilon, S_d, \Lambda_d^{\text{all}}) \leq \left(\lim_{\varepsilon \rightarrow 0} n(\varepsilon, S_1, \Lambda_1^{\text{all}}) \right)^d < \infty.$$

In our case, $d \leq d^*$, and we have strong (T, Ω^{res}) -tractability for all tractability functions T with $t^{\text{str}} = 0$ since

$$n(\varepsilon, S_d, \Lambda_d^{\text{all}}) \leq C := \left(\lim_{\varepsilon \rightarrow 0} n(\varepsilon, S_1, \Lambda_1^{\text{all}}) \right)^{d^*} \quad \text{for all } (\varepsilon, d) \in \Omega^{\text{res}}.$$

Assume then that W_1 has infinitely many positive eigenvalues λ_j which is equivalent to assuming that $\lim_{\varepsilon \rightarrow 0} n(\varepsilon, S_1, \Lambda_1^{\text{all}}) = \infty$. In this case we have the following theorem.

Theorem 4.2. *Let*

$$\Omega^{\text{res}} = [1, \infty) \times [d^*] \quad \text{with } d^* \geq 1, \text{ and } \lim_{\varepsilon \rightarrow 0} n(\varepsilon, S_1, \Lambda_1^{\text{all}}) = \infty.$$

Then the following three statements are equivalent:

(i)

$$A := \liminf_{\varepsilon \rightarrow 0} \frac{\ln T(\varepsilon^{-1}, 1)}{\ln n(\varepsilon, S_1, \Lambda_1^{\text{all}})} \in (0, \infty],$$

(ii) *S is (T, Ω^{res}) -tractable in the class of linear information,*

(iii) *S is strongly (T, Ω^{res}) -tractable in the class of linear information.*

If (i) holds then the exponent of strong (T, Ω^{res}) -tractability and the exponent of (T, Ω^{res}) -tractability satisfy

$$\frac{1}{A} \leq t^{\text{tra}} \leq t^{\text{str}} \leq \frac{d^*}{A}.$$

Proof. It is enough to show that (iii) \Rightarrow (ii) \Rightarrow (i) \Rightarrow (iii).

(iii) \Rightarrow (ii) is obvious.

(ii) \Rightarrow (i). For $d = 1$ we now know that

$$CT(\varepsilon^{-1}, 1)^t \geq n(\varepsilon, S_1, \Lambda_1^{\text{all}})$$

for some positive C and t with $t \geq t^{\text{tra}}$. Taking logarithms we obtain

$$\frac{\ln T(\varepsilon^{-1}, 1)}{\ln n(\varepsilon, S_1, \Lambda_1^{\text{all}})} \geq \frac{1}{t} + \frac{\ln C^{-1}}{t \ln n(\varepsilon, S_1, \Lambda_1^{\text{all}})}.$$

Since $n(\varepsilon, S_1, \Lambda_1^{\text{all}})$ goes to infinity, we conclude that $A \geq 1/t > 0$, as claimed. Furthermore, $t \geq 1/A$ and since t can be arbitrarily close to t^{tra} we have $t^{\text{tra}} \geq 1/A$.

(i) \Rightarrow (iii). We now know that for any $\delta \in (0, A)$ there exists a positive ε_δ such that for all $\varepsilon \in (0, \varepsilon_\delta]$ we have

$$n(\varepsilon, S_1, \Lambda_1^{\text{all}}) \leq T(\varepsilon^{-1}, 1)^{1/(A-\delta)}.$$

Hence, there is a constant $C_\delta \geq 1$ such that

$$n(\varepsilon, S_1, \Lambda_1^{\text{all}}) \leq C_\delta T(\varepsilon^{-1}, 1)^{1/(A-\delta)} \quad \text{for all } \varepsilon \in (0, 1].$$

From (15) we obtain for all $d \in [d^*]$,

$$n(\varepsilon, S_d, \Lambda_d^{\text{all}}) \leq C_\delta^{d^*} T(\varepsilon^{-1}, 1)^{d^*/(A-\delta)} \quad \text{for all } \varepsilon \in (0, 1].$$

This proves strong tractability with the exponent at most $d^*/(A-\delta)$. Since δ can be arbitrarily small, $t^{\text{tra}} \leq t^{\text{str}} \leq d^*/A$, which completes the proof. \square

Theorem 4.2 states that (T, Ω^{res}) -tractability is equivalent to $A > 0$, where A depends only on the behavior of the eigenvalues for $d = 1$. The condition $A > 0$ means that $\ln T(\varepsilon^{-1}, 1)$ goes to infinity at least as fast as $\ln n(\varepsilon, S_1, \Lambda_1^{\text{all}})$. Note that for a finite positive A and $d^* > 1$, we do not have sharp bounds on the exponents. We shall see later that both bounds in Theorem 4.2 may be attained for some multivariate problems and tractability functions T . It may also happen that $A = \infty$. In this case $t^{\text{tra}} = t^{\text{str}} = 0$ which means that for all $d \in [d^*]$, and all positive t we have

$$n(\varepsilon, S_d, \Lambda_d^{\text{all}}) = o(T(\varepsilon^{-1}, d)^t) \quad \text{as } \varepsilon \rightarrow 0.$$

To verify the condition $A > 0$ and find better bounds on the exponents of tractability, we study the different rates of convergence of the sequence $\lambda = \{\lambda_j\}$. We consider exponential, polynomial and logarithmic rates of λ . That is, we assume:

- exponential rate: λ_j is of order $\exp(-\beta j)$ for some positive β , or a little more generally, λ_j is of order $\exp(-\beta j^\alpha)$ for some positive α and β .
- polynomial rate: λ_j is of order $j^{-\beta} = \exp(-\beta \ln j)$, or a little more generally, λ_j is of order $\exp(-\beta(\ln j)^\alpha)$ for some positive α and β .
- logarithmic rate: λ_j is of order $(\ln j)^{-\beta} = \exp(-\beta \ln \ln j)$ for some positive β .

Note that for $\alpha < 1$, we have sub-exponential or sub-polynomial behavior of the eigenvalues, whereas for $\alpha > 1$, we have super-exponential or super-polynomial behavior of the eigenvalues. For the sake of simplicity we omit the prefixes sub and super and talk only about exponential or polynomial rates.

As we shall see, tractability will depend on some limits. We will denote these limits using the subscripts indicating the rate of convergence of λ . Hence, the subscript e indicates an exponential rate, the subscript p a polynomial rate, and the subscript l a logarithmic one.

4.2.1 Exponential Rate

Theorem 4.3. *Let $\Omega^{\text{res}} = [1, \infty) \times [d^*]$ with $d^* \geq 1$. Let S be a linear tensor product problem with $\lambda_1 = 1$ and with exponentially decaying eigenvalues λ_j ,*

$$K_1 \exp(-\beta_1 j^{\alpha_1}) \leq \lambda_j \leq K_2 \exp(-\beta_2 j^{\alpha_2}) \quad \text{for all } j \in \mathbb{N}$$

for some positive numbers $\alpha_1, \alpha_2, \beta_1, \beta_2, K_1$ and K_2 .

Then S is strongly (T, Ω^{res}) -tractable (as well as (T, Ω^{res}) -tractable due to Theorem 4.2) in the class of linear information iff

$$A_e := \liminf_{x \rightarrow \infty} \frac{\ln T(x, 1)}{\ln \ln x} \in (0, \infty].$$

If $A_e > 0$ then the exponent of (T, Ω^{res}) -tractability satisfies

$$\frac{1}{\alpha_1} \max_{d \in [d^*]} \frac{d}{A_{e,d}} \leq t^{\text{tra}} \leq \frac{1}{\alpha_2} \max_{d \in [d^*]} \frac{d}{A_{e,d}}$$

where

$$A_{e,d} = \liminf_{x \rightarrow \infty} \frac{\ln T(x, d)}{\ln \ln x},$$

(clearly, $A_{e,d} \geq A_{e,1} = A_e > 0$), and the exponent of strong (T, Ω^{res}) -tractability satisfies

$$\frac{d^*}{\alpha_1 A_e} \leq t^{\text{str}} \leq \frac{d^*}{\alpha_2 A_e}.$$

Proof. We have

$$n(\varepsilon, S_1, \Lambda_1^{\text{all}}) = \min\{j \mid \lambda_{j+1} \leq \varepsilon^2\}.$$

Using the estimates on λ_j we obtain

$$\min\{j \mid g_1(j) \leq \varepsilon^2\} \leq n(\varepsilon, S_1, \Lambda_1^{\text{all}}) \leq \min\{j \mid g_2(j) \leq \varepsilon^2\},$$

where $g_i(j) = K_i \exp(-\beta_i(j+1)^{\alpha_i})$. This yields

$$\left(\frac{1}{\beta_1} \ln(K_1 \varepsilon^{-2})\right)^{1/\alpha_1} - 1 \leq n(\varepsilon, S_1, \Lambda_1^{\text{all}}) \leq \left(\frac{1}{\beta_2} \ln(K_2 \varepsilon^{-2})\right)^{1/\alpha_2}.$$

For small ε this leads to

$$\frac{\ln \ln \varepsilon^{-1}}{\alpha_1} (1 + o(1)) \leq \ln n(\varepsilon, S_1, \Lambda_1^{\text{all}}) \leq \frac{\ln \ln \varepsilon^{-1}}{\alpha_2} (1 + o(1)).$$

Therefore A from (i) of Theorem 4.2 satisfies $\alpha_2 A_e \leq A \leq \alpha_1 A_e$. Hence, $A > 0$ iff $A_e > 0$, and (i) of Theorem 4.2 yields the first part of Theorem 4.3.

We now find bounds on the exponents assuming that $A_e > 0$. First we estimate $n(\varepsilon, S_d, \Lambda_d^{\text{all}})$. With $x := \ln((K_2^d \varepsilon^{-2})^{1/\beta_2})$ we obtain from (14),

$$n(\varepsilon, S_d, \Lambda_d^{\text{all}}) \leq m_e(x, d) := \left| \left\{ (i_1, \dots, i_d) \mid \sum_{j=1}^d i_j^{\alpha_2} < x \right\} \right|.$$

We now prove that

$$\kappa_{d,x} \left(\left(\frac{x}{d} \right)^{1/\alpha_2} - 1 \right)^d \leq m_e(x, d) \leq x^{d/\alpha_2}. \quad (20)$$

where $\kappa_{d,x} = 1$ for $x \geq d$, and $\kappa_{d,x} = 0$ for $x < d$. We prove (20) by induction on d . Let $\alpha = \alpha_2$. For $d = 1$ we have $m(x, 1) = |\{i \mid i < x^{1/\alpha}\}|$, i.e., $x^{1/\alpha} - 1 \leq m(x, 1) < x^{1/\alpha}$. For $d > 1$, we have

$$m_e(x, d) = \sum_{k < x^{1/\alpha}} m_e(x - k^\alpha, d - 1).$$

From our induction hypothesis we get

$$m_e(x, d) \leq \sum_{k < x^{1/\alpha}} (x - k^\alpha)^{(d-1)/\alpha} \leq \sum_{k < x^{1/\alpha}} x^{(d-1)/\alpha} \leq x^{d/\alpha}.$$

To prove a lower bound, we can assume that $x > d$, and then

$$\begin{aligned} m_e(x, d) &\geq \sum_{k: k^\alpha + d - 1 \leq x} \left(\left(\frac{x - \xi^\alpha}{d - 1} \right)^{1/\alpha} - 1 \right)^{d-1} \\ &\geq \int_1^{(x+1-d)^{1/\alpha}} \left(\left(\frac{x - \xi^\alpha}{d - 1} \right)^{1/\alpha} - 1 \right)^{d-1} d\xi. \end{aligned}$$

Since $x + 1 - d \geq x/d$ and $(x - \xi^\alpha)/(d - 1) \geq x/d$ for $\xi \in [1, (x/d)^{1/\alpha}]$, we have

$$m_e(x, d) \geq \int_1^{(x/d)^{1/\alpha}} \left(\left(\frac{x}{d} \right)^{1/\alpha} - 1 \right)^{d-1} d\xi = \left(\left(\frac{x}{d} \right)^{1/\alpha} - 1 \right)^d,$$

as claimed.

Consequently, we have for all $\varepsilon \in (0, 1]$ and $d \in [d^*]$,

$$n(\varepsilon, S_d, \Lambda_d^{\text{all}}) \leq \left(\frac{\ln(K_2^d \varepsilon^{-2})}{\beta_2} \right)^{d/\alpha_2}. \quad (21)$$

Take $C^* := \sup\{(1/\beta_2)^{d/\alpha_2} \mid d \in [d^*]\}$. It is easy to see that $K_2 \geq 1$. We want to show the existence of some positive C, t such that

$$n(\varepsilon, S_d, \Lambda_d^{\text{all}}) \leq C^* \ln(K_2^{d^*} \varepsilon^{-2})^{d^*/\alpha_2} \leq C T(\varepsilon^{-1}, 1)^t \quad (22)$$

for all $\varepsilon \in (0, 1]$. The right-hand inequality is equivalent to

$$\frac{\ln(C^*/C)}{t \ln \ln \varepsilon^{-1}} + \frac{d^* \ln \ln(K_2^{d^*} \varepsilon^{-2})}{\alpha_2 t \ln \ln \varepsilon^{-1}} \leq \frac{\ln T(\varepsilon^{-1}, 1)}{\ln \ln \varepsilon^{-1}}. \quad (23)$$

Let $\{A_n\}$ be a sequence in $(0, A_e)$ converging to A_e . Hence for every n there exists a positive ε_n such that

$$\frac{\ln T(\varepsilon^{-1}, 1)}{\ln \ln \varepsilon^{-1}} \geq A_n \quad \text{for all } \varepsilon \in (0, \varepsilon_n].$$

Therefore, decreasing ε_n if necessary, we obtain (23) for all $\varepsilon \in (0, \varepsilon_n]$ as long as we choose $C \geq C^*$ and $t > d^*/(\alpha_2 A_n)$. To establish (22) for all $\varepsilon \in (\varepsilon_n, 1]$, we can keep the same t and, if necessary, increase C . Hence, we have strong tractability with the exponent $t^{\text{str}} \leq d^*/(\alpha_2 A_n)$, and with n tending to infinity, we conclude that $t^{\text{str}} \leq d^*/(\alpha_2 A_e)$.

We know that the problem is also tractable. To obtain an upper bound on the exponent of tractability we use (21) and we find positive C and t for which

$$n(\varepsilon, S_d, \Lambda_d^{\text{all}}) \leq \left(\frac{\ln(K_2^{d^*} \varepsilon^{-2})}{\beta_2} \right)^{d/\alpha_2} \leq CT(\varepsilon^{-1}, d)^t \quad \forall d \in [d^*].$$

Proceeding as before, we conclude that $t^{\text{tra}} \leq \max_{d \in [d^*]} d/(\alpha_2 A_{e,d})$.

To obtain lower bounds on the exponents, we use the estimate

$$n(\varepsilon, S_d, \Lambda_d^{\text{all}}) \geq \tilde{m}_e(z, d) := \left| \left\{ (i_1, \dots, i_d) \mid \sum_{j=1}^d i_j^{\alpha_1} < z \right\} \right|,$$

where $z = z(\varepsilon, d) := \ln((K_1^d \varepsilon^{-2})^{1/\beta_1})$. For sufficiently small ε , we can use the left-hand side of (20) with α_2 replaced by α_1 which yields

$$n(\varepsilon, S_d, \Lambda_d^{\text{all}}) \geq c z^{d/\alpha_1} = c \left(\ln((K_1^d \varepsilon^{-2})^{1/\beta_1}) \right)^{d/\alpha_1}$$

for all $d \in [d^*]$, where c is independent of ε and d . Thus, for all $t > t^{\text{str}}$ there exists a $C > 0$ such that for small ε we have the inequality

$$CT(\varepsilon^{-1}, 1)^t \geq c \left(\ln((K_1^{d^*} \varepsilon^{-2})^{1/\beta_1}) \right)^{d^*/\alpha_1},$$

which is equivalent to

$$\frac{\ln T(\varepsilon^{-1}, 1)}{\ln \ln \varepsilon^{-1}} \geq \frac{\ln(c/C)}{t \ln \ln \varepsilon^{-1}} + \frac{d^* \ln(\beta_1^{-1} \ln(K_1^{d^*} \varepsilon^{-2}))}{\alpha_1 t \ln \ln \varepsilon^{-1}}.$$

This implies $A_e \geq d^*/(\alpha_1 t)$, and $t^{\text{str}} \geq d^*/(\alpha_1 A_e)$.

For tractability, we know that there are positive C and t such that

$$CT(\varepsilon^{-1}, d)^t \geq c \left(\ln((K_1^d \varepsilon^{-2})^{1/\beta_1}) \right)^{d/\alpha_1} \quad \forall d \in [d^*].$$

Proceeding as before, we conclude that $t^{\text{tra}} \geq \max_{d \in [d^*]} d/(\alpha_1 A_{e,d})$. This concludes the proof. \square

For exponentially decaying eigenvalues, Theorem 4.3 states that strong tractability (and tractability) are equivalent to the condition $A_e > 0$. If we know the precise order of convergence of λ , i.e., when $\alpha_1 = \alpha_2 = \alpha > 0$, then we know the exponents of tractability,

$$\begin{aligned} t^{\text{tra}} &= \frac{1}{\alpha} \max_{d \in [d^*]} \frac{d}{A_{e,d}}, \\ t^{\text{str}} &= \frac{1}{\alpha} \frac{d^*}{A_e}. \end{aligned}$$

As we shall see it may happen that $t^{\text{str}} > t^{\text{tra}}$.

We now illustrate Theorem 4.3 for a number of tractability functions T .

- Polynomial tractability, $T(x, y) = xy$. Then $A_{e,d} = A_e = \infty$, and we have strong tractability with $t^{\text{tra}} = t^{\text{str}} = 0$.
- Separable restrictive tractability, $T(x, y) = f_1(x)$ for $(x, y) \in \Omega^{\text{res}}$ and a non-decreasing function

$$f_1 : [1, \infty) \rightarrow [1, \infty) \quad \text{with} \quad \lim_{x \rightarrow \infty} \frac{\ln f_1(x)}{x} = 0.$$

Then strong (T, Ω^{res}) -tractability holds iff

$$A_{e,d} = A_e = \liminf_{x \rightarrow \infty} \frac{\ln f_1(x)}{\ln \ln x} \in (0, \infty].$$

Note that $A_e > 0$ iff $f_1(x)$ is at least of order $(\ln x)^\beta$ for some positive β . If we take $f(x) = \lceil \ln(x+1) \rceil$ then we have strong tractability with $A_e = 1$. For $\alpha_1 = \alpha_2 = \alpha > 0$, the exponents are $t^{\text{str}} = t^{\text{tra}} = d^*/\alpha$.

- Non-separable symmetric tractability, $T(x, y) = \exp(f(x)f(y))$ with f as in (13). Then (T, Ω^{res}) -tractability holds iff

$$A_{e,d} = f(d) \liminf_{x \rightarrow \infty} \frac{f(x)}{\ln \ln x} \in (0, \infty].$$

Hence, $A_e = A_{e,1} > 0$ iff $f(x)$ is at least of order $\beta \ln \ln x$ for some positive β . For example, if we take $f(x) = \ln^{1+\alpha}(x+1)$ for $\alpha > -1$ then $A_{e,d} = \infty$ and $t^{\text{str}} = t^{\text{tra}} = 0$. For $f(x) = \beta \ln \ln(x+c)$ with $c > \exp(1) - 1$ and a positive β we have $f(1) > 0$ and

$$A_{e,d} = f(d) \beta = \beta^2 \ln \ln(d+c).$$

For $\alpha_1 = \alpha_2 = \alpha > 0$, we now have

$$t^{\text{str}} = \frac{d^*}{\alpha \beta^2 \ln \ln(1+c)}.$$

Assume for simplicity that $d^* = 2$ and take c close to $\exp(1) - 1$. Then the maximum of the function $d/\ln \ln(d+c)$ is attained for $d = 1$, and we have

$$t^{\text{tra}} = \frac{1}{\alpha \beta^2 \ln \ln(1+c)} = \frac{t^{\text{str}}}{d^*}.$$

4.2.2 Polynomial Rate

Theorem 4.4. *Let $\Omega^{\text{res}} = [1, \infty) \times [d^*]$ with $d^* \geq 1$. Let S be a linear tensor product problem with $\lambda_1 = 1$ and with polynomially decaying eigenvalues λ_j ,*

$$K_1 \exp(-\beta_1(\ln j)^\alpha) \leq \lambda_j \leq K_2 \exp(-\beta_2(\ln j)^\alpha) \quad \text{for all } j \in \mathbb{N}$$

for some positive numbers $\alpha, \beta_1, \beta_2, K_1$ and K_2 .

Then S is strongly (T, Ω^{res}) -tractable (as well (T, Ω) -tractable due to Theorem 4.2) in the class of linear information iff

$$A_p := \liminf_{x \rightarrow \infty} \frac{\ln T(x, 1)}{(\ln x)^{1/\alpha}} \in (0, \infty].$$

If $\alpha \in (0, 1]$ and $A_p > 0$ then the exponents of (T, Ω^{res}) -tractability satisfy

$$\left(\frac{2}{\beta_1}\right)^{1/\alpha} A_p^{-1} \leq t^{\text{tra}} \leq t^{\text{str}} \leq \left(\frac{2}{\beta_2}\right)^{1/\alpha} A_p^{-1}.$$

If $\alpha \in (1, \infty)$ and $A_p > 0$ then the exponent of (T, Ω^{res}) -tractability satisfies

$$\left(\frac{2}{\beta_1}\right)^{1/\alpha} \max_{d \in [d^*]} \frac{d^{1-1/\alpha}}{A_{p,d}} \leq t^{\text{tra}} \leq \left(\frac{2}{\beta_2}\right)^{1/\alpha} \max_{d \in [d^*]} \frac{d^{1-1/\alpha}}{A_{p,d}},$$

where

$$A_{p,d} = \liminf_{x \rightarrow \infty} \frac{\ln T(x, d)}{(\ln x)^{1/\alpha}},$$

(clearly, $A_{p,d} \geq A_{p,1} = A_p > 0$), and the exponent of strong (T, Ω^{res}) -tractability satisfies

$$(d^*)^{1-1/\alpha} \left(\frac{2}{\beta_1}\right)^{1/\alpha} A_p^{-1} \leq t^{\text{str}} \leq (d^*)^{1-1/\alpha} \left(\frac{2}{\beta_2}\right)^{1/\alpha} A_p^{-1}.$$

Proof. We now have

$$\min\{j \mid g_1(j) \leq \varepsilon^2\} \leq n(\varepsilon, S_1, \Lambda_1^{\text{all}}) \leq \min\{j \mid g_2(j) \leq \varepsilon^2\}$$

with $g_i(j) = K_i \exp(-\beta_i(\ln(j+1))^\alpha)$. This yields

$$\exp\left(\left(\beta_1^{-1} \ln(K_1 \varepsilon^{-2})\right)^{1/\alpha}\right) - 1 \leq n(\varepsilon, S_1, \Lambda_1^{\text{all}}) \leq \exp\left(\left(\beta_2^{-1} \ln(K_2 \varepsilon^{-2})\right)^{1/\alpha}\right).$$

For small ε this leads to

$$\left(\frac{2 \ln \varepsilon^{-1}}{\beta_1}\right)^{1/\alpha} (1 + o(1)) \leq \ln n(\varepsilon, S_1, \Lambda_1^{\text{all}}) \leq \left(\frac{2 \ln \varepsilon^{-1}}{\beta_2}\right)^{1/\alpha} (1 + o(1)).$$

Hence, A from (i) of Theorem 4.2 satisfies $(\beta_2/2)^{1/\alpha} A_p \leq A \leq (\beta_1/2)^{1/\alpha} A_p$. Hence, $A > 0$ iff $A_p > 0$, and (i) of Theorem 4.2 yields the first part of Theorem 4.4, and that $t^{\text{tra}} \geq (2/\beta_1)^{1/\alpha} A_p^{-1}$.

We now find bounds on the exponents assuming that $A_p > 0$. First we estimate $n(\varepsilon, S_d, \Lambda_d^{\text{all}})$. With $x = x(\varepsilon, d) := \ln((K_2^d \varepsilon^{-2})^{1/\beta_2})$ we have

$$n(\varepsilon, S_d, \Lambda_d^{\text{all}}) \leq m_p(x, d) := \left| \left\{ (i_1, \dots, i_d) \mid \sum_{j=1}^d (\ln i_j)^\alpha < x \right\} \right|.$$

We now prove the following estimates on $m_p(x, d)$. Let $s > 1$. If $\alpha \in (0, 1]$ then there exists a positive number $C(s, d)$ such that

$$\exp(x^{1/\alpha}) - 1 \leq m_p(x, d) \leq C(s, d) \exp(s x^{1/\alpha}). \quad (24)$$

If $\alpha \in [1, \infty)$ then there exists a positive number $C(s, d)$ such that

$$\left(\exp\left(\left(\frac{x}{d}\right)^{1/\alpha}\right) - 1\right)^d \leq m_p(x, d) \leq C(s, d) \exp(s d^{1-1/\alpha} x^{1/\alpha}). \quad (25)$$

For $d = 1$ we have $m_p(x, 1) = |\{j \mid j < \exp(x^{1/\alpha})\}|$ and $\exp(x^{1/\alpha}) - 1 \leq m_p(x, 1) < \exp(x^{1/\alpha})$.

We start with $\alpha \in (0, 1]$. The lower bound is already proved since $m_p(x, d) \geq m_p(x, 1)$. To obtain an upper bound on $m_p(x, d)$, we modify an argument from the proof of Theorem 3.1(ii) in [8]. Let $\zeta(s) = \sum_{k=1}^{\infty} k^{-s}$ denote the Riemann zeta function. We show by induction on d that

$$m_p(x, d) \leq \zeta(s)^{d-1} \exp(s x^{1/\alpha}).$$

For $d = 1$ this holds. Assume that our claim holds for d . Then

$$\begin{aligned} m_p(x, d+1) &= \sum_{k < \exp(x^{1/\alpha})} m_p(x - (\ln k)^\alpha, d) \\ &\leq \zeta(s)^{d-1} \sum_{k < \exp(x^{1/\alpha})} \exp\left(s(x - (\ln k)^\alpha)^{1/\alpha}\right). \end{aligned}$$

Since $(a - b)^{1/\alpha} \leq a^{1/\alpha} - b^{1/\alpha}$ for all $a \geq b \geq 0$ and $\alpha \in (0, 1]$, we obtain

$$\begin{aligned} m_p(x, d+1) &\leq \zeta(s)^{d-1} \sum_{k < \exp(x^{1/\alpha})} \exp(s x^{1/\alpha}) \exp(-s \ln k) \\ &= \zeta(s)^{d-1} \exp(s x^{1/\alpha}) \sum_{k < \exp(x^{1/\alpha})} k^{-s} \leq \zeta(s)^d \exp(s x^{1/\alpha}). \end{aligned}$$

Let now $\alpha \in [1, \infty)$. Again we proceed with induction on d . For $d = 1$, the estimate (25) holds. Assume that our claim holds for d . Again we have

$$m_p(x, d+1) = \sum_{k < \exp(x^{1/\alpha})} m_p(x - (\ln k)^\alpha, d).$$

To get a lower bound on $m_p(x, d+1)$, we obtain

$$\begin{aligned} m_p(x, d+1) &\geq \int_1^{\exp(x^{1/\alpha})} \left(\exp\left(\left(\frac{x - (\ln \xi)^\alpha}{d}\right)^{1/\alpha}\right) - 1 \right)^d d\xi \\ &\geq \int_1^{\exp((x/(d+1))^{1/\alpha})} \left(\exp\left(\left(\frac{x}{d+1}\right)^{1/\alpha}\right) - 1 \right)^d d\xi \\ &\geq \left(\exp\left(\left(\frac{x}{d+1}\right)^{1/\alpha}\right) - 1 \right)^{d+1}. \end{aligned}$$

We now obtain an upper bound on $m_p(x, d+1)$. Let $r = (1 + s)/2$. Since $r > 1$, we can use the upper bound on $m_p(x, d)$ and obtain

$$\begin{aligned} m_p(x, d+1) &\leq C(r, d) \left\{ \exp(r d^{1-1/\alpha} x^{1/\alpha}) \right. \\ &\quad \left. + \int_1^{\exp(x^{1/\alpha})} \exp(r d^{1-1/\alpha} (x - (\ln \xi)^\alpha)^{1/\alpha}) d\xi \right\}. \end{aligned}$$

The substitution $z = \ln \xi$ leads to

$$\int_1^{\exp(x^{1/\alpha})} \exp(r d^{1-1/\alpha} (x - (\ln \xi)^\alpha)^{1/\alpha}) d\xi \leq \int_0^{x^{1/\alpha}} \exp(rh(z)) dz,$$

where $h(z) = d^{1-1/\alpha}(x - z^\alpha)^{1/\alpha} + z$. Since

$$h'(z) = 1 - d^{1-1/\alpha} \left(\frac{x}{z^\alpha} - 1 \right)^{1/\alpha-1},$$

the function h takes its maximum at $z = (x/(d+1))^{1/\alpha}$, and we get

$$\begin{aligned} m_p(x, d+1) &\leq C(r, d) \left\{ \exp(r d^{1-1/\alpha} x^{1/\alpha}) + x^{1/\alpha} \exp(r(d+1)^{1-1/\alpha} x^{1/\alpha}) \right\} \\ &\leq C(r, d) (1 + x^{1/\alpha}) \exp(r(d+1)^{1-1/\alpha} x^{1/\alpha}). \end{aligned}$$

Since

$$\begin{aligned} a &:= \sup_{x>0} (1 + x^{1/\alpha}) \exp(- (s-r)(d+1)^{1-1/\alpha} x^{1/\alpha}) \\ &= \sup_{x>0} (1 + x^{1/\alpha}) \exp(- (s-1)(d+1)^{1-1/\alpha} x^{1/\alpha}/2) < \infty \end{aligned}$$

we take $C(s, d+1) = a C(r, d)$ and conclude that

$$m_p(x, d+1) \leq C(s, d+1) \exp(s(d+1)^{1-1/\alpha} x^{1/\alpha}),$$

as claimed.

Let $\gamma := \max\{0, 1 - 1/\alpha\}$. Then (24) and (25) yield that for every $s > 1$ there exists a positive C_s such that for all $\varepsilon \in (0, 1]$ and $d \in [d^*]$,

$$n(\varepsilon, S_d, \Lambda_d^{\text{all}}) \leq C_s \exp(s d^\gamma (\ln \varepsilon^{-2/\beta_2})^{1/\alpha}). \quad (26)$$

Knowing that $A_p > 0$, we want to show that

$$C_s \exp(s (d^*)^\gamma (\ln \varepsilon^{-2/\beta_2})^{1/\alpha}) \leq C T(\varepsilon^{-1}, 1)^t \quad (27)$$

for some positive C and t . Let $\{A_n\}$ be a sequence in $(0, A_p)$ converging to A_p . Then for every n there exists a positive ε_n such that

$$\frac{\ln T(\varepsilon^{-1}, 1)}{(\ln \varepsilon^{-1})^{1/\alpha}} \geq A_n \quad \text{for all } \varepsilon \in (0, \varepsilon_n].$$

Observe that (27) is equivalent to

$$\frac{s (d^*)^\gamma \left(\frac{2}{\beta_2}\right)^{1/\alpha}}{t} + \frac{\ln(C_s/C)}{t (\ln \varepsilon^{-1})^{1/\alpha}} \leq \frac{\ln T(\varepsilon^{-1}, 1)}{(\ln \varepsilon^{-1})^{1/\alpha}}.$$

This holds for all $\varepsilon \in (0, \varepsilon_n]$ if $t \geq s(d^*)^\gamma(2/\beta_2)^{1/\alpha}A_n^{-1}$ and $C \geq C_s$. For $\varepsilon > \varepsilon_n$ we can keep the same t and, if necessary, increase C . Hence (27) holds with $t = s(d^*)^\gamma(2/\beta_2)^{1/\alpha}A_n^{-1}$. Thus, S is strongly (T, Ω^{res}) -tractable. Taking s arbitrarily close to 1 and n to infinity, we conclude $t^{\text{str}} \leq (d^*)^\gamma(2/\beta_2)^{1/\alpha}A_p^{-1}$.

We now show that in the case $\alpha \in (1, \infty)$ the exponent of strong tractability fulfills $t^{\text{str}} \geq (d^*)^{1-1/\alpha}(2/\beta_1)^{1/\alpha}A_p^{-1}$. Here we use the estimate

$$n(\varepsilon, S_d, \Lambda_d^{\text{all}}) \geq m_p(z, d),$$

where $z = z(\varepsilon, d) := \ln((K_1^d \varepsilon^{-2})^{1/\beta_1})$. For small ε , the left-hand side of (25) implies that there is a positive $c(d)$ such that

$$n(\varepsilon, S_d, \Lambda_d^{\text{all}}) \geq c(d) \exp\left(d^{1-1/\alpha} \left(\frac{1}{\beta_1} \ln(K_1^d \varepsilon^{-2})\right)^{1/\alpha}\right). \quad (28)$$

Thus for all $t > t^{\text{str}}$ there exists a $C > 0$ such that for small ε we have

$$CT(\varepsilon^{-1}, 1)^t \geq c(d^*) \exp\left((d^*)^{1-1/\alpha} \left(\frac{1}{\beta_1} \ln(K_1^d \varepsilon^{-2})\right)^{1/\alpha}\right),$$

which is equivalent to

$$\frac{\ln T(\varepsilon^{-1}, 1)}{(\ln \varepsilon^{-1})^{1/\alpha}} \geq \frac{\ln(c(d^*)/C)}{t(\ln \varepsilon^{-1})^{1/\alpha}} + \frac{(d^*)^{1-1/\alpha}}{t} \left(\frac{1}{\beta_1} \frac{\ln(K_1^d \varepsilon^{-2})}{\ln \varepsilon^{-1}}\right)^{1/\alpha}.$$

Taking the limit inferior for $\varepsilon \rightarrow 0$, we obtain $A_p \geq (d^*)^{1-1/\alpha}(2/\beta_1)^{1/\alpha}t^{-1}$, and $t^{\text{str}} \geq (d^*)^{1-1/\alpha}(2/\beta_1)^{1/\alpha}A_p^{-1}$.

We finally find estimates on the exponent of tractability for $\alpha \in (1, \infty)$. We proceed similarly as before and obtain that

$$CT(\varepsilon^{-1}, d)^t \geq n(\varepsilon, S_d, \Lambda_d^{\text{all}}) \quad \forall d \in [d^*]$$

implies due to (28) that for small ε ,

$$\frac{t \ln T(\varepsilon^{-1}, d)}{(\ln \varepsilon^{-1})^{1/\alpha}} \geq d^{1-1/\alpha} \left(\frac{2}{\beta_1}\right)^{1/\alpha} (1 + o(1)).$$

This yields that $t^{\text{tra}} \geq (2/\beta_1)^{1/\alpha} \max_{d \in [d^*]} d^{1-1/\alpha}/A_{p,d}$.

To get an upper bound on t^{tra} , we use (26), and conclude that it is enough to find positive C and t such that

$$C_s \exp\left(s d^{1-1/\alpha} (\ln \varepsilon^{-2/\beta_2})^{1/\alpha}\right) \leq CT(\varepsilon^{-1}, d)^t \quad \forall d \in [d^*].$$

This holds for $t \geq s \max_{d \in [d^*]} d^{1-1/\alpha} (2/\beta_2)^{1/\alpha} A_{p,d}^{-1}$. Since s can be arbitrarily close to one, we get that $t^{\text{tra}} \leq \max_{d \in [d^*]} d^{1-1/\alpha} (2/\beta_2)^{1/\alpha} A_{p,d}^{-1}$, which completes the proof. \square

For polynomially decaying eigenvalues, Theorem 4.4 states that strong tractability (and tractability) are equivalent to the condition $A_p > 0$. If we know the precise order of convergence of λ , i.e., when $\beta_1 = \beta_2 = \beta > 0$, then we know the exponents of tractability. For $\alpha \in (0, 1]$ we have

$$t^{\text{tra}} = t^{\text{str}} = \left(\frac{2}{\beta}\right)^{1/\alpha} A_p^{-1},$$

whereas for $\alpha \in (1, \infty)$ we have

$$\begin{aligned} t^{\text{tra}} &= \left(\frac{2}{\beta}\right)^{1/\alpha} \max_{d \in [d^*]} \frac{d^{1-1/\alpha}}{A_{p,d}}, \\ t^{\text{str}} &= (d^*)^{1-1/\alpha} \left(\frac{2}{\beta}\right)^{1/\alpha} A_p^{-1}. \end{aligned}$$

As before, it may happen that $t^{\text{str}} > t^{\text{tra}}$.

We now illustrate Theorem 4.4 for a number of tractability functions T .

- Polynomial tractability, $T(x, y) = xy$. Then $A_{p,d} = A_p$ and its value depends on α . We have $A_p = 0$ for $\alpha < 1$, and $A_p = 1$ for $\alpha = 1$, and $A_p = \infty$ for $\alpha > 1$. Hence, we have strong tractability (and tractability) iff $\alpha \geq 1$. For $\alpha > 1$, we have $t^{\text{tra}} = t^{\text{str}} = 0$, whereas for $\alpha = 1$ and $\beta_1 = \beta_2 = \beta > 0$, we have $t^{\text{tra}} = t^{\text{str}} = 2/\beta$.
- Separable restrictive tractability, $T(x, y) = f_1(x)$ with f_1 as for exponential decaying eigenvalues. Then strong (T, Ω^{res}) -tractability holds iff

$$A_{p,d} = A_p = \liminf_{x \rightarrow \infty} \frac{\ln f_1(x)}{(\ln x)^{1/\alpha}} \in (0, \infty].$$

Note that $A_p > 0$ iff $f_1(x)$ is at least of order $\exp(\eta(\ln x)^{1/\alpha})$ for some positive η . If we take $f_1(x) = \exp(\eta(\ln x)^{1/\alpha})$ then we have strong tractability with $A_p = \eta$. For $\beta_1 = \beta_2 = \beta > 0$, the exponents are $t^{\text{str}} = t^{\text{tra}} = (d^*)^{(1-1/\alpha)_+} (2/\beta)^{1/\alpha} \eta^{-1}$.

- Non-separable symmetric tractability, $T(x, y) = \exp(f(x)f(y))$ with f as in (13). Then (T, Ω^{res}) -tractability holds iff

$$A_{p,d} = f(d) \liminf_{x \rightarrow \infty} \frac{f(x)}{(\ln x)^{1/\alpha}} \in (0, \infty].$$

Hence, $A_p = A_{p,1} > 0$ iff $f(x)$ is at least of order $\eta (\ln x)^{1/\alpha}$ for some positive η . For example, if we take $f(x) = \eta (\ln(x+c))^{1/\alpha}$ with a positive c , then $A_{p,d} = f(d)\eta$. For a given $\alpha \in [1, \infty)$, $\beta_1 = \beta_2 = \beta > 0$, and sufficiently small c , the maximum of the function $d^{1-1/\alpha}/A_{p,d}$ is attained for $d = 1$, and we have

$$t^{\text{tra}} = \frac{2^{1/\alpha}}{\eta^2 (\beta \ln(1+c))^{1/\alpha}} = \frac{t^{\text{str}}}{(d^*)^{1-1/\alpha}}.$$

4.2.3 Logarithmic Rate

Theorem 4.5. *Let $\Omega^{\text{res}} = [1, \infty) \times [d^*]$ with $d^* \geq 1$. Let S be a linear tensor product problem with $\lambda_1 = 1$ and with logarithmically decaying eigenvalues λ_j , $K_1 \exp(-\beta \ln(\ln(j)+1)) \leq \lambda_j \leq K_2 \exp(-\beta \ln(\ln(j)+1))$ for all $j \in \mathbb{N}$ for some positive numbers β, K_1 and K_2 .*

Let $\beta \leq 2$. Then S is not (T, Ω^{res}) -tractable in the class of linear information.

Let $\beta > 2$. Then S is strongly (T, Ω^{res}) -tractable (as well (T, Ω^{res}) -tractable due to Theorem 4.2) in the class of linear information iff

$$A_l := \liminf_{x \rightarrow \infty} \frac{\ln T(x, 1)}{x^{2/\beta}} \in (0, \infty].$$

If $\beta > 2$ and $A_l > 0$ then the exponent of (T, Ω^{res}) -tractability satisfies

$$\max_{d \in [d^*]} \frac{K_1^{d/\beta}}{A_{l,d}} \leq t^{\text{tra}} \leq \max_{d \in [d^*]} \frac{K_2^{d/\beta}}{A_{l,d}},$$

where

$$A_{l,d} := \liminf_{x \rightarrow \infty} \frac{\ln T(x, d)}{x^{2/\beta}},$$

(clearly, $A_{l,d} \geq A_{l,1} = A_l > 0$), and the exponent of strong (T, Ω^{res}) -tractability satisfies

$$\frac{K_1^{1/\beta}}{A_l} \leq t^{\text{str}} \leq \frac{K_2^{d^*/\beta}}{A_l}.$$

(Note that the numbers K_1 and K_2 must satisfy $K_1 \leq 1 \leq K_2$. Thus, if $K_1 = K_2 = 1$, we have also $K_1^{1/\beta} = K_2^{d^*/\beta}$.)

Proof. We now have

$$\min\{j \mid g_1(j) \leq \varepsilon^2\} \leq n(\varepsilon, S_1, \Lambda_1^{\text{all}}) \leq \min\{j \mid g_2(j) \leq \varepsilon^2\}$$

with $g_i(j) = K_i \exp(-\beta \ln(\ln(j+1) + 1))$. This yields

$$\exp(K_1^{1/\beta} \varepsilon^{-2/\beta} - 1) - 1 \leq n(\varepsilon, S_1, \Lambda_1^{\text{all}}) \leq \exp(K_2^{1/\beta} \varepsilon^{-2/\beta} - 1).$$

For small ε this leads to

$$K_1^{1/\beta} \varepsilon^{-2/\beta} (1 + o(1)) \leq \ln n(\varepsilon, S_1, \Lambda_1^{\text{all}}) \leq K_2^{1/\beta} \varepsilon^{-2/\beta} (1 + o(1)).$$

Assume first that $\beta \leq 2$. Then

$$\liminf_{\varepsilon \rightarrow 0} \frac{\ln T(\varepsilon^{-1}, 1)}{\ln n(\varepsilon, S_1, \Lambda_1^{\text{all}})} \leq K_1^{-1/\beta} \liminf_{\varepsilon \rightarrow 0} \frac{\ln T(\varepsilon^{-1}, 1)}{\varepsilon^{-1} + 1} \frac{\varepsilon^{-1} + 1}{\varepsilon^{-2/\beta}} = 0$$

due to (7). Therefore A from (i) of Theorem 4.2 is zero, and we do not have tractability, as claimed.

Assume then that $\beta > 2$. Then $K_2^{-1/\beta} A_l \leq A \leq K_1^{-1/\beta} A_l$. Hence, $A > 0$ iff $A_l > 0$, and (i) of Theorem 4.2 yields the first part of Theorem 4.4, and that $t^{\text{str}} \geq K_1^{1/\beta} A_l^{-1}$.

We now find bounds on the exponents assuming that $A_l > 0$. First we estimate $n(\varepsilon, S_d, \Lambda_d^{\text{all}})$. With $x = x(\varepsilon, d) := \ln((K_2^d/\varepsilon^2)^{1/\beta})$ we get

$$n(\varepsilon, S_d, \Lambda_d^{\text{all}}) \leq m_l(x, d) := \left| \left\{ (i_1, \dots, i_d) \mid \sum_{j=1}^d \ln(\ln(i_j) + 1) < x \right\} \right|.$$

We prove that for every $s > 1$ there exists a positive number $C(s, d)$ such that

$$\exp(\exp(x) - 1) - 1 \leq m_l(x, d) \leq C(s, d) \exp(s(\exp(x) - 1)). \quad (29)$$

Let $\eta := \exp(x)$. Clearly we have $m_l(x, 1) = |\{j \mid j < \exp(\eta - 1)\}|$, which implies

$$\exp(\eta - 1) - 1 \leq m_l(x, 1) \leq \exp(\eta - 1).$$

Let now $d \geq 1$ and assume that (29) holds for d . Then

$$m_l(x, d+1) = \sum_{k < \exp(\eta-1)} m_l(x - \ln(\ln(k) + 1), d).$$

Thus, we get the trivial lower bound estimate

$$m_l(x, d+1) \geq m_l(x, d) \geq \exp(\eta - 1) - 1.$$

We now obtain an upper bound on $m_l(x, d+1)$. Let $r = (1+s)/2$. Then

$$m_l(x, d+1) \leq C(r, d) \left\{ \exp(r(\eta - 1)) + \int_1^{\exp(\eta-1)} \exp\left(r \left(\exp(x - \ln(\ln(\xi) + 1)) - 1\right)\right) d\xi \right\}.$$

The last integral is of the form

$$\int_1^{\exp(\eta-1)} \exp\left(r \left(\frac{\eta}{\ln(\xi) + 1} - 1\right)\right) d\xi = \int_1^\eta \exp(r h(z)) dz,$$

where $z = \ln(\xi) + 1$, and $h : [1, \eta] \rightarrow \mathbb{R}$ with $h(z) = \eta/z + z/r - (1 + 1/r)$. It is easy to check that h takes its maximum $\eta - 1$ at the point $z = 1$. So we have

$$\int_1^\eta \exp(r h(z)) dz \leq (\eta - 1) \exp(r(\eta - 1)).$$

This results in

$$\begin{aligned} m_l(x, d+1) &\leq C(r, d) \eta \exp(r(\eta - 1)) \\ &= C(r, d) \eta \exp((r-s)(\eta - 1)) \exp(s(\eta - 1)) \\ &\leq C(r, d) \left(\sup_{\xi \geq 1} \xi \exp(-(s-1)(\xi - 1)/2) \right) \exp(s(\eta - 1)) \\ &\leq C(s, d+1) \exp(s(\eta - 1)) \end{aligned}$$

for suitably large $C(s, d+1)$, as claimed.

Due to (29), we conclude that

$$n(\varepsilon, S_d, \Lambda_d^{\text{all}}) \leq C(s, d) \exp(s(K_2^{d/\beta} \varepsilon^{-2/\beta} - 1)).$$

For $A_l \in (0, \infty]$, $\varepsilon \in (0, 1]$ and $d \in [d^*]$, we want to show that

$$C(s, d) \exp\left(s \left(K_2^{d/\beta} \varepsilon^{-2/\beta} - 1\right)\right) \leq C T(\varepsilon^{-1}, 1)^t \quad (30)$$

for some positive C, t . Let therefore $\{A_n\}$ be a sequence in $(0, A_l)$ converging to A_l . Thus for every n there exists a positive ε_n such that

$$\frac{\ln T(\varepsilon^{-1}, 1)}{\varepsilon^{-2/\beta}} \geq A_n \quad \text{for all } \varepsilon \in (0, \varepsilon_n].$$

Then (30) is equivalent to

$$\frac{s K_2^{d/\beta}}{t} + \frac{\ln(C(s, d)/C) - s}{t \varepsilon^{-2/\beta}} \leq \frac{\ln T(\varepsilon^{-1}, 1)}{\varepsilon^{-2/\beta}}.$$

This holds for all $\varepsilon \in (0, \varepsilon_n]$ if $t \geq s K_2^{d/\beta} A_n^{-1}$ and $C \geq C(s, d)$. For $\varepsilon \in (\varepsilon_n, 1]$ we can keep the same t and, if necessary, increase C . Taking s tending to 1 and n to infinity, we conclude $t^{\text{str}} \leq K_2^{d^*/\beta} A_l^{-1}$.

Similarly we show bounds on the exponent t^{tra} since $(\ln T(\varepsilon^{-1}, d))/\varepsilon^{-2/\beta}$ is arbitrarily close to $A_{l,d}$ for small ε . This leads to $t^{\text{tra}} \leq \max_{d \in [d^*]} K_2^{d/\beta} / A_{l,d}$. To get a lower bound on t^{tra} , we use the left-hand side inequality in (29) to conclude that

$$n(\varepsilon, S_d, \Lambda_d^{\text{all}}) \geq \exp\left(K_1^{d/\beta} \varepsilon^{-2/\beta} - 1\right) - 1.$$

This yields that $t^{\text{tra}} \geq \max_{d \in [d^*]} K_1^{d/\beta} / A_{l,d}$, and completes the proof. \square

For logarithmically decaying eigenvalues, Theorem 4.5 states that for $\beta \leq 2$, we do not have tractability. This means that the eigenvalues λ_j converge to zero too slowly no matter how we choose a tractability function T . For $\beta > 2$, strong tractability (and tractability) are equivalent to the condition $A_l > 0$. In this case, and for $K_1 = K_2 = 1$, we know the exponents of tractability,

$$t^{\text{tra}} = t^{\text{str}} = A_l^{-1}.$$

We now illustrate Theorem 4.5 for a number of tractability functions T .

- Polynomial tractability, $T(x, y) = xy$. Then for $\beta > 2$, we have $A_{l,d} = A_l = 0$. Hence, strong tractability (and tractability) does *not* hold for an arbitrary positive β .

- Separable restrictive tractability, $T(x, y) = f_1(x)$ with f_1 as for exponential decaying eigenvalues. Let $\beta > 2$. Then strong (T, Ω^{res}) -tractability holds iff

$$A_{l,d} = A_l = \liminf_{x \rightarrow \infty} \frac{\ln f_1(x)}{x^{2/\beta}} \in (0, \infty].$$

Note that $A_l > 0$ iff $\ln f_1(x)$ is at least of order x^α with $\alpha \in [2/\beta, 1)$. If we take $f_1(x) = \exp(x^\alpha)$ then we have strong tractability with $A_l = 0$ for $\alpha \in (2/\beta, 1)$, and then $t^{\text{str}} = t^{\text{tra}} = 0$, whereas $A_l = 1$ for $\alpha = 2/\beta$ and $t^{\text{str}} = t^{\text{tra}} = 1$ for $K_2 = K_1 = 1$.

- Non-separable symmetric tractability, $T(x, y) = \exp(f(x)f(y))$ with f as in (13). For $\beta > 2$, (T, Ω^{res}) -tractability holds iff

$$A_{l,d} = f(d) \liminf_{x \rightarrow \infty} \frac{f(x)}{x^{2/\beta}} \in (0, \infty].$$

Hence, $A_l = A_{l,1} > 0$ iff $f(x)$ is at least of order $x^{2/\beta}$. For example, if we take $f(x) = x^{2/\beta}$ then $A_{l,d} = f(d)$. For $K_2 \leq \exp(1/d^*)$, the maximum of the function $K_2^{d/\beta}/f(d)$ is attained for $d = 1$, and $t^{\text{tra}} \leq K_2^{1/\beta}$ and $t^{\text{str}} \leq \exp(1)^{1/\beta}$.

4.3 Restricted Tractability with $d^* \geq 1$ and $\varepsilon_0 < 1$

Based on the results for restricted tractability in ε and d , it is easy to study restricted tractability with $d^* \geq 1$ and $\varepsilon \in (0, 1)$. In this subsection we denote

$$\Omega^{\text{res}} = \Omega^{\text{res}}(\varepsilon_0, d^*) = [1, \infty) \times [d^*] \cup [1, \varepsilon_0^{-1}) \times \mathbb{N}$$

for $d^* \in \mathbb{N}_0$ and $\varepsilon_0 \in (0, 1]$.

Hence, restricted tractability in ε corresponds to $\Omega^{\text{res}}(\varepsilon_0, 0) = [1, \varepsilon_0^{-1}) \times \mathbb{N}$ with $\varepsilon_0 \in (0, 1)$, and restricted tractability in d corresponds to $\Omega(1, d^*) = [1, \infty) \times [d^*]$ with $d^* \geq 1$.

Since $\Omega^{\text{res}}(\varepsilon_0, d^*) = \Omega^{\text{res}}(\varepsilon_0, 0) \cup \Omega^{\text{res}}(1, d^*)$, it is obvious that strong tractability and tractability for $d^* \geq 1$ and $\varepsilon_0 \in (0, 1)$ are equivalent to restricted strong tractability and tractability in ε and d , respectively. We summarize this simple fact in the following lemma.

Lemma 4.6. *Let $d^* \geq 1$ and $\varepsilon_0 \in (0, 1)$. Let S be a linear tensor product problem with $\lambda_1 = 1$. Then*

- S is strongly $(T, \Omega^{\text{res}}(\varepsilon_0, d^*))$ -tractable in the class of linear information iff S is strongly $(T, \Omega^{\text{res}}(\varepsilon_0, 0))$ - and strongly $(T, \Omega^{\text{res}}(1, d^*))$ -tractable in the class of linear information.
- S is $(T, \Omega^{\text{res}}(\varepsilon_0, d^*))$ -tractable in the class of linear information iff S is $(T, \Omega^{\text{res}}(\varepsilon_0, 0))$ - and $(T, \Omega^{\text{res}}(1, d^*))$ -tractable in the class of linear information.
- The exponents of strong tractability and tractability for $\Omega^{\text{res}}(\varepsilon_0, d^*)$ are the maximum of the respected exponents for $\Omega^{\text{res}}(\varepsilon_0, 0)$ and $\Omega^{\text{res}}(1, d^*)$.

We now combine the results of the previous subsections and present two theorems on tractability of S for $\Omega^{\text{res}}(\varepsilon_0, d^*)$. In these theorems by saying S is strongly tractable we mean that S is strongly $(T, \Omega^{\text{res}}(\varepsilon_0, d^*))$ -tractable in the class of linear information, and by saying S is tractable we mean that S is $(T, \Omega^{\text{res}}(\varepsilon_0, d^*))$ -tractable in the class of linear information.

Theorem 4.7. *Let $d^* \geq 1$ and $\varepsilon_0 \in (0, 1)$. Let S be a linear tensor product problem with $\lambda_1 = 1$.*

- Let $\lambda_2 = 1$. Then S is not tractable.
- Let $\varepsilon_0^2 < \lambda_2 < 1$. Then S is not strongly tractable, and S is tractable iff

$$A = \liminf_{\varepsilon \rightarrow 0} \frac{\ln T(\varepsilon^{-1}, 1)}{\ln n(\varepsilon, S_1, \Lambda_1^{\text{all}})} \in (0, \infty],$$

$$B = \liminf_{d \rightarrow \infty} \inf_{\varepsilon \in [\varepsilon_0, \sqrt{\lambda_2})} \frac{\ln T(\varepsilon^{-1}, d)}{\alpha(\varepsilon) \ln d} \in (0, \infty],$$

where $\alpha(\varepsilon) = \lceil 2 \ln(1/\varepsilon) / \ln(1/\lambda_2) \rceil - 1$.

If $A > 0$ and $B > 0$ then

$$\max(A^{-1}, B^{-1}) \leq t^{\text{tra}} \leq \max(d^* A^{-1}, B^{-1}).$$

- Let $0 < \lambda_2 \leq \varepsilon_0^2$.
Let $\lim_{\varepsilon \rightarrow 0} n(\varepsilon, S_1, \Lambda_1^{\text{all}}) < \infty$. Then S is strongly tractable and $t^{\text{str}} = 0$.
Let $\lim_{\varepsilon \rightarrow 0} n(\varepsilon, S_1, \Lambda_1^{\text{all}}) = \infty$. Then S is strongly tractable iff S is tractable iff

$$A = \liminf_{\varepsilon \rightarrow 0} \frac{\ln T(\varepsilon^{-1}, 1)}{\ln n(\varepsilon, S_1, \Lambda_1^{\text{all}})} \in (0, \infty].$$

If $A > 0$ then

$$A^{-1} \leq t^{\text{tra}} \leq t^{\text{str}} \leq d^* A^{-1}.$$

- Let $\lambda_2 = 0$. Then $n(\varepsilon, S_d, \Lambda_d^{\text{all}}) = 1$ for all $(\varepsilon, d) \in \Omega(\varepsilon_0, d^*)$, and S is strongly tractable with $t^{\text{str}} = 0$,

Proof. For $\lambda_2 = 1$, it is enough to apply the first part of Lemma 3.1.

Let $\varepsilon_0^2 < \lambda_2 < 1$. The lack of strong tractability follows from the second part of Lemma 3.1. Tractability in ε holds iff $B \in (0, \infty]$ due to the second part of Theorem 4.1. Let $\lim_{\varepsilon \rightarrow 0} n(\varepsilon, S_1, \Lambda_1^{\text{all}}) < \infty$. Then tractability in d holds and, in this case, $A \in (0, \infty]$, due to the reasoning before Theorem 4.2. Let $\lim_{\varepsilon \rightarrow 0} n(\varepsilon, S_1, \Lambda_1^{\text{all}}) = \infty$. Then tractability in d holds iff $A \in (0, \infty]$ due to Theorem 4.2. Hence, Lemma 4.6 implies that S is tractable iff both $A, B \in (0, \infty]$. The bounds on t^{tra} are due to Theorems 4.1, 4.2 and Lemma 4.6.

For $0 < \lambda_2 \leq \varepsilon_0^2$ and $\lim_{\varepsilon \rightarrow 0} n(\varepsilon, S_1, \Lambda_1^{\text{all}}) < \infty$, we conclude strong tractability of S due to the first part of Theorem 4.1, the reasoning before Theorem 4.2 and Lemma 4.6. In this case, $t^{\text{str}} = 0$.

For $0 < \lambda_2 \leq \varepsilon_0^2$ and $\lim_{\varepsilon \rightarrow 0} n(\varepsilon, S_1, \Lambda_1^{\text{all}}) = \infty$, strong tractability in ε holds with $t^{\text{str}} = 0$ due to the first part of Theorem 4.1, and strong tractability in d is equivalent to tractability in d and equivalent to $A \in (0, \infty]$ due to Theorem 4.2. This and Lemma 4.6 yield that S is strongly tractable iff S is tractable iff $A \in (0, \infty]$. The bounds on t^{tra} and t^{str} follow from Theorem 4.2.

For $\lambda_2 = 0$, the problem is trivial due to the last part of Lemma 3.1. \square

We now summarize tractability conditions for $\Omega(\varepsilon_0, d^*)$ assuming the specific rates of convergence of the eigenvalues $\lambda = \{\lambda_j\}$ as discussed in Theorems 4.3, 4.4 and 4.5.

Theorem 4.8. *Let $d^* \geq 1$ and $\varepsilon_0 \in (0, 1)$. Let S be a linear tensor product problem with $\lambda_2 < \lambda_1 = 1$.*

- Let $\lambda_j = \Theta(\exp(-\beta j^\alpha))$ converge to zero with an exponential rate for some positive α and β .
 - Let $\varepsilon_0^2 < \lambda_2$. Then S is not strongly tractable, and S is tractable iff $A_e = A_{e,1} \in (0, \infty]$ and $B \in (0, \infty]$ with

$$A_{e,d} = \liminf_{x \rightarrow \infty} \frac{\ln T(x, d)}{\ln \ln x} \in (0, \infty],$$

and B as in Theorem 4.7. If $A_e > 0$ and $B > 0$ then

$$t^{\text{tra}} = \max \left(\frac{1}{\alpha} \max_{d \in [d^*]} \frac{d}{A_{e,d}}, \frac{1}{B} \right).$$

- Let $\lambda_2 \leq \varepsilon_0^2$. Then S is strongly tractable iff $A_e \in (0, \infty]$. If $A_e > 0$ then

$$t^{\text{str}} = \frac{d^*}{\alpha A_e} \quad \text{and} \quad t^{\text{tra}} = \frac{1}{\alpha} \max_{d \in [d^*]} \frac{d}{A_{e,d}}.$$

- Let $\lambda_j = \Theta(\exp(-\beta(\ln j)^\alpha))$ converge to zero with a polynomial rate for some positive α and β .

- Let $\varepsilon_0^2 < \lambda_2$. Then S is not strongly tractable, and S is tractable iff $A_p = A_{p,1} \in (0, \infty]$ and $B \in (0, \infty]$ with

$$A_{p,d} = \liminf_{x \rightarrow \infty} \frac{\ln T(x, d)}{(\ln x)^{1/\alpha}} \in (0, \infty],$$

and B as in Theorem 4.7. If $A_p > 0$ and $B > 0$ then

$$t^{\text{tra}} = \max \left(\left(\frac{2}{\beta} \right)^{1/\alpha} \max_{d \in [d^*]} \frac{d^{(1-1/\alpha)_+}}{A_{p,d}}, \frac{1}{B} \right).$$

- Let $\lambda_2 \leq \varepsilon_0^2$. Then S is strongly tractable iff $A_p \in (0, \infty]$. If $A_p > 0$ then

$$t^{\text{str}} = \left(\frac{2}{\beta} \right)^{1/\alpha} \frac{(d^*)^{(1-1/\alpha)_+}}{A_p} \quad \text{and} \quad t^{\text{tra}} = \left(\frac{2}{\beta} \right)^{1/\alpha} \max_{d \in [d^*]} \frac{d^{(1-1/\alpha)_+}}{A_{p,d}}.$$

- Let $\lambda_j = \exp(-\beta(\ln(\ln(j) + 1)))$ converge to zero with a logarithmic rate for some positive β . For $\beta \leq 2$, S is not tractable. For $\beta > 2$, we have the following.

- Let $\varepsilon_0^2 < \lambda_2$. Then S is not strongly tractable, and S is tractable iff $A_l \in (0, \infty]$ and $B \in (0, \infty]$ with

$$A_l = \liminf_{x \rightarrow \infty} \frac{\ln T(x, 1)}{x^{2/\beta}} \in (0, \infty]$$

and B as in Theorem 4.7. If $A_l > 0$ and $B > 0$ then

$$t^{\text{tra}} = \max \left(\frac{1}{A_l}, \frac{1}{B} \right).$$

- Let $\lambda_2 \leq \varepsilon_0^2$. Then S is strongly tractable iff $A_l \in (0, \infty]$. If $A_l > 0$ then

$$t^{\text{str}} = t^{\text{tra}} = \frac{1}{A_l}.$$

Proof. For the exponential rate and $\varepsilon_0^2 < \lambda_2$, the lack of strong tractability follows from Theorem 4.7, whereas tractability is equivalent to $A_e, B \in (0, \infty]$ due to Theorems 4.3 and 4.1. The formula for t^{tra} also follows from these two theorems and Lemma 4.6.

For the exponential rate and $\lambda_2 \leq \varepsilon_0^2$, strong tractability in ε trivially holds, and strong tractability in d holds iff $A_e > 0$ due to Theorem 4.3. The formulas for t^{str} and t^{tra} are also from Theorem 4.3.

For the polynomial and logarithmic rates, we proceed in the same way and use Theorem 4.4 for the polynomial case, and Theorem 4.5 for the logarithmic case, instead of Theorem 4.3. \square

We illustrate Theorems 4.7 and 4.8 for a number of tractability functions T .

- Polynomial tractability, $T(x, y) = xy$. Then $A_{e,d} = A_{p,d} = \infty$ for $\alpha > 1$, whereas $A_{p,d} = 1$ if $\alpha = 1$, and $A_{p,d} = 0$ for $\alpha < 1$. Finally, $A_{l,d} = 0$ for $\beta > 2$. Hence, for logarithmically and polynomially decaying eigenvalues with $\alpha < 1$, S is not tractable.

Let $\varepsilon_0^2 < \lambda_2$. We have $B = 1/\alpha(\varepsilon_0)$. Then for exponentially and polynomially decaying eigenvalues with $\alpha > 1$, S is not strongly tractable but is tractable with the exponent

$$t^{\text{tra}} = \alpha(\varepsilon_0) = \lceil 2 \ln(1/\varepsilon_0) / \ln(1/\lambda_2) \rceil - 1 ..$$

For polynomially decaying eigenvalues with $\alpha = 1$, S is not strongly tractable but is tractable with the exponent

$$t^{\text{tra}} = \max \left(\frac{2}{\beta}, \alpha(\varepsilon_0) \right).$$

Let $\lambda_2 \leq \varepsilon_0^2$. Then for exponentially and polynomially decaying eigenvalues with $\alpha > 1$, S is strongly tractable with $t^{\text{str}} = 0$. For polynomially decaying eigenvalues with $\alpha = 1$, S is strongly tractable with $t^{\text{str}} = 2/\beta$.

- Separable restrictive tractability, $T(x, y) = f_1(x)$ for $(x, y) \in \Omega(1, d^*)$, and $T(x, y) = f_2(y)$ for $(x, y) \in \Omega(\varepsilon_0, 0)$ with non-decreasing f_1 and f_2 such that $\lim_{t \rightarrow \infty} (\ln f_i(t))/t = 0$.

For simplicity, let us take $f_i(t) = \exp((\ln t)^{\alpha_i})$ for some positive α_i . Then $A_{e,d} = \infty$, whereas $A_{p,d} = \infty$ if $\alpha_1 > 1/\alpha$, and $A_{p,d} = 1$ if $\alpha_1 = 1/\alpha$, and $A_{p,d} = 0$ if $\alpha_1 < 1/\alpha$. Finally, $A_{l,d} = 0$. Hence, for polynomially decaying eigenvalues with $\alpha_1 < 1/\alpha$, and for logarithmically decaying eigenvalues S is not tractable.

Let $\varepsilon_0^2 < \lambda_2$. If $\alpha_2 < 1$, then $B = 0$ and S is not tractable. Let $\alpha_2 \geq 1$. Then for exponentially and polynomially decaying eigenvalues with $\alpha_1 > 1/\alpha$, S is not strongly tractable but S is tractable. The exponent of tractability is $t^{\text{tra}} = \alpha(\varepsilon_0)$ if $\alpha_2 = 1$ and $t^{\text{tra}} = 0$ if $\alpha_2 > 1$. For polynomially decaying eigenvalues with $\alpha_1 = 1/\alpha$, S is not strongly tractable but tractable with exponent

$$t^{\text{tra}} = \max \left\{ \left(\frac{2}{\beta} \right)^{1/\alpha} (d^*)^{(1-1/\alpha)_+}, \alpha(\varepsilon_0) \right\} \quad \text{if } \alpha_2 = 1$$

and

$$t^{\text{tra}} = \left(\frac{2}{\beta} \right)^{1/\alpha} (d^*)^{(1-1/\alpha)_+} \quad \text{if } \alpha_2 > 1.$$

Let $\lambda_2 \leq \varepsilon_0^2$. Then for exponentially and polynomially decaying eigenvalues with $\alpha_1 > 1/\alpha$, S is strongly tractable and $t^{\text{str}} = 0$. For polynomially decaying eigenvalues with $\alpha_1 = 1/\alpha$, S is strongly tractable with

$$t^{\text{tra}} = t^{\text{str}} = \left(\frac{2}{\beta} \right)^{1/\alpha} (d^*)^{(1-1/\alpha)_+}.$$

- Non-separable symmetric tractability, $T(x, y) = \exp(f(x)f(y))$ with f as in (13). For simplicity, let us take $f(x) = (\ln(x+1))^\eta$ for some positive η . Then $A_{e,d} = \infty$, whereas $A_{p,d} = \infty$ for $\eta > 1/\alpha$, and $A_{p,d} = f(d)$ for $\eta = 1/\alpha$, and $A_{p,d} = 0$ for $\eta < 1/\alpha$. Finally, $A_{l,d} = 0$. Hence, S is not tractable for logarithmically and polynomially decaying eigenvalues with $\eta < 1/\alpha$.

Let $\varepsilon_0^2 < \lambda_2$. If $\eta < 1$, then $B = 0$ and S is not tractable. Let $\eta \geq 1$. Then for exponentially and polynomially decaying eigenvalues

with $\eta > 1/\alpha$, S is not strongly tractable but S is tractable. In the case $\eta > 1$ we have $t^{\text{tra}} = 0$. For polynomially decaying eigenvalues with $\eta = 1/\alpha$, S is not strongly tractable but tractable. If we have $\eta > 1$, then $\alpha \in (0, 1)$ and

$$t^{\text{tra}} = \left(\frac{2}{\beta \ln 2} \right)^{1/\alpha}.$$

Let $\lambda_2 \leq \varepsilon_0^2$. Then for exponentially and polynomially decaying eigenvalues with $\eta > 1/\alpha$, S is strongly tractable and $t^{\text{str}} = 0$. For polynomially decaying eigenvalues with $\eta = 1/\alpha$, S is strongly tractable with

$$t^{\text{str}} = (d^*)^{(1-1/\alpha)_+} \left(\frac{2}{\beta \ln 2} \right)^{1/\alpha}, \quad t^{\text{tra}} = \left(\frac{2}{\beta} \right)^{1/\alpha} \max_{d \in [d^*]} \frac{d^{(1-1/\alpha)_+}}{(\ln(d+1))^{1/\alpha}}.$$

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