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by

Jürgen Tolksdorf, and Torsten Thumstädter

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Jürgen Tolksdorf*
Max-Planck Inst. for Mathematics in the Sciences
Leipzig, Germany

Torsten Thumstädter†
University of Mannheim, Germany‡

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Abstract

We discuss the mass of the (physical component of the) Higgs boson in one-loop and top-quark mass approximation. For this the minimal Standard Model is regarded as a specific (parameterized) gauge theory of Dirac type. It is shown that the latter formulation, in contrast to the usual description of the Standard Model, gives a definite value for the Higgs mass. The predicted value for the Higgs mass (in natural units) is given by $\mathrm{m_H} = 188 \pm 15$ GeV. Although the Higgs mass is predicted to be near the upper bound, $\mathrm{m_H}$ is in full accordance with the range $114 < \mathrm{m_H} < 193$ GeV that is allowed by the Standard Model.

We show that the inclusion of (Dirac) massive neutrinos does not alter the results presented. We also briefly discuss how the derived mass value is related to those obtained within the frame of non-commutative geometry.

Keywords: Dirac Type Operators, Quantum Field Theory, Standard Model, Higgs Mass

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*email: juergen.tolksdorf@mis.mpg.de
†tthumstaedter@tuev-nord.de
‡Since Nov. 05 at the TÜV NORD EnSys, Dep. ETB, Hannover, Germany

1 Introduction

The Higgs boson (more precisely, the physical component thereof) is known to be the last outstanding particle predicted by the (minimal) Standard Model (STM). Within the usual description of the STM the expected mass range of the Higgs boson is restricted to the interval $[114, 193)$ GeV. (c.f. [Ros03]). This prediction of the range of the Higgs mass results from including quantum corrections and additional assumptions stemming, for example, from cosmology. Of course, over the last decade there have been many attempts to better specify the value of this mass range using different mathematical approaches to the STM, like, for instance [Ros03]. One particular mathematical approach worth mentioning here is given within the realm of non-commutative geometry, see for instance [Coq89], [Ka91], [Ka92], [Ka93], [GBV93], [Con94] and [KS96]. For similar approaches one may consult, for instance, [HPS91], [MO94], [MO96] and the appropriate references therein.

In this paper we discuss the Higgs mass within the STM using the geometrical frame of (parameterized) Dirac type gauge theories (GTDT). The mathematical background of GTDT is discussed in some detail in [TT05]. However, in order to be self-contained we present a purely local description of GTDT which is also needed to derive the relations for the Higgs mass.

The basic idea of GTDT is to introduce a general geometrical setup to describe (a certain class of) gauge theories in terms of fermions. Hence, the fundamental ingredients of GTDT are so-called "generalized Dirac operators". They basically differ from the usual Dirac operator by a general zero order term. This zero order term in turn may be used to define a gauge potential not only by a single one-form but also by forms of various degrees (please, see below). In this sense, Dirac type operators may be regarded as more general than connections. Physically speaking, generalized Dirac operators permit incorporation of different fields into one single mathematical object, which in turn are physically motivated by the postulated interactions of the fermions considered. Another advantage of describing gauge theories in terms of generalized Dirac operators is that the latter naturally induce specific Lagrangian densities. These densities can be shown to be equivariant with respect to the action of the full gauge group including the gauge group of Yang-Mills, of gravity and the diffeomorphism group of the underlying (space-time) manifold. In this sense one may say that the gauge theories defined by the corresponding Lagrangians have a "square root" in terms of generalized Dirac operators. This is not only conceptually more satisfying than the usual "adding of actions", but may also have some phenomenological consequences. Accordingly, the present paper aims at showing how the geometrical setup of GTDT allows the specification of the range of the Higgs mass of the STM. The calculations presented are similar to those given, for instance, in [KS97]. In particular, we restrict our discussion to one-loop order and top-quark mass approximation.

One basic feature of GTDT is that it is logically inconsistent to assume that space-time is flat. This is because a Dirac type operator generically yields a non-vanishing energy-momentum tensor which in turn implies a non-vanishing curvature of space-time. However, one may still assume that gravitational effects may be negligible in comparison with some given energy scale naturally implied by the gauge theory at hand. In fact, this is our basic assumption as far as the presented calculations of the Higgs mass are concerned (see also the corresponding concluding remarks).

The paper is organized as follows: In the second section we present a purely local description of GTDT as it is needed to follow the line of reasoning involved in the calculation of the Higgs mass. In the third section we summarize the STM as it is described as a special GTDT. There we also present a natural parametrization of the general mathematical scheme that is presented in [TT05]. In the fourth section we discuss the parameter relations between the appropriately parameterized GTDT of the STM with its usual mathematical description. We then discuss the resulting renormalization flow equations for the energy dependence of the coupling constants to one-loop order and in top-quark mass approximation. This is done in the $\overline{MS}-$scheme. Afterwards we discuss the possible changes when a massive neutrino sector is included. We also discuss in this section the principal bounds of the Higgs mass within GTDT. In the fifth section we compare our results with those presented within the geometrical scheme of non-commutative geometry. We conclude with some comments on the results discussed in this paper.

2 GTDT - A Local Description

In this section we present a local description of gauge theories of Dirac type in the case of a four dimensional (parallelizable) Lorentzian manifold. This description will then be applied to the (minimal) Standard Model in the next section in order to obtain some statements about the mass of the (physical component of the) Higgs boson.

Basically, a GTDT is given by the following *universal (Dirac-) Lagrangian*:

$$\mathcal{L}_{\mathrm{D}} := (\bar{\psi} i D\psi + V_{\mathrm{D}})\sqrt{-|g|}\, d^4x, \tag{1}$$

$$V_{\mathrm{D}} \equiv \tfrac{\mathrm{N}}{2}\, r_{\mathrm{M}} + \mathrm{tr}(\gamma^{\mu\nu}[\theta_\mu,\theta_\nu]) + \tfrac{1}{8}\, g_{\mu\nu}\, \mathrm{tr}\Big(\gamma^\sigma\, [\theta_\sigma,\gamma^\mu]\gamma^\lambda\, [\theta_\lambda,\gamma^\nu]\Big)\,. \tag{2}$$

Here, $|g| \equiv \det(g_{\mu\nu})$ and r_{M} denotes the Ricci scalar curvature with respect to the Lorentz metric $g_{\mu\nu}$ of signature -2. The Dirac matrices $\gamma^\mu \in \mathbf{M}_{\mathrm{N}}(\mathbb{C})$ fulfill the Clifford relation $\{\gamma^\mu,\gamma^\nu\} \equiv \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = -2\, g^{\mu\nu}\mathbf{1}_{\mathrm{N}}$ with $g^{\mu\nu}$ being the inverse of $g_{\mu\nu}$. Also, we use the common abbreviation $\gamma^{\mu\nu} \equiv \frac{1}{2}\,[\gamma^\mu,\gamma^\nu]$ for the generators of the (proper orthochroneous) Lorentz transformations in the spin representation. The dimension $\mathrm{N} \equiv 4\mathrm{N}_{\mathrm{F}}$ of the representation space is given by the "fermion representation", i.e. $\psi \in \mathcal{C}^\infty(\mathcal{M},\mathbb{C}^4\otimes\mathbb{C}^{\mathrm{N}_{\mathrm{F}}})$. In the following $\mathcal{M} \subset \mathbb{R}^4$ denotes some open subset such that $T\mathcal{M} \simeq \mathcal{M}\times\mathbb{R}^4$.

The Dirac operator $D = \gamma^\mu\nabla^{\mathrm{D}}_\mu$ is defined in terms of the (Dirac) connection

$$\nabla^{\mathrm{D}}_\mu := \partial_\mu + \omega_\mu + \theta_\mu$$

$$\equiv \nabla^{\mathrm{S}}_\mu + \theta_\mu, \tag{3}$$

with $\omega_\mu \equiv \frac{1}{4}\gamma^{\alpha\beta}\omega_{\mu\alpha\beta} \in \mathcal{C}^\infty(\mathcal{M}, \mathbf{M}_{\mathrm{N}}(\mathbb{C}))$ being the spin-connection form and ∇^{S}_μ the corresponding spin connection with respect to $g_{\mu\nu}$. Also, the one-form $\theta_\mu \in \mathcal{C}^\infty(\mathcal{M}, \mathbf{M}_{\mathrm{N}}(\mathbb{C}))$ denotes a general gauge potential. The connection (3) is called a *Clifford connection* (or, "twisted spin-connection") if the general gauge potential θ_μ fulfills

$$[\theta_\mu, \gamma^\nu] = 0. \tag{4}$$

In this case we write $\theta_\mu = A_\mu$, such that a Clifford connection reads[1]

$$\begin{aligned}\nabla^{\mathrm{D}}_\mu &\equiv \nabla^{\mathrm{Cl}}_\mu := \partial_\mu + \omega_\mu + A_\mu \\ &= \nabla^{\mathrm{S}}_\mu + A_\mu.\end{aligned} \tag{5}$$

Accordingly, the Dirac operator D is then called a "twisted spin Dirac operator". However, for a general gauge potential θ_μ one has $[\theta_\mu, \gamma^\nu] \neq 0$. In this more general situation the appropriate Dirac operator D is known as a *generalized Dirac operator* (or, "operator of Dirac type"), see for example in [ABS64] and [BGV91]. In what follows, however, we will refer to $D = \gamma^\mu \nabla^{\mathrm{D}}_\mu$ simply as a Dirac operator, even in the case where D is defined with respect to general $(g_{\mu\nu}, \theta_\mu)$.

Of course, a general gauge potential θ_μ can be decomposed as $\theta_\mu = A_\mu + (\theta_\mu - A_\mu) \equiv A_\mu + H_\mu$. It can be shown that the *Dirac potential* (2) is independent of such a decomposition. Hence, without loss of generality we may decompose a general connection (3) into a Clifford connection plus a general gauge potential:

$$\begin{aligned}\nabla^{\mathrm{D}}_\mu &= \nabla^{\mathrm{S}}_\mu + A_\mu + H_\mu \\ &= \nabla^{\mathrm{Cl}}_\mu + H_\mu\end{aligned} \tag{6}$$

and thereby substitute θ_μ by H_μ in (2). This general gauge potential H_μ can be expressed also in terms of the Dirac operator D itself:

$$\begin{aligned}H_\mu &= -\tfrac{1}{4}\, g_{\mu\nu}\gamma^\nu (D - \gamma^\sigma \nabla^{\mathrm{Cl}}_\sigma) \\ &\equiv \xi_\mu \Phi_{\mathrm{D}}.\end{aligned} \tag{7}$$

Note that $\gamma^\mu \xi_\mu = -\frac{1}{4}\, g_{\mu\nu}\gamma^\mu\gamma^\nu = \mathbf{1}_{\mathrm{N}}$ and $\Phi_{\mathrm{D}} = \gamma^\mu H_\mu$. Thus, we may decompose any (generalized) Dirac operator as

$$D = \gamma^\sigma \nabla^{\mathrm{Cl}}_\sigma + \Phi_{\mathrm{D}}. \tag{8}$$

Note that for any gauge potential H'_μ which fulfills the two requirements: $\gamma^\mu H'_\mu = \gamma^\mu H_\mu$ and $\xi_\mu \gamma^\nu H'_\nu = H'_\mu$, one infers that $H'_\mu = H_\mu$. Also note that

$$\Phi_{\mathrm{D}} = \sum_{k=0}^{4} \sum_{0 \le \nu_1 < \cdots < \nu_k \le 3} \gamma^{\nu_1} \cdots \gamma^{\nu_k}\, \chi^{(k)}_{\nu_1 \cdots \nu_k}, \tag{9}$$

[1] In [TT05] the covariant derivative of a Clifford connection is denoted by ∂_A. It should not be confounded with the lifted Levi-Civita connection.

with $[\chi^{(k)}_{\nu_1\cdots\nu_k}, \gamma^\mu] = 0$ being considered as k-forms on $\mathcal{M}$ which take values in $\mathbf{M}_\mathrm{N}(\mathbb{C})$. The lowest order contribution $\chi^{(0)}$ is characterized by $[\Phi_\mathrm{D}, \gamma^\mu] = 0$. In contrast, the highest order contribution

$$\sum_{0\leq\nu_1<\cdots<\nu_4\leq 3} \gamma^{\nu_1}\cdots\gamma^{\nu_4}\,\chi^{(4)}_{\nu_1\cdots\nu_4} = \gamma_5\phi, \tag{10}$$

fulfills the condition

$$\{\Phi_\mathrm{D}, \gamma^\mu\} = 0, \tag{11}$$

with $\phi \in \mathcal{C}^\infty(\mathcal{M}, \mathbf{M}_{\mathrm{N_F}}(\mathbb{C}))$ and $\gamma_5 = i\gamma^0\cdots\gamma^3$ the canonical grading operator on the spinor space (such that $\mathbb{C}^4 = S_\mathrm{L} \oplus S_\mathrm{R}$ decomposes into the "left-handed" and "right-handed" spinors). The condition (11) is analogous to (4) for it is equivalent to

$$[\theta_\mu, \gamma^\nu] = \delta^\nu_\mu \gamma^\lambda \theta_\lambda. \tag{12}$$

Moreover, the first order contribution only yields a re-definition of the Yang-Mills gauge potential A_μ. Hence, in the sequel we will omit the first order part in Φ_D.

The *relative curvature* of a Dirac type operator is defined as

$$\begin{aligned} F^\theta_{\mu\nu} &:= \nabla^\mathrm{S}_\mu\theta_\nu - \nabla^\mathrm{S}_\nu\theta_\mu + [\theta_\mu, \theta_\nu] \\ &= \partial_\mu\theta_\nu - \partial_\nu\theta_\mu + [\theta_\mu, \theta_\nu] + [\omega_\mu, \theta_\nu] - [\omega_\nu, \theta_\mu]. \end{aligned} \tag{13}$$

It naturally decomposes as

$$\begin{aligned} F^\theta_{\mu\nu} &= F^\mathrm{A}_{\mu\nu} + F^\mathrm{A,H}_{\mu\nu} \\ &= F^\mathrm{A}_{\mu\nu} + F^\mathrm{H}_{\mu\nu} + \kappa^\mathrm{A,H}_{\mu\nu}. \end{aligned} \tag{14}$$

Here, $\kappa^\mathrm{A,H}_{\mu\nu} := [A_\mu, H_\nu] - [A_\nu, H_\mu]$ abbreviates the "interaction term" between the gauge potentials A_μ and H_μ. The curvature

$$\begin{aligned} F^\mathrm{A,H}_{\mu\nu} &:= F^\mathrm{D}_{\mu\nu} - F^\mathrm{Cl}_{\mu\nu} \\ &= \nabla^\mathrm{Cl}_\mu H_\nu - \nabla^\mathrm{Cl}_\nu H_\mu + [H_\mu, H_\nu] \end{aligned} \tag{15}$$

denotes the relative curvature of H_μ with respect to (ω_μ, A_μ) and $F^\mathrm{D}_{\mu\nu}$ is the curvature with respect to the Dirac connection (3). In the case of $H_\mu = 0$ (i.e. $\theta_\mu = A_\mu$) the relative curvature is called the "twisting curvature" of D. Since $[\omega_\mu, A_\nu] = 0$, the twisting curvature $F^\mathrm{A}_{\mu\nu} = \nabla^\mathrm{S}_\mu A_\nu - \nabla^\mathrm{S}_\nu A_\mu + [A_\mu, A_\nu]$ coincides with the usual Yang-Mills field strength, provided Clifford connections ∇^Cl_μ are identified with Yang-Mills connections $\nabla^\mathrm{A}_\mu \equiv \partial_\mu + A_\mu$.

There is a distinguished class of Dirac type operators called *Dirac operators of Yukawa type* (c.f. [TT05]). These operators are defined by

$$\Phi_\mathrm{D} := \gamma_5\phi \tag{16}$$

Note that D is odd if and only if $\mathbb{C}^{\mathrm{N_F}} = E_{\mathrm{L}} \oplus E_{\mathrm{R}}$ and ϕ is odd. Assuming that $\phi \neq 0$, it can be shown that the field equations for Dirac operators of Yukawa type give rise to the existence of a constant (skew-hermitian) matrix function $\mathcal{D} \in \mathcal{C}^\infty(\mathcal{M}, \mathbf{M}_{\mathrm{N_F}}(\mathbb{C}))$ and a real-valued smooth function $h \in \mathcal{C}^\infty(\mathcal{M}, \mathbb{R})$ such that

$$\phi = h\mathcal{D}. \tag{17}$$

This reduces the gauge symmetry group to the isotropy group of $\mathcal{D}$. Accordingly, a Yukawa type Dirac operator is said to be in the *unitary gauge* if it reads

$$D = \gamma^\sigma \nabla^{\mathrm{Cl}}_\sigma + \gamma_5 \mathcal{D}. \tag{18}$$

A Yukawa-type Dirac operator is said to represent a *fermionic vacuum* if

$$\begin{aligned} D &= \gamma^\mu(\partial_\mu + \omega_\mu) + \gamma_5 \mathcal{D} \\ &\equiv \gamma^\mu \nabla^{\mathrm{S}}_\mu + \gamma_5 \mathcal{D} \end{aligned} \tag{19}$$

and $g_{\mu\nu}$ fulfills the Einstein equations

$$R_{\mu\nu} = \kappa_{\mathrm{gr}} \mathrm{tr}\mathcal{D}^2\, g_{\mu\nu}. \tag{20}$$

A general Yukawa type operator is then considered as a perturbation of a fermionic vacuum. Note that with respect to the latter any Yukawa type operator corresponds to $(h, A_\mu, h_{\mu\nu})$. Here, the metric $h_{\mu\nu}$ is considered as a perturbation of $g_{\mu\nu}$ which satisfies the Einstein equation with the energy-momentum tensor being defined with respect to (ψ, h, A_μ). As already mentioned in the introduction, we will neglect the influence of a non-flat space-time and assume that $g_{\mu\nu} = h_{\mu\nu} \approx \eta_{\mu\nu}$. Some appropriate comments on a justification of this assumption will be given in the conclusion.

To lowest order the metric $g_{\mu\nu}$ is fully determined by the spectrum of $\mathcal{D}^2$. In contrast, the field equations of a Yukawa type operator do not determine either the Yang-Mills connection (i.e. the gauge potential A_μ), or the (physical component of the) Higgs field h. For this one has to slightly enlarge the class of Yukawa type operators, which is referred to as *Pauli-Yukawa* type Dirac operators (PDY). They are defined by Dirac operators of the form

$$\begin{aligned} D &= \begin{pmatrix} \gamma^\mu(\nabla^{\mathrm{S}}_\mu + \theta_\mu) & -\frac{1}{2}\gamma^{\mu\nu}F^\theta_{\mu\nu} \\ \frac{1}{2}\gamma^{\mu\nu}F^\theta_{\mu\nu} & \gamma^\mu(\nabla^{\mathrm{S}}_\mu + \theta_\mu) \end{pmatrix} \\ &\equiv \gamma^\mu(\nabla^{\mathrm{S}}_\mu + \theta_\mu) + \mathcal{I}(\tfrac{1}{2}\gamma^{\mu\nu}F^\theta_{\mu\nu}) \end{aligned} \tag{21}$$

with the fermion representation space being doubled. Here, the *Higgs gauge potential* reads $H_\mu := \xi_\mu \gamma_5 \phi$ and thus $\theta_\mu = A_\mu + \xi_\mu \gamma_5 \phi$. Moreover, $\mathcal{I} := \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ may be regarded as defining an additional complex structure for $\mathcal{I}^2 = -\mathbf{1}_2$. Hence, in the case of a twisted spin Dirac operator (i.e. $\theta_\mu = A_\mu$) a PDY reduces to $D = \gamma^\mu \nabla^{\mathrm{Cl}}_\mu + \mathcal{I}(\frac{1}{2}\gamma^{\mu\nu}F^{\mathrm{A}}_{\mu\nu})$, where the second part formally looks like the well-known “Pauli-term”.

The relative curvature $F^{\rm A,H}_{\mu\nu}$ of the Higgs gauge potential explicitly reads

$$F^{\rm A,H}_{\mu\nu} = \gamma_5(\xi_\mu[\nabla^{\rm A}_\nu, \phi] - \xi_\nu[\nabla^{\rm A}_\mu, \phi]) + [\xi_\mu, \xi_\nu]\phi^2\,. \tag{22}$$

Note that this (relative) curvature depends on the total field content $(g_{\mu\nu}, \phi, A_\mu)$.

With respect to a fermionic vacuum (19) one may consider Clifford connections which are compatible with the vacuum, i.e. gauge potentials A_μ which satisfy

$$[A_\mu, \mathcal{D}] = 0. \tag{23}$$

In this case the interaction term $\kappa^{\rm A,H}_{\mu\nu}$ vanishes identically and the relative curvature (14) reduces to

$$F^\theta_{\mu\nu} = F^{\rm A}_{\mu\nu} + F^{\rm H}_{\mu\nu}. \tag{24}$$

In fact, the latter relation is equivalent to the compatibility of a Clifford connection with a fermionic vacuum.

The interaction term $\kappa^{\rm A,H}_{\mu\nu}$ has a simple physical meaning. With respect to the fermionic vacuum it corresponds to the "Yang-Mills mass matrix". Indeed, the "generalized Yang-Mills Lagrangian" ${\rm tr}F^\theta_{\mu\nu}F^{\mu\nu}_\theta$ clearly yields a term like

$$\begin{aligned} g^{\alpha\beta}g^{\mu\nu}{\rm tr}\kappa^{\rm A,H}_{\mu\alpha}\kappa^{\rm A,H}_{\nu\beta} &\sim (1+h)^2 g^{\mu\nu}{\rm tr}(\mathcal{D}\{A_\mu, A_\nu\}\mathcal{D}) \\ &\sim (1+h)^2\, {\rm M}^2_{ab}\, g^{\mu\nu} A^a_\mu A^b_\nu, \end{aligned} \tag{25}$$

with ${\rm M}^2_{ab} := {\rm tr}(\mathcal{D}\{{\rm T}_a, {\rm T}_b\}\mathcal{D})$ being proportional to the (squared) Yang-Mills mass matrix and $A_\mu = A^a_\mu {\rm T}_a$. Note that the generators ${\rm T}_a \in \mathbf{M}_{\rm N_F}(\mathbb{C})$ refer to the fermion representation. Moreover, ${\rm M}^2_{ab}$ equals zero exactly for those generators which commute with $\mathcal{D}$.

We emphasize that the "generalized Pauli term" $\gamma^{\mu\nu}F^\theta_{\mu\nu}$ in (21) does not contribute to the fermionic part in (1) when restricted to the real sub-space of "particles-anti-particles". It only contributes to the Dirac potential $V_{\rm D}$. More precisely, for D of Pauli-Yukawa type one obtains the total Lagrangian:

$$\mathcal{L}_{\rm D} = (2\bar\psi(i\gamma^\mu\nabla^{\rm Cl}_\mu + i\gamma_5\phi)\psi + V_{\rm D})\sqrt{-|g|}\,d^4x, \tag{26}$$

$$V_{\rm D} = \lambda_{\rm gr}\, r_{\rm M} - (\lambda_{\rm YM}\,{\rm tr}F^{\rm A\dagger}_{\mu\nu}F^{\mu\nu}_{\rm A} + \lambda_{\rm H}\,{\rm tr}\nabla_\mu\phi^\dagger\nabla^\mu\phi - V_{\rm H}). \tag{27}$$

Here, $\nabla_\mu\phi \equiv [\nabla^{\rm A}_\mu, \phi] = \partial_\mu\phi + [A_\mu, \phi]$ and

$$V_{\rm H} = \alpha_{\rm H}({\rm tr}\phi^\dagger\phi)^2 - \beta_{\rm H}{\rm tr}\phi^\dagger\phi \tag{28}$$

is the usual *Higgs potential* of the (minimal) Standard Model and $\lambda_{\rm gr}, \lambda_{\rm YM}, \lambda_{\rm H}, \alpha_{\rm H}, \beta_{\rm H}$ are real parameters to be specified in the next section. For a more detailed discussion, in particular, of the occurrence of the factor 2 and of the grading involution γ_5 in the Yukawa coupling and the geometrical meaning of (25), we again refer to [TT05].

In the next section we will make use of (26) – (27) which formally looks like the total Lagrangian of the (minimal) Standard Model including gravity. In fact, for specific Yukawa type Dirac operators one may appropriately re-write the Dirac-Lagrangian (26) to get exactly the form of the STM-Lagrangian. The scheme proposed may also have some phenomenological consequences since (26) is derived in one stroke by a specific class of Pauli-Yukawa type operators. For this, however, one still has to take into account the different mass dimensions of the various fields and to also include an appropriate parametrization of both the general Dirac-Lagrangian (1) and the specific class of Dirac operators one deals with.

3 The (minimal) STM as a GTDT

As discussed in the previous section the STM-Lagrangian has a natural "square root" in terms of Pauli-Yukawa type Dirac operators (PDY). This holds true in particular for the Higgs sector. However, within the framework of Dirac type gauge theories the Higgs field ϕ transforms with respect to the full fermion representation of the gauge group. This is in contrast to the minimal Standard Model where the Higgs field is supposed to transform with respect to a specific sub-representation of the fermion representation ρ_{F} (see below). These two representations are related by the Yukawa-coupling matrix G_{Y} which can be considered as a linear mapping from the representation space of the Higgs field to the representation space of left-handed fermions. For a general discussion we again refer to [TT05]. In what follows, we will restrict ourselves to the specific case of the minimal Standard Model (see, for instance, [Nac90]).

3.1 Data of the (minimal) STM as a specific GTDT

To specify a GTDT one has to choose a gauge group G, a unitary representation ρ_{F} thereof, as well as some (class of) Dirac operators D. In the case of the (minimal) STM this means:

- G equals $\mathrm{SU}(3) \times \mathrm{SU}(2) \times \mathrm{U}(1)$;
- ρ_{F} equals the fermion representation:

$$\begin{aligned} \mathrm{E_L} &:= \bigoplus_1^3 \left[(1,2,-1/2) \oplus (3,2,1/6)\right], \\ \mathrm{E_R} &:= \bigoplus_1^3 \left[(1,1,-1) \oplus (3,1,-1/3) \oplus (3,1,2/3)\right], \end{aligned} \tag{29}$$

  where (n_3, n_2, n_1) denote the tensor product, respectively, of an n_3 dimensional representation of SU(3), an n_2 dimensional representation of SU(2) and a one dimensional representation of U(1) with "hypercharge" y: $\rho(e^{i\theta}) := e^{iy\theta}$, $y \in \mathbb{Q}$, $\theta \in [0, 2\pi[$.

More explicitly, we have

$$\rho_{\mathrm{F}} := \rho_{\mathrm{L}} \oplus \rho_{\mathrm{R}} : \mathrm{SU}(3) \times \mathrm{SU}(2) \times \mathrm{U}(1) \to \mathrm{Aut}(E) \subset \mathrm{U}(45) \tag{30}$$

with

$$\rho_{\mathrm{L}}(\mathbf{c}, \mathbf{w}, b) \quad := \quad \begin{pmatrix} \mathbf{c} \otimes \mathbf{1}_{\mathrm{N}} \otimes \mathbf{w}\, b^q_{\mathrm{L}} & 0 \\ & \\ 0 & \mathbf{1}_{\mathrm{N}} \otimes \mathbf{w}\, b^l_{\mathrm{L}} \end{pmatrix}, \tag{31}$$

$$\rho_{\mathrm{R}}(\mathbf{c}, \mathbf{w}, b) \quad := \quad \begin{pmatrix} \mathbf{c} \otimes \mathbf{1}_{\mathrm{N}} \otimes \mathbf{B}^q_{\mathrm{R}} & 0 \\ & \\ 0 & \mathbf{B}^l_{\mathrm{R}} \end{pmatrix} \tag{32}$$

and

$$\begin{aligned} E &\equiv E_{\mathrm{L}} \oplus E_{\mathrm{R}} \\ &\simeq \left[(\mathbb{C}^{18}_q \oplus \mathbb{C}^6_l)\right]_{\mathrm{L}} \oplus \left[(\mathbb{C}^9 \oplus \mathbb{C}^9)_q \oplus \mathbb{C}^3_l\right]_{\mathrm{R}}. \end{aligned} \tag{33}$$

- The Dirac operator D is of Pauli-Yukawa type with Yukawa coupling ϕ given by

$$\phi \equiv i \begin{pmatrix} 0 & \tilde{\phi} \\ \tilde{\phi}^\dagger & 0 \end{pmatrix} \tag{34}$$

with

$$\begin{aligned} \tilde{\phi} \equiv G_{\mathrm{Y}}(\varphi) \quad &:= \quad \begin{pmatrix} \mathbf{1}_3 \otimes (\mathbf{g}'^q \otimes \varphi,\, \mathbf{g}^q \otimes \boldsymbol{\epsilon}\overline{\varphi}) & 0 \\ & \\ 0 & \mathbf{g}^l \otimes \varphi \end{pmatrix} \\ &\equiv \quad \begin{pmatrix} \mathbf{1}_3 \otimes \tilde{\varphi}_q & 0 \\ 0 & \tilde{\varphi}_l \end{pmatrix}. \end{aligned} \tag{35}$$

Here, respectively, $\mathbf{g}'^q, \mathbf{g}^q \in \mathbf{M}_{\mathrm{N}}(\mathbb{C})$ denote the matrices of the Yukawa coupling constants for quarks of electrical charge -1/3 and 2/3 (i.e. of quarks of "d"-type, and of "u"-type) and $\mathbf{g}^l \in \mathbf{M}_{\mathrm{N}}(\mathbb{C})$ is the matrix of the Yukawa coupling constants for the leptons of charge -1 (i.e. of leptons of "electron" type). While $\mathbf{g}^q$ and $\mathbf{g}^l$ can be assumed to be diagonal and real, the matrix $\mathbf{g}'^q$ is related to the Kobayashi-Maskawa matrix and therefore is neither diagonal nor real. The "weak hyper-charges" for the left and right handed quarks (indicated by the superscript "q") and leptons (superscript "l") are defined by: $\rho(b) := e^{iy\theta}$, $b \in \mathrm{U}(1)$, $y \in \mathbb{Q}$, $\theta \in [0, 2\pi[$. Then, the two by two diagonal matrices $\mathbf{B}^q_{\mathrm{R}}$ and $\mathbf{B}^l_{\mathrm{R}}$ in (32) are: $\mathbf{B}^q_{\mathrm{R}} := \mathrm{diag}(b^{d'}_{\mathrm{R}}, b^u_{\mathrm{R}})$ and $\mathbf{B}^l_{\mathrm{R}} := b^l_{\mathrm{R}} \mathbf{1}_{\mathrm{N}}$.

In (35) $\varphi \in \mathcal{C}^\infty(\mathcal{M}, \mathbb{C}^2)$ denotes the Higgs field of the minimal Standard Model. It carries a specific sub-representation ρ_{H} of the fermion representation ρ_{F} which is given by

$$\rho_{\mathrm{H}} : \mathrm{SU}(3) \times \mathrm{SU}(2) \times \mathrm{U}(1) \quad \longrightarrow \quad \mathrm{U}(2)$$

$$(\mathbf{c}, \mathbf{w}, b) \quad \mapsto \quad \mathbf{w}\, e^{iy_h\theta}. \tag{36}$$

Finally, $\boldsymbol{\epsilon}$ is the anti-diagonal matrix $\boldsymbol{\epsilon} := \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$ and $\overline{\varphi}$ here means the complex conjugate of φ.

As a sub-representation of ρ_{F} the representation ρ_{H} is fixed by the relations of the hyper-charges of the quarks and the leptons:

$$\begin{aligned} y_h &= y_{\mathrm{L}}^l - y_{\mathrm{R}}^l \\ &= y_{\mathrm{L}}^q - y_{\mathrm{R}}^{d'} \\ &= y_{\mathrm{R}}^u - y_{\mathrm{L}}^q. \end{aligned} \tag{37}$$

Here,

$$\begin{aligned} (y_{\mathrm{L}}^q,\, y_{\mathrm{L}}^l) &= (1/6, -1/2), \\ (\,(y_{\mathrm{R}}^{d'}, y_{\mathrm{R}}^u),\, y_{\mathrm{R}}^l) &= (\,(-1/3, 2/3),\, -1), \end{aligned} \tag{38}$$

according to the fermion representation (29).

The corresponding fermionic vacuum $\mathcal{D}$ is given by

$$\mathcal{D} := i \begin{pmatrix} 0 & \mathbf{M} \\ \mathbf{M}^\dagger & 0 \end{pmatrix}, \tag{39}$$

with

$$\begin{aligned} \mathbf{M} &\equiv \begin{pmatrix} \mathbf{1}_3 \otimes \mathbf{M}_q & 0 \\ 0 & \mathbf{M}_l \end{pmatrix}, \\ \mathbf{M}_q &\equiv \begin{pmatrix} 0 & \mathbf{m}^{d'} \\ \mathbf{m}^u & 0 \end{pmatrix}, \\ \mathbf{M}_l &\equiv \begin{pmatrix} 0 \\ \mathbf{m}^l \end{pmatrix}, \end{aligned} \tag{40}$$

where, respectively, the matrices $\mathbf{m}^l := \frac{v}{\sqrt{2}}\mathbf{g}^l \in \mathrm{M}_{\mathrm{N}}(\mathbb{C})$ and $\mathbf{m}^u := \frac{v}{\sqrt{2}}\mathbf{g}^q \in \mathrm{M}_{\mathrm{N}}(\mathbb{C})$ denote the "mass matrices" of the charged leptons (l) and quarks (q) of "u-type". They can be assumed to be diagonal and real. The corresponding $\mathrm{N}\times\mathrm{N}$ matrix $\mathbf{m}^{d'} := \frac{v}{\sqrt{2}}\mathbf{g}'^q$ of "d-type" quarks is neither diagonal nor real. It is related to the mass matrix of "d-type" quarks $\mathbf{m}^d = \mathrm{diag}(m^{d_1}, \ldots, m^{d_{\mathrm{N}}})$, $m^{d_k} \in \mathbb{R}$, via the Kobayashi-Maskawa matrix $\mathbf{V} \in \mathrm{U(N)}$: $\mathbf{m}^{d'} = \mathbf{V}\,\mathbf{m}^d\,\mathbf{V}^*$. Here and above N is of appropriate size that is defined by the fermion representation (29) (or (33)) and $v > 0$ is the "vacuum expectation value" of the Higgs boson.

With the choice of $(\mathrm{G}, \rho_{\mathrm{F}}, D)$ we have mathematically specified a particular Dirac type gauge theory. However, from a physical perspective we still have to appropriately parameterize this GTDT. Of course, the parametrization cannot be arbitrary. It should be in accordance with the geometrical frame of a GTDT. As we mentioned already, there are basically two objects which can be parameterized: the Dirac-Lagrangian (1) and the general Dirac operator defined by (3). Note that in the specific case at hand the parameters introduced by the Yukawa coupling matrices only arise because of the change of the representation of the Higgs field (i.e. to consider ϕ as a function of φ).

3.2 Geometrical Parametrization

In general, an admissible parametrization of (1) is given by the commutant of the fermion representation $\rho_{\rm F}$. This is similar to the introduction of the Yang-Mills coupling constant for each simple gauge group. To explain this let us consider the case where $\mathrm{G} = \mathrm{SU}(2)$, $\rho_{\rm F}$ some unitary representation thereof and D is specified by (21) with $\theta_\mu = A_\mu$. In this case, the Dirac-Lagrangian (26) reduces to

$$\mathcal{L}_{\rm D} = (2\bar{\psi} i\gamma^\mu \nabla^{\rm Cl}_\mu \psi + V_{\rm D})\sqrt{-|g|}\, d^4x, \tag{41}$$

$$V_{\rm D} = \lambda_{\rm gr}\, r_{\rm M} - \lambda_{\rm YM}\, {\rm tr} F^{\rm A\,\dagger}_{\mu\nu} F^{\mu\nu}_{\rm A}. \tag{42}$$

Here, the constants $\lambda_{\rm gr}, \lambda_{\rm YM}$ are purely numerical and basically fixed by the dimension of space-time and the dimension of the fermion representation. To get started we may re-scale ψ to get rid of the factor 2 in the fermionic part of the Dirac-Lagrangian. Moreover, we may introduce a relative constant $\lambda_{\rm D} \in \mathbb{R}$ and thereby replace the total Lagrangian (41) by

$$\mathcal{L}_{\rm D} = (\bar{\psi} i\gamma^\mu \nabla^{\rm Cl}_\mu \psi + \lambda_{\rm D} V_{\rm D})\sqrt{-|g|}\, d^4x. \tag{43}$$

Of course, the free parameter $\lambda_{\rm D}$ can be absorbed by $\lambda_{\rm gr}, \lambda_{\rm YM}$, which will then be treated as arbitrary free parameters. In particular, $\lambda_{\rm gr}$ will be proportional to Newton's gravitational constant (or the inverse square of the Planck length ℓ_p) after we have introduced an appropriate length-scale to give the various fields their correct physical dimensions (see below). Next, we take into account that all fields are represented in the fermion representation $\rho_{\rm F}$. Accordingly, there is another numerical constant $\lambda_{\rm F}$, which only depends on $\rho_{\rm F}$, such that we may re-write the Yang-Mills part in (42) as

$${\rm tr} F^{\rm A\,\dagger}_{\mu\nu} F^{\mu\nu}_{\rm A} = \lambda_{\rm F} < F^{\rm A}_{\mu\nu}, F^{\mu\nu}_{\rm A} > \tag{44}$$

Here, $<\cdot,\cdot>$ denotes the Killing form on su(2). Since the latter is proportional to the usual trace with respect to the fundamental representation one may re-write again the Yang-Mills part as

$$\lambda_{\rm YM}\, {\rm tr} F^{\rm A\,\dagger}_{\mu\nu} F^{\mu\nu}_{\rm A} = \frac{1}{2 {\rm g}^2_{\rm YM}}\, {\rm tr} F^{\rm A}_{\mu\nu} F^{\mu\nu}_{\rm A} \tag{45}$$

where on the right-hand side $F^{\rm A}_{\mu\nu}$ is supposed to be in the fundamental representation of su(2). It follows that

$$\lambda_{\rm YM} = \lambda_{\rm rep}/{\rm g}^2_{\rm YM} \tag{46}$$

with $\lambda_{\rm rep}$ being a numerical constant which is basically fixed by the fermion representation and the fundamental representation. The constant $g_{\rm YM}$ denotes the usual Yang-Mills coupling constant which parameterizes the most general Killing form on Lie(G).

The specific example discussed so far can be slightly generalized by taking into account the reducibility of $\rho_{\rm F}$. Let $\mathfrak{z} = \mathfrak{z}^\dagger$ be the most general element of the corresponding commutant. In fact, $\mathfrak{z}$ can be regarded as a constant mapping commuting with the action of the Clifford algebra, which can be expressed in terms of the Dirac operator as follows:

$$[D, \mathfrak{z}] = 0. \tag{47}$$

Hence, we may generalize the Yang-Mills part in the Dirac-Lagrangian (41) by

$$\mathrm{tr}(\mathfrak{z} F^{\rm A\,\dagger}_{\mu\nu} F^{\mu\nu}_{\rm A}). \tag{48}$$

The introduction of the commutant with respect to the fermion representation provides us with a natural parametrization of the Dirac-Lagrangian which is compatible with the geometrical scheme of a GTDT. Indeed, the Dirac potential is but a trace of an endomorphism which is uniquely determined by D (c.f. section 2.2 in [TT05]). As a result, the Dirac potential (2) is replaced by

$$V_{{\rm D},\mathfrak{z}} \quad\equiv\quad \mathrm{tr}(\mathfrak{z}\, r_{\rm M}) + \mathrm{tr}(\mathfrak{z}\gamma^{\mu\nu}[H_\mu, H_\nu]) + \tfrac{1}{8}\, g_{\mu\nu}\, \mathrm{tr}\Big(\mathfrak{z}\gamma^\sigma\, [H_\sigma, \gamma^\mu]\gamma^\lambda\, [H_\lambda, \gamma^\nu]\Big)\,. \tag{49}$$

Of course, in the more simple case where $\rho_{\rm F}$ is irreducible, this replacement just leads back to $V_{{\rm D},\mathfrak{z}} = \lambda_{\rm D} V_{\rm D}$.

The parametrization of D depends on its specific form. In general, one may replace A_μ by $A'_\mu \equiv \lambda'_{\rm A} A_\mu$ and (9) by

$$\lambda\Phi_{\rm D} \equiv \sum_{k=0}^{4} \sum_{0\leq\nu_1<\cdots<\nu_k\leq 3} \gamma^{\nu_1}\cdots\gamma^{\nu_k}\, \lambda_{(k)}\, \chi^{(k)}_{\nu_1\cdots\nu_k}\,, \tag{50}$$

with $\lambda_{(k)}$ being appropriate " coupling constants".

For example, in the case of the PDY this corresponds to the replacement:

$$\begin{aligned} \theta_\mu \quad&\mapsto\quad \theta'_\mu \equiv \lambda'\theta_\mu := \lambda'_{\rm A} A_\mu + \lambda'_{\rm H} H_\mu, & (51)\\ F^\theta_{\mu\nu} \quad&\mapsto\quad \lambda^{-1} F^{\theta'}_{\mu\nu} \equiv \lambda^{-1}_{\rm A} F^{\rm A'}_{\mu\nu} + \lambda^{-1}_{\rm H} F^{\rm A',H'}_{\mu\nu}. & (52)\end{aligned}$$

Here, the curvature $F^{\theta'}_{\mu\nu}$ is defined with respect to the parameterized gauge potentials A'_μ and $H'_\mu \equiv \lambda'_{\rm A} H_\mu$. Of course, by re-scaling the gauge potentials one may assume without loss of generality that $\lambda'_{\rm A} = \lambda'_{\rm H} \equiv 1$. Therefore, the parameterized PDY corresponds to the replacement:

$$F^\theta_{\mu\nu} \quad\mapsto\quad \lambda^{-1} F^\theta_{\mu\nu} \equiv \lambda^{-1}_{\rm A} F^{\rm A}_{\mu\nu} + \lambda^{-1}_{\rm H} F^{\rm A,H}_{\mu\nu}. \tag{53}$$

From a geometrical perspective such a parametrization is quite acceptable, for curvatures are always considered as elements of vector spaces in contrast to the corresponding gauge potentials. Moreover, the re-parametrization (53) is known from the usual geometrical description of Yang-Mills gauge theories. However, the constants $\lambda_{\rm A}, \lambda_{\rm H}$ should not be identified with the usual Yang-Mills coupling constants. For example, in the case of a simple gauge group G and an irreducible fermionic representation $\rho_{\rm F}$ thereof, the constant $\lambda_{\rm A}$ turns out only to be proportional to the Yang-Mills coupling constant $g_{\rm YM}$ which parameterizes the most general Killing form of G. More precisely, one obtains

$$\lambda_{\rm A} = \sqrt{\frac{\lambda_{\rm YM}}{\lambda_{\rm rep}}}\, g_{\rm YM} \tag{54}$$

with $\lambda_{\rm YM}$ being an arbitrary free parameter.

Moreover, as it turns out, one may put $\lambda_{\rm H}$ equal to one without loss of generality (c.f. (58) - (59)). However, the constant $\lambda_{\rm A}$ will be crucial for the calculation of the Higgs mass (c.f. relations (76) below). Indeed, this freedom will guarantee the numerical consistence of the presented geometrical description of the STM.

Note that the replacement $D \mapsto \lambda D$ is mathematically inappropriate for D belongs to an affine space. Moreover, since the relative curvature $F^{\rm A,H}_{\mu\nu}$ decomposes into $F^{\rm A,H}_{\mu\nu} = F^{\rm H}_{\mu\nu} + \kappa^{\rm A,H}_{\mu\nu}$, one may introduce the more general parametrization:

$$\lambda_{\rm H}^{-1} F^{\rm A,H}_{\mu\nu} \mapsto \lambda_{\rm H}^{-1} F^{\rm H}_{\mu\nu} + \lambda_{\rm int}^{-1} \kappa^{\rm A,H}_{\mu\nu}. \tag{55}$$

However, for reasons of covariance one has to identify $\lambda_{\rm int}$ with $\lambda_{\rm H}$. Finally, the parametrization of the off-diagonal elements of (21) by the same coupling constants(s) is enforced by quantum field theory. In fact, it is well-known that the occurrence of the Pauli-term in the fermionic part of the Lagrangian spoils the renormalizability of the fermionic theory. It therefore has to drop out in the fermionic action (c.f. section five in [TT05]).

In the *parameterized form* the Dirac-Lagrangian with respect to the above data of the (minimal) STM explicitly reads:

$$\begin{aligned}
\mathcal{L}_{\rm D} &\equiv \mathcal{L}_{\rm D,fer} + \mathcal{L}_{\rm D,bos}\,, \\
\mathcal{L}_{\rm D,fer} &:= (\bar\psi i\gamma^\mu \nabla^{\rm Cl}_\mu \psi)\sqrt{-|g|}\, d^4x + i(\bar\psi\gamma_5\phi\psi)\sqrt{-|g|}\, d^4x, &(56)\\
\mathcal{L}_{\rm D,bos} &:= \lambda_{\rm gr}\, r_{\rm M}\sqrt{-|g|}\, d^4x \;-\; \lambda'_{\rm YM}\,{\rm tr}(\mathfrak{Z} F^{\rm A\,\dagger}_{\mu\nu} F^{\mu\nu}_{\rm A})\sqrt{-|g|}\, d^4x \\
&\quad + \lambda'_{\rm H}\,{\rm tr}(\mathfrak{Z}\nabla_\mu\phi^\dagger\nabla^\mu\phi)\sqrt{-|g|}\, d^4x \\
&\quad - (\,\alpha'_{\rm H}\,{\rm tr}\mathfrak{Z}(\phi^\dagger\phi)^2 \;-\; \beta'_{\rm H}\,{\rm tr}(\mathfrak{Z}\phi^\dagger\phi)\,)\sqrt{-|g|}\, d^4x, &(57)
\end{aligned}$$

$$\alpha'_{\rm H} = \frac{27}{64}\frac{1}{\pi{\rm tr}(\mathfrak{Z})}\left(\frac{\ell}{\ell_{\rm P}}\right)^2\frac{1}{\lambda_{\rm H}^2}, \quad \beta'_{\rm H} = \frac{1}{4}\frac{1}{\pi{\rm tr}(\mathfrak{Z})}\frac{1}{\ell_{\rm P}^2} \tag{58}$$

$$\lambda'_{\rm YM} = \frac{1}{8}\frac{1}{\pi{\rm tr}(\mathfrak{Z})}\left(\frac{\ell}{\ell_{\rm P}}\right)^2\frac{1}{\lambda_{\rm A}^2}, \quad \lambda'_{\rm H} = \frac{9}{32}\frac{1}{\pi{\rm tr}(\mathfrak{Z})}\left(\frac{\ell}{\ell_{\rm P}}\right)^2\frac{1}{\lambda_{\rm H}^2}. \tag{59}$$

Here, ψ is already appropriately re-scaled and $\mathcal{L}_{\mathrm{D,bos}}$ is normalized such that $\lambda_{\mathrm{gr}} = \frac{1}{16\pi\ell_{\mathrm{P}}^2}$. Note that there are two independent length-scales involved which are introduced for quite different reasons. First, one arbitrary length-scale ℓ is introduced to provide the various fields involved with the appropriate physical (length) dimension. As a consequence, one has then to introduce a second length-scale that is given by the Planck length ℓ_{P} (Newton's gravitational constant), to make the Einstein-Hilbert Lagrangian dimensionless. Note that to identify this second length-scale with the Planck scale is mainly motivated by the "Newtonian limit" of Einstein's theory of gravity. Otherwise, this second length-scale is also considered as a free parameter. As it turns out, the length-scale ℓ corresponds to the (inverse of the) Higgs mass (please, see below), contrary to what one might naively expect from (58). We stress again that all fields involved are considered to be represented with respect to the fermion representation ρ_{F}.

4 Higgs Mass Relations

Since the STM-Lagrangian has a natural "square root" in terms of a PDY, one obtains specific relations between the corresponding parameters. In the next two sub-sections it will be demonstrated how these relations yield restrictions to the Higgs mass when (one-loop) quantum corrections are taken into account.

4.1 Parameter Relations between GTDT and the STM

In this sub-section we first re-write the parameterized Dirac-Lagrangian (56) in terms of the ordinary fields of the STM. A comparison with the usual parametrization of the STM-Lagrangian yields some constraints of the parameters which are not known in the usual description of the minimal Standard Model. This discussion is analogous to what has been presented already in [Tol98]. Hence, we will skip the details and present here only the relevant results. With the help of these results, however, we will show how the geometrical description of the STM in terms of GTDT gives rise to bounds of the Higgs mass. This, of course, will be discussed in some detail in the next sub-section.

The bosonic Lagrangian within the usual description of the STM reads:

$$\begin{aligned}\mathcal{L}_{SM} = &-\left(\frac{1}{2\mathrm{g}_3^2}\mathrm{tr}(\mathrm{C}_{\mu\nu}\mathrm{C}^{\mu\nu}) + \frac{1}{2\mathrm{g}_2^2}\mathrm{tr}(\mathrm{W}_{\mu\nu}\mathrm{W}^{\mu\nu}) + \frac{1}{4\mathrm{g}_1^2}\mathrm{B}_{\mu\nu}\mathrm{B}^{\mu\nu}\right)\sqrt{-|g|}\,d^4x \\ &+\frac{1}{2}(\nabla_\mu\varphi)^*(\nabla^\mu\varphi)\sqrt{-|g|}\,d^4x - \left[\lambda(\varphi^*\varphi)^2 - \frac{\mu^2}{2}\varphi^*\varphi\right]\sqrt{-|g|}\,d^4x. \qquad (60)\end{aligned}$$

Here, respectively, $C_{\mu\nu}$, $W_{\mu\nu}$ and $B_{\mu\nu}$ denote the Yang-Mills field strengths with respect to the fundamental representation of SU(3), SU(2) and U(1); φ is the usual Higgs doublet sitting in the fundamental representation of SU(2). Its U(1) representation is defined with respect to the hypercharge relations (37), which turn out to be crucial for re-writing (57) in terms of the physical fields.

In terms of the data of the (minimal) Standard Model the Dirac-Lagrangian of a PDY (57) corresponds to the bosonic STM-Lagrangian (60) provided the following relations are fulfilled (see also [Tol98], [Thu03]):

$$\frac{1}{\mathrm{g}_1^2} = 2A\frac{3Ny_q + y_l\mathrm{trX}}{4N + \mathrm{trX}}, \quad \frac{1}{\mathrm{g}_2^2} = A\frac{3N + \mathrm{trX}}{4N + \mathrm{trX}}, \quad \frac{1}{\mathrm{g}_3^2} = 4A\frac{N}{4N + \mathrm{trX}}, \tag{61}$$

$$1 = 2B\frac{3\mathrm{tr}(\mathbf{g}'^{\mathrm{q}}\mathbf{g}'^{\mathrm{q}*} + \mathbf{g}^{\mathrm{q}}\mathbf{g}^{\mathrm{q}*}) + \mathrm{tr}(\mathrm{X}\mathbf{g}^{\mathrm{l}}\mathbf{g}^{\mathrm{l}*})}{4N + \mathrm{trX}}, \tag{62}$$

$$\lambda = C\frac{3\mathrm{tr}((\mathbf{g}'^{\mathrm{q}}\mathbf{g}'^{\mathrm{q}*})^2 + (\mathbf{g}^{\mathrm{q}}\mathbf{g}^{\mathrm{q}*})^2) + \mathrm{tr}(\mathrm{X}(\mathbf{g}^{\mathrm{l}}\mathbf{g}^{\mathrm{l}*})^2)}{4N + \mathrm{trX}}, \tag{63}$$

$$\mu^2 = \frac{1}{3\pi}\left(\frac{1}{\ell_{\mathrm{P}}}\right)^2 \frac{3\mathrm{tr}(\mathbf{g}'^{\mathrm{q}}\mathbf{g}'^{\mathrm{q}*} + \mathbf{g}^{\mathrm{q}}\mathbf{g}^{\mathrm{q}*}) + \mathrm{tr}(\mathrm{X}\mathbf{g}^{\mathrm{l}}\mathbf{g}^{\mathrm{l}*})}{4N + \mathrm{trX}}. \tag{64}$$

Here, we used the following abbreviations:

$$y_q := 2(y_L^q)^2 + (y_R^{d'})^2 + (y_R^u)^2, \quad y_l := 2(y_L^l)^2 + (y_R^l)^2, \tag{65}$$

$$X := \frac{\lambda^l}{\lambda_q}, \; A := \frac{1}{12\pi}\left(\frac{\ell}{\ell_{\mathrm{P}}}\right)^2 a^2, \; B := \frac{1}{3\pi}\left(\frac{\ell}{\ell_{\mathrm{P}}}\right)^2 b^2, \; C := \frac{1}{2\pi}\left(\frac{\ell}{\ell_{\mathrm{P}}}\right)^2 b^2, \tag{66}$$

with $a := \frac{1}{\lambda_{\mathrm{A}}}$, $b := \frac{3}{4}\frac{1}{\lambda_{\mathrm{H}}}$ and $\lambda_q \in \mathbb{R}^+$, as well as $\lambda^l := \mathrm{diag}(\lambda_{1\mathrm{l}}, \ldots, \lambda_{\mathrm{Nl}})$ and $\lambda_{il} \in \mathbb{R}^+$. We also made use of $\mathfrak{z} = \mathbf{1}_4 \otimes \mathrm{diag}(z_L, z_R)$, with $z_L := \mathrm{diag}(\lambda_q \mathbf{1}_{6\mathrm{N}}, \lambda^l \otimes \mathbf{1_2})$ and $z_R := \mathrm{diag}(\lambda_q \mathbf{1}_{6\mathrm{N}}, \lambda^l)$ and of the relations (37).

From the relations (62) and (64) one immediately infers that

$$\ell_{\mathrm{H}} \equiv \frac{\ell}{\lambda_{\mathrm{H}}} = 2\sqrt{\frac{2}{3}}\,\frac{1}{\mu}. \tag{67}$$

As mentioned already, we may put $\lambda_{\mathrm{H}} \equiv 1$ for it simply re-scales $\ell \mapsto \ell_{\mathrm{H}}$ according to the relations (58) and (59). As a consequence, a PDY is physically parameterized by the two constants $(\ell_{\mathrm{H}}, \lambda_{\mathrm{YMH}})$ instead of the three parameters $(\ell, \lambda_{\mathrm{A}}, \lambda_{\mathrm{H}})$ with $\lambda_{\mathrm{YMH}} \equiv \lambda_{\mathrm{H}}/\lambda_{\mathrm{A}}$. Moreover, as far as the Standard Model is concerned the "relative coupling constant" λ_{YMH} turns out to be numerically fixed (see below).

Because of (67), one has

$$\ell_{\mathrm{H}} \sim 1/\mathrm{m}_{\mathrm{H}}. \tag{68}$$

Hence, the two length-scales involved in the geometrical description of the (minimal) Standard Model as a specific GTDT are determined by the Planck mass and the mass of the Higgs. As one may naively expect, it turns out that $\mathrm{m}_{\mathrm{H}}/\mathrm{m}_{\mathrm{P}} \lll 1$ on one-loop and top quark-mass approximation. Therefore, within these approximations ℓ_{H} is the dominant length-scale and gravitational effects may be fully negligible.

The relations (61) – (64) are derived on "tree-level" by comparing (57) with (60). In this approximation, however, the constraints for the gauge couplings (61) are inconsistent with the known experimental data (c.f. Table 1). More precisely, when taking into account the measured values of the gauge couplings there exists no choice of the model parameters such that all three relations for the gauge couplings are fulfilled. On the other hand, it is well-known that the gauge couplings are running couplings which depend on the considered energy scale. Hence, according to the renormalization group philosophy, the inconsistence of (61) may be interpreted in such a way that (61) – (64) are actually supposed to hold true only at certain critical values of the energy scale. In the following sub-section we will make use of the renormalization flow equations to determine these critical energy values. At the critical values it is then possible to solve the parameter relations with respect to the Higgs self-coupling λ from which we finally obtain the Higgs mass via the ratio

$$\frac{\mathrm{m_H}}{\mathrm{m_W}} = 4\sqrt{2}\,\frac{\sqrt{\lambda}}{\mathrm{g}_2}\ , \tag{69}$$

where the numerical values of the gauge coupling g_2 and the mass of the W-boson are regarded to be known from experiments.

4.2 One-Loop Quantum Corrections and the Higgs Mass

In this section we follow the same strategy as in [CIKS97], [CIS97] to determine the mass of the Higgs boson.

For the STM the renormalization flow equations in one-loop and top quark mass approximation have been derived in [FJSE93] using the $\overline{MS}$-scheme:

$$\dot{\mathrm{g}}_1 = \beta_1(\mathrm{g}_1) := \frac{41}{96\pi^2}\,\mathrm{g}_1^3, \quad \dot{\mathrm{g}}_2 = \beta_2(\mathrm{g}_2) := -\frac{19}{96\pi^2}\,\mathrm{g}_2^3, \quad \dot{\mathrm{g}}_3 = \beta_3(\mathrm{g}_3) := -\frac{7}{16\pi^2}\,\mathrm{g}_3^3, \tag{70}$$

$$\dot{\mathrm{g}}_t = \beta_t(\mathrm{g}_1,\mathrm{g}_2,\mathrm{g}_3,\mathrm{g}_t) := \frac{1}{16\pi^2}\left(9\mathrm{g}_t^3 - \left(8\mathrm{g}_3^2 + \frac{9}{4}\mathrm{g}_2^2 + \frac{17}{12}\mathrm{g}_1^2\right)\mathrm{g}_t\right), \tag{71}$$

$$\begin{aligned} \dot{\lambda} &= \beta_\lambda(\mathrm{g}_1,\mathrm{g}_2,\mathrm{g}_3,\mathrm{g}_t,\lambda) \\ &:= \frac{1}{16\pi^2}\left(96\lambda^2 + \left(24\mathrm{g}_t^2 - 9\mathrm{g}_2^2 - 3\mathrm{g}_1^2\right)\lambda - 6\mathrm{g}_t^4 + \frac{9}{32}\mathrm{g}_2^4 + \frac{3}{32}\mathrm{g}_1^4 + \frac{3}{16}\mathrm{g}_1^2\mathrm{g}_2^2\right). \end{aligned} \tag{72}$$

Here, the derivative is taken with respect to a dimensionless scale parameter $t = \ln(\frac{\Lambda}{E_0})$, with Λ being an arbitrary energy scale and E_0 a reference energy. g_t is the Yukawa coupling of the top quark.

The renormalization flow equations for g_1, g_2, g_3 can be explicitly integrated:

$$\mathrm{g}_1(t) = \frac{1}{\sqrt{A_1 - \frac{41}{48\pi^2}t}}, \quad \mathrm{g}_2(t) = \frac{1}{\sqrt{A_2 + \frac{19}{48\pi^2}t}}, \quad \mathrm{g}_3(t) = \frac{1}{\sqrt{A_3 + \frac{7}{8\pi^2}t}}, \tag{73}$$

Table 1: Gauge Couplings

	value	abs. error
g_1	0.3575	0.0001
g_2	0.6507	0.0007
g_3	1.218	0.0026

Gauge couplings at the energy $E_0 = \mathrm{m_Z} = 91.187$ GeV (c.f. [CIKS97])

where $A_i \equiv 1/\mathrm{g}_i^2(0), \quad i = 1, \ldots, 3$.

We may use these solutions to determine the critical values t_c for which the relations (61) – (64) are fulfilled. When the relations (38) of the hyper-charges are taken into account one actually obtains a unique value of the critical energy (N = 3 generations):

$$\begin{aligned} t_c &= \frac{8\pi^2}{21}(3A_1 - 9A_2 + 4A_3) && (74) \\ &= 18.4734356 \pm 0.264602490, && (75) \end{aligned}$$

which corresponds to $E_c = (0.90 \pm 0.29) \cdot 10^{10}$ GeV. Here, we have again made use of the experimental values summarized in Table 1.

Next, we aim to find an initial value for λ at the critical scale point t_c and integrate the system (71) – (72). This allows to compute the value of λ at any scale point t where the values for g_2 and $\mathrm{m_W}$ are known.

If one divides (63) and the third equation of (61) by (62) one obtains in the top quark mass approximation:

$$\lambda = \frac{3}{4}\mathrm{g}_t^2, \quad \lambda_{\mathrm{YMH}}^2 = \frac{9}{8}\frac{\mathrm{g}_t^2}{\mathrm{g}_3^2}. \tag{76}$$

Note that the parameter λ_{YMH} is numerically fixed by the SU(3) Yang-Mills-coupling constant g_3 and the Yukawa-coupling constant g_t of the top-quark at the critical scale point t_c (resp. at the critical energy E_c).

To proceed, we numerically integrate the system of differential equations (71) – (72). In the first step we integrate the first differential equation for g_t for the initial value at $t = 0$. When $t = 0$ corresponds to the reference energy $E_0 = \mathrm{m_Z} = 91.187$ GeV we have the initial value (c.f. [CIKS97]):

$$\mathrm{g}_t(0) = 0.7087794100 \pm 0.02638699. \tag{77}$$

With this solution the value of g_t can be computed at the scale point t_c. In the second step we integrate the second differential equation numerically with respect to the initial condition:

$$\lambda(t_c) = \frac{3}{4}\,\mathrm{g}_t(t_c). \tag{78}$$

Eventually, this allows one to calculate $\lambda = \lambda(0)$ for the reference energy $E_0 = \mathrm{m_Z} = 91.187$ GeV (c.f. [CIKS97]):

$$\lambda(0) = 0.072726 \pm 0.011663. \tag{79}$$

As a consequence, one obtains for the Higgs mass the value

$$\mathrm{m_H} \quad = \quad 188.3291 \pm 15.1378042 \text{ GeV} \tag{80}$$

for $\mathrm{m_W} = 80.33 \pm 0.15\,\mathrm{GeV}$ (c.f. [CIKS97]). Here, the error is due to the errors of the initial values of g_1, g_2, g_3, t_c and $\mathrm{m_W}$, as well as their influence on the numerical integration of the renormalization flow equations.

According to the STM the Higgs mass can be restricted to the interval (c.f. [Ros03]):

$$\mathrm{m_H} \in (114, 193) \text{ GeV}. \tag{81}$$

It is valid for the STM with one Higgs boson and without super-symmetry. As a result, the predicted value of the Higgs mass within the geometrical frame of GTDT is found to be at the upper bound of the interval (81).

4.3 Model Bounds for the Higgs Mass and Massive Neutrinos

For the sake of completeness (and comparison, see below), we briefly discuss here how the statement (80) may be weakened if one introduces the most general parametrization of the PDY. We stress, however, that such a parametrization is not favored by the geometrical setup of GTDT (c.f. [TT05]). Disregarding geometry, however, such a non-geometrical parametrization may be still of interest for it yields the principal model bounds for the predicted value of the Higgs mass within the mathematical frame presented. In this sub-section, we also briefly discuss how the statement (80) may depend on massive neutrinos.

From the naive point of view of "counting free parameters" one may parameterize (53) also as follows:

$$\begin{aligned}
\lambda^{-1}\, F^{\theta}_{\mu\nu} \quad &\equiv \quad \lambda_{\mathrm{A}}^{-1}\, F^{\mathrm{A}}_{\mu\nu} + \lambda_{\mathrm{H}}^{-1}\, (\partial_\mu H_\nu - \partial_\nu H_\mu + [A_\mu, H_\nu] - [A_\nu, H_\mu]) + \lambda_{\mathrm{self}}^{-1}\, [H_\mu, H_\nu] \\
&= \quad \lambda_{\mathrm{A}}^{-1}\, F^{\mathrm{A}}_{\mu\nu} + \lambda_{\mathrm{H}}^{-1}\, ([\nabla^{\mathrm{A}}_\mu, H_\nu] - [\nabla^{\mathrm{A}}_\nu, H_\mu]) + \lambda_{\mathrm{self}}^{-1}\, [H_\mu, H_\nu].
\end{aligned} \tag{82}$$

It turns out that this non-geometrical parametrization of the PDY does not change the critical scale-point t_c. In particular, (82) does not alter the uniqueness of the critical energy point E_c. Nonetheless, the parametrization (82) gives rise to a "fuzziness" of the predicted Higgs mass, similar to what is obtained within the "(real) Connes-Lott" description of the STM (see the next section).

To obtain the general bounds for the Higgs mass we have to consider the Dirac-Langrangian for PDY's with the Pauli term being parameterized like (82). One gets the same expressions as in (56) and (57), however, with different pre-factors. In the case considered they read:

$$\alpha'_{\rm H} = \frac{27}{64}\frac{1}{\pi \mathrm{tr}(\mathfrak{Z})}\left(\frac{\ell}{\ell_{\rm P}}\right)^2\frac{1}{\lambda^2_{\rm self}}, \quad \beta'_{\rm H} = \frac{1}{4}\frac{1}{\pi \mathrm{tr}(\mathfrak{Z})}\frac{1}{\ell^2_{\rm P}}, \tag{83}$$

$$\lambda'_{\rm YM} = \frac{1}{8}\frac{1}{\pi \mathrm{tr}(\mathfrak{Z})}\left(\frac{\ell}{\ell_{\rm P}}\right)^2\frac{1}{\lambda^2_{\rm A}}, \quad \lambda'_{\rm H} = \frac{9}{32}\frac{1}{\pi \mathrm{tr}(\mathfrak{Z})}\left(\frac{\ell}{\ell_{\rm P}}\right)^2\frac{1}{\lambda^2_{\rm H}}. \tag{84}$$

These relations differ from (58) – (59) only by the self-coupling constant $\alpha'_{\rm H}$ due to the parametrization (82).

By the same analysis as for the geometrically parameterized PDY one obtains parameter relations analogous to (61) – (64). They only differ from the latter in the definition of the constants A, B, C :

$$A := \frac{1}{12\pi}\left(\frac{\ell_{\rm H}}{\ell_{\rm P}}\right)^2\lambda^2_{\rm YMH}, \quad B := \frac{3}{16\pi}\left(\frac{\ell_{\rm H}}{\ell_{\rm P}}\right)^2, \quad C := \frac{9}{32\pi}\left(\frac{\ell_{\rm H}}{\ell_{\rm P}}\right)^2\lambda^2_{\rm H,self} \tag{85}$$

where again $\ell_{\rm H} \equiv \ell/\lambda_{\rm H}, \lambda_{\rm YMH} \equiv \lambda_{\rm H}/\lambda_A$ and $\lambda_{\rm H,self} := \lambda_{\rm H}/\lambda_{\rm self}$.

Doing the same analysis as in the previous section one concludes that the value for the critical scale point does not change. Basically, the reason is that the relations for the gauge couplings (61) remain the same. For this reason one may proceed in the same way as before to end up with

$$\lambda^2_{\rm YMH} = \frac{9}{8}\frac{\mathrm{g}_t^2}{\mathrm{g}_3^2}, \quad \lambda = \frac{3}{4}\lambda^2_{\rm H,self}\,\mathrm{g}_t^2. \tag{86}$$

Therefore, with respect to the more general parametrization $(\ell_{\rm H}, \lambda_{\rm YMH}, \lambda_{\rm H,self})$ the value of Higgs self-coupling constant λ at the critical scale point is not fixed by the appropriate value of $\mathrm{g}_t(t_c)$. In other words, unlike to the geometrical parametrization, essentially defined by the length scale $\ell_{\rm H} \simeq 1/\mathrm{m}_{\rm H}$, the non-geometrical parametrization gives rise to an additional coupling constant $\lambda_{\rm H,self}$, that is also tied to the Higgs mass.

In what follows we abbreviate $\kappa := \frac{3}{4}\lambda^2_{\rm H,self}\,\mathrm{g}_t^2$. We now have to integrate the flow equation (72) with respect to the initial value:

$$\lambda(t_c) = \kappa, \quad \kappa > 0. \tag{87}$$

The model bounds are then determined by the boundary values for (71) with respect to (87), which give rise to the minimal and the maximal values for the Higgs mass.

Due to standard theorems on ordinary differential equations the values of the solutions at a certain scale point t of (72) depend monotonically on the initial value (c.f. [Ama83]).

Hence, the lower bound for the Higgs mass is determined by integrating (71) and calculating $\underline{\mathrm{m}_{\mathrm{H}}}$ with respect to (69) at $\lambda(t_c) = 0$. One gets:

$$\underline{\mathrm{m}_{\mathrm{H}}} = 129.5719 \text{ GeV}. \tag{88}$$

In order to obtain an upper bound for the Higgs mass one looks for a differential equation

$$\dot{\lambda} = \tilde{\beta_\lambda} \tag{89}$$

such that:

1. For the same initial values κ the solutions of this equation are upper bounds of the solutions of (72).
2. The solutions explicitly depend on the initial value κ.

The second property permits to calculate the limit $\kappa \to +\infty$ which yields the upper bound for all possible solutions of (72).

One may define $\tilde{\beta_\lambda}$ in the following way:

$$\begin{aligned}
\tilde{\beta_\lambda} &:= \ell_1\lambda^2 + \ell_2\lambda + \ell_3, \\
\ell_1 &:= \frac{6}{\pi^2}, \quad \ell_2 := \frac{1}{16\pi^2}(24\overline{\mathrm{g}_t}^2(0) - 9\overline{\mathrm{g}_2}^2(t_c + \Delta t_c) - 3\overline{\mathrm{g}_1}^2(0)), \\
\ell_3 &:= \frac{1}{16\pi^2}(-6\underline{\mathrm{g}_t}^4(t_c + \Delta t_c) + \frac{9}{32}\overline{\mathrm{g}_2}^4(0) + \frac{3}{32}\overline{\mathrm{g}_1}^4(t_c + \Delta t_c) \\
&\quad + \frac{3}{16}\overline{\mathrm{g}_2}^2(0)\overline{\mathrm{g}_1}^2(t_c + \Delta t_c)).
\end{aligned} \tag{90}$$

Here, Δt_c is the error of the critical scale point t_c.

Any solution $\tilde{\lambda}$ of (89) with initial value $\kappa > 0$ fulfills $\tilde{\lambda}(t) \geq \lambda(t)$ for $t \geq 0$, with $\lambda(t)$ being the solution of (72) with initial value κ. Using (90) the differential equation (89) can be explicitly solved for arbitrary initial value κ :

$$\begin{aligned}
\tilde{\lambda}(t) &= \sqrt{|\beta|}\coth\left(\frac{\ell_1\beta}{\sqrt{|\beta|}}(t - t_c) + \operatorname{arcoth}\left(\frac{\kappa + \alpha}{\sqrt{|\beta|}}\right)\right) - \alpha, \\
\alpha &:= \frac{1}{2}\frac{\ell_1}{\ell_2}, \quad \beta := -\frac{1}{4}\frac{\ell_2^2}{\ell_1^2} + \frac{\ell_3}{\ell_1}, \quad \beta < 0.
\end{aligned} \tag{91}$$

This permits to calculate $\tilde{\lambda}_{as}(t) := \lim_{\kappa\to\infty}\tilde{\lambda}(t)$:

$$\tilde{\lambda}_{as}(t) \ := \sqrt{|\beta|}\coth\left(\tfrac{\ell_1\beta}{\sqrt{|\beta|}}(t - t_c)\right) - \alpha \tag{92}$$

which finally yields the upper bound for the Higgs mass $\overline{\mathrm{m}_{\mathrm{H}}}$ at the scale point $t = 0$:

$$\tilde{\lambda}_{as}(0) = 0.4566939695 \quad \Rightarrow \quad \overline{\mathrm{m}_{\mathrm{H}}} = 473.3272252 \text{ GeV}. \tag{93}$$

Therefore, we end up with the following range of the predicted value of the Higgs mass:

$$\mathrm{m_H} \in [129.57,\, 437.33)\,\mathrm{GeV}. \tag{94}$$

Obviously, this range has a non-empty intersection with (81). In particular, it has a lower-bound close to the expected value of the Higgs mass.

Next, we discuss a simple modification the STM which takes into account the possibility of *massive neutrinos*. We restrict ourselves to a few remarks concerning so-called "Dirac type mass terms" which seems to fit best with GTDT.

The STM can easily be enhanced with a right handed neutrino sector by replacing $\rho_{F,R}$ in (30) as follows:

$$\rho(c, w, \theta) := \mathrm{diag}(\mathrm{c} \otimes \mathbf{1_N} \otimes \mathrm{diag}(\mathrm{e}^{\mathrm{iy_R^{d'}}\theta}, \mathbf{1_N} \otimes \mathrm{diag}(\mathrm{e}^{\mathrm{iy_R^e}\theta}, \mathrm{e}^{\mathrm{iy_R^\nu}\theta})), \tag{95}$$

with, respectively, $y_R^e, y_R^\nu \in \mathbb{Q}$ being the hyper-charges of the right handed electron and neutrino. Accordingly, the matrix $\tilde{\phi}$ in (35) has to be modified by:

$$\tilde{\phi} := \mathrm{diag}\left(\mathbf{1_3} \otimes \begin{pmatrix} \mathbf{g'}^q\varphi_1 & \mathbf{g}^q\bar{\varphi}_2 \\ \mathbf{g'}^q\varphi_2 & -\mathbf{g}^q\bar{\varphi}_1 \end{pmatrix}, \begin{pmatrix} \mathbf{g'}^l\varphi_1 & \mathbf{g}^l\bar{\varphi}_2 \\ \mathbf{g'}^l\varphi_2 & -\mathbf{g}^l\bar{\varphi}_1 \end{pmatrix}\right), \tag{96}$$

where $\mathbf{g'}^l$ and $\mathbf{g}^l$ are the corresponding leptonic Yukawa coupling matrices.

This simple modification of the STM permits to construct a theory that also contains massive neutrinos. The appropriate neutrino mass terms are generated by so called *Dirac type mass terms*. It is known, however, that there exist other neutrino mass generating mechanisms as well (c.f. [Bil02], [BGGM03]).

Since Dirac type mass terms result by only modifying the (right-handed) representation of the gauge group of the STM, one may perform exactly the same analysis as has been carried out in the foregoing section. It turns out that the critical energy scale t_c (and hence E_c) is identical with (74). Moreover, since the relations (75) are unchanged one ends up with the same predicted value of the Higgs mass (80) as in the STM without massive neutrinos. Note that this is indeed remarkable, for the parameter relations which correspond to (61) – (64) are nonetheless different from the relations obtained in the case of massless neutrinos.

5 A Brief Comparison with NCG

As mentioned already in the introduction there are various different geometrical descriptions of the STM. Some of these where especially addressed to make predictions of the Higgs mass. Therefore, it may be also of interest to briefly discuss how some of these approaches to the STM are related to the frame presented here. In what follows, we will restrict ourselves to two different geometrical descriptions of the STM within the general frame of non-commutative geometry. One of which is usually referred to as "Chamseddine-Connes model

(CCM)" (see, for example [CC97], [CIKS97], [CIS97]) and which has some formal similarity to GTDT. The second approach (which can be actually regarded as the predecessor of CCM) is called the "Connes-Lott model (CLM)" (see, for example [CL90], [SZ95], [KS97], [IKS95a], [IKS95b], [CIS99], [CIS97], [IS96]). This model is based on A. Connes's general ideas of non-commutative geometry as presented, for example, in [Con94], [GBVF01] and [SZ95].

5.1 Comparison with the CCM

The formal similarity in the geometrical description of the STM between CCM and GTDT is that in both approaches Dirac-Yukawa type operators play a basic role. The motivation, however, is very different. Within the frame of CCM these generalized Dirac operators are motivated by non-commutative geometry (via the tensor product of spectral triples). In contrast, in GTDT the Dirac-Yukawa type Dirac operators naturally arise from the Bochner-Lichnerowicz-Weitzenböck decomposition. Physically, these Dirac operators are motivated by perturbation theory and the Yukawa coupling. The basic difference of both approaches lies in the "action". Indeed, CCM postulates what is referred to as *spectral action* which incorporates gravity within non-commutative geometry (c.f. [CC97]). Basically, the evaluation of the spectral action consists of a (sophistically) modified heat kernel asymptotic (see, for instance, in [GBVF01]). For this, however, one deals with (closed compact) Riemannian manifolds instead of (open) Lorentzian manifolds to geometrically model "space-time". In contrast, GTDT only uses (globally defined) densities instead of functionals. Moreover, in the latter scheme a specific Lagrangian (1) is canonically associated with every Dirac operator. This Lagrangian is fully determined by the Dirac operator in question.

The evaluation of the spectral action with respect to the Dirac-Yukawa operator that is defined by (29) – (37) leads to parameter relations which are quite similar to (61) – (64), see, for example, [CIKS97]). In this reference, also the value of the Higgs mass is calculated by a similar analysis to that presented in the previous section. It turns out that the parameter relations for the gauge couplings (61) are equivalent to those derived within CCM. The reason for this is that these relations basically follow from the fermion representation of the gauge fields. This, however, is supposed to hold true in both descriptions of the STM. As a consequence, one obtains the same critical scale point (75) and hence also the same critical energy E_c. On the other hand, all other parameter relations turn out to be essentially different from those presented here. As a consequence, one obtains a different value of the Higgs mass (see again [CIKS97], as well as [CIS99]):

$$m_{\mathrm{H}} = 190 \pm 5 \text{ GeV}. \tag{97}$$

Unfortunately, this prediction of the Higgs mass is of limited value insofar as the CCM approach to the STM is incompatible with certain experimentally known values of the (ratio of the) gauge coupling constants. Indeed, the parameter relations of the CCM (for the so-called "stiff action") imply the following relation at the critical scale-point t_c between the

SU(3)−gauge coupling constant and the Yukawa coupling constant of the top quark:

$$\frac{\mathrm{g}_3^2}{\mathrm{g}_t^2} = \frac{3}{2}. \tag{98}$$

This relation, however, is not compatible with the known experimental data. A similar numerical inconsistency is obtained also in the case of the so-called "soft-action" (see again [CIKS97]; a detailed discussion may be found also in [Thu03]).

In contrast, in the geometrical frame presented the above relation between the Yukawa coupling constant and the SU(3)−gauge coupling constant is replaced by (76) which includes the free "relative coupling constant" $\lambda_{\rm YMH}$. This additional free parameter has its origin in the generalization of the usual Yang-Mills curvature. It is quite remarkable that the tree-level relations (58) - (59) are such that only $\lambda_{\rm H}$, but not $\lambda_{\rm A}$, can be chosen equal to one without loss of generality. This is because of the usual Yang-Mills term in the bosonic Lagrangian (57). It is this subtle interplay between the usual Yang-Mills curvature and its generalization with respect to the Higgs gauge potential which allows the presented geometrical description of the STM to be numerically consistent.

5.2 Comparison with the CLM

The Connes-Lott approach to the Standard Model is clearly conceptually different from the geometrical frame presented here. The CLM essentially incorporates the basic ideas of A. Connes' mathematical theory of non-commutative geometry. Hence, it does not come as a surprise that the appropriate parameter relations obtained from the CLM are basically different from those implied by the GTDT approach to the STM (c.f. [CIS99]). Yet, in both geometrical schemes Dirac type operators of the form (19) play a fundamental role though their geometrical origin and physical interpretation is quite different. Indeed, in the CLM the geometrical role of the operators (19) is two-fold: First, they correspond to total exterior derivatives (in this context $\mathcal{D}$ is referred to as the "inner Dirac operator"); Second, (19) induces the non-commutative analogue of the Riemannian volume measure $\mu_{\rm M}$ in the bosonic action ("Dixmier trace"). However, the fermionic and the bosonic CLM-action are defined in totally different ways, in contrast to the CCM and the frame presented here. In any case, in the CLM the (Riemannian) metric has to be chosen by hand, similar to the case of Yang-Mills gauge theories. Actually, the bosonic action in the CLM frame is a non-commutative generalization of the usual (Euclidean) Yang-Mills action. Of course, as far as the calculation of the value of the Higgs mass is concerned, the metric independence of (19) does not matter. However, from a purely conceptual perspective the arbitrariness of the metric seems unsatisfying (like in the usual Yang-Mills gauge theories) and may serve as the main motivation to change the definition of the bosonic action within non-commutative geometry from the Dixmier trace to the spectral action (c.f. [Con88], [Con96]).

During the last decade different CLM approaches to the STM have been developed. Accordingly, there are also various statements about the predicted value of the mass of the

Higgs within the frame of non-commutative geometry (see, for example, [Con95]). Within Connes' *real geometry* the fermionic representation, considered as an algebra representation, can be chosen differently from (29). As a consequence, the commutant is also differently parameterized. Moreover, one also obtains more freedom to parameterize the appropriate scalar products used to define the fermionic and bosonic actions in the (real) CLM. Note that the introduction of the real structure also yields a doubling of the gauge degrees of fermionic freedom quite similar to what is needed in order to introduce the Pauli-Dirac-Yukawa operator (21) (see also our discussion in [TT05]. Therefore, in contrast to the CCM and GTDT approach to the STM, within the (real) CLM one does not obtain a unique critical scale-point t_c on which the corresponding parameter relations are assumed to hold true but, instead, a whole range of such points. This range corresponds to the energy interval

$$E_c \in [\mathrm{m_Z},\ 2 \cdot 10^5)\ \mathrm{GeV}. \tag{99}$$

Hence, in the (real) CLM the critical energy point E_c is at least by five orders less than in GTDT (and CCM). The predicted values of the Higgs mass are contained in the interval

$$\mathrm{m_H} \in [284, 300)\,\mathrm{GeV}, \tag{100}$$

which has an empty intersection with (81) (c.f. [IKS95a], [IKS95b], [CIS99]). This, in fact, remains true even if one restricts the commutant and thus the parametrization in the CLM. For example, analogous to GTDT there is also a geometrically distinguished parametrization in CLM which gives rise to the following definite value of the Higgs mass (c.f. loc site):

$$\mathrm{m_H} = 289\,\mathrm{GeV}. \tag{101}$$

6 Conclusion

In this article, we discussed the possible values of the Higgs mass as it is predicted by the (minimal) Standard Model when the latter is considered as a specific gauge theory of Dirac type. We have shown that this approach to the STM permits to yield (in a specific approximation including quantum corrections) a definite value of the Higgs mass without referring to additional assumptions coming, for instance, from cosmology. This is quite in contrast to the usual (non-geometrical) description of the STM, which only gives rise to a whole range of possible values of the Higgs mass. Within the GTDT approach to the STM the predicted value of the Higgs mass is in full accordance with the STM range, though it lies on the upper bound of the allowed interval. The presented approach of the STM clearly demonstrates once more the power of a geometrical understanding of physics and, in the case at hand, of the Standard Model of particle physics. This is emphasized by the circumstance, that a non-geometrical parametrization of a geometrical description may usually yield a "fuzziness" of the predictive power. To demonstrate this we also discussed the most general (but non-geometrical) parametrization possible in the GTDT approach to the STM. Similar to the Connes-Lott approach to the STM this gives rise to an interval of possible values of the Higgs mass. In the frame presented, however, this interval has been shown to have a non-empty intersection with the allowed STM range. In particular, the GTDT predicted lower bound is close to the lower bound of the STM range. In the presented geometrical scheme the fuzziness in the prediction of the value of the Higgs mass is originated only in the "self-interaction" $[H_\mu, H_\nu]$ of the Higgs gauge potential. Of course, when expressed in terms of the usual Higgs potential $V_{\rm H}$ it is this self-interaction which gives rise to the mass of the physical Higgs boson.

We also discussed how the predicted value of the Higgs mass may depend on the existence of massive neutrinos. It turns out that the inclusion of Dirac type mass terms to the STM does not alter the results presented.

Since there are some similarities to other geometrical approaches to the STM, we included a brief comparison of our approach, in particular, with the Chamseddine-Connes and the (real) Connes-Lott approach to the STM concerning the Higgs mass.

Some comments on the role of gravity within GTDT may be worth mentioning. Actually, Einstein's theory of gravity is an integral part of GTDT. Although neglected in our discussion of the Higgs mass, it plays a fundamental role in this approach to the STM. This is because it is intimately related to spontaneous symmetry breaking. Indeed, spontaneous symmetry breaking is considered as being due to the Higgs gauge potential

$$H_\mu \sim g_{\mu\nu}\gamma^\nu\gamma_5\phi.$$

Accordingly, the role of the usual Higgs potential $V_{\rm H}$ is regarded as only giving rise to the mass of the Higgs boson. Concerning the numerical calculations of the value of the Higgs mass done in this paper, gravitational effects are assumed to be negligible (similar to other approaches). The physical reason that this can be done without contradictions within the setup of GTDT is that the two length-scales involved, $\ell_{\rm H}$ and $\ell_{\rm P}$, are actually independent of each other. The drawback of this independence, of course, is that GTDT seems not to permit a unification of gravity with the strong and the electroweak interactions of the STM. On the other hand, on the energy scales considered one may not expect such a unification. In fact, one obtains for the critical energy point E_c, on which the parameter relations are shown to hold true, that $E_c/{\rm m}_{\rm P} \lll 1$. Accordingly, one has ${\rm m}_{\rm H}/{\rm m}_{\rm P} \lll 1$ (resp. $\ell_{\rm H}/\ell_{\rm P} \ggg 1$), as it is usually expected (and also very much hoped for). At least, this demonstrates that the GTDT approach to the STM is consistent with the common assumption that gravity is generically negligible within the range of validity of the STM, although Einstein's theory of gravity is naturally included within GTDT.

Irrespective of the concrete value and the geometrical scheme ("commutative" or "non-commutative"), it seems most remarkable that a prediction of a definite value of the Higgs mass can be obtained from the pure Standard Model without additional assumptions, provided the Standard Model is described in geometrical terms. Of course, that the Standard Model can be geometrically described at all is certainly in itself a quite remarkable fact, which one has to take into account in any theory that aims to go beyond the Standard Model.

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