Max-Planck-Institut
für Mathematik
in den Naturwissenschaften
Leipzig

Full field algebras, operads and tensor categories

by

Liang Kong

Preprint no.: 28 2006

Full field algebras, operads and tensor categories

Liang Kong

Abstract

We study the operadic and categorical formulations of (conformal) full field algebras. In particular, we show that a grading-restricted $\mathbb{R}\times\mathbb{R}$-graded full field algebra is equivalent to an algebra over a partial operad constructed from spheres with punctures and local coordinates. This result is generalized to conformal full field algebras over $V^L \otimes V^R$, where V^L and V^R are two vertex operator algebras satisfying certain natural finite and reductive conditions. We also study the geometry interpretation of conformal full field algebras over $V^L \otimes V^R$ equipped with a nondegenerate invariant bilinear form. By assuming slightly stronger conditions on V^L and V^R, we show that a conformal full field algebra over $V^L \otimes V^R$ equipped with a nondegenerate invariant bilinear form exactly corresponds to a commutative Frobenius algebra with a trivial twist in the category of $V^L \otimes V^R$-modules. The so-called diagonal constructions [HK2] of conformal full field algebras are given in tensor-categorical language.

0 Introduction

In [HK2], Huang and the author introduced the notion of conformal full field algebra and some variants of this notion. We also studied their basic properties and gave constructions. We explained briefly without giving details in the introduction of [HK2] that our goal is to construct conformal field theories [BPZ][MS]. It is one of the purpose of this work to explain the connection between conformal full field algebras and genus-zero conformal field theories. [HK2] and this work are actually a part of Huang's program ([H1]-[H13]) of constructing rigorously conformal field theories in the sense of Kontsevich and Segal [S1][S2].

Around 1986, I. Frenkel initiated a program of using vertex operator algebras to construct, in a suitable sense, geometric conformal field theories, the precise mathematical definition of which was actually given independently by Kontsevich and Segal [S1][S2] in 1987. According to Kontsevich and Segal, a conformal field theory is a projective tensor functor from a category consisting of finite many ordered copies of S^1 as objects and the equivalent classes of Riemann surfaces with parametrized boundaries as morphisms to the category of Hilbert spaces. This beautiful and compact definition of conformal field theory encloses enormously rich structures of conformal field theory. In order to construct such theories, it is more fruitful to look at some substructures of conformal field theories at first. In the first category, the set of morphisms which have arbitrary number of copies of S^1 in their domains and only one copy of S^1 in their codomain has a structure of operad [Ma], which is induced by gluing surfaces along their parametrized boundaries. We denoted this operad as $K_{\mathfrak{H}}$. Let H, a Hilbert space, be the image object of S^1. The projective tensor functor in the definition of conformal field theory endows H with a structure of algebra over an operad, which is a $\mathbb{C}$-extension of the operad $K_{\mathfrak{H}}$ [H3].

It is very difficult to study this algebra-over-operad structure on H directly. It was suggested by Huang [H12][H13] that one should first look at a dense subset of H, which carries a structure of an algebra over a partial operad $\tilde{K}^c \otimes \overline{\tilde{K}^{\bar{c}}}$ for $c \in \mathbb{C}$. The so-called sphere partial operad ([H3]) is denoted by K. It contains $K_{\mathfrak{H}}$ as a suboperad. $\tilde{K}^c$ is the $\frac{c}{2}$-power of the determine line bundle over the partial operad K. The restriction of the line bundle $\tilde{K}^c$ on $K_{\mathfrak{H}}$, denoted as $\tilde{K}^c_{\mathfrak{H}}$, is also a suboperad. $\tilde{K}^c \otimes \overline{\tilde{K}^{\bar{c}}}$ is simply the tensor product of the bundle $\tilde{K}^c$ and the complex conjugate of the bundle $\tilde{K}^{\bar{c}}$. It is shown by Huang in [H1][H3] that the category of algebras over partial operad $\tilde{K}^c$ (or $\tilde{K}^c$-algebras), satisfying some natural conditions and the condition that all the correlation functions are rational functions, is isomorphic to the category of vertex operator algebras with central charge c. Importantly, Huang also showed in [H12][H13] that one can obtain algebras over $\tilde{K}^c_{\mathfrak{H}}$, which is a true operad instead of partial operad, by completing the space of vertex operator algebra properly.

The results of Huang suggests that vertex operator algebras (or its proper completion) describe genus-zero conformal field theories whose correlation functions are rational functions. It turns out that vertex operator algebras, in general, are not enough for the genus-one properties of conformal field theories, which require that all the genus-one correlation functions are modular invariant [Z][DLM][Mi1][Mi2][H8]. It is well-known in physics that one needs to combine both the chiral theory [H4][H5][H6] and the antichiral theory to obtain a full conformal field theory. As a result the correlation functions in a conformal field theory, in general, are neither holomorphic nor antiholomorphic. The conformal full field algebras introduced in [HK2] are exactly $\tilde{K}^{c^L} \otimes \overline{\tilde{K}^{\overline{c^R}}}$-algebras with neither holomorphic nor antiholomorphic correlation functions in general, and are capable of producing modular invariant genus-one correlation functions when $c^L = c^R$. We show in Section 1 that the grading-restricted $\mathbb{R} \times \mathbb{R}$-graded full field algebras are exactly smooth algebras over $\widehat{K}$, which is partial suboperad of K, and conformal full field algebra over $V^L \otimes V^R$ are exactly smooth algebras over $\tilde{K}^{c^L} \otimes \overline{\tilde{K}^{\overline{c^R}}}$. The modular invariance property of conformal full field algebras will be studied in [HK3].

In order to cover the entire genus-zero conformal field theories, one still needs to consider the Riemann surfaces with more than one copy of S^1 in the codomain. In chiral theory, it was studied by Hubbard in the framework of so-called vertex operator coalgebras [Hub1][Hub2]. In a theory including both chiral and antichiral parts, it amounts to have a conformal full field algebra equipped with a nondegenerate invariant bilinear form [HK2]. Conversely, a conformal full field algebra with a nondegenerate invariant bilinear form gives a complete set of data needed for a genus-zero conformal field theory (nonunitary). In particular, we show in Section 2 that such conformal full field algebras are just algebraic representations of the sewing operations among spheres with arbitary number of negatively oriented and positively oriented punctures as long as the resulting surfaces after sewing are still of genus-zero. The invariant property of bilinear form used in this work is slightly different from that in [HK2]. Both definitions have clear geometric meanings. But we need the new definition for a reason which is explained later.

Although the notion of conformal full field algebras has the advantage of being a pure algebraic formulation of a part of genus-zero conformal field theories, its axioms are still very hard to check directly. The theory of the tensor products of the modules of vertex operator algebras is developed by Huang and Lepowsky [HL1]-[HL4][H2][H7]. The following Theorem proved by Huang in [H7] is very crucial for our constructions of conformal full field algebras.

Theorem 0.1. *Let V be a vertex operator algebra satisfying the following conditions:*

1. *Every $\mathbb{C}$-graded generalized V-module is a direct sum of $\mathbb{C}$-graded irreducible V-modules,*
2. *There are only finitely many inequivalent $\mathbb{C}$-graded irreducible V-modules,*
3. *Every $\mathbb{R}$-graded irreducible V-module satisfies the C_1-cofiniteness condition.*

Then the direct sum of all (in-equivalent) irreducible V-modules has a natural structure of intertwining operator algebra and the category of V-modules, denoted as $\mathcal{C}_V$ has a natural structure of vertex tensor category. In particular, $\mathcal{C}_V$ has natural structure of braided tensor category.

Assumption 0.2. All the vertex operator algebras appeared in this work, V, V^L and V^R are all assumed to satisfy the conditions in Theorem 0.1 without further announcement. Sometimes we even assume stronger conditions on them, as we will do explicitly in Section 4 and Section 5.

In [HK2], we have studied in detail the properties of conformal full field algebras over $V^L \otimes V^R$. In particular, an equivalent definition of conformal full field algebra over $V^L \otimes V^R$ is also given. We recalled this result in Theorem 3.2. The axiom in this definition are much easier to verify than those in the original definition of conformal full field algebra. Also by Theorem 0.1, the categories of V^L-modules, V^R-modules and $V^L \otimes V^R$-modules, denoted as $\mathcal{C}_{V^L}$, $\mathcal{C}_{V^L}$ and $\mathcal{C}_{V^L\otimes V^R}$ respectively, all have the structures of braided tensor category [H7][HK2]. In this work, the braiding structure in $\mathcal{C}_{V^L\otimes V^R}$ is chosen to be different from the one obtained from Theorem 0.1. Using Theorem 3.2 and this new braiding structure on $\mathcal{C}_{V^L\otimes V^R}$, we can show, without much effort, that a conformal full field algebra over $V^L \otimes V^R$ is equivalent to a commutative associative algebra in $\mathcal{C}_{V^L\otimes V^R}$ with a trivial twist.

We would also like to give a categorical formulation of conformal full field algebra over $V^L \otimes V^R$ equipped with a nondegenerate invariant bilinear form, where V^L and V^R are assumed to satisfy the conditions in Theorem 4.5, which are slightly stronger than those in Theorem 0.1. It turns out that the way we define the invariant property of bilinear form of conformal full field algebra in [HK2] is not easy to work with categorically. This is the reason why a slightly modified notion of invariant bilinear form is introduced in Section 2. This modification leads us to consider on the graded dual space of a module over a vertex operator algebra a module structure which is different from (but equivalent to) the usual contragredient module structure [FHL]. Because of this, we redefine the duality maps [H9][H11] in this new convention and prove the rigidity (in appendix). Then we show in detail, in Section 4, that a conformal full field algebra over $V^L \otimes V^R$ equipped with a nondegenerate invariant bilinear form in the new sense exactly amounts to a commutative Frobenius algebra with a trivial twist in $\mathcal{C}_{V^L\otimes V^R}$.

Once the categorical formulation is known. We can give a categorical construction of conformal full field algebras equipped with a nondegenerate invariant bilinear form. This construction was previously given by Huang and the author in [HK2] by using intertwining operator algebras, and was also known to physicists as diagonal construction (see for example [FFFS] and references therein).

Recently, Fuchs, Runkel, Schweigert and Fjelstad have proposed a very general construction of all correlation functions of boundary conformal conformal field theories using 3-dimensional topological field theories in a series of papers [FRS1]-[FRS4][FjFRS][RFFS][SFR]. In particular, a construction of commutative associative algebras in $\mathcal{C}_{V\otimes V}$ is explicitly given in [RFFS]. Our approach is somewhat complementary to their approach (see [RFFS] for comments on the relation of two approachs). We hope that two approachs can be combined to obtain a rather complete picture of conformal field theory in the near future.

The layout of this paper is as follow. In Section 1, we study the operadic formulation of grading-restricted $\mathbb{R} \times \mathbb{R}$-graded full field algebra and its variants, following the work of Huang [H3]. In Section 2, we give a the geometric description of a conformal full field algebra over $V^L \otimes V^R$ equipped with a nondegenerate invariant bilinear form. In Section 3, we give a categorical formulation of conformal full field algebra over $V^L \otimes V^R$. In Section 4, we give a categorical formulation of conformal full field algebra over $V^L \otimes V^R$ with a nondegenerate invariant bilinear form for V^L and V^R satisfying the conditions in Theorem 4.5. In Section 5, we give a categorical construction of conformal full field algebras over $V^L \otimes V^R$ and prove that such obtained conformal full field algebras over $V^L \otimes V^R$ are naturally equipped with a nondegenerate invariant bilinear form.

For the convenience of readers, the materials in Section 3, 4, 5 are completely independent of those in Section 1, 2. For those who is only interested in categorical formulation of conformal full field algebras over $V^L \otimes V^R$, it is harmless to start from Section 3 directly.

Convention of notations: $\mathbb{N}, \mathbb{R}, \mathbb{R}_+, \mathbb{C}, \mathbb{H}, \hat{\mathbb{C}}, \hat{\mathbb{H}}$ denote the set of natural numbers, real numbers, positive real numbers and complex numbers, and upper half plane, one point compactification of $\mathbb{C}$ and $\mathbb{H} \cup \mathbb{R}$, respectively. We also use I_F to denote the identity map on a vector space F.

Acknowledgment This work grows from a chapter in author's thesis. I want to thank my advisor Yi-Zhi Huang for his constant support and many important suggestions. I also want to thank him for spending incredible amount of time in helping me to improve the writting of my thesis and this work. I thank J. Lepowsky and C. Schweigert for many inspiring conversations related to this work.

1 Operadic formulations of full field alegbras

In this section, we study the operadic formulation of full field algebra. In section 1.1, we recall the notion of sphere partial operad K and its partial suboperad $\widehat{K}$. We introduce the notion of smooth function on K. In section 1.2, we recall the notions of determinant line bundle over K and the $\mathbb{C}$-extensions of K, such as $\tilde{K}^c$ and $\tilde{K}^{c^L} \otimes \overline{\tilde{K}^{\overline{c^R}}}$ for $c, c^L, c^R \in \mathbb{C}$. Section 1.1 and 1.2 are mainly taken from [H3]. The readers who is interested in knowing more on this subject should consult with [H3] for details. In section 1.3, we recall the notion of algebra over partial operad

[H3], and explain what it means for an algebra over $\widehat{K}$ and $\tilde{K}^{c^L}\otimes\overline{\tilde{K}^{\overline{c^R}}}$ to be smooth. Then we give two isomorphism theorems. The first one says that the category of grading-restricted $\mathbb{R}\times\mathbb{R}$-graded full field algebras is isomorphic to the category of smooth $\widehat{K}$-algebras. The second one says that the category of conformal full field algebras over $V^L\otimes V^R$ is isomorphic to the category of smooth $\tilde{K}^{c^L}\otimes\overline{\tilde{K}^{\overline{c^R}}}$-algebras over $V^L\otimes V^R$. We give a selfcontent proof of the first isomorphism theorem. The proof of the second isomorphism theorem is technical, and heavily depends on the results in [H3].

1.1 Sphere partial operad K

A sphere with tubes of type (n_-, n_+) is a sphere S with n_- ordered punctures $p_i, i = 1, \dots, n_-$ and n_+ ordered punctures $q_j, j = 1, \dots, n_+$, together with a negatively oriented local chart (U_i, φ_i) around each p_i and a positively oriented local chart (V_j, ψ_j) around each q_i, where U_i and V_j are neighborhood of p_i and q_j respectively and the local coordinate maps $\varphi_i : U_i \to \mathbb{C}$ and $\psi_j : V_j \to \mathbb{C}$ are conformal maps so that $\varphi(p_i) = \psi_j(q_j) = 0$.

The conformal equivalence of sphere with tubes is defined to be the conformal maps between two spheres so that the germs of local coordinate map φ_i and ψ_j are preserved. We then obtain a moduli space of sphere with tubes of type (n_-, n_+). In this section, We are only interested in spheres with tubes of type $(1, n)$. For this type of spheres with tubes, we label the only negative oriented puncture as the 0-th puncture. We denote the moduli space of sphere with tubes of type $(1, n)$ as $K(n)$.

Using automorphisms of sphere, we can select a canonical representative from each conformal equivalence class in $K(n)$ for all $n > 0$ by fixing the n-th puncture at $0 \in \mathbb{C}$, the 0-th puncture at ∞, and ψ_0, the local coordinate map at ∞, to be so that

$$\lim_{w\to\infty} w\psi_0(w) = -1. \tag{1.1}$$

As a consequence, the moduli space $K(n), n \in \mathbb{Z}_+$ can be identified with

$$K(n) = M^{n-1} \times H \times (\mathbb{C}^\times \times H)^n$$

where

$$M^{n-1} = \{(z_1, \dots, z_{n-1}) | z_i \in \mathbb{C}^\times, z_i \neq z_j, \text{ for } i \neq j\}$$

and

$$\begin{aligned} H \;=\; & \{A = (A_1, A_2, \dots) \in \prod_{i=1}^{\infty} \mathbb{C} \quad |A_i \in \mathbb{C}, \quad e^{\sum_{j=1}^{\infty} A_j x^{j+1}\frac{d}{dx}} x \\ & \text{is an absolute convergent series in some neighborhood of } 0 \;\}. \end{aligned}$$

Similarly, using automorphisms of sphere, we can choose the canonical representatives of conformal equivalence classes of sphere with tube of type (1,0) so that the moduli space $K(0)$ can be identified with the set

$$K(0) = \{A \in \prod_{i=1}^{\infty} \mathbb{C} | A_1 = 0\}.$$

We will denote a general element in $K(n), n \in \mathbb{N}$ as

$$P = (z_1, \ldots, z_{n-1}; A^{(0)}, (a_0^{(1)}, A^{(1)}), \ldots, (a_0^{(n)}, A^{(n)})), \tag{1.2}$$

where $z_1, \ldots, z_{n-1}$ denotes the location of the first $n-1$ punctures and the rest of the data give the local coordinate maps at the negatively oriented (0-th) puncture and positively oriented punctures respectively as follow:

$$f_0(w) = -e^{\sum_{j\in\mathbb{Z}_+} A_j^{(0)} x^{j+1} \frac{d}{dx}} x \Big|_{x=\frac{1}{w}} \tag{1.3}$$

$$f_i(w) = a_0^{(i)} e^{\sum_{j\in\mathbb{Z}_+} A_j^{(i)} w^{j+1} \frac{d}{dw}} w, \quad \forall i = 1, \ldots, n. \tag{1.4}$$

Let $P \in K(m)$ and $Q \in K(n)$. Let $\bar{B}^r$ be the closed ball in $\mathbb{C}$ centered at 0 with radius r, φ_i the germs of local coordinate map at i-th puncture p_i of P, and ψ_0 the germs of local coordinate map at 0-th puncture q_0 of Q. Then we say that the i-th tube of P can be sewn with 0-th tube of Q if there is a $r \in \mathbb{R}_+$ such that p_i and q_0 are the only punctures in $\varphi_i^{-1}(\bar{B}^r)$ and $\psi_0^{-1}(\bar{B}^{1/r})$ respectively. A new sphere with tubes in $K(m+n-1)$, denoted as $P{}_i\infty_0 Q$, can be obtained by cutting out $\varphi_i^{-1}(\bar{B}^r)$ and $\psi_0^{-1}(\bar{B}^{1/r})$ from P and Q respectively, and then identifying the boundary circle via the map

$$\psi^{-1} \circ J_{\hat{\mathbb{H}}} \circ \varphi_i$$

where $J_{\hat{\mathbb{H}}} : w \to \frac{-1}{w}$.

Remark 1.1. $\mathbb{H}$ denotes the upper hand plane. We use it to remind us the fact that $w \to \frac{-1}{w}$ is an automorphism of upper hand plane.

Therefore, we have sewing operations:

$${}_i\infty_0 : K(m) \times K(n) \to K(m+n-1)$$

partially defined on the entire K between two spheres with tubes along two oppositely oriented tubes.

Remark 1.2. Our definition of sewing operation is defined differently from that defined in [H3], where $J_{\hat{\mathbb{C}}} : w \mapsto \frac{1}{w}$ is used in the place of $J_{\hat{\mathbb{H}}}$. Also notice that our convention of local coordinate map (1.3) is also different from that in [H3] by a sign. Combining effect of these two difference is a trivial one. Namely, our definition of sewing operation is equivalent to that in [H3] if one identify each sphere with tube with local coordinate map at ∞ being $f_0(w)$ used in this paper, with the same sphere with tube but with local coordinate map at ∞ be $-f_0(w)$ used in [H3]. This identification actually gives an isomorphism of partial operad.

Remark 1.3. One reason for introducing this new convention is to make sure that the invariant bilinear form on conformal full field algebra introduced in Section 2 has a clear geometric meaning. The notion of invariant bilinear form on a conformal full field algebra used in this work is slightly different from that in [HK2] for a categorical reason (see also the Remark 2.1).

Remark 1.4. Another reason for our choice being geometrically natural is that $J_{\hat{\mathbb{H}}}$ is an automorphism of $\hat{\mathbb{H}}$ while $J_{\hat{\mathbb{C}}}$ is not. This will be important in boundary conformal field theories, where we study algebras over a partial operad consisting of disks with strips [HK1], which is often modeled as upper half planes. Hence it is more natural to use $J_{\hat{\mathbb{H}}}$ instead of $J_{\hat{\mathbb{C}}}$ there. Moreover, in the study of boundary conformal field theories, it is necessary to embed disks with strips into spheres with tubes by certain doubling maps ([HK1][Ko]). In order for the embedding becoming a morphism of partial operad, it is also natural to define the sewing operations in K using $J_{\hat{\mathbb{H}}}$ instead of $J_{\hat{\mathbb{C}}}$.

Let $\mathbf{0}$ be the sequence $(0, 0, \dots)$. The element

$$I_K := (\mathbf{0}, (1, \mathbf{0})) \in K(1)$$

is called identity element. Its tubes are sewable with any oppositely oriented tubes in other sphere. Moreover, it satisfies the following identity property:

$$Q\,{}_i\infty_0\, I = I\,{}_1\infty_0\, Q = Q$$

for any $Q \in K(n), n \in \mathbb{N}$. The subset

$$\{(\mathbf{0}, (a, \mathbf{0}) \in H \times (\mathbb{C}^\times \times H)\} \tag{1.5}$$

of $K(1)$ together with the sewing operation ${}_1\infty_0$ is a group isomorphic to $\mathbb{C}^\times$. There is also an obvious action of permutation group S_n on $K(n)$. The following result is proved in [H3].

Proposition 1.5. *The collection of sets*

$$K = \{K(n)\}_{n \in \mathbb{N}}$$

together with I_K, sewing operations, the actions of S_n on $K(n)$ and the group (1.5), is a $\mathbb{C}^\times$-rescalable partial operad.

Let $A(a;i) = \{A_j | A_i = a, A_j = 0, j \in \mathbb{Z}_+, j \neq i\}$. For simplicity, we will also use $A(a;i)$ to denote the element in $K(0)$ such that the local coordinate map at ∞ is given by

$$-\exp\left(a\left(\frac{1}{w}\right)^{i+1}\frac{d}{d\frac{1}{w}}\right)\frac{1}{w} = -\exp\left(-aw^{-i+1}\frac{d}{dw}\right)\frac{1}{w}.$$

Let $\widehat{K}(n)$ be the subset of $K(n)$ consisting of elements of the form

$$(z_1, \dots, z_{n-1}; A(a;1), (a_0^{(1)}, \mathbf{0}), \dots, (a_0^{(n)}, \mathbf{0})). \tag{1.6}$$

Then $\widehat{K} = \{\widehat{K}(n)\}_{n \in \mathbb{N}}$ is a partial suboperad of K [H3].

We use "overline" to denote complex conjugation. A function f on $K(n), n \in \mathbb{N}$ is called *smooth* if there is a $N \in \mathbb{N}$ such that f can be written as

$$\sum_{k=1}^{N} g_k(A^{(0)}, a_0^{(1)}, A^{(1)}, \dots, a_0^{(n)}, A^{(n)}; \overline{A^{(0)}}, \overline{a_0^{(1)}}, \overline{A^{(1)}}, \dots, \overline{a_0^{(n)}}, \overline{A^{(n)}})$$
$$h_k(z_1, \dots, z_{n-1}; \bar{z}_1, \dots, \bar{z}_{n-1}) \tag{1.7}$$

where g_k are polynomial functions of $A_j^{(i)}$ and $\overline{A_j^{(i)}}$, for $i = 0, \dots, n, j \in \mathbb{Z}_+$ and linear combination of $\prod_{i=1}^{n}(a_0^{(i)})^{r_i}\overline{a_0^{(i)}}^{s_i}$ for some $r_i, s_i \in \mathbb{R}$, and h_k are smooth functions of $z_i, i = 1, \dots, n-1$. It is clear that there is also a naturally induced notion of smooth function on $\widehat{K}$.

A tangent space $TK(n)$ of $K(n), n \in \mathbb{N}$ can be defined naturally. Let $\epsilon \in \mathbb{C}$. We define two tangent vectors in $T_I K(1)$ as follow: for any smooth function f on $K(1)$,

$$\begin{aligned} \mathcal{L}_I(z)f &:= \left.\frac{\partial}{\partial \epsilon}\right|_{\epsilon=0} f(P(z)_1\infty_0 A(\epsilon;2)), \\ \bar{\mathcal{L}}_I(\bar{z})f &:= \left.\frac{\partial}{\partial \bar{\epsilon}}\right|_{\epsilon=0} f(P(z)_1\infty_0 A(\epsilon;2)). \end{aligned} \tag{1.8}$$

The following proposition is a generalization of Proposition 3.2.5 in [H6].

Proposition 1.6.

$$\begin{aligned}
\mathcal{L}_I(z) &= z^{-2}\frac{\partial}{\partial a^{(1)}}\bigg|_I + \sum_{i=0}^{1}\sum_{j=1}^{\infty} z^{-(2i-1)j-2}\frac{\partial}{\partial A_j^{(i)}}\bigg|_I, \\
\bar{\mathcal{L}}_I(\bar{z}) &= \bar{z}^{-2}\frac{\partial}{\partial \overline{a^{(1)}}}\bigg|_I + \sum_{i=0}^{1}\sum_{j=1}^{\infty} \bar{z}^{-(2i-1)j-2}\frac{\partial}{\partial \overline{A_j^{(i)}}}\bigg|_I.
\end{aligned} \tag{1.9}$$

Proof. Let $P = P(z)_1\infty_0 A(\epsilon, 2)$ with its coordinates in moduli space given by

$$(z; A^{(0)}, (a_0^{(1)}, A^{(1)}), (a_0^{(2)}, A^{(2)})).$$

It is shown in [H3] that z, $A_j^{(0)}$, $a_0^{(1)}$, $A_j^{(1)}$, $a_0^{(2)}$ and $A_j^{(2)}$, $j \in \mathbb{Z}_+$ are holomorphic functions of ϵ. Hence their complex conjugation $\bar{z}$, $\overline{A_j^{(0)}}$, $\overline{a_0^{(1)}}$, $\overline{A_j^{(1)}}$, $\overline{a_0^{(2)}}$ and $\overline{A_j^{(2)}}$ are holomorphic with respect to $\bar{\epsilon}$. Therefore (1.9) follows from Proposition 3.2.5 in [H3]. ∎

1.2 Determinant line bundle over K

The determinant line bundle over K and the $\mathbb{C}$-extensions of K are studied in [H3]. For each $n \in \mathbb{N}$, the determinant line bundle $\mathrm{Det}(n)$ over $K(n)$ is a trivial bundle over $K(n)$. We denote the fiber at $Q \in K(n)$ as Det_Q. There is a canonical section of $\mathrm{Det}(n)$, denoted by ψ_n, for each $n \in \mathbb{N}$. For any element $Q \in K(n)$, let $\mu_n(Q)$ be the element of the fiber over Q given by

$$\psi_n(Q) = (Q, \mu_n(Q)).$$

Then there is a λ_Q for each element $\tilde{Q}$ of Det_Q such that $\tilde{Q} = (Q, \lambda_Q)$ and $\lambda_Q = \alpha\mu_n(Q)$ for some $\alpha \in \mathbb{C}$.

Consider the following two general elements in $K(m)$ and $K(n)$.

$$\begin{aligned}
P &= (z_1, \dots, z_m; A^{(0)}, (a_0^{(1)}, A^{(1)}), \dots, (a_0^{(m)}, A^{(m)})) \\
Q &= (\xi_1, \dots, \xi_n; B^{(0)}, (b_0^{(1)}, B^{(1)}), \dots, (b_0^{(n)}, B^{(n)}))
\end{aligned} \tag{1.10}$$

If $P_i\infty_0 Q$ exists, then there is a canonical isomorphism:

$$l_{P,Q}^i : \mathrm{Det}_P \otimes \mathrm{Det}_Q \to \mathrm{Det}_{P_i\infty_0 Q},$$

given as:

$$\begin{aligned}
&l_{P,Q}^i(a_1\mu_m(P) \otimes a_2\mu_n(Q)) \\
&\qquad = a_1 a_2 e^{2\Gamma(A^{(i)}, B^{(0)}, a_0^{(i)})}\mu_{m+n-1}(P_i\infty_0 Q),
\end{aligned} \tag{1.11}$$

where Γ is a $\mathbb{C}$-valued analytic function of complex variables $A_j^{(i)}, B_k^{(0)}, a_0^{(i)}, j, k \in \mathbb{N}$. If we expand Γ as formal series, we have

$$\Gamma(A^{(i)}, B^{(0)}, \alpha) \in \mathbb{Q}[\alpha, \alpha^{-1}][[A^{(i)}, B^{(0)}]].$$

For a detailed discussion of Γ, see chapter 4 in [H3]. In particular, we have

$$\overline{\Gamma(A^{(i)}, B^{(0)}, a_0^{(i)})} = \Gamma(\overline{A^{(i)}}, \overline{B^{(0)}}, \overline{a_0^{(i)}}). \tag{1.12}$$

For $(P, \lambda_P) \in \mathrm{Det}(m)$ and $(Q, \lambda_Q) \in \mathrm{Det}(n)$ such that $P_i\infty_0 Q$ exists, then we define a partially defined map

$$_i\widetilde{\infty}_0^2 : \mathrm{Det}(m) \times \mathrm{Det}(n) \to \mathrm{Det}(m+n-1)$$

by

$$(P, \lambda_P)_i\widetilde{\infty}_0^2(Q, \lambda_Q) = (P_i\infty_0 Q, l^i_{P,Q}(\lambda_P \otimes \lambda_Q)).$$

Using this partial operation, one obtain a $\mathbb{C}^\times$-rescalable partial operad structure on $\mathrm{Det} = \{\mathrm{Det}(n)\}_{n\in\mathbb{N}}$.

For $c \in \mathbb{C}$, the so-called *vertex partial operad of central charge c*, denoted as $\tilde{K}^c$, is the $\frac{c}{2}$-th power of the determinant line bundle Det over K. $\tilde{K}^c$ also has a structure of partial operad. The construction of sewing operations on $\tilde{K}^c$ is same as that on Det except replacing (1.11) by

$$\begin{aligned}(l^i_{P,Q})^c&(a_1\mu_m(P) \otimes a_2\mu_n(Q)) \\ &= a_1a_2e^{\Gamma(A^{(i)}, B^{(0)}, a_0^{(i)})c}\mu_{m+n-1}(P_i\infty_0 Q).\end{aligned} \tag{1.13}$$

We denote the corresponding sewing operation as $_i\widetilde{\infty}_0^c$. It is proved in [H3] that $\tilde{K}^c$ is also $\mathbb{C}^\times$-rescalable associative partial operad and a $\mathbb{C}$-extension of K (see [H3]).

The complex conjugation $\overline{\tilde{K}^{\bar{c}}}$ of holomorphic line bundle $\tilde{K}^{\bar{c}}$ also has a natural structure of partial operad. The section of ψ on $\tilde{K}^{\bar{c}}$ canonically gives a section $\bar{\psi}$ on $\overline{\tilde{K}^{\bar{c}}}$. Using the global section $\bar{\psi}$, the canonical isomorphism

$$\bar{l}^i_{P,Q} : \overline{\mathrm{Det}^{\bar{c}}}_P \otimes \overline{\mathrm{Det}^{\bar{c}}}_Q \to \overline{\mathrm{Det}^{\bar{c}}}_{P_i\infty_0 Q},$$

for any pair of $P, Q \in K$ so that $P_i\infty_0 Q$ exists, can be written as

$$\bar{l}^i_{P,Q}(\lambda_1 \otimes \lambda_2) = \lambda_1\lambda_2 e^{\Gamma(\overline{A^{(i)}}, \overline{B^{(0)}}, \overline{a_0^{(i)}})c} \tag{1.14}$$

where $(a_0^{(i)}, A^{(i)})$ gives the local coordinate map at i-th positively oriented puncture in P and $B^{(0)}$ gives the local coordinate map at ∞ in Q. Observe that (1.14) implies that $\overline{\tilde{K}^{\bar{c}}} = \overline{\tilde{K}}^c$ as partial operads.

In this work, we are also interested in the tensor product bundle $\tilde{K}^{c^L} \otimes \overline{\tilde{K}^{\overline{c^R}}}$ for $c^L, c^R \in \mathbb{C}$. It is clear that it is a $\mathbb{C}^\times$-rescalable partial operad as well. The natural section induced from $\tilde{K}^{c^L}$ and $\overline{\tilde{K}^{\overline{c^R}}}$ is simply $\psi \otimes \bar{\psi}$. We will denote the sewing operation on $\tilde{K}^{c^L} \otimes \overline{\tilde{K}^{\overline{c^R}}}$ simply as $\widetilde{\infty}$ without making its dependents on c^L, c^R explicit.

A function on $\tilde{K}^c$ or $\tilde{K}^{c^L} \otimes \overline{\tilde{K}^{\overline{c^R}}}$ is called *smooth* if it is smooth on the base space K and linear on fiber.

1.3 Two isomorphism theorems

In this subsection we discuss the operadic formulations of grading-restricted $\mathbb{R}\times\mathbb{R}$-graded full field algebras and conformal full field algebras over V. We apologize for not recalling the definitions of these two notions here. They can be found in [HK2].

We first recall the definition of algebra over partial operad. Let G be a group and U a complete reducible G-module and W a G-submodule of U. We will use $\overline{U}$ to denote the algebraic completion of U. It should be distinguishable with complex conjugation from the context. Let $\mathcal{H}^G_{UW}(n)$ be the set of multilinear maps $U^{\otimes n} \to \overline{U}$ such that image of $W^{\otimes n}$ is in $\overline{W}$. $\mathcal{H}^G_{UW} := \{\mathcal{H}^G_{UW}(n)\}_{n\in\mathbb{N}}$ has a natural structure of partial pseudo-operad [H3], which satisfies all the axioms of partial operad except the associativity.

Definition 1.7. Let $\mathcal{P}$ be a partial operad with rescaling group G. A $\mathcal{P}$-pseudo-algebra is a triple (U, W, ν), where U is a completely reducible G-module

$$U = \coprod_{M\in A} U_{(M)}$$

in which A is the set of equivalent class of irreducible G-modules, W is a submodule of U, and ν is a morphism from $\mathcal{P}$ to the partial pseudo-operad $\mathcal{H}^G_{U,W}$ ([H4]), satisfying the following conditions

1. $\dim U_{(M)} < \infty$ for all $M \in A$.
2. The submodule of U generated by the homogeneous components of the elements of $\nu_0(\mathcal{P}(0))$ is W [1].
3. The map from G to $\mathcal{H}^G_{U,W}(1)$ induced from ν_1 is the given representation of G on U.

[1]Here, ν_n for $n \in \mathbb{N}$ is simply the restriction of ν on $\mathcal{P}(n)$.

If $\mathcal{P}$ is rescalable [H3], we call a $\mathcal{P}$-pseudo-algebra a $\mathcal{P}$-algebra.

In this work, we are interested in studying $\widehat{K}$-algebras and $\tilde{K}^{c^L} \otimes \overline{\tilde{K}^{\overline{c^R}}}$-algebras, both of which are $\mathbb{C}^\times$-rescalable partial operads.

Definition 1.8. A $\widehat{K}$-algebra (or $\tilde{K}^{c^L} \otimes \overline{\tilde{K}^{\overline{c^R}}}$-algebra) (F, W, ν) is called *smooth* if it satisfies the following conditions:

1. $F_{(m,n)} = 0$ if the real part of m or n is sufficiently small.
2. For any $n \in \mathbb{N}$, $w' \in F', w_1, \ldots, w_{n+1} \in F$, an element Q in $K(n)$ (or $\tilde{K}^{c^L} \otimes \overline{\tilde{K}^{\overline{c^R}}}(n)$), the function
$$Q \mapsto \langle w', \nu(Q)(w_1 \otimes \cdots \otimes w_{n+1})\rangle$$
is smooth on $K(n)$ (or on $\tilde{K}^{c^L} \otimes \overline{\tilde{K}^{\overline{c^R}}}(n)$).

We choose the branch cut of logarithm as follow
$$\log z = \log |z| + \mathrm{Arg} z, \quad 0 \le \mathrm{Arg} z < 2\pi. \tag{1.15}$$
We define the power functions, z^m and $\bar{z}^n$ for $m, n \in \mathbb{R}$, to be $e^{m \log z}$ and $e^{n\overline{\log z}}$ respectively. This convention is used throughout this work.

The following lemma must be well-known. But we are not aware of any source of reference. So we give a proof here.

Lemma 1.9. *If $\rho : \mathbb{C}^\times \to \mathbb{C}^\times$ is an irreducible representation of group $\mathbb{C}^\times$, then $\rho(z) = z^{-m}\bar{z}^{-n}$ for some $m, n \in \mathbb{C}$ and $m - n \in \mathbb{Z}$.*

Proof. We view $\mathbb{C}^\times$ as a topological group. All the irreducible representations of $\mathbb{C}^\times$ are one-dimensional. They are continuous maps $\mathbb{C}^\times \to \mathbb{C}^\times$ as group homomorphisms.

$\mathbb{C}$ is the universal covering space of $\mathbb{C}^\times$ with the projection map π given by $z \mapsto e^z$. By the lifting property of covering space, there is a unique continuous map $\tilde{\rho} : \mathbb{C} \to \mathbb{C}$ such that $\rho\pi = \pi\tilde{\rho}$ if we choose $\tilde{\rho} : 0 \mapsto 0$. Let $s_1, s_2 \in \mathbb{C}$. Since ρ is a group homomorphism, hence we have
$$e^{\tilde{\rho}(s_1+s_2)} = \rho(e^{s_1+s_2}) = \rho(e^{s_1})\rho(e^{s_2}) = e^{\tilde{\rho}(s_1)}e^{\tilde{\rho}(s_2)} = e^{\tilde{\rho}(s_1)+\tilde{\rho}(s_2)}. \tag{1.16}$$
It implies that $\tilde{\rho}(s_1 + s_2) = \tilde{\rho}(s_1) + \tilde{\rho}(s_2) + k2\pi i$ for some $k \in \mathbb{Z}$. Notice that k must be unique because $\tilde{\rho}$ is continuous. Recall that we have already chosen $\tilde{\rho}$ to

be so that $\tilde{\rho}(0) = 0$. Therefore $k = 0$. $\tilde{\rho}$ is actually a linear map from $\mathbb{C} \to \mathbb{C}$. Namely, $\tilde{\rho}$ can be written as

$$\tilde{\rho} : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \tag{1.17}$$

where $x, y \in \mathbb{R}$ are the real part and the imaginary part of a complex number in $\mathbb{C}$.

Moreover, the group homomorphism preserves the identity, i.e. $\rho : 1 \mapsto 1$. It implies that $\tilde{\rho}(2\pi i) = l 2\pi i$ for some $l \in \mathbb{Z}$. Applying this result to (1.17), we obtain that $b = 0$ and $d = l \in \mathbb{Z}$. Conversely, it is easy to see that every $\tilde{\rho}$ of form (1.17) with $b = 0$ and $d \in \mathbb{Z}$ gives arise to a group homomorphism $\rho : \mathbb{C}^\times \to \mathbb{C}^\times$ of the following form:

$$z = e^{x+iy} \mapsto e^{ax+i(cx+dy)} = e^{(a+ic)x} e^{diy} = |z|^{a+ic} \left(\frac{z}{\bar{z}}\right)^{d/2} = z^{-m} \bar{z}^{-n}. \tag{1.18}$$

where $m = -\frac{a+ic}{2} - \frac{d}{2}$ and $n = -\frac{a+ic}{2} + \frac{d}{2}$ and $m - n = -d \in \mathbb{Z}$. ∎

Now we study the basic properties of a smooth $\widehat{K}$-algebras. We fix a smooth $\widehat{K}$-algebras (F, W, ν).

The set $I := \{(m, n) \in \mathbb{C} \times \mathbb{C} | m - n \in \mathbb{Z}\}$ together with the usual addition operation gives an abelian group. By Lemma 1.9 any $\widehat{K}$-algebras must be I-graded. Namely,

$$F = \coprod_{(m,n) \in I} F_{(m,n)}.$$

Since $\nu_1((\mathbf{0}, (a, \mathbf{0})))$ for $a \in \mathbb{C}^\times$ gives the representation of $\mathbb{C}^\times$ on F by the definition of algebra over operad, we must have $\nu_1((0, a))u = a^{-m} \bar{a}^{-n} u$ for $u \in F_{(m,n)}$. Let $\mathbf{d}^L$ and $\mathbf{d}^R$ be the grading operators such that $\mathbf{d}^L u = mu$ and $\mathbf{d}^R u = nu$ for $u \in F_{(m,n)}$. We call m the left weight of $u \in F_{(m,n)}$ and n the right weight of u, and denote them as $\mathrm{wt}\ {}^L u$ and $\mathrm{wt}\ {}^R u$ respectively. We also define $\mathrm{wt}\ u := \mathrm{wt}\ {}^L u + \mathrm{wt}\ {}^R u$ which is called total weight. These two grading operators can also be obtained from $\nu_1((\mathbf{0}, (a, \mathbf{0})))$ as follow:

$$\begin{aligned} \mathbf{d}^L &= \left.\frac{\partial}{\partial a}\right|_{a=1} \nu_1((\mathbf{0}, (a, \mathbf{0}))), \\ \mathbf{d}^R &= \left.\frac{\partial}{\partial \bar{a}}\right|_{a=1} \nu_1((\mathbf{0}, (a, \mathbf{0}))). \end{aligned} \tag{1.19}$$

Conversely, using $\mathbf{d}^L$ and $\mathbf{d}^R$, we can also express the action of $\nu_1((\mathbf{0}, (a, \mathbf{0})))$ on F as

$$\nu_1((\mathbf{0}, (a, \mathbf{0}))) = a^{-\mathbf{d}^L} \bar{a}^{-\mathbf{d}^R}. \tag{1.20}$$

By the definition of smooth function on K, the correlation functions are linear combination of $\prod_i (a_0^{(i)})^{r_i} \overline{a_0^{(i)}}^{s_i}$ where $r_i, s_i \in \mathbb{R}$. Therefore, by (1.20), F must be $\mathbb{R}\times\mathbb{R}$-graded.

Remark 1.10. When we define the smoothness, we only allow the real powers of $a_0^{(i)}$ and $\bar{a}_0^{(i)}$ to appear exactly for the sake of restricting F to a $\mathbb{R}\times\mathbb{R}$-grading instead of a $\mathbb{C}\otimes\mathbb{C}$-grading. This is a physically natural condition because the operator $\mathbf{d}^L + \mathbf{d}^R$ is the Hamiltonian in physics. It can only has positive real eigenvalues in a unitary theory. If the total weights are real, then by Lemma 1.9, both the left weights and the right weights must be real as well.

The image of $(\mathbf{0}) \in \widehat{K}(0)$ under the morphism ν gives arise to a very special element $\mathbf{1} \in \overline{F}$. Namely,

$$\mathbf{1} := \nu_0((\mathbf{0})). \tag{1.21}$$

We call $\mathbf{1}$ the vacuum state. We have $(\mathbf{0}, (a, \mathbf{0}))_1\infty_0(\mathbf{0}) = (\mathbf{0})$ for all $a \in \mathbb{C}^\times$. This implies $\mathbf{1} \in F_{(0,0)}$. Since $\widehat{K}(0) = \{(\mathbf{0})\}$, W is just $\mathbb{C}\mathbf{1}$.

Let $P(z) = (z; \mathbf{0}, (1, \mathbf{0}), (1, \mathbf{0})) \in \widehat{K}(2)$. We denote the linear map

$$\nu_2(P(z)) : F \otimes F \to \overline{F}$$

as $\mathbb{Y}(\cdot; z, \bar{z})\cdot$. Here we use both z and its complex conjugation $\bar{z}$ in $\mathbb{Y}$ to emphasis that ν is not holomorphic in general. Because

$$P(z)_1\infty_0(\mathbf{0}) = (\mathbf{0}, (1, \mathbf{0})),$$

we have

$$\mathbb{Y}(\mathbf{1}; z, \bar{z}) = I_F. \tag{1.22}$$

The equation (1.22) is called *vacuum property*. Moreover, since

$$\lim_{z\to 0} P(z)_2\infty_0(\mathbf{0}) = (\mathbf{0}, (1, \mathbf{0})) = I_K$$

and ν maps identity to identity, we have

$$\lim_{z\to 0} \mathbb{Y}(\mathbf{1}; z, \bar{z}) = I_F, \tag{1.23}$$

which is called *creation property*.

Proposition 1.11. *For $a, z \in \mathbb{C}^\times$ and $u \in F$, we have*

$$a^{\mathbf{d}^L}\bar{a}^{\mathbf{d}^R}\mathbb{Y}(u; z, \bar{z})a^{-\mathbf{d}^L}\bar{a}^{-\mathbf{d}^R} = \mathbb{Y}(a^{\mathbf{d}^L}\bar{a}^{\mathbf{d}^R}u; az, \bar{a}\bar{z}). \tag{1.24}$$

Proof. First, for $u, v \in F$, we have

$$\begin{aligned}
&\nu_2((z; \mathbf{0}, (a_1, \mathbf{0}), (a_2, \mathbf{0}))(u \otimes v) \\
&\quad = \nu_2((P(z)_1\infty_0(\mathbf{0}, (a_1, \mathbf{0})))_2\infty_0(\mathbf{0}, (a_2, \mathbf{0})))(u \otimes v) \\
&\quad = (\nu_2(P(z))\ {}_1 *_0 \nu_1((\mathbf{0}, (a_1, \mathbf{0}))))_2 *_0 \nu_1((\mathbf{0}, (a_2, \mathbf{0})))(u \otimes v) \\
&\quad = \mathbb{Y}(a_1^{-\mathbf{d}^L} \bar{a}_1^{-\mathbf{d}^R} u; z, \bar{z})\, a_2^{-\mathbf{d}^L} \bar{a}_2^{-\mathbf{d}^R} v. \qquad (1.25)
\end{aligned}$$

On the other hand, we also have

$$(\mathbf{0}, (a^{-1}, \mathbf{0}))_1\infty_0 P(z) = (az; \mathbf{0}, (a^{-1}, \mathbf{0}), (a^{-1}, \mathbf{0})). \qquad (1.26)$$

Using (1.25), the image of (1.26) under the morphism ν gives

$$a^{\mathbf{d}^L} \bar{a}^{\mathbf{d}^R} \mathbb{Y}(u; z, \bar{z}) = \mathbb{Y}(a^{\mathbf{d}^L} \bar{a}^{\mathbf{d}^R} u; az, \bar{a}\bar{z})\, a^{\mathbf{d}^L} \bar{a}^{\mathbf{d}^R},$$

which implies (1.24). ∎

Corollary 1.12. *For $z \in \mathbb{C}^\times$ and $u \in F$, we have*

$$\begin{aligned}
\left[\mathbf{d}^L, \mathbb{Y}(u; z, \bar{z})\right] &= z\frac{\partial}{\partial z}\mathbb{Y}(u; z, \bar{z}) + \mathbb{Y}(\mathbf{d}^L u; z, \bar{z}) \qquad (1.27) \\
\left[\mathbf{d}^R, \mathbb{Y}(u; z, \bar{z})\right] &= \bar{z}\frac{\partial}{\partial \bar{z}}\mathbb{Y}(u; z, \bar{z}) + \mathbb{Y}(\mathbf{d}^R u; z, \bar{z}), \qquad (1.28)
\end{aligned}$$

Proof. Replace a and $\bar{a}$ in (1.24) by e^s and $e^{\bar{s}}$ respectively, and then take derivative of $\frac{\partial}{\partial s}|_{s=0}$ and $\frac{\partial}{\partial \bar{s}}|_{s=0}$ on both sides of (1.24), we immediate obtain (1.27) and (1.28). ∎

We further define two operators $D^L, D^R \in \mathrm{Hom}(F, \overline{F})$ to be

$$\begin{aligned}
D^L &:= -\frac{\partial}{\partial a}\bigg|_{a=0} \nu_1((A(a; 1), (1, \mathbf{0})), \\
D^R &:= -\frac{\partial}{\partial \bar{a}}\bigg|_{a=0} \nu_1((A(a; 1), (1, \mathbf{0})). \qquad (1.29)
\end{aligned}$$

Proposition 1.13. $[\mathbf{d}^L, D^L] = D^L, [\mathbf{d}^R, D^R] = D^R$, $[\mathbf{d}^L, D^R] = [\mathbf{d}^R, D^L] = 0$ *and* $[D^L, D^R] = 0$. *As a consequence* $D^R, D^L \in \mathrm{End}\ F$ *and have weights* $(1, 0)$ *and* $(0, 1)$ *respectively.*

Proof. First we prove $[\mathbf{d}^L, D^L] = D^L$. Consider the sewing identity:

$$(A(a; 1), (a_1, \mathbf{0}))_1\infty_0(A(b; 1), (b_1, \mathbf{0})) = (A(a + b/a_1; 1), (a_1 b_1, \mathbf{0})).$$

It implies the following identity:

$$
\begin{aligned}
&\langle w', (\nu_1((A(a;1),(1,\mathbf{0})))\ {}_1*_0\ \nu_1((\mathbf{0},(a_1,\mathbf{0})))\\
&\qquad {}_1*_0\ (\nu_1((A(b;1),(1,\mathbf{0})))\ {}_1*_0\ \nu_1((\mathbf{0},(b_1,\mathbf{0})))(w)\rangle\\
&= \langle w', \nu_1((A(a+b/a_1;1),(1,\mathbf{0})))\ {}_1*_0\ \nu_1((\mathbf{0},(a_1b_1,\mathbf{0})))(w)\rangle
\end{aligned}
\tag{1.30}
$$

for all $w \in F, w' \in F'$. Apply

$$
\left(-\frac{\partial}{\partial a_1}\right)\bigg|_{a_1=1}\left(-\frac{\partial}{\partial b}\right)\bigg|_{b=0,a=0,b_1=1} - \left(-\frac{\partial}{\partial a}\right)\bigg|_{a=0}\left(-\frac{\partial}{\partial b_1}\right)\bigg|_{b=0,a_1=1,b_1=1}
$$

to both sides of equation (1.30). The left hand side of (1.30) gives

$$
\sum_{(m,n)\in I}\langle w', (\mathbf{d}^L P_{(m,n)} D^L - D^L\mathbf{d}^L)w\rangle,
$$

while the right hand of (1.30) gives

$$
\begin{aligned}
&\left(-\frac{\partial}{\partial a_1}\right)\bigg|_{a_1=1}\left(-\frac{\partial}{\partial b}\right)\bigg|_{b=0}\langle w', \nu_1((A(b/a_1;1),(1,\mathbf{0})))a_1^{-\mathbf{d}^L}\bar{a}_1^{-\mathbf{d}^R}w\rangle\\
&\qquad - \left(-\frac{\partial}{\partial a}\right)\bigg|_{a=0}\left(-\frac{\partial}{\partial b_1}\right)\bigg|_{b_1=1}\langle w', \nu_1((A(a;1),(1,\mathbf{0})))b_1^{-\mathbf{d}^L}\bar{b}_1^{-\mathbf{d}^R}w\rangle\\
&= \left(-\frac{\partial}{\partial a_1}\right)\langle w', D^L a_1^{-\mathbf{d}^L-1}\bar{a}_1^{-\mathbf{d}^R}w\rangle\bigg|_{a_1=1} - \langle w', D^L\mathbf{d}^L w\rangle\\
&= \langle w', D^L w\rangle.
\end{aligned}
\tag{1.31}
$$

Compare above two equations. We obtain $[\mathbf{d}^L, D^L] = D^L$.

Second we show $[\mathbf{d}^R, D^L] = 0$. Apply

$$
\left(-\frac{\partial}{\partial \bar{a}_1}\right)\bigg|_{a_1=1}\left(-\frac{\partial}{\partial b}\right)\bigg|_{b=0,a=0,b_1=1} - \left(-\frac{\partial}{\partial a}\right)\bigg|_{a=0}\left(-\frac{\partial}{\partial \bar{b}_1}\right)\bigg|_{b=0,a_1=1,b_1=1}
$$

to both sides of (1.30). The left hand side of (1.30) gives

$$
\sum_{(m,n)\in I}\langle w', (\mathbf{d}^R P_{(m,n)} D^L - D^L\mathbf{d}^R)w\rangle,
$$

while the right hand of (1.30) gives

$$
\left(-\frac{\partial}{\partial a_1}\right)\bigg|_{a_1=1}\langle w', D^L\frac{1}{a_1}a_1^{-\mathbf{d}^L}\bar{a}_1^{-\mathbf{d}^R}w\rangle - \langle w', D^L\mathbf{d}^R w\rangle = 0.
$$

Compare above two equations. It is clear that $[\mathbf{d}^R, D^L] = 0$.

Combining above two results, we conclude that $D^L \in \mathrm{End}V$ and has weight $(1,0)$. Similarly, we can show $[\mathbf{d}^L, D^R] = 0$ and $[\mathbf{d}^R, D^R] = D^R$. Therefore, D^R is in $\mathrm{End}V$ as well and has weight $(0,1)$.

Next we show that $[D^L, D^R] = 0$. The identity

$$\begin{aligned}(A(a+b;1),(1,\mathbf{0})) &= (A(a;1),(1,\mathbf{0}))_1\infty_0(A(b;1),(1,\mathbf{0})) \\ &= (A(b;1),(1,\mathbf{0}))_1\infty_0(A(a;1),(1,\mathbf{0}))\end{aligned} \tag{1.32}$$

in $\widehat{K}$ implies that for $w', w \in F$,

$$\begin{aligned}&\langle w', \nu_1((A(a;1),(1,\mathbf{0})))_1 *_0 \nu_1((A(b;1),(1,\mathbf{0})))w\rangle \\ &\quad= \langle w', \nu_1((A(b;1),(1,\mathbf{0})))_1 *_0 \nu_1((A(a;1),(1,\mathbf{0})))w\rangle.\end{aligned} \tag{1.33}$$

Apply

$$\left(-\frac{\partial}{\partial a}\right)\bigg|_{a=0}\left(-\frac{\partial}{\partial \bar{b}}\right)\bigg|_{b=0}$$

to both sides of (1.33). We obtain $[D^L, D^R] = 0$. ∎

Proposition 1.14. *The following D^L- and D^R-bracket-derivative formula*

$$\left[D^L, \mathbb{Y}(u;z,\bar{z})\right] = \mathbb{Y}(D^L u;z,\bar{z}) = \frac{\partial}{\partial z}\mathbb{Y}(u;z,\bar{z}), \tag{1.34}$$

$$\left[D^R, \mathbb{Y}(u;z,\bar{z})\right] = \mathbb{Y}(D^R u;z,\bar{z}) = \frac{\partial}{\partial \bar{z}}\mathbb{Y}(u;z,\bar{z}), \tag{1.35}$$

holds for all $u \in F$ and $z \in \mathbb{C}^\times$.

Proof. We have the following sewing identity:

$$P(a)_1\infty_0(A(-z;1),(1,\mathbf{0})) = P(a+z). \tag{1.36}$$

Notice that

$$\mathbb{Y}(D^L u, z, \bar{z}) = \frac{\partial}{\partial z}\bigg|_{z=0}\nu_2(P(a)_1\infty_0(A(-z;1),(1,\mathbf{0}))), \tag{1.37}$$

and

$$\frac{\partial}{\partial z}\bigg|_{z=0}\mathbb{Y}(u, z+a, \bar{z}+\bar{a}) = \frac{\partial}{\partial z}\bigg|_{z=0}\nu_2(P(z+a)). \tag{1.38}$$

Hence it is clear that

$$\mathbb{Y}(D^L u, z, \bar{z}) = \frac{\partial}{\partial z}\mathbb{Y}(u, z, \bar{z}). \tag{1.39}$$

Similarly, we can show that the D^L-bracket property follows from the following sewing identity:

$$(A(-z;1),(1,\mathbf{0}))_1\infty_0(P(a)_2\infty_0(A(z;1),(1,\mathbf{0}))) = P(z+a).$$

We omit the detail.

The proof of D^R-bracket and D^R-derivative properties is similar. ∎

Proposition 1.15. *A smooth $\widehat{K}$-algebra (F,ν) has a natural structure of grading-restricted $\mathbb{R}\times\mathbb{R}$-graded full field algebra.*

Proof. We have already proved the $\mathbb{R}\times\mathbb{R}$-grading. The grading-restriction condition on F are automatic by the definition of smooth $\widehat{K}$-algebra.

The correlation function maps:

$$m_n(u_1,\dots,u_n;z_1,\bar{z},\dots,z_n,\bar{z}_n) \tag{1.40}$$

can be defined as

$$\nu_{n+1}((z_1,\dots,z_n;\mathbf{0},(1,\mathbf{0}),\dots,(1,\mathbf{0})))(u_1\otimes\cdots\otimes u_n\otimes\mathbf{1}) \tag{1.41}$$

if $z_i\neq 0$, and as

$$\nu_n((z_1,\dots,\widehat{z_i},\dots,z_n;\mathbf{0},(1,\mathbf{0}),\dots,(1,\mathbf{0})))(u_1\otimes\dots\widehat{u_i}\cdots\otimes u_n\otimes u_i) \tag{1.42}$$

if $z_i=0$. By the definition of $\mathbf{1}$ given in (1.21) and the fact that

$$\begin{aligned}\lim_{z_i\to 0}&(z_1,\dots,z_n;\mathbf{0},(1,\mathbf{0}),\dots,(1,\mathbf{0}))_i\infty_0(\mathbf{0})\\ &= (z_1,\dots,\widehat{z_i},\dots,z_n;\mathbf{0},(1,\mathbf{0}),\dots,(1,\mathbf{0})),\end{aligned} \tag{1.43}$$

we see that (1.40) are smooth with respect to $z_1,\dots,z_n$. Moreover, we also have $m_1(u;0,0)=\nu_1((1,\mathbf{0}))(u)=I_F(u)=u$. Another identity property of full field algebra:

$$\begin{aligned}m_{n+1}&(u_1,\dots,u_n,\mathbf{1};z_1,\bar{z}_1,\dots,z_{n+1},\bar{z}_{n+1})\\ &= m_n(u_1,\dots,u_n;z_1,\bar{z}_1,\dots,z_n,\bar{z}_n).\end{aligned} \tag{1.44}$$

also holds because the following identity:

$$Q_i\infty_0(\mathbf{0}) = (z_1,\dots,\widehat{z_i},\dots,z_{n-1};A(a;1),(a_0^{(1)},\mathbf{0}),\dots,\widehat{(a_0^{(i)},\mathbf{0})},\dots,(a_0^{(n)},\mathbf{0})),$$

where Q is of form (1.6), holds in $\widehat{K}$.

For the convergence property of $\mathbb{R} \times \mathbb{R}$-graded full field algebra, we use the weaker version of convergence property discussed in the remark 1.2 in [HK2]. Then it is clear that this weaker version of convergence property is automatically true by the definition of algebra over partial operad. The permutation axiom of full field algebra follows automatically from that of partial operad.

The single-valuedness property follows from Lemma 1.9. The rest of axioms follows from (1.24), (1.27), (1.28), (1.34) and (1.35). ∎

Lemma 1.16. *Given a grading-restricted* $\mathbb{R} \times \mathbb{R}$*-graded full field algebra* F*, we have*

$$m_1(u; z, \bar{z}) = e^{zD^L + \bar{z}D^R} u, \tag{1.45}$$

for $u \in F$*, and*

$$\begin{aligned} & e^{aD^L + \bar{a}D^R} m_n(u_1, \ldots, u_n; z_1, \bar{z}_1, \ldots, z_n, \bar{z}_n) \\ & \quad = m_n(u_1, \ldots, u_n; z_1 + a, \overline{z_1 + a}, \ldots, z_n + a, \overline{z_n + a}). \end{aligned} \tag{1.46}$$

for all $(z_1, \ldots, z_n) \in M^n$ *and* $a \in \mathbb{C}$*.*

Proof. Recall the following formula

$$\mathbb{Y}(u; z, \bar{z})\mathbf{1} = e^{zD^L + \bar{z}D^R} u$$

which is proved in [HK2]. Then we have

$$m_1(u; z, \bar{z}) = m_2(u, \mathbf{1}, z, \bar{z}, 0, 0) = \mathbb{Y}(u; z, \bar{z})\mathbf{1} = e^{zD^L + \bar{z}D^R} u.$$

Thus we have proved (1.45). (1.46) can be proved as follow:

$$\begin{aligned} & e^{aD^L + \bar{a}D^R} m_n(u_1, \ldots, u_n; z_1, \bar{z}_1, \ldots, z_n, \bar{z}_n) \\ & \quad = m_1(m_n(u_1, \ldots, u_n; z_1, \bar{z}_1, \ldots, z_n, \bar{z}_n), a) \\ & \quad = m_n(u_1, \ldots, u_n; z_1 + a, \overline{z_1 + a}, \ldots, z_n + a, \overline{z_n + a}). \end{aligned} \tag{1.47}$$

∎for all $(z_1, \ldots, z_n) \in M^n$ and $a \in \mathbb{C}$.

Theorem 1.17. *The category of grading-restricted* $\mathbb{R} \times \mathbb{R}$*-graded full field algebras is isomorphic to the category of smooth* $\widehat{K}$*-algebras.*

Proof. The proof is similar to that of Theorem 5.4.5 in [H3]. Our case is much simpler.

Given a grading-restricted $\mathbb{R} \times \mathbb{R}$-graded full field algebra

$$(F, m, \mathbf{1}, \mathbf{d}^L, \mathbf{d}^R, D^L, D^R).$$

We define a map ν from $\widehat{K}$ to $\mathcal{H}_{F\mathbb{C}\mathbf{1}}^{\mathbb{C}^\times}$ as follow. We define

$$\nu_0((\mathbf{0})) := \mathbf{1}. \tag{1.48}$$

For $n > 0$, $u_i \in F, a, z_i \in \mathbb{C}, i = 1, \dots, n$ and $Q \in \widehat{K}(n)$ as (1.6), we define

$$\begin{aligned}\nu(Q)(u_1 \otimes \cdots \otimes u_n) := e^{-aD^L - \bar{a}D^R} m_n((a_0^{(1)})^{-\mathbf{d}^L}(\overline{a_0^{(1)}})^{-\mathbf{d}^R}u_1, \dots, \\ (a_0^{(n)})^{-\mathbf{d}^L}(\overline{a_0^{(n)}})^{-\mathbf{d}^R}u_n; z_1, \overline{z_1}, \dots, z_{n-1}, \overline{z_{n-1}}, 0, 0).\end{aligned} \tag{1.49}$$

In particular,

$$\begin{aligned}\nu_1((\mathbf{0}, (a, \mathbf{0})) &= m_1((a_0^{(1)})^{-\mathbf{d}^L}(\overline{a_0^{(1)}})^{-\mathbf{d}^R}u_1; 0, 0) \\ &= (a_0^{(1)})^{-\mathbf{d}^L}(\overline{a_0^{(1)}})^{-\mathbf{d}^R}u_1.\end{aligned} \tag{1.50}$$

This gives the representation of rescaling group $\mathbb{C}^\times$ according to the $\mathbb{R}\times\mathbb{R}$-grading of F. The $\mathbb{R}\times\mathbb{R}$-grading, together with the single-valuedness property, guarantees that the grading is in the set $\{(m,n)|m,n \in \mathbb{R}, m-n \in \mathbb{Z}\}$ as required by the axioms of smooth $\widehat{K}$-algebra. The grading restriction conditions and the smoothness of all correlation functions are all automatically satisfied.

Let $P \in \widehat{K}(m)$ and $Q \in \widehat{K}(n)$ given as follow:

$$\begin{aligned}P &= (z_1, \dots, z_{m-1}; A(a; 1), (a_0^{(1)}, \mathbf{0}), \dots, (a_0^{(m)}, \mathbf{0})) \\ Q &= (\xi_1, \dots, \xi_{n-1}; A(b; 1), (b_0^{(1)}, \mathbf{0}), \dots, (b_0^{(n)}, \mathbf{0})).\end{aligned} \tag{1.51}$$

We assume that $P_i\infty_0 Q$ exists. Then we have

$$\begin{aligned}P_i\infty_0 Q = (z_1, \dots, z_{i-1}, \frac{\xi_1 - a}{a_0^{(i)}} + z_i, \dots, \frac{\xi_m - a}{a_0^{(i)}} + z_i, z_{i+1}, \dots, z_n; \\ A(a; 1), (a_0^{(1)}, \mathbf{0}), \dots, (a_0^{(i-1)}, \mathbf{0}), (a_0^{(i)}b_0^{(1)}, \mathbf{0}), \dots, (a_0^{(i)}b_0^{(m)}, \mathbf{0}), \\ (a_0^{(i-1)}, \mathbf{0}), \dots, (a_0^{(n)}, \mathbf{0})\ ).\end{aligned} \tag{1.52}$$

By (1.46), we have

$$
\begin{aligned}
&\nu_m(P)_i *_0 \nu_n(Q)\\
&\quad= \sum_{k,l\in\mathbb{R}} e^{-aD^L-\bar{a}D^R} m_m((a_0^{(1)})^{-\mathbf{d}^L}\overline{(a_0^{(1)})}^{-\mathbf{d}^R}u_1,\dots,\\
&\qquad\qquad (a_0^{(i)})^{-\mathbf{d}^L}\overline{(a_0^{(i)})}^{-\mathbf{d}^R}P_{k,l}m_n((b_0^{(1)})^{-\mathbf{d}^L}\overline{(b_0^{(1)})}^{-\mathbf{d}^R}v_1,\dots,(b_0^{(n)})^{-\mathbf{d}^L}\overline{(b_0^{(n)})}^{-\mathbf{d}^R}v_n;\\
&\qquad\qquad\qquad \xi_1-b,\overline{\xi_1-b},\dots,\xi_m-b,\overline{\xi_m-b}),\\
&\qquad\qquad\qquad\qquad \dots,(a_0^{(m)})^{-\mathbf{d}^L}\overline{(a_0^{(m)})}^{-\mathbf{d}^R}u_m;z_1,\bar{z}_1,\dots,z_m,\bar{z}_m)\\
&\quad= \sum_{k,l\in\mathbb{R}} e^{-aD^L-\bar{a}D^R} m_m((a_0^{(1)})^{-\mathbf{d}^L}\overline{(a_0^{(1)})}^{-\mathbf{d}^R}u_1,\dots,\\
&\qquad\qquad P_{k,l}m_n((a_0^{(i)}b_0^{(1)})^{-\mathbf{d}^L}\overline{(a_0^{(i)}b_0^{(1)})}^{-\mathbf{d}^R}v_1,\dots,(a_0^{(i)}b_0^{(n)})^{-\mathbf{d}^L}\overline{(a_0^{(i)}b_0^{(n)})}^{-\mathbf{d}^R}v_n;\\
&\qquad\qquad (\xi_1-b)/a_0^{(i)},\overline{(\xi_1-b)/a_0^{(i)}},\dots,(\xi_m-b)/a_0^{(i)},\overline{(\xi_m-b)/a_0^{(i)}}),\\
&\qquad\qquad\qquad\qquad \dots,(a_0^{(m)})^{-\mathbf{d}^L}\overline{(a_0^{(m)})}^{-\mathbf{d}^R}u_m;z_1,\bar{z}_1,\dots,z_m,\bar{z}_m)\\
&\quad= e^{-aD^L-\bar{a}D^R} m_{m+n-1}((a_0^{(1)})^{-\mathbf{d}^L}\overline{(a_0^{(1)})}^{-\mathbf{d}^R}u_1,\dots,(a_0^{(i)}b_0^{(1)})^{-\mathbf{d}^L}\overline{(a_0^{(i)}b_0^{(1)})}^{-\mathbf{d}^R}v_1,\\
&\qquad\qquad \dots,(b_0^{(n)})^{-\mathbf{d}^L}\overline{(b_0^{(n)})}^{-\mathbf{d}^R}v_n,\dots,(a_0^{(m)})^{-\mathbf{d}^L}\overline{(a_0^{(m)})}^{-\mathbf{d}^R}u_m;z_1,\bar{z}_1,\dots,\\
&\qquad\qquad \frac{\xi_1-b}{a_0^{(i)}}+z_i,\overline{\frac{\xi_1-b}{a_0^{(i)}}+z_i},\dots,\frac{\xi_m-b}{a_0^{(i)}}+z_i,\overline{\frac{\xi_m-b}{a_0^{(i)}}+z_i},\dots,z_m,\bar{z}_m)\\
&\quad= \nu_{m+n-1}(P_i\infty_0 Q).
\end{aligned}
$$

We have checked all the axioms of smooth $\widehat{K}$-algebra. Therefore the triple $(F,\mathbb{C}\mathbf{1},\nu)$ gives a smooth $\widehat{K}$-algebra. Thus we obtain a functor from the category of $\mathbb{R}\times\mathbb{R}$-graded full field algebras to that of smooth $\widehat{K}$-algebras.

On the other hand, by Proposition 1.15, we also have a functor from the category of smooth $\widehat{K}$-algebras to the category of of grading-restricted $\mathbb{R}\times\mathbb{R}$-graded full field algebras.

Now we show that these two functors give isomorphisms between two categories. Let $(F,\tilde{m},\tilde{\mathbf{1}},\tilde{D}^L,\tilde{D}^R)$ be full field algebra obtained according to Proposition (1.15) from a smooth $\widehat{K}$-algebra, which is further obtained from a grading-restricted $\mathbb{R}\times\mathbb{R}$-graded full field algebra $(F,m,\mathbf{1},D^L,D^R)$ according to (1.49). First $\tilde{\mathbf{1}}=\mathbf{1}$ is obvious by our constructions (1.48) and (1.21).

By construction (1.41) and (1.42), for $z_i \neq 0, i = 1, \dots, n$, we have

$$
\begin{aligned}
&\tilde{m}_n(u_1, \dots u_n; z_1, \bar{z}_1, \dots, z_n, \bar{z}_n) \\
&\qquad = \nu_{n+1}((z_1, \dots, z_n; \mathbf{0}, (1, \mathbf{0}), \dots, (1, \mathbf{0})))(u_1 \otimes u_n \otimes \mathbf{1}) \\
&\qquad = m_{n+1}(u_1, \dots, u_n, \mathbf{1}; z_1, \bar{z}_1, \dots, z_n, \bar{z}_n, 0, 0) \\
&\qquad = m_n(u_1, \dots, u_n; z_1, \bar{z}_1, \dots, z_n, \bar{z}_n). \qquad (1.53)
\end{aligned}
$$

The cases when $z_i = 0$ for some $i = 1, \dots, n$ follows from smoothness. Therefore $\tilde{m} = m$.

By 1.29, we also have

$$
\begin{aligned}
\tilde{D}^L &= -\frac{\partial}{\partial a}\bigg|_{a=0} v_1((A(a;1), (1, \mathbf{0})) = -\frac{\partial}{\partial a}\bigg|_{a=0} e^{-aD^L - \bar{a}D^R} = D^L, \\
\tilde{D}^R &= -\frac{\partial}{\partial \bar{a}}\bigg|_{a=0} v_1((A(a;1), (1, \mathbf{0})) = -\frac{\partial}{\partial a}\bigg|_{a=0} e^{-aD^L - \bar{a}D^R} = D^R. \qquad (1.54)
\end{aligned}
$$

The coincidence of two gradings is also obvious. Therefore, we have proved that one way of composing two functors gives the identity functor on the category of grading restricted $\mathbb{R} \times \mathbb{R}$-graded full field algebra.

Similarly, one can show that the opposite way of composing these two functors also gives the identity functor on the category of smooth $\widehat{K}$-algebras. ∎

Let us now turn our attention to the conformal case. Using the result of Huang [H3], it is easy to see that the tensor product $V^L \otimes V^R$ of two vertex operator algebras V^L and V^R with central charge c^L and c^R respectively, has a canonical structure of smooth $\tilde{K}^{c^L} \otimes \overline{\tilde{K}^{\overline{c^R}}}$-algebras. If a smooth $\tilde{K}^{c^L} \otimes \overline{\tilde{K}^{\overline{c^R}}}$-algebras (F, W, ν) is equipped with an embedding $\rho : V^L \otimes V^R \hookrightarrow F$ as a morphism of smooth $\tilde{K}^{c^L} \otimes \overline{\tilde{K}^{\overline{c^R}}}$-algebra, we call it a smooth $\tilde{K}^{c^L} \otimes \overline{\tilde{K}^{\overline{c^R}}}$-algebras over $V^L \otimes V^R$ and denote it as (F, ν, ρ).

We consider a conformal full field algebra F over $V^L \otimes V^R$, denoted as a triple (F, m, ρ) where $\rho : V^L \otimes V^R \hookrightarrow F$ is a monomorphism of conformal full field algebra. By Assumption 0.2, V^L, V^R satisfy the conditions in Theorem 0.1. In this case, the products and iterates of intertwining operators of V^L, V^R satisfies very nice convergence and analytic extension properties [H7]. As a consequence, m_n also have certain analytic extension properties [HK2]. Namely, for $u_1, \dots, u_n \in F, w' \in F'$,

$$\langle w', \mathbb{Y}(u_1; z_1, \zeta_1) \dots \mathbb{Y}(u_n; z_n, \zeta_n)\mathbf{1}\rangle$$

is absolutely convergent when $|z_1| > \cdots > |z_n| > 0$ and $|\zeta_1| > \cdots > |\zeta_n| > 0$ and can be analytically extended to a multivalued analytic function in the region given by $z_i \neq z_j$, $z_i \neq 0$, $\zeta_i \neq \zeta_j$, $\zeta_i \neq 0$. We use

$$E(m)_n(w', u_1, \dots, u_n; z_1, \zeta_1, \dots, z_n, \zeta_n)$$

to denote this function.

For the simplicity of notation, we will not distinguish $L^L(n)$ with $L^L(n)\otimes 1$ and $L^R(n)$ with $1\otimes L^R(n)$ in this work.

Proposition 1.18. A conformal full field algebra over $V^L\otimes V^R$, (F, m, ρ), has a canonical structure of smooth $\tilde{K}^{c^L}\otimes\overline{\tilde{K}^{\overline{c^R}}}$-algebras over $V^L\otimes V^R$.

Proof. We first define a map $\nu_n : \tilde{K}^{c^L}\otimes\overline{\tilde{K}^{\overline{c^R}}}(n)\to \mathrm{Hom}(F^{\otimes n},\overline{F})$ for $n\in\mathbb{N}$. For $(A^{(0)})\in K(0)$, we define

$$\nu_0(\psi\otimes\bar{\psi}((A^{(0)}))) := e^{-L^L_-(A^{(0)})-L^R_-(A^{(0)})}\mathbf{1}. \tag{1.55}$$

We will define other cases indirectly. Recall that we choose the canonical representative of $Q\in K(n)$ to have 0-th puncture at ∞, n-th puncture at 0 and satisfy the condition (1.1). We relax the condition a little. We call all the representatives of an equivalent class Q with 0-th puncture sitting at ∞ and satisfying the condition (1.1) as quasi-canonical representatives. The set of quasi-canonical representatives are clearly parametrized by z_n the location of the n-th puncture. When $z_n=0$, it is nothing but the canonical representative. In order to distinguish the notation from those of canonical representative (1.2), we use

$$((\infty;1,A^{(0)});(z_1;a_0^{(0)},A^{(1)}),\dots,(z_n;a_0^{(n)},A^{(n)})) \tag{1.56}$$

to denote a quasi-canonical representative of a sphere with tubes with punctures at ∞ and $z_1,\dots,z_n\in\mathbb{C}, z_i\neq z_j$ with local coordinate maps giving by (1.3) and (1.4) respectively, and $z_n\neq 0$ in general.

For $\tilde{Q}=\lambda\psi\otimes\bar{\psi}(Q)\in\tilde{K}^{c^L}\otimes\overline{\tilde{K}^{\overline{c^R}}}(n), n\geq 1$ where $Q\in K(n)$ given by a quasi-canonical representative of form (1.56), we define ν_n, the restriction of ν on $\tilde{K}^{c^L}\otimes\overline{\tilde{K}^{\overline{c^R}}}(n)$, as follow:

$$\begin{aligned}\nu_n(\tilde{Q}) := \lambda e^{-L^L_-(A^{(0)})-L^R_-(\overline{A^{(0)}})}m_n(&e^{-L^L_+(A^{(1)})-L^R_+(\overline{A^{(1)}})}(a_0^{(1)})^{-L^L(0)}\overline{a_0^{(1)}}^{-L^R(0)}u_1,\dots,\\ &e^{-L^L_+(A^{(n)})-L^R_+(\overline{A^{(n)}})}(a_0^{(1)})^{-L^L(0)}\overline{a_0^{(1)}}^{-L^R(0)}u_n;z_1,\bar{z}_1,\dots,z_{n-1},\bar{z}_{n-1},z_n,\bar{z}_n),\end{aligned} \tag{1.57}$$

where

$$L^L_\pm(A)=\sum_{j=1}^{\infty}A_jL^L(\pm j),\quad L^R_\pm(A)=\sum_{j=1}^{\infty}A_jL^R(\pm j)$$

for $A\in\prod_{n\in\mathbb{N}}\mathbb{C}$.

Of course, we have to show that ν is well-defined on $\tilde{K}^{c^L} \otimes \overline{\tilde{K}^{\overline{c^R}}}$. It follows immediately from the identity (1.46) and Huang's proof of case 1 (as we will recall later) in the Proof of the sewing axiom in Proposition 5.4.1 in [H3][2].

The advantage of this way of defining ν is that we see immdiately that ν_n is S_n-equivariant because of the permutation axiom of full field algebra. Another advantage of this definition is that we can always assume that $z_i, i = 1, \dots, n$ have distinct absolute values. Indeed, if for the quasi-canonical representative we start with there are some z_k having same absolute values, then we can always choose another quasi-canonical representative such that $|z_i| \neq |z_j|$ for $i \neq j$. This fact will be useful later.

We will show that such defined (F, ν, ρ) gives a smooth $\tilde{K}^{c^L} \otimes \overline{\tilde{K}^{\overline{c^R}}}$-algebra over $V^L \otimes V^R$.

First of all, by the construction (1.57), we have

$$\nu_1((\mathbf{0}, (a, \mathbf{0}))) = a^{-L^L(0)} \bar{a}^{-L^R(0)}.$$

Since $L^L(0)$ and $L^R(0)$ are two grading operators on F, F is exactly graded by inequivalent irreducible representations of the rescaling group $\mathbb{C}^\times$. The grading restriction conditions are automatic.

From the construction (1.57), $\nu_1((\mathbf{0}, (1, \mathbf{0})))(u) = m_1(u; 0, 0) = u$. Thus ν maps the identity I_K to the identity I_F. We also see that $\nu_0(\tilde{K}^{c^L} \otimes \overline{\tilde{K}^{\overline{c^R}}}(0))$ is the subspace of F generated by the actions of $\{L^L(n), L^R(n)\}_{n \in \mathbb{Z}}$ on $\mathbf{1}^L \otimes \mathbf{1}^R$. We denote this space as $\langle \omega^L, \omega^R \rangle$.

Notice that (1.57) is completely compactible with the smooth $\tilde{K}^{c^L} \otimes \overline{\tilde{K}^{\overline{c^R}}}$-algebra structure on $V^L \otimes V^R$. In other words, if $(F, \langle \omega^L, \omega^R \rangle, \nu)$ indeed gives a smooth $\tilde{K}^{c^L} \otimes \overline{\tilde{K}^{\overline{c^R}}}$-algebra, then ρ must be an embedding of $V^L \otimes V^R$ into F as a smooth $\tilde{K}^{c^L} \otimes \overline{\tilde{K}^{\overline{c^R}}}$-algebra monomorphism.

It remains to prove the sewing properties of ν, i.e.

$$\nu_{m+n-1}(\tilde{P}\, {}_i\widetilde{\infty}_0\, \tilde{Q}) = \nu_m(\tilde{P})\, {}_i *_0\, \nu_n(\tilde{Q}). \tag{1.58}$$

for all $\tilde{P} \in \tilde{K}^{c^L} \otimes \overline{\tilde{K}^{\overline{c^R}}}(m)$ and $\tilde{Q} \in \tilde{K}^{c^L} \otimes \overline{\tilde{K}^{\overline{c^R}}}(n)$ as long as $\tilde{P}\, {}_i\widetilde{\infty}_0\, \tilde{Q}$ exists. The proof is essentially same as that of the sewing axiom in Proposition 5.4.1 in [H3]. We assume that readers are familar with the proof of Proposition 5.4.1 in [H3]. We will only point out where the differences are.

In the rest of the proof, we always use $\tilde{P}, \tilde{Q}$ to denote $\psi \otimes \bar{\psi}(P), \psi \otimes \bar{\psi}(Q)$ respectively for $P \in K(m), Q \in K(n), m \in \mathbb{Z}_+, n \in \mathbb{N}$.

[2] Actually, it is only a trivial case of the case 1 in Huang's proof of Proposition 5.4.1 in [H3], namely when P and Q in (1.59) is so that $a_0 = 1, A^{(1)} = B^{(0)} = \mathbf{0}$.

In the proof of Proposition 5.4.1 in [H3], the sewing axiom is proved in cases. The first case is when $m = n = 1$. Let

$$\begin{aligned} P &:= (A^{(0)}, (a_0, A^{(1)})), \\ Q &:= (B^{(0)}, (b_0, B^{(1)})). \end{aligned} \tag{1.59}$$

Assume that $P{}_1\infty_0 Q$ exists. In this case, only Virasoro algebras are involved. Since two Virasoro algebras generated by $\{L^L(n)\}_{n\in\mathbb{Z}}$ and $\{L^R(n)\}_{n\in\mathbb{Z}}$ are mutually commutative, we can study these two Virasoro algebras seperately. The left Virasoro algebra is completely same as the Virasoro algebra in the proof of Proposition 5.4.1 in [H3]. For the right Virasoro algebra, we need one more piece of fact. Let $\mathcal{A} = \{\mathcal{A}_1, \dots\}$ and $\mathcal{B} = \{\mathcal{B}_1, \dots,\}$ be two sequences of formal variables. It was proved in [H3] that,

$$e^{-L^R_-(\mathcal{A})}\alpha^{-L^L(0)}e^{L^R_-(\mathcal{B})} = e^{L^R_-(\Psi_-)}e^{L^R_+(\Psi_+)}e^{\Psi_0 L^R(0)}\alpha^{-L^R(0)}e^{\Gamma(\mathcal{A},\mathcal{B},\alpha)c^R}, \tag{1.60}$$

where $\Psi_{\pm,0} = \Psi_{\pm,0}(\mathcal{A}, \mathcal{B}, \alpha) \in \mathbb{C}[\alpha, \alpha^{-1}][[\mathcal{A}, \mathcal{B}]]$. From Huang's study of $\Psi_{\pm,0}$, it is easy to see that they are actually in $\mathbb{Q}[\alpha, \alpha^{-1}][[\mathcal{A}, \mathcal{B}]]$. Hence we have

$$\overline{\Psi_{\pm,0}(A^{(1)}, B^{(0)}, a)} = \Psi_{\pm,0}(\overline{A^{(1)}}, \overline{B^{(0)}}, \bar{a}). \tag{1.61}$$

Using this fact, we see that (1.58) holds in this case.

Moreover, let $P_t := (A^{(0)}, (t^{-1}a_0, A^{(1)}))$. Then $P_t\,{}_1\infty_0 Q$ also exists for all $1 \geq |t| > 0$. We have $\nu_1(\tilde{P}_t)\,{}_1*_0\,\nu_1(\tilde{Q})$ equals to the following double series

$$\begin{aligned} &(\nu_1(\tilde{P})t_1^{L^L(0)}s_1^{L^R(0)})\,{}_1*_0\,\nu_1(\tilde{Q}) \\ &\quad = \sum_{m,n}(e^{-L^L_-(A^{(0)})}e^{-L^L_+(A^{(1)})}a^{-L^L(0)}) \otimes (e^{-L^R_-(\overline{A^{(0)}})}e^{-L^R_+(\overline{A^{(1)}})}\bar{a}^{-L^R(0)}) \\ &\qquad P_{m,n}(e^{-L^L_-(B^{(0)})}e^{-L^L_+(B^{(1)})}b^{-L^L(0)} \otimes e^{-L^R_-(B^{(0)})}e^{-L^R_+(B^{(1)})}\bar{b}^{-L^R(0)})t_1^m s_1^n \end{aligned} \tag{1.62}$$

when $t_1 = t, s_1 = \bar{t}$. Hence Huang's results in [H3] implies that (1.62) is absolutely convergent to a multivalued analytic function of t_1, s_1 when $1 \geq |t_1|, |s_1| > 0$.

The second case is when $i = 2$,

$$P = P(z), \quad Q = (B^{(0)}, (1, \mathbf{0}))$$

or $i = 1$,

$$P = (\mathbf{0}, (a_0^{(1)}, A^{(1)})), \quad Q = P(z).$$

By the definition of ν in (1.57), $\nu_2(P(z)) = \mathbb{Y}(\cdot; z, \bar{z})\cdot$. In our case, $\mathbb{Y}$ is an intertwining operator for $V^L \otimes V^R$ viewed as vertex operator algebra. It is proved in [HK2] that $\mathbb{Y}$ split as follow:

$$\mathbb{Y} = \sum_{i=1}^{N} \mathcal{Y}_i^L \otimes \mathcal{Y}_i^R, \quad \text{for some } N \in \mathbb{Z}_+, \tag{1.63}$$

where $\mathcal{Y}_i^L$ and $\mathcal{Y}_i^R$ are intertwining operators for V^L and V^R respectively. Using this splitting property of $\mathbb{Y}$, we can again consider the left and right seperately. Moreover, it is harmless to only consider the case $N = 1$ because $\mathcal{Y}_i^L$ and $\mathcal{Y}_i^R$ do not change when $L^L(m), L^R(n)$ exchange positions with $\mathcal{Y}_i^L$ and $\mathcal{Y}_i^R$. Only the arguments of $\mathcal{Y}_i^L$ and $\mathcal{Y}_i^R$ have changed. Then Huang's results in [H3] immediately imply that (1.58) holds in this case. Moreover, since F is a module over $V^L \otimes V^R$, the grading of F is of form $(S^L + \mathbb{N}, S^R + \mathbb{N})$ where S^L and S^R are two subsets of $\mathbb{R}$ with finite cardinalities. Then for $u_1, u_2 \in F$, $(\nu(\tilde{P}) t_1^{L^L(0)} s_1^{L^R(0)})_i *_0 \nu(\tilde{Q})(u_1 \otimes u_2)$ is of following form:

$$\sum_{j=1}^{M} \sum_{m,n \in \mathbb{N}} c_{m,n}^{(j)}\, t_1^{m + r_1^{(j)}} s_1^{n + r_2^{(j)}}$$

where $r_1^{(j)} \in S^L, r_2^{(j)} \in S^R$ for some $M \in \mathbb{N}$. We will called such series as generalized power series. Such series share the same convergence and analytic properties as ordinary series except for multivaluedness. Then Huang's result also implies that the generalized power series $(\nu(\tilde{P}) t_1^{L^L(0)} s_1^{L^R(0)})_i *_0 \nu(\tilde{Q})$, is absolutely convergent for $1 \geq |t_1|, |s_1| > 0$.

The third case is when $i = 1, P = P(z)$ and $Q = (B^{(0)}, (1, \mathbf{0}))$. This case is proved in [H3] by using the skew-symmetry of vertex operator algebra to reduce it to the second case. For us, a similar skew-symmetry ((1.41) in [HK2]) holds for conformal full field algebra. Hence we can again reduce the third case to the second case.

We do slightly differently in the fourth case. Let P and Q be as follow

$$\begin{aligned} P &= ((\infty; 1, A^{(0)}); (z_1; a_0^{(1)}, A^{(1)}), \ldots, (z_m; a_0^{(m)}, A^{(m)})), \\ Q &= (z; \mathbf{0}, (b_0^{(1)}, B^{(1)}), (b_0^{(2)}, B^{(2)})). \end{aligned} \tag{1.64}$$

Notice that P is quasi-canonical, while Q is canonical. By the definition of ν in (1.57), it is clear that

$$\nu_m(\tilde{P}) = \nu_m(\tilde{P}')_i *_0 \nu_1(\tilde{P}_i),$$

where

$$\begin{aligned} P' &= ((\infty; 1, A^{(0)}); (z_1; a_0^{(1)}, A^{(1)}), \ldots, (z_i; 1, \mathbf{0}), \ldots, (z_m; a_0^{(m)}, A^{(m)})), \\ P_i &= ((\infty; 1, \mathbf{0}), (0; a_0^{(i)}, A^{(i)})). \end{aligned} \tag{1.65}$$

Then $\nu_m(\tilde{P})\ {}_i*_0\ \nu_2(\tilde{Q})$ equals to the following iterate series

$$\begin{aligned}&\big((\nu_m(\tilde{P}')t_1^{L^L(0)}s_1^{L^R(0)})\ {}_i*_0\ \nu_1(\tilde{P}_i)t_2^{L^L(0)}s_2^{L^R(0)}\big)\ {}_i*_0\ \nu_2(\tilde{Q})\\ &= \sum_{p_2,q_2}\left(\sum_{p_1,q_1}\big((\nu_m(\tilde{P}')\ {}_i*_0\ P_{p_1,q_1}\nu_1(\tilde{P}_i)\big)\ {}_i*_0\ P_{p_2,q_2}\nu_2(\tilde{Q})\ t_1^{p_1}s_1^{q_1}t_2^{p_2}s_2^{q_2}\right)\end{aligned} \tag{1.66}$$

when $t_1 = s_1 = t_2 = s_2 = 1$. We want to show that above iterate series is absolutely convergent.

We first consider a different iterate series

$$\sum_{p_1,q_1}\left(\sum_{p_2,q_2}\big(\nu_m(\tilde{P}')\ {}_i*_0\ P_{p_1,q_1}\nu_1(\tilde{P}_i)\big)\ {}_i*_0\ P_{p_2,q_2}\nu_2(\tilde{Q})\ t_1^{p_1}s_1^{q_1}t_2^{p_2}s_2^{q_2}\right) \tag{1.67}$$

which is obtained by switching order of multiple sum in (1.66). Since $\nu_1(\tilde{P}_i) \in \mathrm{End}F$, we can apply associativity for each term of the iterate sum (1.67). We obtain the following series:

$$\sum_{p_1,q_1}\left(\sum_{p_2,q_2}\nu_m(\tilde{P}')\ {}_i*_0\ P_{p_1,q_1}\big(\nu_1(\tilde{P}_i)\ {}_1*_0\ P_{p_2,q_2}\nu_2(\tilde{Q})\big)\ t_1^{p_1}s_1^{q_1}t_2^{p_2}s_2^{q_2}\right). \tag{1.68}$$

Solving the sewing equation for the case $P_i\ {}_1\infty_0 Q$, we obtain that the canonical representative of $Q' := P_i\ {}_1\infty_0 Q$ can be written in the following form

$$(f_i^{-1}(z); \mathbf{0}, (c_0^{(1)}(a_0^{(i)}), C^{(1)}(a_0^{(i)})), (c_0^{(2)}(a_0^{(i)}), C^{(2)}(a_0^{(i)}))$$

where $c_0^{(k)}(a_0^{(i)}), k = 1, 2$ and $C^{(k)}(a_0^{(i)}), k = 1, 2$ are analytic functions of $a_0^{(i)}$ valued in $\mathbb{C}^\times$ and $\mathbb{C}$ respectively, and f_i is the local coordinate map at $0 \in P_i$. Using the results proved in the case 2, we see that, for $u_1, u_2 \in F$, the following series

$$\sum_{p_2,q_2}\nu_1(\tilde{P}_i)\ {}_1*_0\ P_{p_2,q_2}\nu_2(\tilde{Q})(u_1\otimes u_2)\ t_2^{p_2}s_2^{q_2} \tag{1.69}$$

is absolutely convergent when $1 \geq |t_2|, |s_2| > 0$ to a multivalued analytic function

$$\mathbb{Y}(T_iu_1; f_i^{-1}(t_2z), \overline{f_i^{-1}}(s_2\bar{z}))T_{i+1}u_2 \tag{1.70}$$

where

$$T_k = e^{-L_+^L(C^{(k)}(a_0^{(i)}t_2^{-1}))-L_+^R(\overline{C^{(k)}}(\overline{a_0^{(i)}}s_2^{-1}))}(c_0^{(k)}(a_0^{(i)}t_2^{-1}))^{-L^L(0)}(\overline{c_0^{(k)}}(\overline{a_0^{(i)}}s_2^{-1}))^{-L^R(0)} \tag{1.71}$$

for $k = i, i+1$. When $t_2 = s_2 = 1$, the series (1.69) simply converges to $\nu_2(Q')$.

As we mentioned before, we can always assume $z_1, \dots, z_m$ to have distinct absolute values by choosing a suitable quasi-canonical representative of P'. In this case, $\nu_m(\tilde{P}')$ can be written as a product of $\mathbb{Y}$ which is an intertwining operator of $V^L \otimes V^R$. Since f_i^{-1} map neighborhoods of 0 to neighborhoods of 0, for fixed z, $|f_i^{-1}(t_2 z)|$ is sufficiently small as long as $|t_2|$ is sufficiently small. Similarly, $|\overline{f_i^{-1}}(s_2\bar{z})|$ is sufficiently small as long as $|s_2|$ is sufficiently small. Therefore, we can always find $r > 0$ so that $|z_j| - |z_i| > |f_i^{-1}(t_2 z)|$ and $|z_j| - |z_i| > |\overline{f_i^{-1}}(s_2\bar{z})|$ for all $j \neq i$ and $r > |t_2|, |s_2| > 0$. By the convergence property of intertwining operators of V^L, V^R, it is clear that the following series, for $u_1, \dots, u_{m+1} \in F$,

$$\sum_{p_1, q_1} \nu_m(\tilde{P}')\ {}_i*_0\ P_{p_1,q_1}(\mathbb{Y}(T_i \cdot; f_i^{-1}(t_2 z), \overline{f_i^{-1}}(s_2\bar{z}))T_{i+1}\cdot)(u_1 \otimes \cdots \otimes u_{m+1})\ t_1^{p_1} s_1^{q_1}, \tag{1.72}$$

is absolutely convergent, when $r > |t_2|, |s_2| > 0$ and $1 \geq |t_1|, |s_1| > 0$, to an multivalued analytic functions of t_1, s_1, t_1, s_2, whose restriction on $s_1 = \bar{t}_1, s_2 = \bar{t}_2$ is

$$\begin{aligned} &T_0 m_{m+1}(T_1 u_1, \dots, t_1^{L^L(0)} s_1^{L^R(0)} T_i u_i|_{s_1=\bar{t}_1}, t_1^{L^L(0)} s_1^{L^R(0)} T_{i+1} u_{i+1}|_{s_1=\bar{t}_1}, \dots, T_{m+1} u_{m+1}; \\ &\quad z_1, \bar{z}_1, \dots, z_i + t_1 f_i^{-1}(t_2 z), \bar{z}_i + s_1 \overline{f_i^{-1}}(s_2\bar{z})|_{s_1=\bar{t}_1, s_2=\bar{t}_2}, z_i, \bar{z}_i, \dots, z_m, \bar{z}_m) \end{aligned} \tag{1.73}$$

where T_i, T_{i+1} are given by (1.71) and

$$\begin{aligned} T_0 &= e^{-L_-^L(A^{(0)}) - L_-^R(\overline{A^{(0)}})}, \\ T_j &= e^{-L_+^L(A^{(j)}) - L_+^R(\overline{A^{(j)}})} (a_0^{(j)})^{-L^L(0)} (\overline{a_0^{(j)}})^{-L^R(0)}, \qquad j < i, \\ T_j &= e^{-L_+^L(A^{(j-1)}) - L_+^R(\overline{A^{(j-1)}})} (a_0^{(j-1)})^{-L^L(0)} (\overline{a_0^{(j-1)}})^{-L^R(0)}, \quad j > i+1. \end{aligned} \tag{1.74}$$

By the general property of analytic functions, we obtain that the multiple series (1.67) is absolutely convergent when $r > |t_2|, |s_2| > 0$ and $1 \geq |t_1|, |s_1| > 0$. Hence the series (1.66) is also absolutely convergent when $r > |t_2|, |s_2| > 0$ and $1 \geq |t_1|, |s_1| > 0$. As a special case ($t_1 = s_1 = 1$), we obtain that the following series:

$$\sum_{p_2, q_2} \nu_m(\tilde{P})\ {}_i*_0\ P_{p_2,q_2} \nu_2(\tilde{Q})(u_1 \otimes \dots u_{m+1})\ t_2^{p_2} s_2^{q_2} \tag{1.75}$$

is absolutely convergent, when $r > |t_2|, |s_2| > 0$, to a multivalued analytic function, denoted as $G(t_2, s_2)$. In particular, when $s_2 = \bar{t}_2$, (1.75) converges to

$$\begin{aligned} &T_0 m_{m+1}(T_1 u_1, \dots, T_{m+1} u_{m+1}; z_1, \bar{z}_1, \dots, z_{i-1}, \bar{z}_{i-1}, \\ &\qquad z_i + f_i^{-1}(t_2 z), \bar{z}_i + \overline{f_i^{-1}}(s_2\bar{z})|_{s_2=\bar{t}_2}, z_i, \bar{z}_i, \dots, z_m, \bar{z}_m). \end{aligned} \tag{1.76}$$

Let $g : t_2 \mapsto z_i + f_i^{-1}(t_2 z)$. By the definition of sewing operation, for fixed $z_1, \dots, z_m, z$, $P_i\infty_0 Q$ exists is equivalent to the statement that g is well-defined on the unit disk $B(0;1) = \{1 \geq |t_2| > 0\}$ and $z_j \notin g(B(0;1))$ (or equivalently $\bar{z}_j \notin \bar{g}(B(0,1)))$) for all $j \neq i$. Therefore, the analytic function $G(t_2, s_2)$ is free of singularities in $\{(t_2, s_2) | 1 \geq |t_2|, |s_2| > 0\}$. By the property of generalized power series, (1.75) must be absolutely convergent when $1 \geq |t_2|, |s_2| > 0$. In particular, when $t_2 = s_2 = 1$, the series (1.75) converges absolutely to

$$T_0 m_{m+1}(T_1 u_1, \dots, T_{m+1} u_{m+1}; z_1, \bar{z}_1, \dots, z_{i-1}, \bar{z}_{i-1}, \\ z_i + f_i^{-1}(z), \bar{z}_i + \overline{f_i^{-1}(z)}, z_i, \bar{z}_i, \dots, z_m, \bar{z}_m), \tag{1.77}$$

which is nothing but $\nu_{m+1}(\tilde{P}_i \widetilde{\infty}_0 \tilde{Q})$. We have then finished the proof in this case.

The proofs of general cases are essentially same as that of Proposition 5.4.1 in [H3]. Only difference is that the analytic family with respect to the variables t_1, t_2 there is replaced by t_1, s_1, t_2, s_2 here, where t_1, t_2 are for chiral part and s_1, s_2 are for antichiral part just as we have done in the fourth case. ∎

Theorem 1.19. *The category of conformal full field algebra over $V^L \otimes V^R$ is isomorphic to the category of smooth $\tilde{K}^{c^L} \otimes \overline{\tilde{K}^{\overline{c^R}}}$-algebras over $V^L \otimes V^R$.*

Proof. The proof is again similar to that of the Theorem 5.4.5 in [H3]. One can use the proof of Proposition 1.15 and Theorem 1.17 as a guidance. We only outline the proof here.

Given a conformal full field algebras over $V^L \otimes V^R$, denoted as (F, m, ρ), we have a smooth $\tilde{K}^{c^L} \otimes \overline{\tilde{K}^{\overline{c^R}}}$-algebra over $V^L \otimes V^R$ given by Proposition 1.18. Moreover, this construction (1.55) and (1.57) is functorial. Hence we obtain a functor from the category of conformal full field algebras over $V^L \otimes V^R$ to the category of smooth $\tilde{K}^{c^L} \otimes \overline{\tilde{K}^{\overline{c^R}}}$-algebras over $V^L \otimes V^R$.

Conversely, the canonical section $\psi \otimes \bar{\psi}$ gives a natural embedding from $\widehat{K}$ to $\tilde{K}^{c^L} \otimes \overline{\tilde{K}^{\overline{c^R}}}$ as partial operads. Hence any smooth $\tilde{K}^{c^L} \otimes \overline{\tilde{K}^{\overline{c^R}}}$-algebra over $V^L \otimes V^R$, denoted as (F, ν, ρ), automatically gives a smooth $\widehat{K}$-algebra $(F, \mathbb{C}\mathbf{1}, \nu)$, where $\mathbf{1}$ is the vacuum state in F defined by

$$\mathbf{1} = \nu_0(\psi \otimes \bar{\psi}(\mathbf{0})). \tag{1.78}$$

By Proposition 1.15, there is a natural structure of grading-restricted $\mathbb{R} \times \mathbb{R}$-graded full field algebra on F.

In this case, we also have two elements $\omega^L, \omega^R \in \overline{F}$ given by

$$\omega^L = -\frac{\partial}{\partial \epsilon} \nu(\psi \otimes \bar{\psi}(A(\epsilon; 2)))\bigg|_{\epsilon=0}; \quad \omega^R = -\frac{\partial}{\partial \bar{\epsilon}} \nu(\psi \otimes \bar{\psi}(A(\epsilon; 2)))\bigg|_{\epsilon=0}. \tag{1.79}$$

We define, for $n > 0$,

$$\begin{aligned}
L^L(-n) &:= -\frac{\partial}{\partial A_n^{(0)}}\nu_1(\psi\otimes\bar{\psi}(A^{(0)},(1,\mathbf{0})))\\
L^R(-n) &:= -\frac{\partial}{\partial \overline{A_n^{(0)}}}\nu_1(\psi\otimes\bar{\psi}(A^{(0)},(1,\mathbf{0})))\\
L^L(0) &:= -\frac{\partial}{\partial a}\nu_1(\psi\otimes\bar{\psi}(\mathbf{0},(a,\mathbf{0})))\\
L^R(0) &:= -\frac{\partial}{\partial \bar{a}}\nu_1(\psi\otimes\bar{\psi}(\mathbf{0},(a,\mathbf{0})))\\
L^L(n) &:= -\frac{\partial}{\partial A_n^{(1)}}\nu_1(\psi\otimes\bar{\psi}(\mathbf{0},(1,A^{(1)})))\\
L^R(n) &:= -\frac{\partial}{\partial \overline{A_n^{(1)}}}\nu_1(\psi\otimes\bar{\psi}(\mathbf{0},(1,A^{(1)})))
\end{aligned} \tag{1.80}$$

Using the method in the proof of Proposition 5.4.4 in [H3] (or equivalently that in the proof of Proposition 2.9), it is easily to show that $[L^L(m), L^R(n)] = 0$ for $m, n \in \mathbb{Z}$ and the set $\{L^L(n)\}_{n\in\mathbb{Z}}$ ($\{L^R(n)\}_{n\in\mathbb{Z}}$) generates a Virasoro algebra of central charge c^L (c^R). Moreover, following the proof of Proposition 5.4.4 in [H3] and using (1.9), we can show that

$$\begin{aligned}
\mathbb{Y}(\omega^L; z, \bar{z}) &= \sum_{n\in\mathbb{Z}} L^L(n) z^{-n-2},\\
\mathbb{Y}(\omega^R; z, \bar{z}) &= \sum_{n\in\mathbb{Z}} L^R(n) \bar{z}^{-n-2},
\end{aligned} \tag{1.81}$$

where $\mathbb{Y}(\cdot; z, \bar{z})\cdot = \nu_2(\psi\otimes\bar{\psi}(P(z)))$. The definition of $L^L(0)$ and $L^R(0)$ in (1.80) exactly coincide with $\mathbf{d}^L$ and $\mathbf{d}^R$ in (1.19) respectively. Hence $L^L(0)$ and $L^R(0)$ are exactly the left and the right grading operators, respectively. Moreover, the definition of $L^L(-1), L^R(-1)$ coincides with D^L, D^R in (1.29) respectively. Therefore, F has a structure of conformal full field algebra.

Since these two Virasoro elements are completely determined by a distinguished sphere with tube, they must coincide with the Virasoro elements of V^L and V^R respectively. Hence F is a conformal full field algebra over $V^L \otimes V^R$.

Thus we obtain a functor from the category of smooth $\tilde{K}^{c^L}\otimes\overline{\tilde{K}^{\overline{c^R}}}$-algebras over $V^L\otimes V^R$ to the category of conformal full field algebras over $V^L\otimes V^R$.

Now we prove that these two functors give isomorphisms. Given a conformal full field algebra over $V^L\otimes V^R$, denoted as (F, m, ρ). We obtain another such algebraic structure $(F, \tilde{m}, \tilde{\rho})$ from (F, ν, ρ), which is a smooth $\tilde{K}^{c^L}\otimes\overline{\tilde{K}^{\overline{c^R}}}$-algebras over $V^L \otimes V^R$ induced from (F, m, ρ). $\tilde{m} = m$ is proved in (1.47), and $\tilde{\rho} = \rho$ is automatic. Hence one way of composing these two functors gives the identity functor.

Given a smooth $\tilde{K}^{c^L} \otimes \overline{\tilde{K}^{\overline{c^R}}}$-algebras over $V^L \otimes V^R$, denoted as (F, ν, ρ). We have another such algebra $(F, \tilde{\nu}, \tilde{\rho})$ given by a conformal full field algebra over $V^L \otimes V^R$ (F, m, ρ) which is further induced from (F, ν, ρ). Let $\tilde{Q} = \lambda\psi \otimes \bar{\psi}(Q) \in \tilde{K}^{c^L} \otimes \overline{\tilde{K}^{\overline{c^R}}}(n)$, where Q is given in (1.2). We have

$$\begin{aligned}
\tilde{\nu}_n(\tilde{Q}) : \;&=\; \lambda e^{-L_-^L(A^{(0)})-L_-^R(\overline{A^{(0)}})} m_n\Big(e^{-L_+^L(A^{(1)})-L_+^R(\overline{A^{(1)}})} (a_0^{(1)})^{-L^L(0)} \overline{a_0^{(1)}}^{-L^R(0)} u_1, \dots, \\
&\qquad e^{-L_+^L(A^{(n)})-L_+^R(\overline{A^{(n)}})} (a_0^{(1)})^{-L^L(0)} \overline{a_0^{(1)}}^{-L^R(0)} u_n; z_1, \bar{z}_1, \dots, z_{n-1}, \bar{z}_{n-1}, 0, 0\Big), \\
&=\; \nu_1(\tilde{Q}_0)\, {}_1{*}_0 \Big((\dots (\nu_n(\tilde{P})\, {}_1{*}_0\, \nu_1(\tilde{Q}_1)) \dots)\, {}_n{*}_0\, \nu_1(\tilde{Q}_n) \Big),
\end{aligned} \tag{1.82}$$

where

$$\begin{gathered}
\tilde{P} = \lambda\psi \otimes \bar{\psi}\big((z_1, \dots, z_{n-1}, 0; \mathbf{0}, (1, \mathbf{0}), \dots (1, \mathbf{0}))\big), \\
\tilde{Q}_0 = \psi \otimes \bar{\psi}(A^{(0)}, (1, \mathbf{0})), \quad \tilde{Q}_i = \psi \otimes \bar{\psi}(\mathbf{0}, (a_0^{(i)}, A^{(i)})), \quad i = 1, \dots, n.
\end{gathered} \tag{1.83}$$

Using the defining property of ν, and the fact that

$$\tilde{Q} = (\tilde{Q}_0)\, {}_1\widetilde{\infty}_0 (\dots (\tilde{P}\, {}_1\widetilde{\infty}_0 Q_1) \dots {}_n\widetilde{\infty}_0 Q_n),$$

we obtain $\tilde{\nu}_n(\tilde{Q}) = \nu_n(\tilde{Q})$ immediately. $\tilde{\rho} = \rho$ is obvious. Hence we have shown that the opposite way of composing these two functors also gives the identity functor.

Therefore, the two categories are isomorphic. ∎

2 Invariant bilinear forms

In this section, we study the invariant bilinear form of conformal full field algebra, and give a geometric interpretation of a conformal full field algebra over $V^L \otimes V^R$ equipped with a nondegenerate invariant bilinear form.

An invariant bilinear form $(\cdot, \cdot)$ on a conformal full field algebra F is a bilinear form on F such that, for any $u, w_1, w_2 \in F$,

$$\begin{aligned}
&(w_2, \mathbb{Y}_f(u, x, \bar{x}) w_1) \\
&\quad = (\mathbb{Y}_f(e^{-xL^L(1)} x^{-2L^L(0)} \otimes e^{-\bar{x}L^R(1)} \bar{x}^{-2L^R(0)} u, e^{\pi i} x^{-1}, e^{-\pi i} \bar{x}^{-1}) w_2, w_1).
\end{aligned} \tag{2.1}$$

or equivalently,

$$\begin{aligned}
&(\mathbb{Y}_f(u, e^{\pi i} x, e^{-\pi i} \bar{x}) w_2, w_1) \\
&\quad = (w_2, \mathbb{Y}_f(e^{xL^L(1)} x^{-2L^L(0)} \otimes e^{\bar{x}L^R(1)} \bar{x}^{-2L^R(0)} u, x^{-1}, \bar{x}^{-1}) w_1).
\end{aligned} \tag{2.2}$$

Remark 2.1. The invariance property of bilinear form defined in (2.1) and (2.2) is different from that in [HK2]. Its geometric meaning is shown in (2.17). The main reason for the new definition is that the old definition can not be formulate categorically. With this new definition, we can show in Section 4 that a conformal full field algebra over $V^L \otimes V^R$, where V^L and V^R satisfy certain natural conditions, has a very nice categorical formulation.

Proposition 2.2. *For all $m, n \in \mathbb{Z}$ and $w_1, w_2 \in F$,*

$$((L^L(m) \otimes L^R(n))w_2, w_1) = (w_2, (-1)^{m+n}(L^L(-m) \otimes L^R(-n))w_1). \tag{2.3}$$

Proof. Replace u in (2.1) by $\omega^L \otimes \mathbf{1}^R$ and use the fact that $L^L(1)\omega^L = 0$, we obtain

$$(w_2, \mathbb{Y}_f(\omega^L \otimes \mathbf{1}^R, x, \bar{x})w_1) = (\mathbb{Y}_f(x^{-4}\omega^L \otimes \mathbf{1}^R, e^{\pi i}x^{-1}, e^{-\pi i}\bar{x}^{-1})w_2, w_1).$$

Expanding above equation by components, we obtain

$$\sum_{n\in\mathbb{Z}}(w_2, (L^L(n) \otimes 1)w_1)x^{-n-2} = \sum_{n\in\mathbb{Z}}((L^L(n) \otimes 1)w_2, w_1)x^{n-2}e^{\pi i(-n-2)},$$

which implies that

$$((L^L(n) \otimes 1)w_2, w_1) = (w_2, (-1)^n(L^L(-n) \otimes 1)w_1).$$

Similarly, we can show that

$$((1 \otimes L^R(n))w_2, w_1) = (w_2, (-1)^n(1 \otimes L^R(-n))w_1).$$

Then it is clear that (2.3) is true. ∎

We will use (2.3) in many places in this section without pointing it out explicitly.

Proposition 2.3. *An invariant bilinear form $(\cdot, \cdot)$ on F is automatically symmetric. Namely, For $w_1, w_2 \in F$, we have*

$$(w_1, w_2) = (w_2, w_1) \tag{2.4}$$

Proof. The proof is similar to that of Proposition 5.3.6. in [FHL]. By (2.1), skew symmetry and (2.2), we have

$$\begin{aligned}
&(w_2, \mathbb{Y}_f(u, x, \bar{x})w_1)\\
&= (\mathbb{Y}_f(e^{-xL^L(1)}x^{-2L^L(0)} \otimes e^{-\bar{x}L^R(1)}\bar{x}^{-2L^R(0)}\, u, e^{\pi i}x^{-1}, e^{-\pi i}\bar{x}^{-1})w_2, w_1)\\
&= (e^{-x^{-1}L^L(-1)} \otimes e^{-\bar{x}^{-1}L^R(-1)}\mathbb{Y}_f(w_2, x^{-1}, \bar{x}^{-1})\\
&\qquad\qquad\qquad\qquad e^{-xL^L(1)}x^{-2L^L(0)} \otimes e^{-\bar{x}L^R(1)}\bar{x}^{-2L^R(0)}\, u, w_1)\\
&= (e^{-xL^L(1)}x^{-2L^L(0)} \otimes e^{-\bar{x}L^R(1)}\bar{x}^{-2L^R(0)}\, u,\\
&\qquad\qquad \mathbb{Y}_f(e^{-x^{-1}L^L(1)}x^{2L^L(0)} \otimes e^{-\bar{x}^{-1}L^R(1)}\bar{x}^{2L^R(0)}w_2,\\
&\qquad\qquad\qquad\qquad e^{\pi i}x, e^{-\pi i}\bar{x})e^{x^{-1}L^L(1)} \otimes e^{\bar{x}^{-1}L^R(1)}w_1)\\
&= (x^{-2L^L(0)} \otimes \bar{x}^{-2L^R(0)}\, u, \mathbb{Y}_f(e^{x^{-1}L^L(1)} \otimes e^{\bar{x}^{-1}L^R(1)}w_1, x, \bar{x})\\
&\qquad\qquad\qquad e^{-x^{-1}L^L(1)}x^{2L^L(0)} \otimes e^{-\bar{x}^{-1}L^R(1)}\bar{x}^{2L^R(0)}w_2)\\
&= (\mathbb{Y}_f(e^{-xL^L(1)}x^{-2L^L(0)}e^{x^{-1}L^L(1)} \otimes e^{-\bar{x}L^R(1)}\bar{x}^{-2L^R(0)}e^{\bar{x}^{-1}L^R(1)}w_1, e^{\pi i}x^{-1}, e^{-\pi i}\bar{x}^{-1})\\
&\qquad\qquad x^{-2L^L(0)} \otimes \bar{x}^{-2L^R(0)}\, u, e^{-x^{-1}L^L(1)}x^{2L^L(0)} \otimes e^{-\bar{x}^{-1}L^R(1)}\bar{x}^{2L^R(0)}w_2)\\
&= (\mathbb{Y}_f(x^{-2L^L(0)} \otimes \bar{x}^{-2L^R(0)}\, u, x^{-1}, \bar{x}^{-1})x^{-2L^L(0)} \otimes \bar{x}^{-2L^R(0)}w_1, x^{2L^L(0)} \otimes \bar{x}^{2L^R(0)}w_2)\\
&= (\mathbb{Y}_f(u, x, \bar{x})w_1, w_2) \qquad (2.5)
\end{aligned}$$

∎

Now we consider conformal full field algebra over $V^L \otimes V^R$. In this case, F has a countable basis. We choose it to be $\{e_i\}_{i\in\mathbb{N}}$. If an invariant bilinear form $(\cdot, \cdot)$ on F is also nondegenerate, we also have the dual basis $\{e^i\}_{i\in\mathbb{N}}$. In this case, we can define a linear map $\Delta : \mathbb{C} \to \overline{F \otimes F}$ as follow:

$$\Delta : 1 \mapsto \sum_{i\in\mathbb{R}} e_i \otimes e^i. \qquad (2.6)$$

A conformal full field algebra over $V^L \otimes V^R$ with a nondegenerate invariant bilinear form has a very nice geometric interpretation. This is what we will discuss in the remaining part of this section.

Recall the notion of sphere with tubes of type (n_-, n_+), where n_- (n_+) is the number of negatively (positively) oriented tubes. In section 1, we have only studied the moduli space of spheres with tubes of type $(1, n)$, the structure on which is captured in a notion called sphere partial operad. In this section, we would like to consider sphere with tubes of type (n_-, n_+) for all $n_-, n_+ \in \mathbb{N}$.

In the case $n_- = 1$, we have described the moduli space of conformal equivalent classes of sphere with tubes of type $(1, n)$ by choosing a canonical representative for each conformal equivalent class. In particular, we choose the only negatively oriented puncture to sit at $\infty \in \hat{\mathbb{C}}$. For general $n_- \in \mathbb{N}$, there is no canonical choice for the positions of the negatively oriented punctures. We will take a different approach. We label the i-th negatively oriented puncture as $-i$-th puncture and j-th positively oriented puncture as j-th puncture.

Let us use

$$Q = (\,(z_{-n_-}; a_0^{(-n_-)}, A^{(-n_-)}), \dots, (z_{-1}; a_0^{(-1)}, A^{(-1)})| \\ (z_1; a_0^{(1)}, A^{(1)}), (z_{-n_-}; a_0^{(n_+)}, A^{(n_+)})\,) \quad (2.7)$$

to denote a sphere $\hat{\mathbb{C}}$ with positively (negative) oriented punctures at $z_i \in \hat{\mathbb{C}}$ for $i = 1, \dots, n_+$ $(i = -1, \dots, -n_-)$, and with local coordinate map f_i around each puncture z_i:

$$f_i(w) = e^{\sum_j A_j^{(i)} x^{j+1} \frac{d}{dx}} (a_0^{(i)})^{x \frac{d}{dx}} x \Big|_{x = w - z_i} \quad \text{if } z_i \in \mathbb{C}\ , \quad (2.8)$$

$$= e^{\sum_j A_j^{(i)} x^{j+1} \frac{d}{dx}} (a_0^{(i)})^{x \frac{d}{dx}} x \Big|_{x = \frac{-1}{w}} \quad \text{if } z_i = \infty. \quad (2.9)$$

Remark 2.4. Notice that (2.9) is different from (1.3), even when we set $a_0^{(i)} = 1$, by a factor -1. It simply means that the parametrizations of the corresponding tubes, given by (2.9) and (1.3), are different. So we expect that their algebraic realizations are also different, as one can see later by comparing (1.57) with (2.14), (2.15), (2.16) and keeping in mind of (2.3)! Therefore, nothing is really changed except one should distinguish the notation (2.7) with (1.2).

We denote the set of all such Q as $\mathcal{T}(n_-, n_+)$, and their disjoint union as

$$\mathcal{T} = \{\mathcal{T}(n_-, n_+)\}_{n_-, n_+ \in \mathbb{N}}.$$

It is clear that there is an action of $SL(2, \mathbb{C})$, the group of Mobius transformations, on $\mathcal{T}(n_-, n_+)$. Each orbit of this action represents a single conformal equivalent class of sphere with tubes. We denote the set of orbits as

$$K(n_-, n_+) = \mathcal{T}(n_-, n_+)/SL(2, \mathbb{C}).$$

Let $K(0, 0)$ be the one element set consisting of the standard sphere $\hat{\mathbb{C}}$ with no additional structure. Then we denote the disjoint union as:

$$\mathbb{K} = \{K(n_-, n_+)\}_{n_-, n_+ \in \mathbb{N}}.$$

For a pair of elements $P \in K(m_-, m_+)$ and $Q \in K(n_-, n_+)$, a partial sewing operation ${}_i\infty_{-j}$ between the i-th positively oriented tube in P and the j-th negatively oriented tube in Q can be defined same as in Section 1. We denote the sphere with tube after sewing as

$$P_i\infty_{-j} Q$$

where $m_+ \geq i > 0 > -j \geq -n_-$. This sewing operation is associative. Notice that if we sew two spheres with tubes along more than one pair of tubes with opposite orientations, we obtain a surface of higher genus. We will not discuss this type of multiple sewing operations between two spheres with tubes in this work. By restricting to only single sewing operation between any pair of elements in $\mathbb{K}$, it is clear that $\mathbb{K}$ is closed under these sewing operations. Moreover, $\mathbb{K}$ is generated from the following elements:

$$
\begin{aligned}
&(\,(\infty; 1, \mathbf{0})|\,) \\
Q_2 &= (\,|(\infty; 1, \mathbf{0}), (0; 1, \mathbf{0})\,) \\
Q_{-2} &= (\,(\infty; 1, \mathbf{0}), (0; 1, \mathbf{0})|\,) \\
P(z) &= (\,(\infty; 1, \mathbf{0})|(z; 1, \mathbf{0}), (0; 1, \mathbf{0})\,)
\end{aligned} \tag{2.10}
$$

and elements in $K(1,1)$ by sewing operations.

There is also a natural $S_{n_-} \times S_{n_+}$-action on $K(n_-, n_+)$. Namely, S_{n_+} (S_{n_-}) permute the positively (negatively) oriented punctures among each other. We refer to the action of permutation group on $\mathbb{K}$ for all $n_-, n_+ \in \mathbb{N}$ as $S_- \times S_+$-action.

The structure on $\mathbb{K}$ induced from sewing operations and $S_- \times S_+$-actions are richer than the structure of a partial operad. As we can see that it contains the sphere partial operad K as a substructure.

The determinant line bundle on $\mathbb{K}$, denoted as $\mathrm{Det}_{\mathbb{K}}$, is a trivial bundle on $\mathbb{K}$. The determinant line bundle on $K(0,0)$ is just a complex line. We fix an element of Det_{Q_2} as $\mu_2(Q_2)$ and an element of Det_{Q_2} as $\mu_{-2}(Q_{-2})$. These choices will determine a canonical section on the determinant line bundle $\mathrm{Det}_{\mathbb{K}}$ over $\mathbb{K}$ (see the section 6.5 in [H3] for detail). The sewing operations and $S_- \times S_+$ actions on $\mathbb{K}$ can be extended to those on the determinant line bundle. Moreover, these sewing operations on $\mathrm{Det}_{\mathbb{K}}$ is associative and compatible with $S_{n_-} \times S_{n_+}$-action on $\mathrm{Det}_{\mathbb{K}}$ for all $n_+, n_- \in \mathbb{N}$. We denote the sewing operation on $\mathrm{Det}_{\mathbb{K}}$ as $\widetilde{\infty}$.

We denote the $c/2$ power of the determinant line bundle on $\mathbb{K}$ as $\tilde{\mathbb{K}}^c$. We are mainly interested in the line bundle $\tilde{\mathbb{K}}^{c^L} \otimes \overline{\tilde{\mathbb{K}}^{\overline{c^R}}}$. We denote the canonical section in $\tilde{\mathbb{K}}^{c^L} \otimes \overline{\tilde{\mathbb{K}}^{\overline{c^R}}}$ as

$$
\psi^L \otimes \psi^R : \mathbb{K} \to \tilde{\mathbb{K}}^{c^L} \otimes \overline{\tilde{\mathbb{K}}^{\overline{c^R}}}.
$$

and the sewing operation as $\widetilde{\infty}$.

We can extend the definition of smooth function on $\tilde{K}^{c^L} \otimes \overline{\tilde{K}^{\overline{c^R}}}$ to that of smooth function on $\tilde{\mathbb{K}}^{c^L} \otimes \overline{\tilde{\mathbb{K}}^{\overline{c^R}}}$ in the obvious way.

When we want to emphasis the structures on $\tilde{\mathbb{K}}^{c^L} \otimes \overline{\tilde{\mathbb{K}}^{\overline{c^R}}}$, which include partially defined sewing operations $\widetilde{\infty}$ and $S_{n_-} \times S_{n_+}$-actions, we will denote them as a triple

$$
(\tilde{\mathbb{K}}^{c^L} \otimes \overline{\tilde{\mathbb{K}}^{\overline{c^R}}}, \widetilde{\infty}, S_- \times S_+).
$$

We will be interested in some algebraic realization of $(\tilde{\mathbb{K}}^{c^L} \otimes \overline{\tilde{\mathbb{K}}^{\overline{c^R}}}, \widetilde{\infty}, S_- \times S_+)$. Consider the set

$$\mathbb{F} = \{\mathrm{Hom}(F^{\otimes m_+}, \overline{F^{\otimes m_-}})\}_{m_+, m_- \in \mathbb{N}}.$$

where $F^{\otimes 0} = \mathbb{C}$. Let $f \in \mathrm{Hom}(F^{\otimes m_+}, \overline{F^{\otimes m_-}})$ and $g \in \mathrm{Hom}(F^{\otimes n_+}, \overline{F^{\otimes n_-}})$. The map g can be expanded as follow:

$$g : v_1 \otimes \cdots \otimes v_{n_+} \mapsto \sum_{l_1, \ldots l_{n_-} \in \mathbb{N}} g_{l_1 \ldots l_{n_-}}(v_i, \ldots, v_{n_+})\ e_{l_1} \otimes \cdots \otimes e_{l_{n_-}}.$$

We can define $f_i *_{-j} g$ for $1 \leq i \leq m_+, 1 \leq j \leq n_-$ as

$$\begin{aligned} &\sum_{k \in \mathbb{R}} \sum_{l_1, \ldots l_{n_-}} g_{l_1 \ldots l_{n_-}}(u_i, \ldots, u_{i+n_+}) e_{l_1} \otimes \cdots \otimes e_{l_{j-1}} \\ &\quad \otimes f(u_1, \ldots, u_{i-1}, e_{l_j}, u_{i+n_++1}, \ldots, u_{m_++n_+-1}) \otimes e_{l_{j+1}} \otimes \cdots \otimes e_{l_{n_-}} \end{aligned} \tag{2.11}$$

whenever the sum is absolutely convergent. $S_{m_-} \times S_{m_+}$-action on $\mathrm{Hom}(F^{\otimes m_+}, \overline{F^{\otimes m_-}})$ can be defined in the obvious way.

Similar to algebras over partial operads, what we are looking for here is a smooth map

$$\Psi : \tilde{\mathbb{K}}^{c^L} \otimes \overline{\tilde{\mathbb{K}}^{\overline{c^R}}} \to \mathbb{F}$$

such that Ψ is equivariant with respective to $S_- \otimes S_+$-actions and satisfies the following conditions:

$$\Psi(\tilde{P}_i \widetilde{\infty}_{-j} \tilde{Q}) = \Psi(\tilde{P})_i *_{-j} \Psi(\tilde{Q}) \tag{2.12}$$

for any $\tilde{P}, \tilde{Q} \in \tilde{\mathbb{K}}^{c^R} \otimes \overline{\tilde{\mathbb{K}}^{\overline{c^R}}}$ whenever $\tilde{P}_i \widetilde{\infty}_{-j} \tilde{Q}$ exists. We will show that a conformal full field algebra F together with a nondegenerate invariant bilinear form $(\cdot, \cdot)$ canonically give a map Ψ which is $S_- \times S_+$-equivariant and satisfies (2.12).

Let (F, m, ρ) be a conformal full field algebra over $V^L \otimes V^R$ equipped with a nondegenerate bilinear form $(\cdot, \cdot)$. Let $\{e^i\}_{i \in \mathbb{N}}$ be the dual basis of $\{e_i\}_{i \in \mathbb{N}}$.

For $Q \in \mathcal{T}(n_-, n_+)$ of the form (2.7) and $\lambda \in \mathbb{C}^\times$, we define

$$\Psi(\lambda(\psi^L \otimes \psi^R)(Q))(u_{-n_-} \otimes \cdots \otimes u_{n_+}) \tag{2.13}$$

in the following three cases:

1. If $z_k \neq \infty$ for all $k = -n_-, \dots, -1, 1, \dots, n_+$,

$$\lambda \sum_{i_1,\dots,i_{n_-}\in\mathbb{N}} \big(\mathbf{1},\, m_{n_-+n_+}(e^{-L_+^L(A^{(-n_-)})-L_+^R(\bar{A}^{(-n_-)})}(a_0^{(-n_-)})^{-L^L(0)}\overline{a_0^{(-n_-)}}^{-L^R(0)} e_{i_{n_-}},$$
$$\dots, e^{-L_+^L(A^{(-1)})-L_+^R(\bar{A}^{(-1)})}(a_0^{(-1)})^{-L^L(0)}\overline{a_0^{(-1)}}^{-L^R(0)} e_{i_1},$$
$$e^{-L_+^L(A^{(1)})-L_+^R(\bar{A}^{(1)})}(a_0^{(1)})^{-L^L(0)}\overline{a_0^{(1)}}^{-L^R(0)} u_1,$$
$$\dots, e^{-L_+^L(A^{(n_+)})-L_+^R(\bar{A}^{(n_+)})}(a_0^{(n_+)})^{-L^L(0)}\overline{a_0^{(n_+)}}^{-L^R(0)} u_{n_+};$$
$$z_{-n_-}, \bar{z}_{-n_-}, \dots, z_{-1}, \bar{z}_{-1}, z_1, \bar{z}_1, \dots, z_{n_+}, \bar{z}_{n_+})\big)\, e^{i_1} \otimes \cdots \otimes e^{i_{n_-}}; \quad (2.14)$$

2. If $\exists\, k \in \{-n_-, \dots, -1\}$ such that $z_k = \infty$ (recall (2.9)), we define (2.13) to be the formula obtained from (2.14) by switching the first $\mathbf{1}$ with

$$e^{-L_+^L(A^{(k)})-L_+^R(\bar{A}^{(k)})}(a_0^{(k)})^{-L^L(0)}\overline{a_0^{(k)}}^{-L^R(0)} e_{i_{-k}}; \quad (2.15)$$

3. If $\exists\, k \in \{1, \dots, n_+\}$ such that $z_k = \infty$ (recall (2.9)), we define (2.13) to be the formula obtained from (2.14) by switching the first $\mathbf{1}$ with the

$$e^{-L_+^L(A^{(k)})-L_+^R(\bar{A}^{(k)})}(a_0^{(k)})^{-L^L(0)}\overline{a_0^{(k)}}^{-L^R(0)} u_k. \quad (2.16)$$

We have finished the definition of Ψ in all cases. Some interesting cases are listed explicitly below:

$$\begin{array}{rcl} \Psi(\psi^L \otimes \psi^R(\hat{\mathbb{C}})) & = & (\mathbf{1}, \mathbf{1}) I_{\mathbb{C}}, \\ \Psi(\psi^L \otimes \psi^R(\,(\infty, 1, \mathbf{0})|\,)) & = & \mathbf{1} \\ \Psi(\psi^L \otimes \psi^R(Q_2))(u \otimes v) & = & (u, v) \\ \Psi(\psi^L \otimes \psi^R(Q_{-2})) & = & \Delta. \\ \Psi(\psi^L \otimes \psi^R(P(z)))(u \otimes v) & = & \mathbb{Y}(u, z, \bar{z})v \end{array} \quad (2.17)$$

where $\hat{\mathbb{C}}$ is the single element in $K(0,0)$. We can always choose $(\cdot,\cdot)$ to be so that $(\mathbf{1}, \mathbf{1}) = 1$.

First question we want to ask is whether such defined Ψ induces a well-defined map, still denoted as Ψ, on $\tilde{\mathbb{K}}^{c^L} \otimes \overline{\tilde{\mathbb{K}}^{\overline{c^R}}}$.

Lemma 2.5. *Ψ is well-defined on $\tilde{\mathbb{K}}^{c^L} \otimes \overline{\tilde{\mathbb{K}}^{\overline{c^R}}}$.*

Proof. Firstly, we need to show that the coefficient of each $e^{i_1} \otimes \cdots \otimes e^{i_{n_-}}$ in (2.14) is invariant under the action of $SL(2,\mathbb{C})$ on Q. It is equivalent to prove Lemma for $Q \in \mathcal{T}(0, n_- + n_+)$. Hence we only need to consider the case when $n_- = 0$.

Secondly, if $z_k \neq \infty, 0$ for all $k = -n_-, \ldots, n_+$, then (2.14) equals to

$$\Psi(\lambda(\psi^L \otimes \psi^R)(P))(u_{-n_-} \otimes \cdots \otimes u_{n_+} \otimes \mathbf{1} \otimes \mathbf{1}),$$

where P is obtained from Q by adding the $n_+ + 1$-th and $n_+ + 2$-th positively oriented tubes at ∞ and 0 respectively with arbitrary local coordinates maps. Hence we conclude that it is sufficient to consider the case $z_k = \infty$ and $z_l = 0$ for some $-n_- \leq k, l \leq n_+$. For simplicity, we can assume $k = 1$ and $l = n_+$. The proof for general k, l is exactly the same.

The group $SL(2,\mathbb{C})$ is generated by the following three Mobius transformations:

$$\begin{aligned} w &\mapsto w - a \\ w &\mapsto aw \\ w &\mapsto -\frac{1}{w} \end{aligned} \tag{2.18}$$

where $a \in \mathbb{C}^\times$. Hence it is enough to show the invariance of Ψ with respect to above three Mobius transformations.

We first consider the transformation $w \mapsto w - a$. It maps a $Q \in \mathcal{T}(0, n_+)$ of form:

$$(\,|(\infty, a_0^{(1)}, A^{(1)}), (z_2, a_0^{(2)}, A^{(2)}), \ldots, (z_{n_+}, a_0^{(n_+)}, A^{(n_+)})\,)$$

$(z_{n_+} = 0)$ to

$$Q' = (\,|(\infty, b_0, B), (z_2 - a, a_0^{(2)}, A^{(2)}), \ldots, (z_{n_+} - a, a_0^{(n_+)}, A^{(n_+)})\,),$$

where b_0, B is so that the local coordinate map at ∞ is given by $f_1(w + a)$, where $f_1(w)$ is given by (2.9). We have

$$\begin{aligned} f_1(w+a) &= e^{a\frac{d}{dw}} f_1(w) \\ &= e^{ax^2\frac{d}{dx}} e^{\sum_j A_j^{(1)} x^{j+1}\frac{d}{dx}} (a_0^{(1)})^{x\frac{d}{dx}} x \Big|_{x=\frac{-1}{w}} \end{aligned} \tag{2.19}$$

By the definition of Ψ given in (2.14), we see that

$$\begin{aligned} \Psi \circ \psi^L \otimes \psi^R(Q') = \big(& e^{-aL^L(1) - \bar{a}L^R(1)} u_1', \\ & m_{n_+-1}(u_2', \ldots, u_{n_+}'; z_2 - a, \overline{z_2 - a}, \ldots, z_{n_+} - a, \overline{z_{n_+} - a})\,\big) \end{aligned} \tag{2.20}$$

where

$$u'_j = e^{-L^L_+(A^{(k)})-L^R_+(\bar{A}^{(k)})}(a_0^{(k)})^{-L^L(0)}\overline{a_0^{(k)}}^{-L^R(0)}u_j \tag{2.21}$$

for $j = 1, \dots, n_+$. Apply (2.3) and (1.46) to (2.20), we obtain that

$$\Psi \circ \psi^L \otimes \psi^R(Q') = \Psi \circ \psi^L \otimes \psi^R(Q).$$

The transformation $w \mapsto aw$ maps Q in the form (2.7) to Q'', which has only positively oriented punctures at $\infty, az_2, \dots, az_{n_+}$ with local coordinate maps given by $f_i(a^{-1}w)$. We have

$$f_1(a^{-1}w) = a^{-w\frac{d}{dw}}f_1(w) = a^{x\frac{d}{dx}}e^{\sum_j A_j^{(1)}x^{j+1}\frac{d}{dx}}(a_0^{(1)})^{x\frac{d}{dx}}x\Big|_{x=\frac{-1}{w}}$$

and $f_i(a^{-1}w) = a^{-w\frac{d}{dw}}f_i(w)$ for $i = 2, \dots, n_+$. By the definition (2.14), we obtain

$$\begin{aligned}
&\Psi \circ \psi^L \otimes \psi^R(Q'') \\
&\quad = \big(a^{-L^L(0)}\bar{a}^{-L^R(0)}u'_1, m_{n_+-1}(a^{L^L(0)}\bar{a}^{L^R(0)}u'_2, \dots a^{L^L(0)}\bar{a}^{L^R(0)}u'_{n_+}; \\
&\qquad\qquad az_2, \overline{az_2}, \dots, az_{n_+}, \overline{az_{n_+}})\big).
\end{aligned} \tag{2.22}$$

Using (1.24), it is clear that

$$\Psi \circ \psi^L \otimes \psi^R(Q'') = \Psi \circ \psi^L \otimes \psi^R(Q).$$

Now we consider the last Mobius transformation $w \to -\frac{1}{w}$. It transforms Q to Q''', which has punctures at $0, \frac{-1}{z_2}, \dots, \frac{-1}{z_{n_+-1}}, \infty$ with local coordinate maps $f_i(\frac{-1}{w})$. In particular, for $i = 1$, we have

$$f_1\left(-\frac{1}{w}\right) = e^{\sum_{j=1}^{\infty} A_j^{(1)}w^{j+1}\frac{d}{dw}}w;$$

and for $i = 2, \dots, n_+ - 1$,

$$\begin{aligned}
f_i\left(\frac{-1}{w}\right) &= e^{\sum_j A_j^{(i)}x^{j+1}\frac{d}{dx}}(a_0^{(i)})^{x\frac{d}{dx}}x\Big|_{x=\frac{-1}{w}-z_i} \\
&= e^{z_i x^2\frac{d}{dx}}(z_i)^{2x\frac{d}{dx}}e^{\sum_j A_j^{(i)}x^{j+1}\frac{d}{dx}}(a_0^{(i)})^{x\frac{d}{dx}}x\Big|_{x=w+1/z_i};
\end{aligned}$$

and for $i = n_+$,

$$f_{n_+}\left(-\frac{1}{w}\right) = e^{\sum_j A_j^{(n_+)}x^{j+1}\frac{d}{dx}}(a_0^{(n_+)})^{x\frac{d}{dx}}x\Big|_{x=-\frac{1}{w}}.$$

By the definition of Ψ, we obtain

$$\begin{aligned}
&\Psi \circ \psi^L \otimes \psi^R(Q''') \\
&\quad = \big(u'_{n_+}, m_{n_+-1}(e^{-z_2 L^L(1) - \bar{z}_2 L^R(1)} z_2^{-2L^L(0)} \bar{z}_2^{-2L^R(0)} u'_2, \\
&\qquad\qquad e^{-z_{n_+-1} L^L(1) - \bar{z}_{n_+-1} L^R(1)} z_{n_+-1}^{-2L^L(0)} \bar{z}_{n_+-1}^{-2L^R(0)} u'_{n_+-1}, u'_1; \\
&\qquad -1/z_2, \overline{-1/z_2}, \ldots, -1/z_{n_+-1}, \overline{-1/z_{n_+-1}}, 0, 0)\big).
\end{aligned} \tag{2.23}$$

In the cases that $z_2, \ldots, z_{n_+-1}$ have pairwise distinct absolute values, we can expand m_{n_+-1} as a $n_+ - 1$-products of $\mathbb{Y}$s in a proper order. Then using the fact that the bilinear form $(\cdot, \cdot)$ is symmetric and invariant, it is easy to see that

$$\Psi \circ \psi^L \otimes \psi^R(Q''') = \Psi \circ \psi^L \otimes \psi^R(Q) \tag{2.24}$$

holds in these cases. The remaining cases are the complement of an open dense subset, all the cases in which have been proved. By the smoothness of m_{n_+-1} with respect to $z_2, \ldots, z_{n_+-1}$, (2.24) holds in all remaining cases. ∎

The following Lemma follows immediately from the definition of Ψ.

Lemma 2.6. *Let Q be an element of $K(n_-, n_+)$ of the form (2.7). $Q_{\pm 2}$ and Q are always sewable with Q along oppositely oriented tubes. Denote $\psi^L \otimes \psi^R(Q_{\pm 2})$ and $\psi^L \otimes \psi^R(Q)$ as $\tilde{Q}_{\pm 2}$ and $\tilde{Q}$ respectively. Then for all $1 \le k \le n_-$ and $1 \le l \le n_+$, we have*

$$\begin{aligned}
\Psi(\tilde{Q}_2)_i \widetilde{\infty}_{-k} \tilde{Q} &= \Psi(\tilde{Q}_2)_i *_{-k} \Psi(\tilde{Q}), \quad \forall i = 1, 2, \\
\Psi(\tilde{Q}_l \widetilde{\infty}_i \tilde{Q}_{-2}) &= \Psi(\tilde{Q})_l *_{-i} \Psi(\tilde{Q}_{-2}), \quad \forall i = 1, 2.
\end{aligned} \tag{2.25}$$

Theorem 2.7. *Ψ is smooth $S_- \times S_+$-equivariant and satisfies (2.12).*

Proof. The smoothness of Ψ follows from that of m_n and our construction (2.14). It is also clear that Ψ is $S_- \times S_+$-equivariant because of the permutation property of m_n.

It remains to show (2.12). Let $P \in K(m_-, m_+)$ and $Q \in K(n_-, n_+)$ have the following form:

$$\begin{aligned}
P &= ((\xi_{-m_-}, a_0^{(-m_-)}, A^{(-m_-)}), \ldots, (\xi_{-1}, a_0^{(-1)}, A^{(-1)})| \\
&\qquad (\xi_1, a_0^{(1)}, A^{(1)}), \ldots, (\xi_{m_+}, a_0^{(m_+)}, A^{(m_+)})), \\
Q &= ((\eta_{-n_-}, a_0^{(-n_-)}, A^{(-n_-)}), \ldots, (\eta_{-1}, a_0^{(-1)}, A^{(-1)})| \\
&\qquad (\eta_1, a_0^{(1)}, A^{(1)}), \ldots, (\eta_{n_+}, a_0^{(n_+)}, A^{(n_+)})).
\end{aligned} \tag{2.26}$$

Let $\tilde{P} = \psi^L \otimes \psi^R(P)$ and $\tilde{Q} = \psi^L \otimes \psi^R(Q)$. Assume that $\tilde{P}_i \widetilde{\infty}_{-j} \tilde{Q}$ exists for $1 \le i \le m_+$ and $1 \le j \le n_-$. Because Ψ is $S_- \times S_+$-equivariant, we can further assume that $i = m_+$ and $j = n_-$. By Lemma 2.5, we can choose $\xi_{-m_-} = \infty, \xi_{m_+} = 0$ and $\eta_{-n_-} = \infty, \eta_{n_+} = 0$.

Let $P' \in K(1, m_- + m_+ - 1)$ and $Q' \in K(1, n_- + n_+ - 1)$ be given as

$$P' = ((\infty, a_0^{(-m_-)}, A^{(-m_-)})|(\xi_{-m_-+1}, a_0^{(-m_-+1)}, A^{(-m_-+1)}), \dots, (0, a_0^{(m_+)}, A^{(m_+)}),$$
$$Q' = (\,(\infty, a_0^{(-n_-)}, A^{(-n_-)})|(\eta_{-n_-+1}, a_0^{(-n_-+1)}, A^{(-n_-+1)}), \dots, (0, a_0^{(n_+)}, A^{(n_+)})\,).$$

Then it is clear that

$$\begin{aligned} P &= (\dots(P'_1\infty_{-1}Q_{-2})_1\infty_{-1}\dots)_1\infty_{-1}Q_{-2}, \\ Q &= (\dots(Q'_1\infty_{-1}Q_{-2})_1\infty_{-1}\dots)_1\infty_{-1}Q_{-2}. \end{aligned} \tag{2.27}$$

Let $\tilde{P}'$ and $\tilde{Q}'$ be elements in the fiber over P and Q respectively such that

$$\begin{aligned} \tilde{P} &= (\dots(\tilde{P}'_1\widetilde{\infty}_{-1}\tilde{Q}_{-2})_1\widetilde{\infty}_{-1}\dots)_1\widetilde{\infty}_{-1}\tilde{Q}_{-2}, \\ \tilde{Q} &= (\dots(\tilde{Q}'_1\widetilde{\infty}_{-1}\tilde{Q}_{-2})_1\widetilde{\infty}_{-1}\dots)_1\widetilde{\infty}_{-1}\tilde{Q}_{-2}. \end{aligned} \tag{2.28}$$

By (2.25), we obtain

$$\begin{aligned} \Psi(\tilde{P}) &= (\dots(\Psi(\tilde{P}')_1 *_{-1} \Psi(\tilde{Q}_{-2}))_1 *_{-1} \dots)_1 *_{-1} \Psi(\tilde{Q}_{-2}), \\ \Psi(\tilde{Q}) &= (\dots(\Psi(\tilde{Q}')_1 *_{-1} \Psi(\tilde{Q}_{-2}))_1 *_{-1} \dots)_1 *_{-1} \Psi(\tilde{Q}_{-2}). \end{aligned} \tag{2.29}$$

Then we have

$$\begin{aligned} \Psi(\tilde{P})_i *_{-j} \Psi(\tilde{Q}) = &\; (\,(\dots(\Psi(\tilde{P}')_1 *_{-1} \Psi(\tilde{Q}_{-2}))_1 *_{-1} \dots)_1 *_{-1} \Psi(\tilde{Q}_{-2})\,) \\ &\; {}_i *_{-j} (\,(\dots(\Psi(\tilde{Q}')_1 *_{-1} \Psi(\tilde{Q}_{-2}))_1 *_{-1} \dots)_1 *_{-1} \tilde{Q}_{-2})\,). \end{aligned}$$

One can check easily that the right hand side of above equation can be obtained equivalently by first doing the contraction $\Psi(\tilde{P}')_{m_-+m_+-1} *_{-1} \Psi(\tilde{Q}')$ then doing the remaining contractions with $\Psi(\tilde{Q}_{-2})$. By Theorem 1.19, we have

$$\Psi(\tilde{P}')_{m_-+m_+-1} *_{-1} \Psi(\tilde{Q}') = \Psi(\tilde{P}'_{m_-+m_+-1}\widetilde{\infty}_{-1}\tilde{Q}'). \tag{2.30}$$

By (2.25) again, the remaining contractions between both sides of (2.30) and $\Psi(\tilde{Q}_{-2})$ give exactly $\Psi(\tilde{P})_i *_{-j} \Psi(\tilde{Q})$ and $\Psi(\tilde{P}\,_i\widetilde{\infty}_{-j}\tilde{Q})$ respectively. Therefore, we obtain the identity (2.12). ∎

3 Commutative associate algebras in $\mathcal{C}_{V^L\otimes V^R}$

In this section, we study the categorical formulation of conformal full field algebra over $V^L \otimes V^R$.

Let V be a vertex operator algebra, which satisfies the condition in Theorem 0.1 by our assumption. Then $\mathcal{C}_V$, the category of V-module, has a structure of vertex tensor category. We review some of the ingredients of vertex tensor category $\mathcal{C}_V$, and set our notations along the way.

There is a tensor product bifunctor $\boxtimes_{P(z)} : \mathcal{C}_V \times \mathcal{C}_V \to \mathcal{C}_V$ for each $P(z), z \in \mathbb{C}^\times$ in the sphere partial operad K. We also denote $\boxtimes_{P(1)}$ simply as $\boxtimes$.

For each V-module W, there is a left unit isomorphism $l_W : V \boxtimes W \to W$ defined by

$$l_W(v \boxtimes w) = Y_W(v,1)w, \qquad \forall v \in V, w \in W, \tag{3.1}$$

where Y_W is the vertex operator which defines the module structure on W, and a right unit isomorphism $r_W : W \boxtimes V \to W$ defined by

$$r_W(w \boxtimes v) = e^{L(-1)} Y_W(v,-1)w, \qquad \forall v \in V, w \in W. \tag{3.2}$$

Let W_1 and W_2 be V-modules. For a given path $\gamma \in \mathbb{C}^\times$ from a point z_1 to z_2, there is a parallel isomorphism associated to this path

$$\mathcal{T}_\gamma : W_1 \boxtimes_{P(z_1)} W_2 \longrightarrow W_1 \boxtimes_{P(z_2)} W_2.$$

Let $\mathcal{Y}$ be the intertwining operator corresponding to the intertwining map $\boxtimes_{P(z_2)}$ and $l(z_1)$ the value of the logarithm of z_1 determined by $\log z_2$ and analytic continuation along the path γ. For $w_1 \in W_1, w_2 \in W_2$, the natural extension of $\mathcal{T}_\gamma$, denoted as $\overline{\mathcal{T}_\gamma}$, is defined as

$$\overline{\mathcal{T}_\gamma}(w_1 \boxtimes_{P(z_1)} w_2) = \mathcal{Y}(w_1, e^{l(z_1)})w_2,$$

which uniquely determines $\mathcal{T}_\gamma$ since the homogeneous components of $w_1 \boxtimes_{P(z_1)} w_2$ span the module $W_1 \boxtimes_{P(z_2)} W_2$. Moreover, the parallel isomorphism depends only on the homotopy class of γ.

A remark of our notation, we use "overline" for both complex conjugation and natural extension of morphisms in tensor categories. There should be no confusion because they act on completely different things.

There is an associativity isomorphism, for each triple of V-modules W_1, W_2, W_3,

$$\mathcal{A}_{P(z_1),P(z_2)}^{P(z_1-z_2),P(z_2)} : W_1 \boxtimes_{P(z_1)} (W_2 \boxtimes_{P(z_2)} W_3) \to (W_1 \boxtimes_{P(z_1-z_2)} W_2) \boxtimes_{P(z_2)} W_3,$$

which is characterized by

$$\overline{\mathcal{A}_{P(z_1),P(z_2)}^{P(z_1-z_2),P(z_2)}}(w_{(1)} \boxtimes_{P(z_1)} (w_{(2)} \boxtimes_{P(z_2)} w_{(3)})) = (w_{(1)} \boxtimes_{P(z_1-z_2)} w_{(2)}) \boxtimes_{P(z_2)} w_{(3)} \tag{3.3}$$

for $w_{(i)} \in W_i, i = 1,2,3$. Let $z_1 > z_2 > z_1 - z_2 > 0$. The associativity isomorphism $\mathcal{A}$ of the braided tensor category is

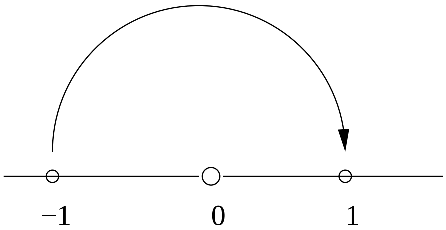
$$\mathcal{A} = \mathcal{T}_{\gamma_3} \circ (\mathcal{T}_{\gamma_4} \boxtimes_{P(z_2)} I) \circ \mathcal{A}_{P(z_1),P(z_2)}^{P(z_1-z_2),P(z_2)} \circ (I \boxtimes_{P(z_1)} \mathcal{T}_{\gamma_2}) \circ \mathcal{T}_{\gamma_1}. \tag{3.4}$$

where γ_1 and γ_2 are paths in $\mathbb{R}_+$ from 1 to z_1 and z_2, respectively; and γ_3 and γ_4 are paths in $\mathbb{R}_+$ from z_2 and $z_1 - z_2$ to 1, respectively.

There is also a braiding isomorphism $\mathcal{R}^+_{W_1W_2} : W_1 \boxtimes W_2 \to W_2 \boxtimes W_1$ for each pair of V-modules W_1, W_2, defined as

$$\overline{\mathcal{R}^+_{W_1W_2}}(w_1 \boxtimes w_2) = e^{L(-1)}\overline{\mathcal{T}}_{\gamma_+}(w_2 \boxtimes_{P(-1)} w_1), \tag{3.5}$$

where γ_+ is a path from -1 to 1 inside upper half plane as shown in the following diagram.

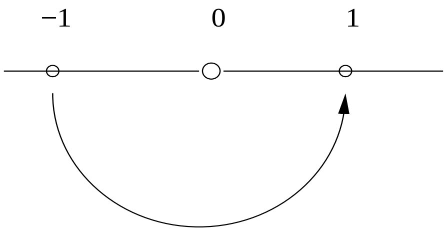


(3.6)

The inverse of $\mathcal{R}^+_{W_1W_2}$ is denoted as $\mathcal{R}^-_{W_1W_2}$, which is characterized by

$$\overline{\mathcal{R}^-_{W_1W_2}}(w_1 \boxtimes w_2) = e^{L(-1)}\overline{\mathcal{T}}_{\gamma_-}(w_2 \boxtimes_{P(-1)} w_1), \tag{3.7}$$

where γ_- is a path in the lower half plane as shown in the following picture.



(3.8)

For simplicity, we often write $\mathcal{R}^+_{W_1W_2}$ and $\mathcal{R}^-_{W_1W_2}$ as $\mathcal{R}_+$ and $\mathcal{R}_-$ respectively. Notice that we change the superscripts to subscripts because we need save space for some superscripts needed later.

Let $W_i, i = 1, 2, 3$ be V-modules and $\mathcal{V}^{W_3}_{W_1W_2}, \mathcal{V}^{W_3}_{W_2W_1}$ be the space of intertwining operators of type $\binom{W_3}{W_1W_2}$ and $\binom{W_3}{W_2W_1}$. We have isomorphisms $\Omega_r : \mathcal{V}^{W_3}_{W_1W_2} \to \mathcal{V}^{W_3}_{W_2W_1}, r \in \mathbb{Z}$ for V-modules $W_i, i = 1, 2, 3$, given as follow:

$$\Omega_r(\mathcal{Y})(w_2, z)w_1 = e^{zL(-1)}\mathcal{Y}(w_1, e^{(2r+1)\pi i}z)w_2, \tag{3.9}$$

for $\mathcal{Y} \in \mathcal{V}^{W_3}_{W_1W_2}$.

For any $\mathcal{Y} \in \mathcal{V}^{W_3}_{W_1W_2}$ and $z \in \mathbb{C}^\times$, by the universal property of tensor product[HL1]-[HL4], there is a unique module map $W_1 \boxtimes_{P(z)} W_2 \to W_3$ associated with $\mathcal{Y}$. We denote it as $m_{\mathcal{Y}}^{P(z)}$. We also denote $m_{\mathcal{Y}}^{P(1)}$ simply as $m_{\mathcal{Y}}$. For $z_1, z_2 > 0$, let γ be a path in $\mathbb{R}_+$ from z_1 to z_2, then we have

$$m_{\mathcal{Y}}^{P(z_2)} \circ \mathcal{T}_\gamma = m_{\mathcal{Y}}^{P(z_1)}. \tag{3.10}$$

Proposition 3.1. $m_{\Omega_0(\mathcal{Y})} = m_{\mathcal{Y}} \circ \mathcal{R}_+$, $m_{\Omega_{-1}(\mathcal{Y})} = m_{\mathcal{Y}} \circ \mathcal{R}_-$.

Proof. Let $u_1 \in W_1, u_2 \in W_2$ and $\mathcal{Y}_0$ be the intertwining operator associated with intertwining map $\boxtimes : W_1 \otimes W_2 \to W_1 \boxtimes W_2$. Let γ_+ be the path given in (3.6). Then we have

$$\begin{aligned}
\overline{m_{\mathcal{Y}}} \circ \overline{\mathcal{R}_+}(u_2 \boxtimes u_1) &= \overline{m_{\mathcal{Y}}}(e^{L(-1)}\overline{\mathcal{T}}_{\gamma_+}(u_1 \boxtimes_{P(-1)} u_2)) \\
&= e^{L(-1)}\overline{m_{\mathcal{Y}}}(\mathcal{Y}_0(u_1, e^{\pi i})u_2) \\
&= e^{L(-1)}\overline{m_{\mathcal{Y}}}(e^{\pi i L(0)}(e^{-\pi i L(0)}u_1 \boxtimes e^{-\pi i L(0)}u_2) \\
&= e^{L(-1)}e^{\pi i L(0)}\mathcal{Y}(e^{-\pi i L(0)}u_1, 1)e^{-\pi i L(0)}u_2 \\
&= e^{L(-1)}\mathcal{Y}(u_1, e^{\pi i})u_2 \\
&= \Omega_0(\mathcal{Y})(u_2, 1)u_1 \\
&= m_{\Omega_0(\mathcal{Y})}(u_2 \boxtimes u_1).
\end{aligned} \tag{3.11}$$

Hence we obtain the first identity. The proof of the second identity is exactly same. ∎

The following Theorem is proved in [HK2].

Theorem 3.2. *A conformal full field algebra over $V^L \otimes V^R$ is equivalent to a module F for the vertex operator algebra $V^L \otimes V^R$ equipped with an intertwining operator $\mathcal{Y}$ of type $\binom{F}{FF}$ and an injective linear map $\rho : V^L \otimes V^R \to F$, satisfying the following conditions:*

1. *The identity property:* $\mathcal{Y}(\rho(\mathbf{1}^L \otimes \mathbf{1}^R), x) = I_F$.

2. *The creation property: For $u \in F$,* $\lim_{x\to 0} \mathcal{Y}(u, x)\rho(\mathbf{1}^L \otimes \mathbf{1}^R) = u$.

3. *The associativity: for $u, v, w \in F$ and $w' \in F'$,*

$$\begin{aligned}
&\langle w', \mathcal{Y}(u; z_1, \zeta_1)\mathcal{Y}(v; z_1, \zeta_1)w\rangle \\
&\qquad = \langle w', \mathcal{Y}(\mathcal{Y}(u; z_1 - z_2, \zeta_1 - \zeta_2)v; z_2, \zeta_2)w\rangle
\end{aligned} \tag{3.12}$$

   when $|z_1| > |z_2| > 0$ and $|\zeta_1| > |\zeta_2| > 0$.

4. *The single-valuedness property:*

$$e^{2\pi i(L^L(0) - L^R(0))} = I_F. \tag{3.13}$$

5. *The skew symmetry:*

$$\mathbb{Y}^{\mathcal{Y}}(u,1,1)v = e^{L^L(-1)+L^R(-1)}\mathbb{Y}^{\mathcal{Y}}(v,e^{\pi i},e^{-\pi i})u. \tag{3.14}$$

Corollary 3.3. *The category of conformal full field algebras over $V^L \otimes V^R$ is isomorphic to the category of open-string vertex operator algebras which contain the vertex operator algebra $V^L \otimes V^R$ in its meromorphic center and satisfy the single-valuedness condition (3.13) and the skew symmetry (3.14).*

Proof. The proof is almost obvious, except that the associativity axiom in Theorem 3.2 might looks stronger than that of open-string vertex operator algebra, which only requires (3.12) to hold for $z_1 = \zeta_1 > 0$ and $z_2 = \zeta_2 > 0$. However, under the strong condition that both V^L and V^R satisfy the condition in Theorem 0.1 and the splitting property of $\mathcal{Y}$ in this case, one can show that these two associativities are actually equivalent. ∎

The category of V^L- (V^R-) modules, denoted as $\mathcal{C}_{V^L}$ ($\mathcal{C}_{V^R}$), has a natural structure of vertex tensor category and semisimple braided tensor category. The unit isomorphisms, the associativity isomorphisms and the braiding isomorphisms in $\mathcal{C}_{V^L}$ and $\mathcal{C}_{V^R}$ are given in the same way as (3.1), (3.2), (3.3), (3.5) and (3.7). We denote them as $l^L, r^L, \mathcal{A}^L, \mathcal{R}^L_{\pm}$ and $l^R, r^R, \mathcal{A}^R, \mathcal{R}^R_{\pm}$ respectively.

Now we consider the category of $V^L \otimes V^R$-modules, denoted as $\mathcal{C}_{V^L\otimes V^R}$. It was proved in [HK2] that $V^L \otimes V^R$ also satisfies the condition in Theorem 0.1. Thus by Theorem 0.1, the category $\mathcal{C}_{V^L\otimes V^R}$ has a structure of semisimple braided tensor category.

We would like to take a closer look at the tensor product in $\mathcal{C}_{V^L\otimes V^R}$. Let $A = A^L \otimes A^R$, $B = B^L \otimes B^R$ and $C = C^L \otimes C^R$ be $V^L \otimes V^R$-modules. Let $\mathcal{Y}^L \in \mathcal{V}_{A^LB^L}^{C^L}$ and $\mathcal{Y}^R \in \mathcal{V}_{A^RB^R}^{C^R}$. Let $\sigma : A \otimes B \to (A^L \otimes B^L) \otimes (A^R \otimes B^R)$ be the map defined as

$$\sigma : (a^L \otimes a^R) \otimes (b^L \otimes b^R) \mapsto (a^L \otimes b^L) \otimes (a^R \otimes b^R)$$

for $a^L \in A^L, a^R \in A^R, b^L \in B^L, b^R \in B^R$. The following result is due to Dong, Mason and Zhu [DMZ].

Proposition 3.4. $(\mathcal{Y}^L \otimes \mathcal{Y}^R) \circ \sigma \in \mathcal{V}_{AB}^C$ *and* $\mathcal{V}_{A^LB^L}^{C^L} \otimes \mathcal{V}_{A^RB^R}^{C^R} \cong \mathcal{V}_{AB}^C$ *canonically.*

For $z \in \mathbb{C}^\times$, the $P(z)$-tensor product ([HL1]-[HL4]) of any two $V^L\otimes V^R$-modules A and B, is a $V^L \otimes V^R$-module $A \boxtimes_{P(z)} B$, together with a $P(z)$-intertwining map $\boxtimes_{P(z)}$ from $A \otimes B$ to $\overline{A \boxtimes_{P(z)} B}$, satisfying the universal property that given any $V^L \otimes V^R$-modules C and a $P(z)$-intertwining map $G : A \otimes B \to \overline{C}$, there is a unique module map $\eta : A \boxtimes_{P(z)} B \to C$ such that $G = \overline{\eta} \circ \boxtimes_{P(z)}$.

Lemma 3.5. *Let $A = A^L \otimes A^R$ and $B = B^L \otimes B^R$. Then $(A^L \boxtimes_{P(z)} B^L) \otimes (A^R \boxtimes_{P(z)} B^R)$ together with the $P(z)$-intertwining map $(\boxtimes_{P(z)} \otimes \boxtimes_{P(z)}) \circ \sigma$ from $A \otimes B$ to $\overline{(A^L \boxtimes_{P(z)} B^L) \otimes (A^R \boxtimes_{P(z)} B^R)}$ is a $P(z)$-tensor product of A and B.*

Proof. We will only show the case $z = 1$. The proof of general case is the same. Given any $V^L \otimes V^R$-module C and a decomposition $C = \coprod_{i=1}^n C_i^L \otimes C_i^R$ or equivalently a family of morphism $\pi_i : C \to C_i^L \otimes C_i^R$ and $\iota_i : C_i^L \otimes C_i^R \to C$ satisfying the following conditions

$$\pi_i \circ \iota_i = I_{C_i^L \otimes C_i^R}; \quad \pi_j \iota_i = 0, \text{for } i \neq j; \quad \sum_i \iota_i \pi_i = I_C. \tag{3.15}$$

Assume f is a $P(1)$-intertwining map from $(A^L \otimes A^R) \otimes (B^L \otimes B^R)$ to $\overline{C}$. It induces a $P(1)$-intertwining map $f_i : A \otimes B \to \overline{C_i^L \otimes C_i^R}$ for each i. By Proposition 3.4, $f_i = \sum_k (f_k^L \otimes f_k^R) \circ \sigma$ where f_k^L, f_k^R are intertwining maps $A^L \otimes B^L \to \overline{C_i^L}$, $A^R \otimes B^R \to \overline{C_i^R}$ respectively. By the universal property of $\boxtimes$, we know that there is a unique module map module map $\eta_i = \sum_k \eta_{i;k}^L \otimes \eta_{i;k}^R$, where $\eta_{i;k}^L : A^L \boxtimes B^L \to C_i^L$ and $\eta_{i;k}^R : A^R \boxtimes B^R \to C_i^R$, such that the following diagram

$$\begin{tikzcd}
A \otimes B \arrow[r, "(\boxtimes\otimes\boxtimes)\circ\sigma"] \arrow[d, "f"'] \arrow[dr, "f_i"] & (A^L \boxtimes B^L) \otimes (A^R \boxtimes B^R) \arrow[d, dashed, "\exists!\, \eta_i"] \\
C \arrow[r, "\pi_i"] & C_i^L \otimes C_i^R
\end{tikzcd} \tag{3.16}$$

is commutative. By the universal property of direct product, there is a unique module map η from $(A^L \boxtimes B^L) \otimes (A^R \boxtimes B^R)$ to C such that $\pi_i \circ \eta = \eta_i$. Hence we have

$$\pi_i \circ f = \pi_i \circ \eta \circ (\boxtimes \otimes \boxtimes) \circ \sigma \tag{3.17}$$

Composing two sides of above equation with ι_i from left and sum up all i, we obtain the equality

$$f = \eta \circ (\boxtimes \otimes \boxtimes) \circ \sigma. \tag{3.18}$$

Moreover, the solution of η satisfying (3.18) is unique because each $\pi_i \eta$ gives the unique solution of (3.17) and C as a direct product satisfies the universal property.

We have proved that $(A^L \boxtimes B^L) \otimes (A^R \boxtimes B^R)$ together with $(\boxtimes \otimes \boxtimes) \circ \sigma$ satisfies the universal properties of tensor product, thus gives a $P(1)$-tensor product of A and B. ∎

There are only finite number of equivalent classes of simple objects in $\mathcal{C}_{V^L \otimes V^R}$. Simple objects in $\mathcal{C}_{V^L \otimes V^R}$ are objects of form $W^L \otimes W^R$, where W^L and W^R are simple V^L-module and simple V^R-module respectively [FHL]. Every object in

$\mathcal{C}_{V^L\otimes V^R}$ is a direct sum of simple objects. If two objects A and B has the following decomposition:

$$A = \coprod_{i=1}^{m} A_i^L \otimes A_i^R, \quad B = \coprod_{j=1}^{n} B_j^L \otimes B_j^R, \tag{3.19}$$

then by Lemma 3.5, $\coprod_{i,j}(A_i^L \boxtimes B_j^L) \otimes (A_i^L \boxtimes B_j^L)$ is a tensor product of A and B. Therefore the universal property of tensor product provides a canonical isomorphism:

$$\coprod_{i,j}(A_i^L \boxtimes B_j^L) \otimes (A_i^L \boxtimes B_j^L) \cong A \boxtimes B. \tag{3.20}$$

By Huang's construction given in Theorem 0.1, there is a braiding structure in $\mathcal{C}_{V^L\otimes V^R}$. However, in this work, we are interested in a different braiding structure in $\mathcal{C}_{V^L\otimes V^R}$. For each A and B as in (3.19), we define $\mathcal{R}_{+-}$ by the following commutative diagram:

$$\begin{tikzcd}
\coprod_{i=1}^{M} \coprod_{j=1}^{N} (A_i^L \boxtimes B_j^L) \otimes (A_i^R \boxtimes B_j^R) \arrow[r, "\cong"] \arrow[d, "\mathcal{R}_+^L\otimes\mathcal{R}_-^R"'] & A \boxtimes B \arrow[d, dotted, "\exists!\ \mathcal{R}_{+-}"] \\
\coprod_{i=1}^{M} \coprod_{j=1}^{N} (B_j^L \boxtimes A_i^L) \otimes (B_j^R \boxtimes A_i^R) \arrow[r, "\cong"] & B \boxtimes A,
\end{tikzcd} \tag{3.21}$$

where the two horizontal maps are the canonical isomorphisms as given in (3.20). $\mathcal{R}_+^L \otimes \mathcal{R}_-^R$ is an isomorphism. So is $\mathcal{R}_{+-}$.

Remark 3.6. If we replace the $\mathcal{R}_+^L \otimes \mathcal{R}_-^R$ in the diagram above by $\mathcal{R}_+^L \otimes \mathcal{R}_+^R$, or $\mathcal{R}_-^L \otimes \mathcal{R}_+^R$, or $\mathcal{R}_-^L \otimes \mathcal{R}_-^R$, we will obtain a new morphism in each case. We denoted them as $\mathcal{R}_{++}$, $\mathcal{R}_{-+}$ and $\mathcal{R}_{--}$ respectively. They are all isomorphisms.

Remark 3.7. The horizontal isomorphisms in the diagram (3.21) are induced from universal property of tensor product. As a consequence, such defined $\mathcal{R}_{\pm\pm}$ and $\mathcal{R}_{\pm\mp}$ are all independent of decompositions.

Proposition 3.8. *Each of $\mathcal{R}_{++}$, $\mathcal{R}_{+-}$, $\mathcal{R}_{-+}$ and $\mathcal{R}_{--}$ gives $\mathcal{C}_{V^L\otimes V^R}$ a structure of braided tensor category.*

Proof. It is amount to show that all four isomorphisms satisfy both the functorial properties and the hexagon relations. Because the braiding isomorphism is naturally induced from $\mathcal{R}_+^L \otimes \mathcal{R}_-^R$, it is routine to check that both the functorial properties and the hexagon relations follow from those properties of $\mathcal{R}_+^L$ and $\mathcal{R}_-^R$. ∎

Remark 3.9. It is not hard to see that the braiding isomorphism constructed in Theorem 0.1 is just $\mathcal{R}_{++}$.

From now on, we will only consider the braiding tensor category structure of $\mathcal{C}_{V^L\otimes V^R}$ given by $\mathcal{R}_{+-}$. In order to emphasis our choice of braiding, sometimes we will denote the category $\mathcal{C}_{V^L\otimes V^R}$ as $(\mathcal{C}_{V^L\otimes V^R}, \mathcal{R}_{+-})$.

For any object $A \in \mathcal{C}_{V^L\otimes V^R}$, we also define an isomorphism $\theta_A : A \to A$, called twist, as follow

$$\theta_A = e^{2\pi i L^L(0)} \otimes e^{-2\pi i L^R(0)}. \tag{3.22}$$

Let $A = \coprod_i A_i^L \otimes A_i^R$, $B = \coprod_j B_j^L \otimes B_j^R$, $C = \coprod_k C_k^L \otimes C_k^R$ be $V^L \otimes V^R$-modules. $\Omega_0 \otimes \Omega_{-1}$ acts on the space $\coprod_{i,j,k} \mathcal{V}_{A_i^L B_j^L}^{C_k^L} \otimes \mathcal{V}_{A_i^R B_j^R}^{C_k^R}$ naturally as an automorphism. By the canonical isomorphism given in Proposition 3.4, $\coprod_{i,j,k} \mathcal{V}_{A_i^L B_j^L}^{C_k^L} \otimes \mathcal{V}_{A_i^R B_j^R}^{C_k^R}$ also canonically isomorphic to $\mathcal{V}_{AB}^C$. Therefore, we obtain an action of $\Omega_0 \otimes \Omega_{-1}$ on $\mathcal{V}_{AB}^C$ as an automorphism.

Let $\mathcal{Y} \in \mathcal{V}_{AB}^C$ and $m_{\mathcal{Y}}$ the unique module map $A \boxtimes B \to C$ associated with $\mathcal{Y}$. By Proposition 3.1, it is easy to obtain the following Lemma:

Lemma 3.10. $m_{\Omega_0\otimes\Omega_{-1}(\mathcal{Y})} = m_{\mathcal{Y}} \circ \mathcal{R}_{+-}$.

Theorem 3.11. *The following two notions are equivalent in the sense that the categories given by these notions are isomorphic:*

1. *A conformal full field algebra over $V^L \otimes V^R$, (F, m, ρ).*

2. *A commutative associative algebra (F, m, ι) in the braided tensor category $(\mathcal{C}_{V^L\otimes V^R}, \mathcal{R}_{+-})$ satisfying $\theta_F = I_F$.*

Proof. By the Corollary 3.3 and the result in [HK1], the first three conditions in Theorem 3.2 exactly amounts to an associative algebra in $\mathcal{C}_{V^L\otimes V^R}$. In particular, data in the these two structures are related as follow: $\rho = \iota$ and $m_{\mathbb{Y}} = m$, where $\mathbb{Y}_f$ is the formal vertex operator associated with a conformal full field algebra and is an intertwining operator of $V^L \otimes V^R$ in this case.

It is also obvious that single-valuedness property (3.13) is equivalent to $\theta_F = I_F$. Notice that the skew symmetry (3.14) is equivalent to the following condition:

$$\Omega_0 \otimes \Omega_{-1}(\mathbb{Y}_f) = \mathbb{Y}_f.$$

By Lemma 3.10, it is manifest that the skew symmetry (3.14) exactly amounts to the commutativity of the corresponding associative algebra F in $\mathcal{C}_{V^L\otimes V^R}$. ∎

4 Frobenius algebras in $\mathcal{C}_{V^L\otimes V^R}$

In this section, we will give a categorical formulation of a conformal full field algebra over $V^L \otimes V^R$ equipped with a nondegenerate invariant bilinear form.

4.1 $\tilde{A}_r$ and $\hat{A}_r$

In this subsection, we fix a vertex operator algebra V.

Given an intertwining operator $\mathcal{Y}$ of type $\binom{W_3}{W_1 W_2}$ of V and an integer $r \in \mathbb{Z}$, the so called *r-contragredient operator of $\mathcal{Y}$* ([HL2]) was defined to be the linear map

$$\begin{aligned} W_1 \otimes W_3' &\to W_2'\{z\} \\ w_{(1)} \otimes w'_{(3)} &\mapsto A_r(\mathcal{Y})(w_{(1)}, x)w'_{(3)} \end{aligned} \tag{4.1}$$

given by

$$\begin{aligned} &\langle A_r(\mathcal{Y})(w_{(1)}, x)w'_{(3)}, w_{(2)}\rangle = \\ &\quad = \langle w'_{(3)}, \mathcal{Y}(e^{xL(1)}e^{(2r+1)\pi i L(0)}x^{-2L(0)}w_{(1)}, x^{-1})w_{(2)}\rangle, \end{aligned} \tag{4.2}$$

where $w_{(1)} \in W_1$, $w_{(2)} \in W_2$, $w'_{(3)} \in W'_{(3)}$. The following proposition was proved in [HL4].

Proposition 4.1. *The r-contragredient operator $A_r(\mathcal{Y})$ of an intertwining operator $\mathcal{Y}$ of type $\binom{W_3}{W_1 W_2}$ is an intertwining operator of type $\binom{W_2'}{W_1 W_3'}$. Moreover,*

$$A_{-r-1}(A_r(\mathcal{Y})) = A_r(A_{-r-1}(\mathcal{Y})) = \mathcal{Y}. \tag{4.3}$$

In particular, the correspondence $\mathcal{Y} \mapsto A_r(\mathcal{Y})$ defines a linear isomorphism from $\mathcal{V}_{W_1 W_2}^{W_3}$ to $\mathcal{V}_{W_1 W_3'}^{W_2'}$, and we have $N_{W_1 W_2}^{W_3} = N_{W_1 W_3'}^{W_2'}$.

It turns out that A_r is very hard to work with in the tensor category because they are simply not categorical. To fix the problem, we will introduce two slightly different operators.

Given an intertwining operator $\mathcal{Y}$ of type $\binom{W_3}{W_1 W_2}$ and an integer $r \in \mathbb{Z}$, we define two operators $\tilde{A}_r(\mathcal{Y})$ and $\hat{A}_r(\mathcal{Y})$ as

$$\begin{aligned} \tilde{A}_r(\mathcal{Y})(\cdot, x) &= e^{-(2r+1)\pi i L(0)}A_r(\mathcal{Y})(\cdot, x)e^{(2r+1)\pi i L(0)}, \\ \hat{A}_r(\mathcal{Y})(\cdot, x) &= e^{-(2r+1)\pi i L(0)}A_{-r-1}(\mathcal{Y})(\cdot, x)e^{(2r+1)\pi i L(0)}, \end{aligned} \tag{4.4}$$

or equivalently,

$$\begin{aligned} \langle \tilde{A}_r(\mathcal{Y})(w_1, e^{(2r+1)\pi i}x)w_3', w_2\rangle &= \langle w_3', \mathcal{Y}(e^{xL(1)}x^{-2L(0)}w_1, x^{-1})w_2\rangle, \\ \langle \hat{A}_r(\mathcal{Y})(w_1, x)w_3', w_2\rangle &= \langle w_3', \mathcal{Y}(e^{-xL(1)}x^{-2L(0)}w_1, e^{(2r+1)\pi i}x^{-1})w_2\rangle, \end{aligned} \tag{4.5}$$

for $w_1 \in W_1$, $w_2 \in W_2$, $w_3' \in W_3'$. In particular, in the case $W_1 = V$ and $W_2 = W_3 = W$, $\tilde{A}_r(Y_W)$ and $\hat{A}_r(Y_W)$ related to contragredient vertex operator Y_W' for any $r \in \mathbb{Z}$ as follow

$$\tilde{A}_r(Y_W)(\cdot, x) = \hat{A}_r(Y_W)(\cdot, x) = (-1)^{-L(0)}Y_W'(\cdot, x)(-1)^{L(0)}. \tag{4.6}$$

Lemma 4.2. *Let (W, Y_W) be a V-module. Let $\tilde{Y}_W$ be defined as*

$$\tilde{Y}_W(\cdot, x) = a^{-L(0)} Y_W(\cdot, x) a^{L(0)}.$$

for $a \in \mathbb{C}^\times$. Then $(W, \tilde{Y}_W)$ gives W another module structure which is isomorphic to (W, Y_W). Moreover the isomorphism is given by $w \mapsto a^{-L(0)} w$.

Proof. It is clear that $\tilde{Y}_W(\mathbf{1}, x) = Y_W(\mathbf{1}, a^{-1}x) = I_W$. For any $u, v \in V$ and $N \in \mathbb{N}$ large enough, we have

$$\begin{aligned}
&(x_0 + x_2)^N \tilde{Y}_W(u, x_0 + x_2) \tilde{Y}_W(v, x_2) w \\
&\quad = (x_0 + x_2)^N a^{-L(0)} Y_W(u, x_0 + x_2) Y_W(v, x_2) a^{L(0)} w \\
&\quad = (x_0 + x_2)^N a^{-L(0)} Y_W(Y(u, x_0)v, x_2) a^{L(0)} w \\
&\quad = (x_0 + x_2)^N \tilde{Y}_W(Y(u, x_0)v, x_2) w.
\end{aligned} \tag{4.7}$$

The Jacobi identity follows from the weak associativity ([LL]). Hence $(W, \tilde{Y}_W)$ is a V-module. Now we show that it is isomorphic to (W, Y). Let $f : W \to W$ be so that $f(w) = a^{-L(0)} w$ for $w \in W$. Then we have

$$f(Y_W(u, x)w) = a^{-L(0)} Y_W(u, x) w = a^{-L(0)} Y_W(u, x) a^{L(0)} f(w) = \tilde{Y}(u, x) f(w).$$

Hence f gives an isomorphism between two module structures on W. ∎

By the above Proposition, we see that $(W', \tilde{A}_r(Y_W))$ (or equivalently $(W', \hat{A}_r(Y_W))$) gives on W' another module structure , which is isomorphic to (W', Y'). A difference of these two module structures on W' worth of knowing and frequently used in the later sections is the following relation:

$$\langle w', L(n) w \rangle = (-1)^n \langle L'(n) w', w \rangle, \quad n \in \mathbb{Z}, \tag{4.8}$$

for $w \in W, w' \in W'$.

The notion of intertwining operator depends on the choices of module structures on the three modules involved. The following results is an analogue of Proposition 4.1.

Proposition 4.3. *If $\mathcal{Y}$ is an intertwining operator of type $\binom{(W_3, Y_3)}{(W_1, Y_1)(W_2, Y_2)}$, then $\tilde{A}_r(\mathcal{Y})$, $\hat{A}_r(\mathcal{Y})$ are intertwining operators of type $\binom{(W_2', \tilde{A}_r(Y_2))}{(W_1, Y_1)(W_3', \tilde{A}_r(Y_3))}$ for each $r \in \mathbb{Z}$. Moreover, we have*

$$\tilde{A}_r \circ \hat{A}_r(\mathcal{Y}) = \hat{A}_r \circ \tilde{A}_r(\mathcal{Y}) = \mathcal{Y}. \tag{4.9}$$

Proof. It is routine to show that $\tilde{A}_r(\mathcal{Y})$, $\hat{A}_r(\mathcal{Y})$ are intertwining operators of type $\binom{(W_2', \tilde{A}_r(Y_2))}{(W_1, Y_1)(W_3', \tilde{A}_r(Y_3))}$ for each $r \in \mathbb{Z}$.

We only prove (4.9) here. First, we have

$$\begin{aligned}\langle \hat{A}_r(\tilde{A}_r(\mathcal{Y})(u,x)w',w\rangle &= \langle w', \tilde{A}_r(\mathcal{Y})(e^{-xL(1)}x^{-2L(0)}u, e^{(2r+1)\pi i}x^{-1})w\rangle \\ &= \langle \mathcal{Y}(e^{x^{-1}L(1)}x^{2L(0)}e^{-xL(1)}x^{-2L(0)}u, x)w', w\rangle \\ &= \langle \mathcal{Y}(u,x)w', w\rangle.\end{aligned}$$

Second, we have

$$\begin{aligned}&\langle \tilde{A}_r(\hat{A}_r(\mathcal{Y}))(u, e^{(2r+1)\pi i}x)w', w\rangle \\ &\quad = \langle w', \hat{A}_r(\mathcal{Y})(e^{xL(1)}x^{-2L(0)}u, x^{-1})w\rangle \\ &\quad = \langle \mathcal{Y}(e^{-x^{-1}L(1)}x^{2L(0)}e^{xL(1)}x^{-2L(0)}u, e^{(2r+1)\pi i}x)w', w\rangle \\ &\quad = \langle \mathcal{Y}(u, e^{(2r+1)\pi i}x)w', w\rangle.\end{aligned}$$

Therefore we obtain (4.9). ∎

The correspondence

$$\mathcal{Y} \mapsto \tilde{A}_r(\mathcal{Y}), \qquad \mathcal{Y} \mapsto \hat{A}_r(\mathcal{Y})$$

defines two linear isomorphisms:

$$\mathcal{V}^{(W_3,Y_{W_3})}_{(W_1,Y_{W_1})(W_2,Y_{W_2})} \to \mathcal{V}^{(W_2',\tilde{A}_r(Y_{W_2}))}_{(W_1,Y_{W_1})(W_3',\tilde{A}_r(Y_{W_3}))}. \tag{4.10}$$

Obviously, we have

$$N^{(W_3,Y_{W_3})}_{(W_1,Y_{W_1})(W_2,Y_{W_2})} = N^{(W_2',\tilde{A}_r(Y_{W_2}))}_{(W_1,Y_{W_1})(W_3',\tilde{A}_r(Y_{W_3}))}. \tag{4.11}$$

In this work, we only use $\tilde{A}_r$ or $\hat{A}_r$ instead of A_r. Therefore, to simplify notations, we simply denote $(W', \tilde{A}_r(Y_W))$ as W'.

In [MS][H10], the S_3-action on the space of intertwining operators $\mathcal{V}$ are used. In this work, we will use Ω_r, $\tilde{A}_r$ and $\hat{A}_r$ instead. So we won't see a S_3 action anymore. However, the cyclic subgroup $\mathbb{Z}_3$ of S_3 still appear here. Let $\mathcal{Y} \in \mathcal{V}_{a_1a_2}^{a_3}$. It is easy to see that

$$\begin{aligned}&\langle e^{-xL(-1)}\Omega_r(\tilde{A}_r(\mathcal{Y}))(w_{a_3}', x)w_{a_1}, w_{a_2}\rangle \\ &\qquad\qquad = \langle w_{a_3}', \mathcal{Y}(e^{xL(1)}x^{-2L(0)}w_{a_1}, x^{-1})w_{a_2}\rangle \end{aligned}\tag{4.12}$$

for $w_{a_1} \in W_{a_1}, w_{a_2} \in W_{a_2}, w_{a_3}' \in W_{a_3}'$. It is clear that $\Omega_r \circ \tilde{A}_r$ is independent of $r \in \mathbb{Z}$. We denote it as σ_{123}.

Proposition 4.4. $\sigma_{123}^3 = I_{\mathcal{V}}$.

Proof. Keep in mind the relation (4.8). For $w_3' \in W_3', w_1 \in W_1, w_2 \in W_2$ and $\mathcal{Y} \in \mathcal{V}_{a_1a_2}^{a_3}$, we have

$$\begin{aligned}
&\langle w_3', \sigma_{123}^3(\mathcal{Y})(w_1, x)w_2\rangle \\
&\quad = \langle \sigma_{123}^2(\mathcal{Y})(e^{xL(1)}x^{-2L(0)}w_2, x^{-1})e^{-xL(1)}w_3', w_1\rangle \\
&\quad = \langle e^{xL(1)}x^{-2L(0)}w_2, \sigma_{123}(\mathcal{Y})(e^{x^{-1}L(1)}x^{2L(0)}e^{-xL(1)}w_3', x)e^{-x^{-1}L(1)}w_1\rangle \\
&\quad = \langle e^{xL(1)}x^{-2L(0)}w_2, \sigma_{123}(\mathcal{Y})(x^{2L(0)}w_3', x)e^{-x^{-1}L(1)}w_1\rangle \\
&\quad = \langle \mathcal{Y}(e^{xL(1)}x^{-2L(0)}e^{-x^{-1}L(1)}w_1, x^{-1})e^{-xL(1)}e^{xL(1)}x^{-2L(0)}w_2, x^{2L(0)}w_3'\rangle \\
&\quad = \langle \mathcal{Y}(x^{-2L(0)}w_1, x^{-1})x^{-2L(0)}w_2, x^{2L(0)}w_3'\rangle \\
&\quad = \langle \mathcal{Y}(w_1, x)w_2, w_3'\rangle. \qquad (4.13)
\end{aligned}$$

∎

The inverse of σ_{123} can be expressed as $\hat{A}_r \circ \Omega_{-r-1}$. We denote it as σ_{132}. Both σ_{123} and σ_{132} play similar roles as those in [H10].

4.2 Modular tensor categories

In this subsection, we review Huang's construction of duality maps [H11] in terms of our new conventions.

From now on, we assume V satisfies the condition in Theorem 4.5. By assumption, $V' \cong V$, i.e. $e' = e$. From [FHL], there is a nondegenerate invariant bilinear form $(\cdot, \cdot)$ on V such that $(\mathbf{1}, \mathbf{1}) = 1$. In the rest of this work, we will simply identify V' with V without making the isomorphism explicit.

Let $\{\mathcal{Y}_{ea}^a\}$ be a basis of $\mathcal{V}_{ea}^a$ for all $a \in \mathcal{I}$ such that it coincides with the vertex operator Y_{W^a}, which defines the V-module structure on W^a, i.e. $\mathcal{Y}_{ea}^a = Y_{W^a}$. We choose a basis $\{\mathcal{Y}_{ae}^a\}$ of $\mathcal{V}_{ae}^a$ as

$$\mathcal{Y}_{ae}^a = \Omega_{-1}(\mathcal{Y}_{ea}^a). \qquad (4.14)$$

We also choose a basis $\{\mathcal{Y}_{aa'}^e\}$ of $\mathcal{V}_{aa'}^e$ as

$$\mathcal{Y}_{aa'}^e = \mathcal{Y}_{aa'}^{e'} = \hat{A}_0(\mathcal{Y}_{ae}^a) = \sigma_{132}(\mathcal{Y}_{ea}^a). \qquad (4.15)$$

Notice that these choices are made for all $a \in \mathcal{I}$. In particular, we have

$$\mathcal{Y}_{a'e}^{a'} = \Omega_{-1}(\mathcal{Y}_{ea'}^{a'}), \qquad \mathcal{Y}_{a'a}^e = \mathcal{Y}_{a'a}^{e'} = \hat{A}_0(\mathcal{Y}_{a'e}^{a'}).$$

A remark of our notations, for an arbitrary basis of $\mathcal{V}_{a_1a_2}^{a_3}$, which is priori different from above specific choices, we will use notation $\{\mathcal{Y}_{a_1a_2;i}^{a_3;(p)}\}_{i=1}^{N_{a_1a_2}^{a_3}}$ (with additional subscript and superscript!) for $p \in \mathbb{N}$. For example $\{\mathcal{Y}_{ea;1}^{a;(1)}\}$.

We denote the matrix entries of a fusing matrix with respect to some arbitrary basis as

$$F(\mathcal{Y}^{a_4;(1)}_{a_1a_5;i} \otimes \mathcal{Y}^{a_5;(2)}_{a_2a_3;j}, \mathcal{Y}^{a_4;(3)}_{a_6a_3;k} \otimes \mathcal{Y}^{a_6;(4)}_{a_1a_2;l}).$$

We also use the following simple notation:

$$F_a = F(\mathcal{Y}^{a}_{ae} \otimes \mathcal{Y}^{e}_{a'a}; \mathcal{Y}^{a}_{ea} \otimes \mathcal{Y}^{e}_{aa'}), \quad a \in \mathcal{I}.$$

It is proved in [H10] that $F_a \neq 0$ for all $a \in \mathcal{I}$.

Now we are ready to give the construction of the duality maps [H11]. Since $\mathcal{C}_V$ is semisimple, we only need to discuss irreducible modules. We start with the right duals. For $a \in \mathcal{I}$, we need define maps $e_a : (W^a)' \boxtimes W^a \to V$ for all $a \in \mathcal{I}$. Our choice is

$$e_a = \frac{1}{F_a} m_{\mathcal{Y}^{e}_{a'a}}. \tag{4.16}$$

We also need to define map $i_a : V \to W^a \boxtimes (W^a)'$. Although there is a submodule of $W^a \boxtimes (W^a)'$ isomorphic to V, there is no canonical isomorphism. But we know that $\dim_{\mathbb{C}} \mathrm{Hom}_V(V, W^a \boxtimes (W^a)') = 1$. Hence we choose i_a to be the unique morphism such that

$$m_{\mathcal{Y}^{e}_{aa'}} \circ i_a = I_V. \tag{4.17}$$

It is obvious that $m_{\mathcal{Y}^{b;(p)}_{aa';i}} \circ i_a = 0$ for all $b \neq e$.

For left duals, we define the map $e'_a : W^a \boxtimes (W^a)' \to V$ to be

$$e'_a = \frac{1}{F_a} m_{\mathcal{Y}^{e}_{aa'}}, \tag{4.18}$$

and the map $i'_a : V \to (W^a)' \boxtimes W^a$ to be the unique morphism such that

$$m_{\mathcal{Y}^{e}_{a'a}} \circ i'_a = I_V. \tag{4.19}$$

It is clear that $m_{\mathcal{Y}^{b;(p)}_{a'a;i}} \circ i'_a = 0$ for all $b \neq e$.

Huang's main results in [H9][H10][H11] is stated as follow:

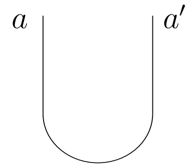
Theorem 4.5. *Let V be a simple vertex operator algebra satisfying the following conditions:*

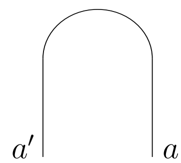
1. *$V_n = 0$ for $n < 0$, $V_{(0)} = \mathbb{C}\mathbf{1}$ and V' is isomorphic to V as V-module,*
2. *Every $\mathbb{N}$-gradable weak V-module is completely reducible,*
3. *V is C_2-cofinite.*

Then $\mathcal{C}_V$, together with its monoidal structure and above duality maps, is a rigid braided tensor category and $\dim a = \frac{1}{F_a}$ *for $a \in \mathcal{I}$. Moreover, $\mathcal{C}_V$, together with the twist $\theta_W = e^{2\pi i L(0)}$ for each object W, has a structure of modular tensor category [T][BK].*

Remark 4.6. Our choice of duality maps is slightly different from that of Huang because in this work the module structure on W' for each V-module is different from that used in [H11]. In the appendix, we will give a detailed proof of rigidity.

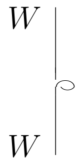
There are powerful tools, called graphic calculus, in a modular tensor category. In particular, the right duality maps i_a and e_a can be denoted by the following graphs:

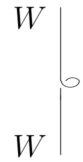
$i_a =$ 

, $e_a =$ 

,

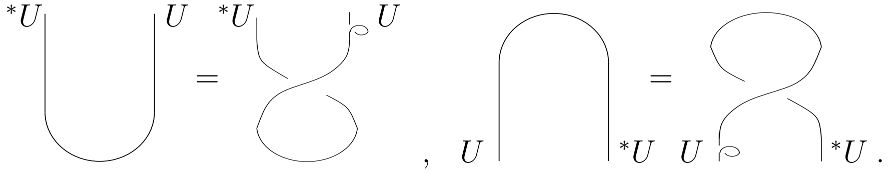
and the twist and its inverse, for any object W, are denoted by the following graphs

$\theta_W =$ 

, $\theta_W^{-1} =$ 

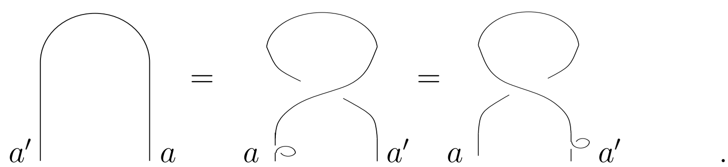
.

Remark 4.7. In a ribbon category, given a right dual of an object U^*, there is automatically a left dual ${}^*U$ given as follows:



(4.20)

In [H11] and this work, the left duals and the right duals are constructed at the same time. It is certainly true that the left duals obtained from (4.20) are compatible with our construction because the left (or right) duals are unique up to isomorphisms. One can also see directly that the identity (A.70) proved in the appendix is nothing but the following identity:



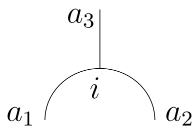
(4.21)

The formula (4.21) is implicitly used in many graphic calculations in this work.

4.3 Categorical formulation of $\tilde{A}_0$ and $\tilde{A}_{-1}$

Since $\mathcal{C}_V$ is a modular tensor category, we have a powerful tool available. It is called graph calculus [T][BK]. For this reason, we would like to express Ω_r and $\tilde{A}_r$ in terms of graphs.

A basis $\{\mathcal{Y}_{a_1 a_2; i}^{a_3;(1)}\}_{i=1}^{N_{a_1 a_2}^{a_3}}$ of $\mathcal{V}_{a_1 a_2}^{a_3}$ for $a_1, a_2, a_3 \in \mathcal{I}$ induces a basis in $\mathrm{Hom}(a_1 \boxtimes a_2, a_3)$, denoted as $e_{a_1 a_2; i}^{a_3}$ and as the following graph:

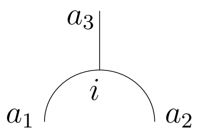


.

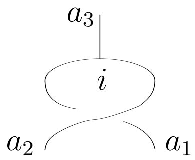
For simplicity, we always use a to represent W^a and a' to represent $(W^a)'$ in graphs.

The map $\Omega_0 : \mathcal{V}_{a_1 a_2}^{a_3} \to \mathcal{V}_{a_2 a_1}^{a_3}$ induces a map $\mathrm{Hom}_V(W^{a_1} \boxtimes W^{a_2}, W^{a_3}) \to \mathrm{Hom}_V(W^{a_2} \boxtimes W^{a_1}, W^{a_3})$, still denoted as Ω_0. By Proposition 3.1, we have

Ω_0 :



$\mapsto$



. (4.22)

Lemma 4.8.

$$e^{(z_1 - z_2)L(1)} z_1^{-2L(0)} e^{z_1 L(-1)} e^{z_1^{-1} L(1)} = z_2^{-2L(0)} e^{z_2 L(-1)} e^{z_2^{-1} L(1)} \tag{4.23}$$

Proof. It follows immediately from the following two identities:

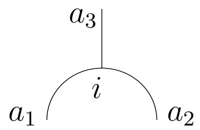
$$\begin{aligned} e^{-(z_1 - z_2) x^2 \frac{d}{dx}} z_1^{2x \frac{d}{dx}} e^{-z_1 \frac{d}{dx}} e^{-z_1^{-1} x^2 \frac{d}{dx}} x &= \frac{z_2 x - 1}{x} \\ z_2^{2x \frac{d}{dx}} e^{-z_2 \frac{d}{dx}} e^{-z_2^{-1} x^2 \frac{d}{dx}} x &= \frac{z_2 x - 1}{x}. \end{aligned} \tag{4.24}$$

∎

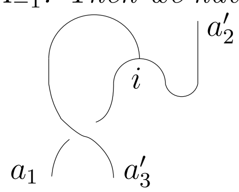
Proposition 4.9. *By the universal property of tensor product, $\tilde{A}_0, \tilde{A}_{-1} : \mathcal{V}_{a_1 a_2}^{a_3} \to \mathcal{V}_{a_1 a_3'}^{a_2'}$ defined in (4.5) also induce two morphisms from* $\mathrm{Hom}_V(W^{a_1} \boxtimes W^{a_2}, W^{a_3})$ *to*

$\mathrm{Hom}_V(W^{a_1} \boxtimes (W^{a_3})', (W^{a_2})')$, *still denoted as* $\tilde{A}_0, \tilde{A}_{-1}$. *Then we have*

$\tilde{A}_0 :$

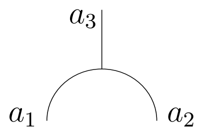


$\mapsto \quad \frac{\dim a_2}{\dim a_3}$

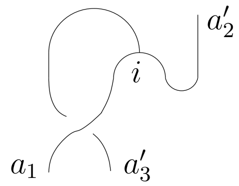


, (4.25)

$\tilde{A}_{-1} :$

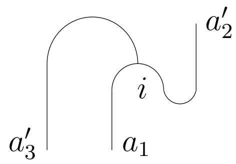


$\mapsto \quad \frac{\dim a_2}{\dim a_3}$



. (4.26)

Proof. Let $\{\mathcal{Y}^{a_2';(2)}_{a_3'a_1;i}\}_{i=1}^{N^{a_3}_{a_1a_2}}$ be a basis of $\mathcal{V}^{(W^{a_2})'}_{(W^{a_3})'W^{a_1}}$ for $a_1, a_2, a_3 \in \mathcal{I}$ such that the induced basis of $\mathrm{Hom}((W^{a_3})' \boxtimes W^{a_1}, (W^{a_2})')$ is given by the following graph:

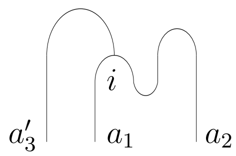


. (4.27)

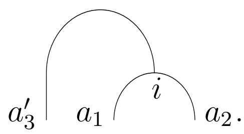
It is enough to show that

$$\mathcal{Y}^{a_2';(2)}_{a_3'a_1;i} = \frac{\dim a_3}{\dim a_2}\, \Omega_0(\tilde{A}_0(\mathcal{Y}^{a_3;(1)}_{a_1a_2;i})). \tag{4.28}$$

The following identity is obvious.



$=$



. (4.29)

It can be rewritten as

$$\dim a_2\ m_{\mathcal{Y}^{e}_{a_2'a_2}} \circ (m_{\mathcal{Y}^{a_2';(2)}_{a_3'a_1;i}} \boxtimes I_{W^{a_2}}) \circ \mathcal{A} = \dim a_3\ m_{\mathcal{Y}^{e}_{a_3'a_3}} \circ (I_{W^{a_3'}} \boxtimes m_{\mathcal{Y}^{a_3;(1)}_{a_1a_2;i}}), \tag{4.30}$$

where we have used the fact that $e_{a_k} = \dim a_k m_{\mathcal{Y}^{e}_{a_k'a_k}}, k = 2, 3$. Using (3.4) and (3.10), it is easy to see that

$$\begin{aligned} \dim a_2\ m^{P(z_2)}_{\mathcal{Y}^{e}_{a_2'a_2}} \circ (m^{P(z_1-z_2)}_{\mathcal{Y}^{a_2';(2)}_{a_3'a_1;i}} \boxtimes_{P(z_2)} I_{W^{a_2}}) \circ \mathcal{A}^{P(z_1-z_2),P(z_2)}_{P(z_1),P(z_2)} \\ = \dim a_3\ m^{P(z_1)}_{\mathcal{Y}^{e}_{a_3'a_3}} \circ (I_{W^{a_3'}} \boxtimes_{P(z_1)} m^{P(z_2)}_{\mathcal{Y}^{a_3;(1)}_{a_1a_2;i}}) \end{aligned} \tag{4.31}$$

for $z_1 > z_2 > z_1 - z_2 > 0$. Then (4.31) immediately implies the following identity:

$$\begin{aligned}
\dim a_2\, \mathcal{Y}^{e}_{a_2' a_2}(\mathcal{Y}^{a_2';(2)}_{a_3' a_1;i}(w_{a_3'}, z_1 - z_2)w_{a_1}, z_2)w_{a_2} \\
= \dim a_3\, \mathcal{Y}^{e}_{a_3' a_3}(w_{a_3'}, z_1)\mathcal{Y}^{a_3;(1)}_{a_1 a_2;i}(w_{a_1}, z_2)w_{a_2}. \qquad (4.32)
\end{aligned}$$

for $w_{a_1} \in W^{a_1}, w_{a_2} \in W^{a_2}, w_{a_3'} \in (W^{a_3})'$.

On the one hand, we have

$$\begin{aligned}
&(\mathcal{Y}^{e}_{a_2' a_2}(\mathcal{Y}^{a_2';(2)}_{a_3' a_1;i}(w_{a_3'}, z_1 - z_2)w_{a_1}, z_2)w_{a_2}, \mathbf{1}) \\
&\quad = \langle \hat{A}_0(\mathcal{Y}^{a_2'}_{a_2' e})(\mathcal{Y}^{a_2';(2)}_{a_3' a_1;i}(w_{a_3'}, z_1 - z_2)w_{a_1}, z_2)w_{a_2}, \mathbf{1}\rangle \\
&\quad = \langle w_{a_2}, \mathcal{Y}^{a_2'}_{a_2' e}(e^{-z_2 L(1)} z_2^{-2L(0)} \mathcal{Y}^{a_2';(2)}_{a_3' a_1;i}(w_{a_3'}, z_1 - z_2)w_{a_1}, e^{\pi i} z_2^{-1})\mathbf{1}\rangle \\
&\quad = \langle w_{a_2}, e^{-z_2^{-1} L(-1)} e^{-z_2 L(1)} z_2^{-2L(0)} \mathcal{Y}^{a_2';(2)}_{a_3' a_1;i}(w_{a_3'}, z_1 - z_2)w_{a_1}\rangle. \qquad (4.33)
\end{aligned}$$

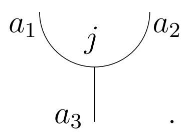
On the other hand, we have

$$\begin{aligned}
&(\mathcal{Y}^{e}_{a_3' a_3}(w_{a_3'}, z_1)\mathcal{Y}^{a_3;(1)}_{a_1 a_2;i}(w_{a_1}, z_2)w_{a_2}, \mathbf{1}) \\
&\quad = \langle \hat{A}_0(\mathcal{Y}^{a_3'}_{a_3' e})(w_{a_3'}, z_1)\mathcal{Y}^{a_3;(1)}_{a_1 a_2;i}(w_{a_1}, z_2)w_{a_2}, \mathbf{1}\rangle \\
&\quad = \langle \mathcal{Y}^{a_3;(1)}_{a_1 a_2;i}(w_{a_1}, z_2)w_{a_2}, \mathcal{Y}^{a_3'}_{a_3' e}(e^{-z_1 L(1)} z_1^{-2L(0)} w_{a_3'}, e^{\pi i} z_1^{-1})\mathbf{1}\rangle \\
&\quad = \langle \mathcal{Y}^{a_3;(1)}_{a_1 a_2;i}(w_{a_1}, z_2)w_{a_2}, e^{-z_1^{-1} L(-1)} e^{-z_1 L(1)} z_1^{-2L(0)} w_{a_3'}\rangle \\
&\quad = \langle z_1^{-2L(0)} e^{z_1 L(-1)} e^{z_1^{-1} L(1)} \mathcal{Y}^{a_3;(1)}_{a_1 a_2;i}(w_{a_1}, z_2)w_{a_2}, w_{a_3'}\rangle \\
&\quad = \langle \mathcal{Y}^{a_3;(1)}_{a_1 a_2;i}(e^{(z_1 - z_2)L(1)}(z_1 - z_2)^{-2L(0)} w_{a_1}, (z_1 - z_2)^{-1}) \\
&\qquad\qquad \cdot z_1^{-2L(0)} e^{z_1 L(-1)} e^{z_1^{-1} L(1)} w_{a_2}, w_{a_3'}\rangle \\
&\quad = \langle z_1^{-2L(0)} e^{z_1 L(-1)} e^{z_1^{-1} L(1)} w_{a_2}, \tilde{A}_0(\mathcal{Y}^{a_3;(1)}_{a_1 a_2;i})(w_{a_1}, e^{\pi i}(z_1 - z_2))w_{a_3'}\rangle \\
&\quad = \langle e^{(z_1 - z_2)L(1)} z_1^{-2L(0)} e^{z_1 L(-1)} e^{z_1^{-1} L(1)} w_{a_2}, \Omega_0(\tilde{A}_0(\mathcal{Y}^{a_3;(1)}_{a_1 a_2;i}))(w_{a_3'}, z_1 - z_2)w_{a_1}\rangle \\
&\quad = \langle z_2^{-2L(0)} e^{z_2 L(-1)} e^{z_2^{-1} L(1)} w_{a_2}, \Omega_0(\tilde{A}_0(\mathcal{Y}^{a_3;(1)}_{a_1 a_2;i}))(w_{a_3'}, z_1 - z_2)w_{a_1}\rangle, \qquad (4.34)
\end{aligned}$$

where we have used (4.23) in the last step.

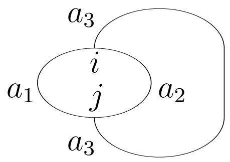
Combining (4.32), (4.33) and (4.34), we obtain (4.28) immediately. ∎

Let us choose a basis $\{f^{a_1 a_2}_{a_3;j}\}_{j=1}^{N^{a_3}_{a_1 a_2}}$ of $\mathrm{Hom}_V(W^{a_3}, W^{a_1} \boxtimes W^{a_2})$, denoted as

$f^{a_1 a_2}_{a_3;j} =$ 

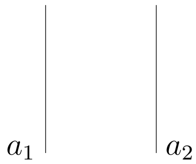
. (4.35)

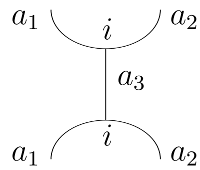
so that

$$\frac{1}{\dim a_3}$$ 

$$= \quad \delta_{ij}. \tag{4.36}$$

Proposition 4.10.



$$= \quad \sum_{a_3\in\mathcal{I}} \sum_{i=1}^{N_{a_1a_2}^{a_3}}$$ 

(4.37)

Proof. The basis $\{e_{a_1a_2;i}^{a_3}\}_{i=1}^{N_{a_1a_2}^{a_3}}$ gives arise to an isomorphism ([HL1]-[HL4])

$$\phi_1 := \sum_{a_3} \sum_{i=1}^{N_{a_1a_2}^{a_3}} e_{a_1a_2;i}^{a_3}: \quad W^{a_1} \boxtimes W^{a_2} \to \coprod_{a\in\mathcal{I}} \coprod_{i=1}^{N_{a_1a_2}^{a}} W^{a;(i)},$$

where $W^{a;(i)}$ denotes the i-th copy of W^a and $e_{a_1a_2;i}^{a} : W^{a_1} \otimes W^{a_2} \to W^{a;(i)}$.

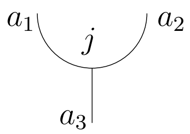
By the condition (4.36), we have $e_{a_1a_2;i}^{a_3} \circ f_{a_3;j}^{a_1a_2} = \delta_{ij} I_{W^{a_3}}$. Let

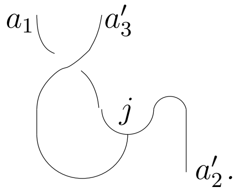
$$\phi_2 := \sum_{a_3} \sum_{j=1}^{N_{a_1a_2}^{a_3}} f_{a_3;j}^{a_1a_2}: \quad \coprod_{a\in\mathcal{I}} \coprod_{j=1}^{N_{a_1a_2}^{a}} W^{a;(j)} \to W^{a_1} \boxtimes W^{a_2}.$$

It is easy to see that we have $\phi_1 \circ \phi_2 = I_{\coprod_{a\in\mathcal{I}}\coprod_{i=1}^{N_{a_1a_2}^{a}} W^{a;(i)}}$. Hence $\phi_2 = \phi_1^{-1}$. Then we also have $\phi_2 \circ \phi_1 = I_{W^{a_1}\boxtimes W^{a_2}}$, which implies (4.37). ∎

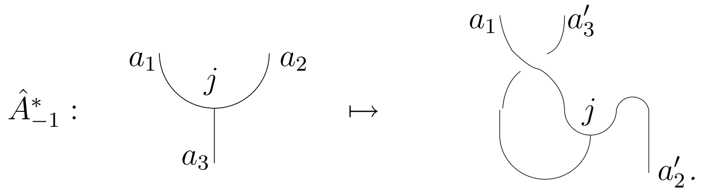
Therefore $\{f_{a_3;i}^{a_1a_2}\}$ can be viewed as the dual basis of $\{\mathcal{Y}_{a_1a_2;i}^{a_3}\}_{i=1}^{N_{a_1a_2}^{a_3}}$. It is useful to figure out the action on $f_{a_3;j}^{a_1a_2;j}$ of $\hat{A}_0$ and $\hat{A}_{-1}$, which are the inverse of $\tilde{A}_0$ and $\tilde{A}_{-1}$ respectively. The result is given in the following Proposition.

Proposition 4.11.

$$\hat{A}_0^* :$$ 

$$\mapsto$$ 

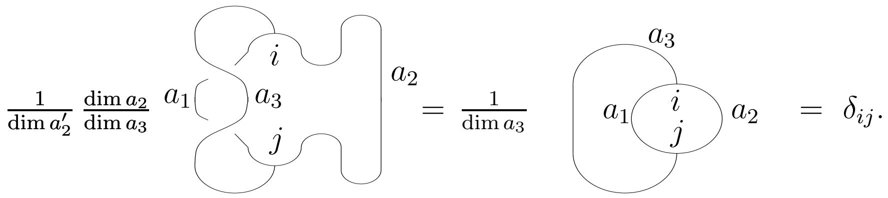
(4.38)



(4.39)

Proof. We will only prove (4.38). The proof of (4.39) is completely an analogue.

It is enough to show that the pairing between the image of (4.25) and (4.38) still gives δ_{ij}. It is proved as follow:



(4.40)

∎

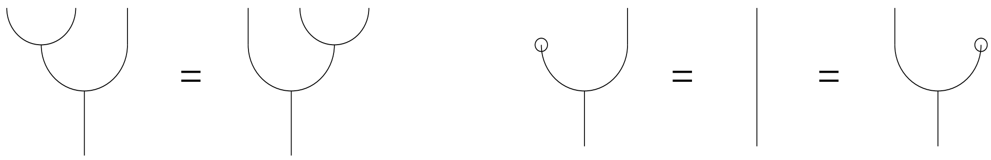
4.4 Frobenius algebra in $\mathcal{C}_{V^L \otimes V^R}$

First let us recall the notion of coalgebra and Frobenius algebra ([FS]) in a tensor category.

Definition 4.12. A coalgebra A in a tensor category $\mathcal{C}$ is an object with a coproduct $\Delta \in \mathrm{Mor}(A, A \otimes A)$ and a counit $\epsilon \in \mathrm{Mor}(A, \mathbf{1}_{\mathcal{C}})$ such that

$$(\Delta \otimes I_A) \circ \Delta = (I_A \otimes \Delta) \circ \Delta, \qquad (\epsilon \otimes I_A) \circ \Delta = I_A = (I_A \otimes \epsilon) \circ \Delta, \tag{4.41}$$

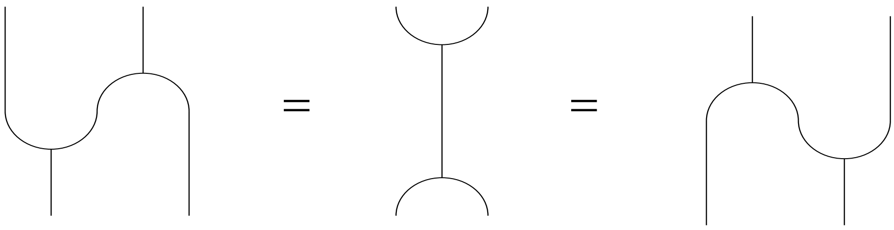
which can also be expressed in term of the following graphic equations:



Definition 4.13. Frobenius algebra in $\mathcal{C}$ is an object that is both an algebra and a co-algebra and for which the product and coproduct are related by

$$(I_A \otimes m) \circ (\Delta \otimes I_A) = \Delta \circ m = (m \otimes I_A) \circ (I_A \otimes \Delta), \tag{4.42}$$

or as the following graphic equations,



. (4.43)

Let $(F, \mathbb{Y}, \rho)$ be a conformal full field algebra over $V^L \otimes V^R$. Then $\mathbb{Y}_f(\cdot, x, x) : F \to \mathrm{End}\ F\{x\}$ is an intertwining operator of type $\binom{F}{FF}$ of $V^L \otimes V^R$ viewed as vertex operator algebra. Let $\mathbb{Y}_f|_{V^L\otimes V^R}$ denote the restriction of $\mathbb{Y}_f(\cdot, x, x)$ on $\rho(V^L \otimes V^R)$. It defines the module structure on F. By Lemma 4.2, we know that $\tilde{A}_0 \otimes \tilde{A}_{-1}(\mathbb{Y}_f|_{V^L\otimes V^R})$ gives a module structure on F'. By Proposition 4.3, $\tilde{A}_0 \otimes \tilde{A}_{-1}(\mathbb{Y}_f)$ is an intertwining operator of type $\binom{F'}{FF'}$.

Let $W_i, i = 1, 2, 3, 4, 5$ be $V^L \otimes V^R$-modules and $\mathcal{Y}$ be an intertwining operator of type $\binom{W_3}{W_1 W_2}$. Let $f : W_2 \to W_4$ and $g : W_3 \to W_5$ are two $V^L \otimes V^R$-module maps. If we define

$$\big(g \circ \mathcal{Y} \circ (I_{W_1} \otimes f)\big)(u_1, x)u_2 := g(\mathcal{Y}(u_1, x)f(u_2)) \tag{4.44}$$

for $u_1 \in W_1, u_2 \in W_2$, then it is clear that $g \circ \mathcal{Y} \circ (I_{W_1} \otimes f)$ is an intertwining operator of type $\binom{W_5}{W_1 W_4}$.

Lemma 4.14. *For $(F, \mathbb{Y}, \rho)$ to be equipped with an invariant bilinear form $(\cdot, \cdot)$ is equivalent to give a homomorphism $\varphi : F \to F'$ as a $V^L \otimes V^R$-module map such that the following condition:*

$$\mathbb{Y}_f = \varphi^{-1} \circ \tilde{A}_0 \otimes \tilde{A}_{-1}(\mathbb{Y}_f) \circ (I_F \otimes \varphi)) \tag{4.45}$$

is satisfied.

Proof. The invariant bilinear form $(\cdot, \cdot)$ and φ are related as follow:

$$\langle \varphi(v), u\rangle = (v, u). \tag{4.46}$$

One can use above equation to obtain one from the other. Moreover, the invariant property of bilinear form given in (2.1) exactly amounts to φ being a $V^L \otimes V^R$-module map and satisfying the condition (4.45). ∎

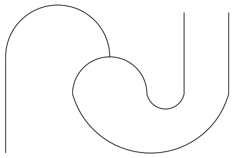
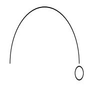
Theorem 4.15. *The following two notions are equivalent in the sense that corresponding categories are isomorphic.*

1. A conformal full field algebra F over $V^L \otimes V^R$ with a nondegenerate invariant bilinear form.

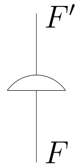
2. A commutative Frobenius algebra F in $(\mathcal{C}_{V^L \otimes V^R}, \mathcal{R}_{+-})$ with $\theta_F = I_F$.

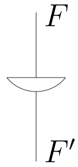
Proof. By Theorem 3.11, a full field algebra F over $V^L \otimes V^R$ gives an commutative associative algebra with $\theta_F = I_F$. An invariant bilinear form on F amounts to give a morphism from $\varphi : F \to F'$ so that (4.45) holds.

F' have a natural coalgebra structure with comultiplication $\Delta_{F'}$ and counit $\epsilon_{F'}$ defined as

$\Delta_{F'} =$  , $\epsilon_{F'} =$  . (4.47)

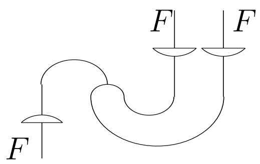
When the bilinear form is nondegenerate, φ is invertible. We will use the following graphic notation for φ and φ^{-1}:

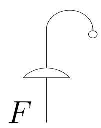
$\varphi =$ 

, $\varphi^{-1} =$ 

. (4.48)

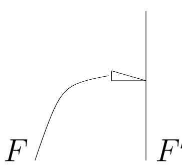
Using the map φ and its inverse, we can obtain a natural coalgebra structure on F defined as follow:

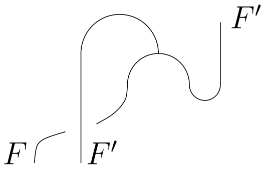
$\Delta_F =$ 

$\epsilon_F =$ 

. (4.49)

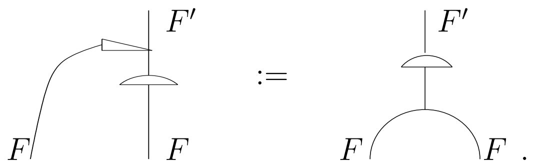
We claim that the multiple $(F, m, \iota, \Delta, \epsilon)$ gives a Frobenius algebra in $(\mathcal{C}, \mathcal{R}_{+-})$. We will prove the defining property (4.43) of Frobenius algebra below. First, we define a left action of F on F' as follow:



:= 

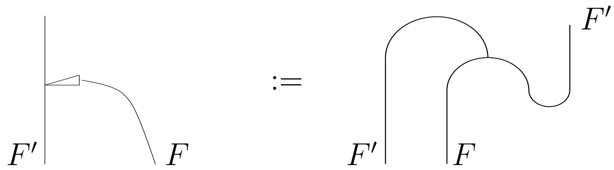
. (4.50)

By (4.25), (4.26) and our choice of braiding $\mathcal{R}_{+-}$, the right hands side of (4.50) is nothing but $\tilde{A}_0 \otimes \tilde{A}_{-1}(m)$. Then it not hard to see that (4.45) exactly amounts to the following relation:



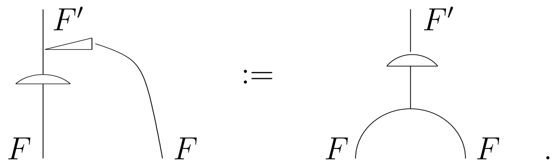
(4.51)

We also introduce a right action of F on F' as follow:



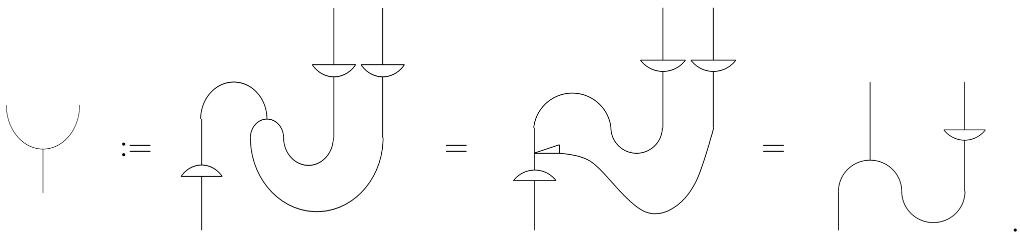
(4.52)

Using (4.51) and the commutativity of F, it is easy to obtain that



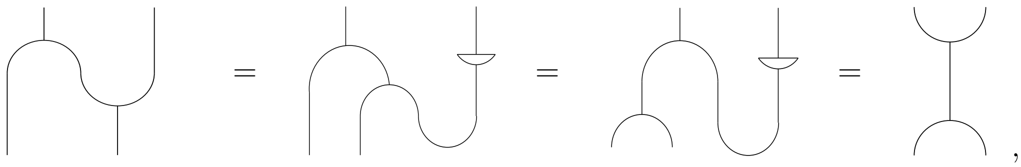
(4.53)

Using (4.53), we see that the Δ can be rewritten as follow:

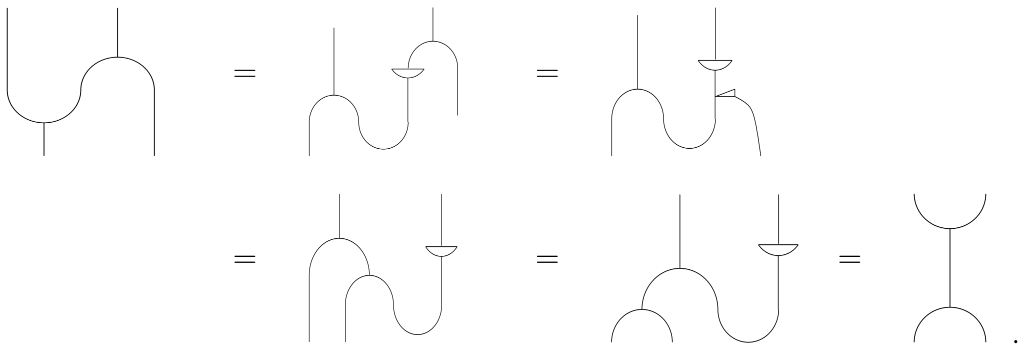


(4.54)

By (4.54) and the associativity of F, we have

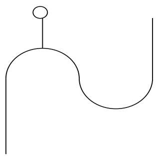
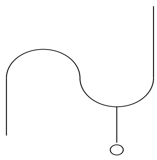


and

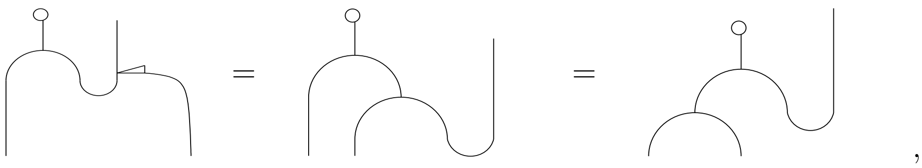


Hence we have proved that F is commutative Frobenius algebra with $\theta_F = I_F$ in $(\mathcal{C}_{V^L \otimes V^R}, \mathcal{R}_{+-})$.

Conversely, given a Frobenius algebra F, it is shown in [FRS2] that there is an isomorphism Φ from F to F'. Φ and its inverse Φ^{-1} is given in the following diagrams:

$\Phi \quad =$  , $\quad \Phi^{-1} \quad =$  . (4.55)

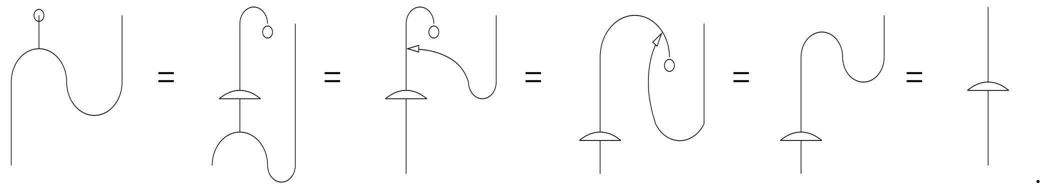
Φ induces a nondegenerate bilinear from on F. We want to show that this bilinear form is also invariant. Let $\varphi := \Phi$. Adopting the notation of φ in (4.48), we see that to prove the invariance property of Φ or φ amounts to prove that (4.51) holds. We first prove (4.53). Indeed, we have



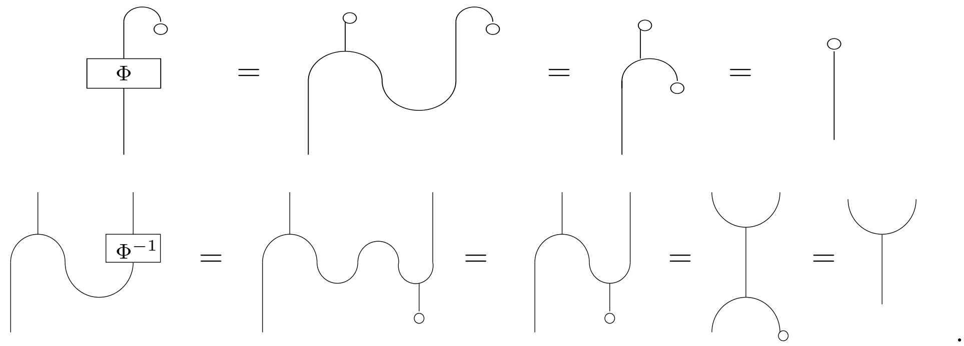
which is nothing but (4.53). Then (4.51) follows from (4.53) by the commutativity of F.

So far, we have constructed two functors between the two categories. It remains to show that these two functors give isomorphisms between two categories.

One can check that if a Frobenius algebra F is obtained from an associative algebra F with an isomorphism $\varphi : F \to F'$ using (4.49), then $\Phi = \varphi$. Indeed, we have



Similarly, given a Frobenius algebra F. One have the isomorphism $\Phi : F \to F'$ given as (4.55). We claim that the Frobenius algebra structure on F induced from Φ using the construction (4.49) is exactly same as the original one. The proof is the given in the following picture.



Remark 4.16. So far we have completely reformulated the genus-zero conformal field theory in term of algebras in tensor category. At least, at the level of tensor category, the conformal field theories are the essentially same as that of closed topological field theories. They are both commutative Frobenius algebras, however, in very different categories. We will show in [HK3] that genus-one conformal field theories are very different from genus-one topological field theories even on the level of tensor category.

5 Construction

In this section, we consider the case $V^L = V^R = V$, where V is assumed to satisfy the conditions in Theorem 4.5. We will give a construction of commutative Frobenius algebra F in the category $(\mathcal{C}_{V\otimes V}, \mathcal{R}_{+-})$ satisfying $\theta_F = I_F$. We assume all the notations used in Section 4.

Let F is be the object in $\mathcal{C}_{V\otimes V}$ given as follow

$$F = \coprod_{a\in\mathcal{I}} W^a \otimes (W^a)'. \tag{5.56}$$

The decomposition of F as a direct sum gives a natural embedding $V \otimes V \hookrightarrow F$. We denote this embedding as ι_F. Now we need select a single morphism $m_F \in \mathrm{Hom}_{V\otimes V}(F \boxtimes F, F)$.

By the universal property of tensor product, $\{\mathcal{Y}^{a_3;(1)}_{a_1a_2;i}\}_{i=1}^{N^{a_3}_{a_1a_2}}$, a basis of $\mathcal{V}^{a_3}_{a_1a_2}$, gives arise to $\{e^{a_3}_{a_1a_2;i}\}_{i=1}^{N^{a_3}_{a_1a_2}}$, a basis of $\mathrm{Hom}_V(W^{a_1} \boxtimes W^{a_2}, W^{a_3})$, as given in the graph (4.27). And the dual basis $\{f^{a_1a_2}_{a_3;j}\}_{j=1}^{N^{a_3}_{a_1a_2}}$ is given in (4.35), satisfying the duality condition (4.36).

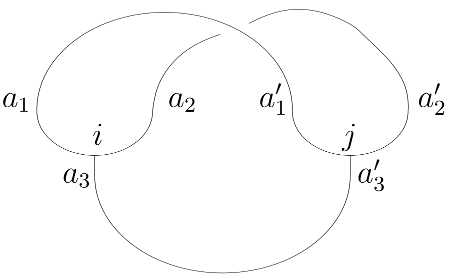
In general, m_F can always be written in the following form

$$m_F = \sum_{a_1,a_2,a_3\in\mathcal{A}} \sum_{i,j=1}^{N^{a_3}_{a_1a_2}} \langle f^{a_1a_2}_{a_3;i}, f^{a_1'a_2'}_{a_3';j}\rangle e^{a_3}_{a_1a_2;i} \otimes e^{a_3'}_{a_1'a_2';j}, \tag{5.57}$$

where $\langle\cdot,\cdot\rangle$ is a bilinear pairing between $\mathrm{Hom}_V(a_1 \boxtimes a_2, a_3)$ and $\mathrm{Hom}_V(a_1' \boxtimes a_2', a_3')$. Actually, using the duality (4.36), m_F can be identified with the pairing $\langle\cdot,\cdot\rangle$ as an element in

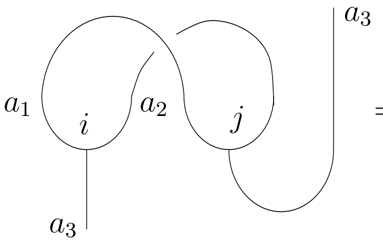
$$(\mathrm{Hom}_V(a_3, a_1 \boxtimes a_2))^* \otimes (\mathrm{Hom}_V(a_3', a_1' \boxtimes a_2'))^*.$$

In particular, it also means that m_F defined in (5.57) is independent of the choice of basis. We define the pairing $\langle f^{a_1a_2}_{a_3;i}, f^{a_1'a_2'}_{a_3';j}\rangle$ by

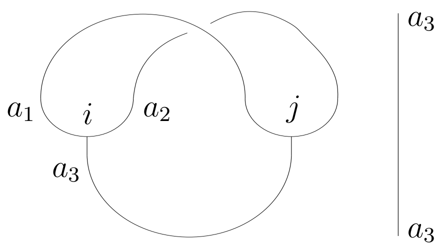
$\frac{1}{\dim a_3}$ 

. (5.58)

The following identity is manifest by itself, and will be useful later.



$= \frac{1}{\dim a_3}$



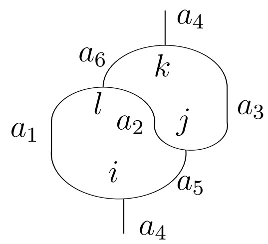
(5.59)

The following Lemma is also manifest by itself. It has appeared in [FFFS].

Lemma 5.1.

$$F^{-1}(\mathcal{Y}^{a_4;(1)}_{a_6a_3;k}\otimes\mathcal{Y}^{a_6;(1)}_{a_1a_2;l},\mathcal{Y}^{a_4;(1)}_{a_1a_5;i}\otimes\mathcal{Y}^{a_5;(1)}_{a_2a_3;j})$$

$=$



(5.60)

Notice that F' has the same contents as F. They are isomorphic as $V\otimes V$-modules. There is, however, no canonical isomorphism. The space of isomorphisms between F and F' is a three dimensional vector space. Now we choose a particular isomorphism $\varphi_F : F \to F'$ given by

$$\varphi_F = \oplus_{a\in\mathcal{I}}\ \dim a\, e^{-2\pi i h_a} I_{W^a\otimes(W^a)'}. \tag{5.61}$$

The isomorphism φ_F induces a nondegenerate invariant bilinear form on F viewed as $V\otimes V$-module.

Theorem 5.2. *The triple (F, m_F, ι_F), together with the bilinear form induced from φ_F, is a commutative Frobenius algebra in $\mathcal{C}_{V\otimes V}$ satisfying $\theta_F = I_F$.*

Proof. The left and right unit properties follow from the facts that $\langle l^{-1}_{W^{a_1}}, l^{-1}_{(W^{a_1})'}\rangle = 1$ and $\langle r^{-1}_{W^{a_1}}, r^{-1}_{(W^{a_1})'}\rangle = 1$. The condition $\theta_F = I_F$ is obvious by the definition of F in (5.56).

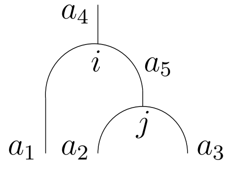
The commutativity amounts to the invariant property of the pairing (5.58) with respect to braiding isomorphisms. This invariant property of the pairing is rather easy to see, and was pointed out by Kirillov in ([Kr]).

Now we prove the associativity:

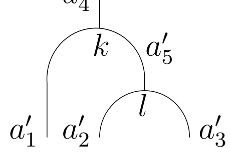
$$m_F\circ(I_F\boxtimes m_F) = m_F\circ(m_F\boxtimes I_F)\circ\mathcal{A}. \tag{5.62}$$

Using (5.59), we have

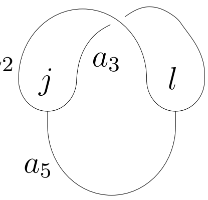
$$m_F\circ(I_F\boxtimes m_F) = \sum_{a_1,a_2,a_3,a_4}\sum_{a_5}\sum_{i,j,k,l}$$



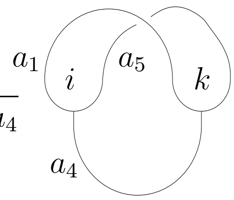
$\otimes$



$$\cdot \frac{1}{\dim a_5}$$



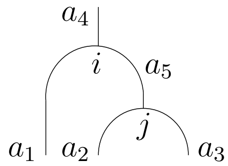
$$\frac{1}{\dim a_4}$$



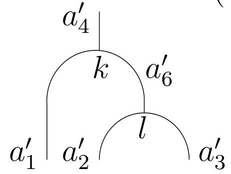
.

(5.63)

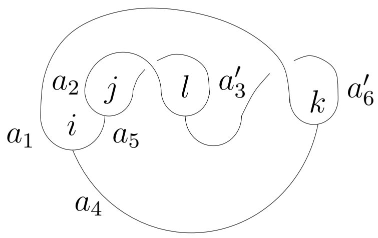
$$= \sum_{a_1,a_2,a_3,a_4} \sum_{a_5,a_6} \sum_{i,j,k,l}$$



$$\otimes$$



$$\cdot \frac{1}{\dim a_4}$$

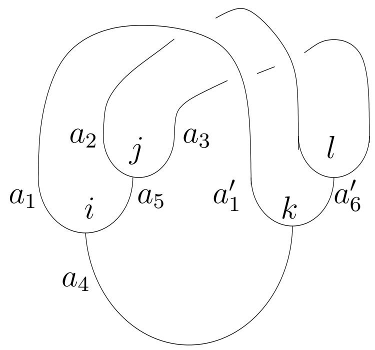


,

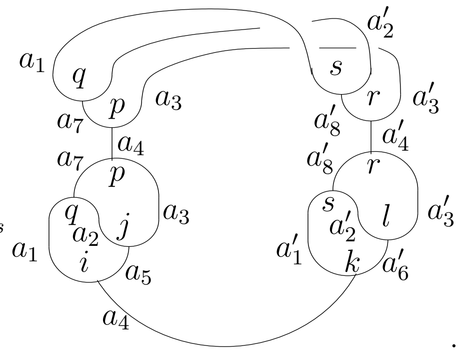
(5.64)

where we have added a sum $\sum_{a_6}$ in the second step. It is harmless because the graph in the last line of (5.64) is nonzero if and only if $a_6 = a_5$.

Now we can first deform the graph in the last line of (5.64), then apply (4.37) to obtain the following identity:



$$= \sum_{a_7,a_8} \sum_{p,q,r,s}$$



.

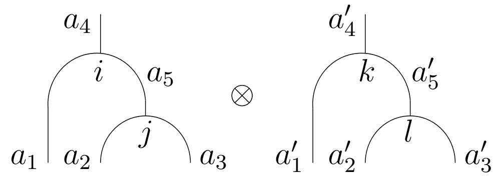
(5.65)

Notice that when we apply (4.37) we need sum up the intermediate index, such as the index a_3 in (4.37). However, the second graph in (5.65) takes nonzero value if and only if the intermediate indices are a_4 and a'_4 respectively.

Now using (5.60), we can move the following two factors

$$F^{-1}(\mathcal{Y}^{a_4;(1)}_{a_7 a_3;p} \otimes \mathcal{Y}^{a_7;(1)}_{a_1 a_2;q}, \mathcal{Y}^{a_4;(1)}_{a_1 a_5;i} \otimes \mathcal{Y}^{a_5;(1)}_{a_2 a_3;j}) \quad ,$$
$$F^{-1}(\mathcal{Y}^{a_4';(1)}_{a_8' a_3';r} \otimes \mathcal{Y}^{a_8';(1)}_{a_1' a_2';s}, \mathcal{Y}^{a_4';(1)}_{a_1' a_6';k} \otimes \mathcal{Y}^{a_6';(1)}_{a_2' a_3';l}) \tag{5.66}$$

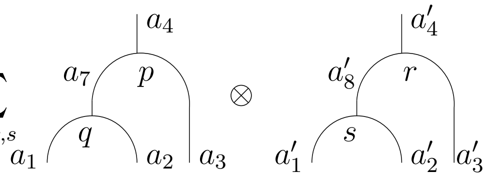
out of the second graph in (5.65), and combine them with



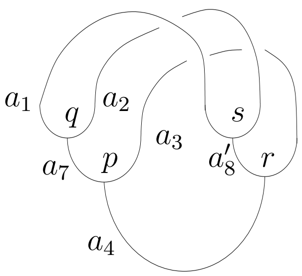
.

Then we compose the resulting morphism from right by $\mathcal{A}^{-1}$, and sum up the indices a_5, a_6, i, j, k, l (using (A.94)). As a result, we obtain the following identity:

$$m_F \circ (I_F \boxtimes m_F) \circ \mathcal{A}^{-1} = \sum_{a_1,a_2,a_3,a_4} \sum_{a_7,a_8} \sum_{p,q,r,s}$$



$$\cdot \frac{1}{\dim a_4}$$



. (5.67)

Notice again that the graph in the second line of (5.67) is nonzero if and only if $a_7 = a_8$. It is easy to show that the right hand side of (5.67) is nothing but $m_F \circ (m_F \boxtimes I_F)$. Therefore, we obtain the associativity (5.62).

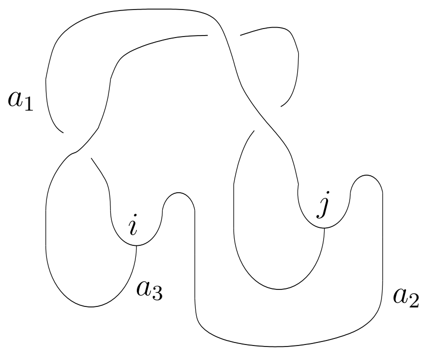
In summary, we have proved that F is a commutative associative algebra with $\theta_F = I_F$. It remains to show that the bilinear form induced from φ_F is invariant. This amounts to show that the following equation

$$m_F = \varphi_F^{-1} \circ \tilde{A}_0 \otimes \tilde{A}_{-1}(m_F) \circ (I_F \otimes \varphi_F) \tag{5.68}$$

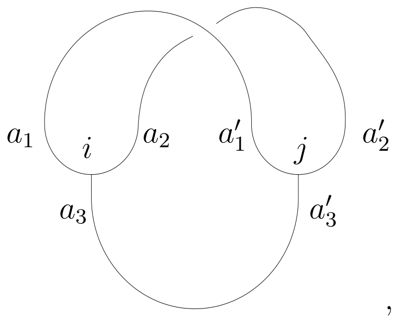
holds. Since the m_F given in (5.57) is independent of choice of basis, we have

$$m_F = \sum_{a_1,a_2,a_3 \in \mathcal{I}} \sum_{i,j} \langle \hat{A}_0^*(f^{a_1 a_2}_{a_3;i}), \hat{A}_{-1}^*(f^{a_1' a_2'}_{a_3';j}) \rangle \tilde{A}_0(e^{a_3}_{a_1 a_2;i}) \otimes \tilde{A}_{-1}(e^{a_3'}_{a_1' a_2';j}). \tag{5.69}$$

Replacing the left hand side of (5.68) by the right hand side of (5.69) and the m_F in the right hand side of (5.68) by the right hand side of (5.57), and using (4.38), (4.39) and (5.61), we see that the validity of (5.68) follows from the following identity:



$= e^{2\pi i(h_{a_2}-h_{a_3})}$



,

which is manifestly true. Hence this bilinear form induced from φ_F is invariant. ∎

A The Proof of rigidity

In this section, we prove that our new choice of duality maps, defined in Section 4.2, satisfies the rigidity axioms.

Lemma A.1.

$$
\begin{aligned}
\mathcal{Y}^e_{a'a} &= e^{2\pi i h_a}\Omega_0(\mathcal{Y}^e_{aa'}) = e^{-2\pi i h_a}\Omega_{-1}(\mathcal{Y}^e_{aa'}), && \text{(A.70)}\\
\tilde{A}_0(\mathcal{Y}^{a'}_{a'e}) &= e^{2\pi i h_a}\mathcal{Y}^{e'}_{a'a}, && \text{(A.71)}\\
\tilde{A}_0(\mathcal{Y}^{a'}_{ea'}) &= \mathcal{Y}^a_{ea}. && \text{(A.72)}
\end{aligned}
$$

Proof. For $w_{a'} \in (W^a)'$ and $w_a \in W^a$, we have

$$
\begin{aligned}
(\mathbf{1}, \mathcal{Y}^e_{a'a}(w_{a'}, x)w_a) &= \langle \mathbf{1}, \hat{A}_0(\mathcal{Y}^{a'}_{a'e})(w_{a'}, x)w_a\rangle\\
&= \langle \mathcal{Y}^{a'}_{a'e}(e^{-xL(1)}x^{-2L(0)}w_{a'}, e^{\pi i}x^{-1})\mathbf{1}, w_a\rangle\\
&= \langle e^{-x^{-1}L(-1)}e^{-xL(1)}x^{-2L(0)}w_{a'}, w_a\rangle, && \text{(A.73)}
\end{aligned}
$$

and

$$
\begin{aligned}
(\mathbf{1}, \Omega_0(\mathcal{Y}^e_{aa'})(w_{a'}, x)w_a) &= \langle \mathbf{1}, e^{xL(-1)}\hat{A}_0(\mathcal{Y}^a_{ae})(w_a, e^{\pi i}x)w_{a'}\rangle\\
&= \langle \mathcal{Y}^a_{ae}(e^{xL(1)}(e^{\pi i}x)^{-2L(0)}w_a, x^{-1})\mathbf{1}, w_{a'}\rangle\\
&= \langle e^{x^{-1}L(-1)}e^{xL(1)}(e^{\pi i}x)^{-2L(0)}w_a, w_{a'}\rangle\\
&= \langle x^{-2L(0)}e^{xL(-1)}e^{x^{-1}L(1)}e^{-2\pi i L(0)}w_a, w_{a'}\rangle\\
&= \langle e^{-2\pi i L(0)}w_a, e^{-x^{-1}L(-1)}e^{-xL(1)}x^{-2L(0)}w_{a'}\rangle. && \text{(A.74)}
\end{aligned}
$$

Combining (A.73) and (A.74), we obtain the first equality of (A.70). The second equality of (A.70) can be proved similarly.

Now we prove (A.71). On the one hand, we have

$$\begin{aligned}\langle \tilde{A}_0(\mathcal{Y}_{a'e}^{a'})(w_{a'}, e^{\pi i}x)w_a, \mathbf{1}\rangle &= \langle w_a, \mathcal{Y}_{a'e}^{a'}(e^{xL(1)}x^{-2L(0)}w_{a'}, x^{-1})\mathbf{1}\rangle \\ &= \langle w_a, e^{x^{-1}L(-1)}e^{xL(1)}x^{-2L(0)}w_{a'}\rangle. \end{aligned} \tag{A.75}$$

On the other hand, we have

$$\begin{aligned}\langle \mathcal{Y}_{a'a}^{e'}(w_{a'}, e^{\pi i}x)w_a, \mathbf{1}\rangle &= \langle \hat{A}_0(\mathcal{Y}_{a'e}^{a'})(w_{a'}, e^{\pi i}x)w_a, \mathbf{1}\rangle \\ &= \langle w_a, \mathcal{Y}_{a'e}^{a'}(e^{zL(1)}(e^{\pi i}z)^{-2L(0)}w_{a'}, x^{-1})\mathbf{1}\rangle \\ &= \langle w_a, e^{x^{-1}L(-1)}e^{xL(1)}x^{-2L(0)}w_{a'}\rangle e^{-2\pi i h_a}. \end{aligned} \tag{A.76}$$

Combining (A.75) and (A.76), we obtain (A.71).

The last identity (A.72) can be proved as follow:

$$\begin{aligned}\langle \tilde{A}_0(\mathcal{Y}_{ea'}^{a'})(\mathbf{1}, e^{\pi i}x)w_a, w_{a'}\rangle &= \langle w_a, \mathcal{Y}_{ea'}^{a'}(e^{xL(1)}x^{-2L(0)}\mathbf{1}, x^{-1})w_{a'}\rangle \\ &= \langle w_a, w_{a'}\rangle \\ &= \langle \mathcal{Y}_{ea}^{a}(\mathbf{1}, e^{\pi i}x)w_a, w_{a'}\rangle.\end{aligned}$$

∎

The following formula is proved in [FHL].

Lemma A.2. *Let $\mathcal{Y}$ be an intertwining operator of type $\binom{W_3}{W_1W_2}$ and $v \in W_1$. We have*

$$e^{xL(1)}\mathcal{Y}(v, x_0)e^{-xL(1)} = \mathcal{Y}(e^{x(1-xx_0)L(1)}(1-xx_0)^{-2L(0)}v, x_0/(1-xx_0)). \tag{A.77}$$

The fusing matrices and their symmetries under the S_3 actions are studied in detail in [H10]. Here similar results hold. They are stated in the following Lemma:

Lemma A.3.

$$\begin{aligned}&F(\mathcal{Y}_{a_1a_5;i}^{a_4;(1)} \otimes \mathcal{Y}_{a_2a_3;j}^{a_5;(2)}, \mathcal{Y}_{a_6a_3;k}^{a_4;(3)} \otimes \mathcal{Y}_{a_1a_2;l}^{a_6;(4)}) \\ &\quad = F^{-1}(\Omega_0(\mathcal{Y}_{a_1a_5;i}^{a_4;(1)}) \otimes \Omega_0(\mathcal{Y}_{a_2a_3;j}^{a_5;(2)}), \Omega_0(\mathcal{Y}_{a_6a_3;k}^{a_4;(3)}) \otimes \Omega_0(\mathcal{Y}_{a_1a_2;l}^{a_6;(4)})) \quad (A.78)\\ &\quad = F(\tilde{A}_0(\mathcal{Y}_{a_2a_3;j}^{a_5;(2)}) \otimes \tilde{A}_0(\mathcal{Y}_{a_1a_5;i}^{a_4;(1)}), \tilde{A}_0(\mathcal{Y}_{a_6a_3;k}^{a_4;(3)}) \otimes \Omega_{-1}(\mathcal{Y}_{a_1a_2;l}^{a_6;(4)})) \quad (A.79)\end{aligned}$$

Proof. (A.78) is proved in [H10]. We prove (A.79) here. Let $w_i \in W^{a_i}, i = 1, 2, 3$ and $w_4' \in (W^{a_4})'$. For $|z_1| > |z_2| > |z_1 - z_2| > 0$, we have

$$
\begin{aligned}
&\langle w_4', \mathcal{Y}_{a_1a_5;i}^{a_4;(1)}(w_1,z_1)\mathcal{Y}_{a_2a_3;j}^{a_5;(2)}(w_2,z_2)w_3\rangle \\
&= \langle \tilde{A}_0(\mathcal{Y}_{a_2a_3;j}^{a_5;(2)})(e^{-z_2L(1)}z_2^{-2L(0)}w_2, e^{\pi i}z_2^{-1}) \\
&\qquad\qquad \tilde{A}_0(\mathcal{Y}_{a_1a_5;i}^{a_4;(1)})(e^{-z_1L(1)}z_1^{-2L(0)}w_1, e^{\pi i}z_1^{-1})w_4', w_3\rangle \\
&= \sum_{a_6\in\mathcal{I}}\sum_{k,l} F\big(\tilde{A}_0(\mathcal{Y}_{a_2a_3;j}^{a_5;(2)})\otimes \tilde{A}_0(\mathcal{Y}_{a_1a_5;i}^{a_4;(1)}), \tilde{A}_0(\mathcal{Y}_{a_6a_3;k}^{a_4;(3)})\otimes \Omega_{-1}(\mathcal{Y}_{a_1a_2;l}^{a_6;(4)})\big) \\
&\qquad \langle \tilde{A}_0(\mathcal{Y}_{a_6a_3;k}^{a_4;(3)})\big(\Omega_{-1}(\mathcal{Y}_{a_1a_2;l}^{a_6;(4)})(e^{-z_2L(1)}z_2^{-2L(0)}w_2, e^{\pi i}(z_2^{-1}-z_1^{-1})) \\
&\qquad\qquad \cdot e^{-z_1L(1)}z_1^{-2L(0)}w_1, e^{\pi i}z_1^{-1}\big)w_4', w_3\rangle \\
&= \sum_{a_6\in\mathcal{I}}\sum_{k,l} F\big(\tilde{A}_0(\mathcal{Y}_{a_2a_3;j}^{a_5;(2)})\otimes \tilde{A}_0(\mathcal{Y}_{a_1a_5;i}^{a_4;(1)}), \tilde{A}_0(\mathcal{Y}_{a_6a_3;k}^{a_4;(3)})\otimes \Omega_{-1}(\mathcal{Y}_{a_1a_2;l}^{a_6;(4)})\big) \\
&\qquad \langle \tilde{A}_0(\mathcal{Y}_{a_6a_3;k}^{a_4;(3)})\big(\mathcal{Y}_{a_1a_2;l}^{a_6;(4)}(e^{-z_1L(1)}z_1^{-2L(0)}w_1, z_2^{-1}-z_1^{-1}) \\
&\qquad\qquad \cdot e^{-z_2L(1)}z_2^{-2L(0)}w_2, e^{\pi i}z_2^{-1}\big)w_4', w_3\rangle. \qquad (A.80)
\end{aligned}
$$

Applying (A.77), we have the follow result:

$$
\begin{aligned}
&\mathcal{Y}_{a_1a_2;l}^{a_6;(4)}(e^{-z_1L(1)}z_1^{2L(0)}w_1, z_2^{-1}-z_1^{-1})e^{-z_2L(1)}z_2^{-2L(0)} \\
&= e^{-z_2L(1)}\mathcal{Y}_{a_1a_2;l}^{a_6;(4)}(e^{\frac{z_2^2}{z_1}L(1)}\left(\frac{z_2}{z_1}\right)^{-2L(0)}e^{-z_1L(1)}z_1^{-2L(0)}w_1, \frac{z_1-z_2}{z_2^2})z_2^{-2L(0)} \\
&= e^{-z_2L(1)}z_2^{-2L(0)}\mathcal{Y}_{a_1a_2;l}^{a_6;(4)}(e^{z^{-1}L(1)}z_1^{2L(0)}e^{-z_1L(1)}z_1^{-2L(0)}w_1, z_1-z_2) \\
&= e^{-z_2L(1)}z_2^{-2L(0)}\mathcal{Y}_{a_1a_2;l}^{a_6;(4)}(w_1, z_1-z_2). \qquad (A.81)
\end{aligned}
$$

Applying it to the right hand side of (A.80), we obtain

$$
\begin{aligned}
&\langle w_4', \mathcal{Y}_{a_1a_5;i}^{a_4;(1)}(w_1,z_1)\mathcal{Y}_{a_2a_3;j}^{a_5;(2)}(w_2,z_2)w_3\rangle \\
&= \sum_{a_6\in\mathcal{I}}\sum_{k,l} F\big(\tilde{A}_0(\mathcal{Y}_{a_2a_3;j}^{a_5;(2)})\otimes \tilde{A}_0(\mathcal{Y}_{a_1a_5;i}^{a_4;(1)}), \tilde{A}_0(\mathcal{Y}_{a_6a_3;k}^{a_4;(3)})\otimes \Omega_{-1}(\mathcal{Y}_{a_1a_2;l}^{a_6;(4)})\big) \\
&\qquad \langle \tilde{A}_0(\mathcal{Y}_{a_6a_3;k}^{a_4;(3)})\big(e^{-z_2L(1)}z_2^{-2L(0)}\mathcal{Y}_{a_1a_2;l}^{a_6;(4)}(w, z_1-z_2)w_2, e^{\pi i}z_2\big)w_4', w_3\rangle \\
&= \sum_{a_6\in\mathcal{I}}\sum_{k,l} F\big(\tilde{A}_0(\mathcal{Y}_{a_2a_3;j}^{a_5;(2)})\otimes \tilde{A}_0(\mathcal{Y}_{a_1a_5;i}^{a_4;(1)}), \tilde{A}_0(\mathcal{Y}_{a_6a_3;k}^{a_4;(3)})\otimes \Omega_{-1}(\mathcal{Y}_{a_1a_2;l}^{a_6;(4)})\big) \\
&\qquad \langle w_4', \mathcal{Y}_{a_6a_3;k}^{a_4;(3)}(\mathcal{Y}_{a_1a_2;l}^{a_6;(4)}(w_1, z_1-z_2)w_2, z_2)w_3\rangle. \qquad (A.82)
\end{aligned}
$$

By the definition of fusing matrices, we see that (A.79) must be true. ∎

Lemma A.4.

$$
F_a = F_{a'} = F^{-1}(\mathcal{Y}_{ea}^{a}\otimes\mathcal{Y}_{aa'}^{e};\mathcal{Y}_{ae}^{a}\otimes\mathcal{Y}_{a'a}^{e}) = F^{-1}(\mathcal{Y}_{ea'}^{a'}\otimes\mathcal{Y}_{a'a}^{e};\mathcal{Y}_{a'e}^{a'}\otimes\mathcal{Y}_{aa'}^{e}). \quad (A.83)
$$

Proof. Using (A.78) and Lemma A.1, we have

$$\begin{aligned}F^{-1}(\mathcal{Y}_{ea}^{a}\otimes\mathcal{Y}_{aa'}^{e};\mathcal{Y}_{ae}^{a}\otimes\mathcal{Y}_{a'a}^{e}) &= F(\mathcal{Y}_{ae}^{a}\otimes\Omega_0(\mathcal{Y}_{aa'}^{e}),\mathcal{Y}_{ea}^{a}\otimes\Omega_0(\mathcal{Y}_{a'a}^{e}))\\ &= F(\mathcal{Y}_{ae}^{a}\otimes e^{-2\pi i h_a}\mathcal{Y}_{a'a}^{e},\mathcal{Y}_{ea}^{a}\otimes e^{-2\pi i h_a}\mathcal{Y}_{aa'}^{e})\\ &= F_a\end{aligned}$$

Similarly we have $F^{-1}(\mathcal{Y}_{ea'}^{a'}\otimes\mathcal{Y}_{a'a}^{e};\mathcal{Y}_{a'e}^{a'}\otimes\mathcal{Y}_{aa'}^{e}) = F_{a'}$. It remains to show that $F_a = F_{a'}$. Using (A.79) and Lemma A.1, we have

$$\begin{aligned}F_{a'} &= F(\mathcal{Y}_{a'e}^{a'}\otimes\mathcal{Y}_{aa'}^{e},\mathcal{Y}_{ea'}^{a'}\otimes\mathcal{Y}_{a'a}^{e})\\ &= F(\tilde{A}_0(\mathcal{Y}_{aa'}^{e})\otimes\tilde{A}_0(\mathcal{Y}_{a'e}^{a'}),\tilde{A}_0(\mathcal{Y}_{ea'}^{a'})\otimes\Omega_{-1}(\mathcal{Y}_{a'a}^{e}))\\ &= F(\mathcal{Y}_{ae}^{a}\otimes e^{2\pi i h_a}\mathcal{Y}_{a'a}^{e},\mathcal{Y}_{ea}^{a}\otimes e^{2\pi i h_a}\mathcal{Y}_{aa'}^{e})\\ &= F_a.\end{aligned}$$

∎

To prove the rigidity [T][BK], it amounts to prove the following Proposition [H12].

Proposition A.5.

$$I_{W^a} = r_{W^a}\circ I_{W^a}\boxtimes e_a\circ(\mathcal{A})^{-1}\circ i_a\boxtimes I_{W^a}\circ l_{W^a}^{-1}, \tag{A.84}$$
$$I_{(W^a)'} = l_{(W^a)'}\circ e_a\boxtimes I_{(W^a)'}\circ\mathcal{A}\circ I_{W^a}\boxtimes i_a\circ r_{(W^a)'}^{-1}, \tag{A.85}$$
$$I_{W^a} = l_a\circ e'_a\boxtimes I_{W^a}\circ\mathcal{A}\circ I_{W^a}\boxtimes i'_a\circ r_{W^a}^{-1}, \tag{A.86}$$
$$I_{(W^a)'} = r_{(W^a)'}\circ I_{(W^a)'}\boxtimes e'_a\circ(\mathcal{A})^{-1}\circ i'_a\boxtimes I_{W^a}\circ l_{(W^a)'}^{-1}. \tag{A.87}$$

Proof. Let $z_1 > z_2 > z_1 - z_2 > 0$. By the universal property of tensor product, we have the following canonical isomorphisms

$$\begin{aligned}&\mathcal{V}_{a_1a_5}^{a_4}\cong\mathrm{Hom}(W^{a_1}\boxtimes_{P(z_1)}W^{a_5},W^{a_4}),\quad \mathcal{V}_{a_2a_3}^{a_5}\cong\mathrm{Hom}_V(W^{a_2}\boxtimes_{P(z_2)}W^{a_3},W^{a_5}),\\ &\mathcal{V}_{a_6a_3}^{a_4}\cong\mathrm{Hom}(W^{a_6}\boxtimes_{P(z_2)}W^{a_3},W^{a_4}),\quad \mathcal{V}_{a_1a_2}^{a_6}\cong\mathrm{Hom}_V(W^{a_1}\boxtimes_{P(z_1-z_2)}W^{a_2},W^{a_6}).\end{aligned} \tag{A.88}$$

We also have the following canonical isomorphisms

$$\begin{aligned}&\coprod_{a_5\in\mathcal{I}}\mathrm{Hom}_V(W^{a_1}\boxtimes_{P(z_1)}W^{a_5},W^{a_4})\otimes\mathrm{Hom}_V(W^{a_2}\boxtimes_{P(z_2)}W^{a_3},W^{a_5})\\ &\qquad\cong\mathrm{Hom}(W^{a_1}\boxtimes_{P(z_1)}(W^{a_2}\boxtimes_{P(z_2)}W^{a_3}),W^{a_4})\\ &\coprod_{a_5\in\mathcal{I}}\mathrm{Hom}(W^{a_6}\boxtimes_{P(z_2)}W^{a_3},W^{a_4})\otimes\mathrm{Hom}_V(W^{a_1}\boxtimes_{P(z_1-z_2)}W^{a_2},W^{a_6})\\ &\qquad\cong\mathrm{Hom}((W^{a_1}\boxtimes_{P(z_1-z_2)}W^{a_2})\boxtimes_{P(z_2)}W^{a_3},W^{a_4})\end{aligned} \tag{A.89}$$

defined by the compositions of maps. Moreover, let γ_1, γ_2 and γ_3 be paths in $\mathbb{R}_+$ from z_1, z_2, z_1-z_2 to 1 respectively. By the naturalness of the parallel isomorphisms $\mathcal{T}_{\gamma_i}, i = 1,2,3$ [HL3] [HKL][HK1][H11], we also have the following commutative diagram:

$$\begin{array}{ccc}
\mathrm{Hom}(W^{a_1} \boxtimes_{P(z_1)} (W^{a_2} \boxtimes_{P(z_2)} W^{a_3}), W^{a_4}) & \xrightarrow{\cong} & \mathrm{Hom}_V(W^{a_1} \boxtimes (W^{a_2} \boxtimes W^{a_3}), W^{a_4}) \\
\Big\downarrow {\scriptstyle ((\mathcal{A}^{P(z_1-z_2),P(z_2)}_{P(z_1),P(z_2)})^{-1})^*} & & \Big\downarrow {\scriptstyle (\mathcal{A}^{-1})^*} \\
\mathrm{Hom}((W^{a_1} \boxtimes_{P(z_1-z_2)} W^{a_2}) \boxtimes_{P(z_2)} W^{a_3}, W^{a_4}) & \xrightarrow{\cong} & \mathrm{Hom}_V((W^{a_1} \boxtimes W^{a_2}) \boxtimes W^{a_3}, W^{a_4})
\end{array} \tag{A.90}$$

where the top and bottom horizontal isomorphisms are given by $\mathcal{T}_{\gamma_1} \circ (I_{W^{a_1}} \boxtimes_{P(z_1)} \mathcal{T}_{\gamma_2})$ and $\mathcal{T}_{\gamma_2} \circ (\mathcal{T}_{\gamma_3} \boxtimes_{P(z_2)} I_{W^{a_3}})$ respectively.

Combining (A.88), (A.89) and (A.90), we obtain the following commutative diagram

$$\begin{array}{ccc}
\coprod_{a_5 \in \mathcal{I}} \mathcal{V}^{a_4}_{a_1 a_5} \otimes \mathcal{V}^{a_5}_{a_2 a_3} & \xrightarrow{\cong} & \mathrm{Hom}_V(W^{a_1} \boxtimes (W^{a_2} \boxtimes W^{a_3}), W^{a_4}) \\
\Big\downarrow {\scriptstyle \mathcal{F}} & & \Big\downarrow {\scriptstyle (\mathcal{A}^{-1})*} \\
\coprod_{a_6 \in \mathcal{I}} \mathcal{V}^{a_4}_{a_6 a_3} \otimes \mathcal{V}^{a_6}_{a_1 a_2} & \xrightarrow{\cong} & \mathrm{Hom}_V((W^{a_1} \boxtimes W^{a_2}) \boxtimes W^{a_3}, W^{a_4})
\end{array} \tag{A.91}$$

with two horizontal maps being canonical isomorphisms. In terms of any basis of intertwining operators, the fusing isomorphism $\mathcal{F}$ can be written as:

$$\begin{aligned}
&\mathcal{F}(\mathcal{Y}^{a_4;(1)}_{a_1 a_5;i} \otimes \mathcal{Y}^{a_5;(2)}_{a_2 a_3;j}) \\
&\quad = \sum_{a_6 \in \mathcal{I}} \sum_{k,l} F(\mathcal{Y}^{a_4;(1)}_{a_1 a_5;i} \otimes \mathcal{Y}^{a_5;(2)}_{a_2 a_3;j}, \mathcal{Y}^{a_4;(3)}_{a_6 a_3;k} \otimes \mathcal{Y}^{a_6;(4)}_{a_1 a_2;l}) \mathcal{Y}^{a_4;(3)}_{a_6 a_3;k} \otimes \mathcal{Y}^{a_6;(4)}_{a_1 a_2;l}
\end{aligned} \tag{A.92}$$

By (A.91), we must have

$$\begin{aligned}
&m_{\mathcal{Y}^{a_4;(1)}_{a_1 a_5;i}} \circ (I_{W^{a_1}} \boxtimes m_{\mathcal{Y}^{a_5;(2)}_{a_2 a_3;j}}) \circ \mathcal{A}^{-1} \\
&\quad = \sum_{a_6 \in \mathcal{I}} \sum_{k,l} F(\mathcal{Y}^{a_4;(1)}_{a_1 a_5;i} \otimes \mathcal{Y}^{a_5;(2)}_{a_2 a_3;j}, \mathcal{Y}^{a_4;(3)}_{a_6 a_3;k} \otimes \mathcal{Y}^{a_6;(4)}_{a_1 a_2;l}) m_{\mathcal{Y}^{a_4;(3)}_{a_6 a_3;k}} \circ (m_{\mathcal{Y}^{a_6;(4)}_{a_1 a_2;l}} \boxtimes I_{W^{a_3}}),
\end{aligned} \tag{A.93}$$

or equivalently

$$\begin{aligned}
&m_{\mathcal{Y}^{a_4;(3)}_{a_6 a_3;k}} \circ (m_{\mathcal{Y}^{a_6;(4)}_{a_1 a_2;l}} \boxtimes I_{W^{a_3}}) \circ \mathcal{A} \\
&\quad = \sum_{a_5 \in \mathcal{I}} \sum_{i,j} F^{-1}(\mathcal{Y}^{a_4;(3)}_{a_6 a_3;k} \otimes \mathcal{Y}^{a_6;(4)}_{a_1 a_2;l}; \mathcal{Y}^{a_4;(1)}_{a_1 a_5;i} \otimes \mathcal{Y}^{a_5;(2)}_{a_2 a_3;j}) m_{\mathcal{Y}^{a_4;(1)}_{a_1 a_5;i}} \circ (I_{W^{a_1}} \boxtimes m_{\mathcal{Y}^{a_5;(2)}_{a_2 a_3;j}})
\end{aligned} \tag{A.94}$$

Notice that $r_{W^a} = m_{\mathcal{Y}^a_{ae}}$, $e_a = m_{\mathcal{Y}^e_{a'a}}$ and $l_{W^a} = m_{\mathcal{Y}^a_{ea}}$ by our construction. Hence we have

$$
\begin{aligned}
& r_a \circ I_{W^a} \boxtimes e_a \circ (\mathcal{A})^{-1} \circ i_a \boxtimes I_{W^a} \circ l_a^{-1} \\
& \quad = \frac{1}{F_a} m_{\mathcal{Y}^a_{ae}} \circ (I_{W^a} \boxtimes m_{\mathcal{Y}^e_{a'a}}) \circ (\mathcal{A})^{-1} \circ i_a \boxtimes I_{W^a} \circ m^{-1}_{\mathcal{Y}^a_{ea}} \\
\text{by (A.93)} & \quad = \sum_{b\in\mathcal{I}} \sum_{i,j} \frac{1}{F_a} F(\mathcal{Y}^a_{ae} \otimes \mathcal{Y}^e_{a'a}; \mathcal{Y}^{a;(1)}_{ba;i} \otimes \mathcal{Y}^{b;(2)}_{aa';j}) \\
& \qquad\qquad m_{\mathcal{Y}^{a;(1)}_{ba;i}} \circ m_{\mathcal{Y}^{b;(2)}_{aa';j}} \boxtimes I_{W^a} \circ i_a \boxtimes I_{W^a} \circ m^{-1}_{\mathcal{Y}^a_{ea}} \\
\text{by (4.17)} & \quad = \frac{1}{F_a} F(\mathcal{Y}^a_{ae} \otimes \mathcal{Y}^e_{a'a}; \mathcal{Y}^a_{ea} \otimes \mathcal{Y}^e_{aa'})\; m_{\mathcal{Y}^a_{ea}} \circ m^{-1}_{\mathcal{Y}^a_{ea}} \\
& \quad = I_{W^a}.
\end{aligned}
$$

which is just (A.84). We prove the remaining three identities below.

$$
\begin{aligned}
& l_{(W^a)'} \circ e_a \boxtimes I_{(W^a)'} \circ \mathcal{A} \circ I_{W^a} \boxtimes i_a \circ r^{-1}_{(W^a)'} \\
& \quad = \frac{1}{F_a} m_{\mathcal{Y}^{a'}_{ea'}} \circ (m_{\mathcal{Y}^e_{a'a}} \boxtimes I_{(W^a)'}) \circ \mathcal{A} \circ I_{W^a} \boxtimes i_a \circ m^{-1}_{\mathcal{Y}^{a'}_{a'e}} \\
\text{by (A.94)} & \quad = \sum_{b\in\mathcal{I}} \sum_{ij} \frac{1}{F_a} F^{-1}(\mathcal{Y}^{a'}_{ea'} \otimes \mathcal{Y}^e_{a'a}; \mathcal{Y}^{a',(3)}_{a'b;i} \otimes \mathcal{Y}^{b;(4)}_{aa';j}) \\
& \qquad\qquad m_{\mathcal{Y}^{a',(3)}_{a'b;i}} \circ (I_{W^a} \boxtimes m_{\mathcal{Y}^{b;(4)}_{aa';j}}) \circ I_{W^a} \boxtimes i_a \circ m^{-1}_{\mathcal{Y}^{a'}_{a'e}} \\
\text{by (4.17)} & \quad = \frac{1}{F_a} F^{-1}(\mathcal{Y}^{a'}_{ea'} \otimes \mathcal{Y}^e_{a'a}; \mathcal{Y}^{a'}_{a'e} \otimes \mathcal{Y}^e_{aa'})\, m_{\mathcal{Y}^{a'}_{a'e}} \circ m^{-1}_{\mathcal{Y}^{a'}_{a'e}} \\
\text{by (A.83)} & \quad = I_{(W^a)'}
\end{aligned}
\tag{A.95}
$$

and

$$
\begin{aligned}
& l_{W^a} \circ e'_a \boxtimes I_{W^a} \circ \mathcal{A} \circ I_{W^a} \boxtimes i'_a \circ r^{-1}_{W^a} \\
& \quad = \frac{1}{F_a} m_{\mathcal{Y}^a_{ea}} \circ (m_{\mathcal{Y}^e_{aa'}} \boxtimes I_{W^a}) \circ \mathcal{A} \circ I_{W^a} \boxtimes i'_a \circ m^{-1}_{\mathcal{Y}^e_{ae}} \\
\text{by (A.94)} & \quad = \sum_{b\in\mathcal{I}} \sum_{ij} \frac{1}{F_a} F^{-1}(\mathcal{Y}^a_{ea} \otimes \mathcal{Y}^e_{aa'}; \mathcal{Y}^{a,(5)}_{ab;i} \otimes \mathcal{Y}^{b;(6)}_{a'a;j}) \\
& \qquad\qquad m_{\mathcal{Y}^{a,(5)}_{ab;i}} \circ (I_{W^a} \boxtimes m_{\mathcal{Y}^{b;(6)}_{a'a;j}}) \circ I_{W^a} \boxtimes i'_a \circ m^{-1}_{\mathcal{Y}^a_{ae}} \\
\text{by (4.19)} & \quad = \frac{1}{F_a} F^{-1}(\mathcal{Y}^a_{ea} \otimes \mathcal{Y}^e_{aa'} \mathcal{Y}^a_{ae} \otimes \mathcal{Y}^e_{a'a})\, m_{\mathcal{Y}^a_{ae}} \circ m^{-1}_{\mathcal{Y}^a_{ae}} \\
\text{by (A.83)} & \quad = I_{W^a}
\end{aligned}
\tag{A.96}
$$

and

$$\begin{aligned}
&r_{(W^a)'} \circ I_{(W^a)'} \boxtimes e'_a \circ \mathcal{A}^{-1} \circ i'_a \boxtimes I_{W^a} \circ l^{-1}_{(W^a)'} \\
&\quad = \frac{1}{F_a} m_{\mathcal{Y}^{a'}_{a'e}} \circ (I_{(W^a)'} \boxtimes m_{\mathcal{Y}^{e}_{aa'}}) \circ \mathcal{A}^{-1} \circ i'_a \boxtimes I_{W^a} \circ m^{-1}_{\mathcal{Y}^{a'}_{ea'}} \\
\text{by (A.93)} &\quad = \sum_{b\in\mathcal{I}} \sum_{ij} \frac{1}{F_a} F(\mathcal{Y}^{a'}_{a'e} \otimes \mathcal{Y}^{e}_{aa'}, \mathcal{Y}^{a';(7)}_{ba';i} \otimes \mathcal{Y}^{b;(8)}_{a'a;j}) \\
&\qquad\qquad m_{\mathcal{Y}^{a';(7)}_{ba';i}} \circ m_{\mathcal{Y}^{b;(8)}_{a'a;j}} \boxtimes I_{W^a} \circ i'_a \boxtimes I_{W^a} \circ m^{-1}_{\mathcal{Y}^{a'}_{ea'}} \\
\text{by (4.19)} &\quad = \frac{1}{F_a} F(\mathcal{Y}^{a'}_{a'e} \otimes \mathcal{Y}^{e}_{aa'}, \mathcal{Y}^{a'}_{ea'} \otimes \mathcal{Y}^{e}_{a'a})\, m_{\mathcal{Y}^{a'}_{ea'}} \circ m^{-1}_{\mathcal{Y}^{a'}_{ea'}} \\
\text{by (A.83)} &\quad = I_{(W^a)'} \qquad\qquad (A.97)
\end{aligned}$$

∎

References

[BK] B. Bakalov and A. Kirillov Jr., *Lectures on Tensor Categories and Modular Functors*, University Lecture Series, Vol. 21, Amer. Math. Soc., Providence, 2001.

[BPZ] A. A. Belavin, A. M. Polyakov and A. B. Zamolodchikov, Infinite conformal symmetries in two-dimensional quantum field theory, *Nucl. Phys.* **B241** (1984), 333–380.

[DLM] C.-Y. Dong, H.-S. Li and G. Mason, Modular-invariance of trace functions in orbifold theory and generalized Moonshine, *Comm. Math. Phys.* **214** (2000), 1–56.

[DMZ] C.-Y. Dong, G. Mason and Y.-C. Zhu, Discrete series of the Virasoro algebra and the moonshine module, in *Algebraic Groups and Their Generalizations: Quantum and infinite-dimensional Methods, Proc. 1991 American Math. Soc. Summer Research Institute*, ed. by W. J. Haboush and B. J. Parshall, Proc. Symp. Pure Math. **56**, Part 2, Amer. Math. Soc., Providence, 1994, 295-316.

[FFFS] G. Felder, J. Fröhlich, J. Fuchs and C. Schweigert, Correlation functions and boundary conditions in rational conformal field theory and three-dimensional topology, *Composition Math.* **131** (2002), 189–237.

[FjFRS] J. Fjelstad, J. Fuchs, I. Runkel and C. Schweigert, TFT construction of RCFT correlators V: Proof of modular invariance and factorisation, hep-th/0503194.

[FHL] I. B. Frenkel, Y.-Z. Huang and J. Lepowsky, On axiomatic approaches to vertex operator algebras and modules, preprint, 1989; *Memoirs Amer. Math. Soc.* **104**, 1993.

[FRS1] J. Fuchs, I. Runkel and C. Schweigert, Conformal correlation functions, Frobenius algebras and triangulations, *Nucl. Phys.* **B624** (2002), 452–468.

[FRS2] J. Fuchs, I. Runkel and C. Schweigert, TFT construction of RCFT correlators I: Partition functions, *Nucl. Phys.* **B678** (2004) 511.

[FRS3] J. Fuchs, I. Runkel and C. Schweigert, TFT construction of RCFT correlators III: Simple currents, *Nucl. Phys.* **B694** (2004) 277

[FRS4] J. Fuchs, I. Runkel and C. Schweigert, TFT construction of RCFT correlators IV: Structure constants and correlation functions, preprint hep-th/0412290, to appear in Nucl. Phys. B.

[FS] J. Fuchs and C. Schweigert, Category theory for conformal boundary conditions, *Fields Institute Commun.* 39 (2003) 25.

[H1] Y.-Z. Huang, Geometric interpretation of vertex operator algebras, *Proc. Natl. Acad. Sci. USA* **88** (1991), 9964–9968.

[H2] Y.-Z. Huang, A theory of tensor products for module categories for a vertex operator algebra, IV, *J. Pure Appl. Alg.* **100** (1995), 173-216.

[H3] Y.-Z. Huang, Two-dimensional conformal geometry and vertex operator algebras, *Progress in Mathematics*, Vol. 148, 1997, Birkhäuser, Boston.

[H4] Y.-Z. Huang, Intertwining operator algebras, genus-zero modular functors and genus-zero conformal field theories, in: *Operads: Proceedings of Renaissance Conferences*, ed. J.-L. Loday, J. Stasheff, and A. A. Voronov, Contemporary Math., Vol. 202, Amer. Math. Soc., Providence, 1997, 335–355.

[H5] Y.-Z. Huang, Genus-zero modular functors and intertwining operator algebras, *Internat. J. Math.* **9** (1998), 845–863.

[H6] Y.-Z. Huang, Generalized rationality and a "Jacobi identity" for intertwining operator algebras, *Selecta Math. (N. S.)*, **6** (2000), 225–267.

[H7] Y.-Z. Huang, Differential equations and intertwining operators, *Comm. Contemp. Math.* **7** (2005), 375–400.

[H8] Y.-Z. Huang, Differential equations, duality and modular invariance, *Comm. Contemp. Math.* **7** (2005), 649–706.

[H9] Y.-Z. Huang, Vertex operator algebras, the Verlinde conjecture and modular tensor categories, *Proc. Natl. Acad. Sci. USA* **102** 2005, 5352–5356.

[H10] Y.-Z. Huang, Vertex operator algebras and the Verlinde conjecture, preprint, arXiv: math.QA/0406291, to appear.

[H11] Y.-Z. Huang, Rigidity and modularity of vertex tensor categories, preprint, arXiv: math.QA/0502533, to appear.

[H12] Y.-Z. Huang, A functional-analytic theory of vertex (operator) algebras, I, *Commun. Math. Phys.* **204** (1999), 61-84.

[H13] Y.-Z. Huang, A functional-analytic theory of vertex (operator) algebras, II, *Commun. Math. Phys.* **242** (2003), 425-444.

[HK1] Y.-Z. Huang and L. Kong, Open-string vertex algebras, tensor categories and operads, *Commun. Math. Phys.* **250**, 433-471 (2004).

[HK2] Y.-Z. Huang and L. Kong, Full field algebras, arXiv:math.QA/0511328.

[HK3] Y.-Z. Huang and L. Kong, Modular invariance of full field algebras, in preparation.

[HKL] Y.-Z. Huang, A. Kirillov Jr. and J. Lepowsky, Braided tensor categories and extensions of vertex operator algebra, to appear.

[HL1] Y.-Z. Huang and J. Lepowsky, A theory of tensor products for module categories for a vertex operator algebra, I, *Selecta Math. (N.S.)* **1** (1995), 699-756.

[HL2] Y.-Z. Huang and J. Lepowsky, A theory of tensor products for module categories for a vertex operator algebra, II, *Selecta Math. (N.S.)* **1** (1995), 757–786.

[HL3] Y.-Z. Huang and J. Lepowsky, Tensor products of modules for a vertex operator algebra and vertex tensor categories, in: *Lie Theory and Geometry, in honor of Bertram Kostant,* ed. R. Brylinski, J.-L. Brylinski, V. Guillemin, V. Kac, Birkhäuser, Boston, 1994, 349–383.

[HL4] Y.-Z. Huang and J. Lepowsky, A theory of tensor products for module categories for a vertex operator algebra, III, *J. Pure Appl. Alg.* **100** (1995), 141-171.

[Hub1] K. Hubbard, The notion of vertex operator coalgebra and a geometric interpretation. math.QA/0405461.

[Hub2] K. Hubbard, Constructions of vertex operator coalgebras via vertex operator algebras, math.QA/0406035.

[Ko] L. Kong, "Open-closed conformal field algebras", in preparation.

[Kr] A. Kirillov Jr. On an inner product in modular tensor categories, *J. Amer. Math. Soc* **9** (1996), no 4, 1135–1169.

[LL] J. Lepowsky, H.-S. Li, Introduction to Vertex operator algebras and their representations (Birkhäuser, Boston 2004).

[Ma] J. P. May, The geometry of iterated loop spaces, *Lecture Notes in Mathematics*, No. 271, Springer-Verlag, Berlin, 1972.

[Mi1] M. Miyamoto, Modular invariance of vertex operator algebras satisfying C_2-cofiniteness. *Duke Math. J.* 122 (2004), no. 1, 51-91.

[Mi2] M. Miyamoto, Intertwining operators and modular invariance, preprint, arXiv: math.QA/0010180.

[MS] G. Moore and N. Seiberg, Classical and quantum conformal field theory, *Comm. Math. Phys.* **123** (1989), 177–254.

[RFFS] I. Runkel, J. Fjelstad, J. Fuchs and C. Schweigert, "Topological and conformal field theory as Frobenius algebras", math.arXiv: math.CT/0512076.

[S1] G. Segal, The definition of conformal field theory, in: *Differential geometrical methods in theoretical physics (Como, 1987)*, NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci., 250, Kluwer Acad. Publ., Dordrecht, 1988, 165–171.

[S2] G. B. Segal, The definition of conformal field theory, *Topology, Geometry and quantum field theory*, Proceeding of the 2002 Oxford Symposium in Honor of the 60th Birthday of Graeme Segal. London Mathematical Society Lecture Note Series (No. 308).

[SFR] C. Schweigert, J. Fuchs and I. Runkel, Categorification and correlation functions in conformal field theory, math.CT/0602079.

[T] Turaev, *Quantum invariant of knots and 3-manifolds*, de Gruyter Studies in Mathematics, vol. 18, Walter de Gruyter, Berline, 1994.

[Z] Y.-C. Zhu, Modular invariance of vertex operator algebras, *J. Amer. math. Soc.* **9** (1996), 237-302.

Max Planck Institute for Mathematics in the Sciences, Inselstrasse 22, D-04103 Leipzig, Germany
E-mail address: lkong@mis.mpg.de