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Stability of invariant manifolds in one and two dimensions

by

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Stability of invariant manifolds in one and two dimensions

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Abstract

We consider the gradient flow associated with a non local free energy functional and extend to such a case results obtained for the Allen-Cahn equation on "slow motions on invariant manifolds". The manifolds in question are time-invariant one-dimensional curves in a L^2 space which connect the two ground states (interpreted as the pure phases of the system) to the first excited state (interpreted as a diffuse interface). Local and structural stability of the manifolds are proved and applications to the characterization of optimal tunnelling are discussed.

AMS (MOS) subject classification: 82C05

1 Introduction

In this paper we consider the evolution equation for $u(x,t)$, $|x| \le L/2$, $t \ge 0$

$$u_t = f_L(u) \tag{1.1}$$

with u_t the t-derivative of u and the "velocity field" $f_L(u)$ given by

$$f_L(u) = J^{\text{neum}} * u - \frac{1}{\beta}\,\text{arctanh}(u), \quad \beta > 1 \tag{1.2}$$

$*$ denotes convolution and

$$J^{\text{neum}}(x,y) = J(x,y) + J(x, R_L(y)) + J(x, R_{-L}(y))$$

$R_L(y) = \frac{L}{2} + (\frac{L}{2} - y)$, $R_{-L}(y) = -\frac{L}{2} - (y - \frac{L}{2}))$, namely $R_{\pm L}(y)$ are the reflections of y around $\pm L$. We suppose that J is a smooth, symmetric, translational invariant probability kernel supported in $|y-x| \le 1$ and that $J'(0,x) \le 0$ for $x \in (0,1)$.

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Later in the paper, see Section 7, we will extend the result to $d = 2$.

We are here mainly interested in the analysis of the stationary points of (1.1) and, more generally, in its invariant sets (manifolds). m is stationary if it solves the "non local mean field equation"

$$m = \tanh\{\beta J^{\text{neum}} * m\} \tag{1.3}$$

As we will see some of the motivations of our work are related to the fact that (1.1) is the gradient flow equation $u_t = -\dfrac{\delta F_L(u)}{\delta u}$ of the "free energy" functional

$$F_L(m) = \int_{-\frac{L}{2}}^{\frac{L}{2}} \phi_\beta(m) dx + \frac{1}{4} \int_{-\frac{L}{2}}^{\frac{L}{2}} \int_{-\frac{L}{2}}^{\frac{L}{2}} J^{\text{neum}}(x, y)[m(x) - m(y)]^2 dx\, dy \tag{1.4}$$

where

$$\phi_\beta(m) = \tilde{\phi}_\beta(m) - \tilde{\phi}_\beta(m_\beta), \qquad \tilde{\phi}_\beta(m) = -\frac{m^2}{2} - \frac{1}{\beta} S(m)$$

$$S(m) = -\frac{1-m}{2} \log \frac{1-m}{2} - \frac{1+m}{2} \log \frac{1+m}{2}$$

The terminology comes from statistical mechanics where $F_L(m)$ is the large deviations rate function as $\gamma \to 0$ of an Ising system with Kac potential J_γ which is obtained from J by a rescaling by γ; m represents a magnetization density profile and $F_L(m)$ its free energy; in (1.2) $\beta = 1/(kT)$ with T the absolute temperature and k the Boltzmann constant. We will not pursue here the connection with the Ising model and take $F_L(m)$ and (1.1) as primitive notions of a theory which lives at a mesoscopic level intermediate between microscopics (statistical mechanics) and macroscopics (thermodynamics and continuum theory).
Notice that the stationary points of (1.1) are at the same time the critical points of $F_L(m)$ so that the analysis of stationarity and of variational problems for F_L are intimately related. Indeed the minimizers of $F_L(m)$ are the two functions $m^{(\pm)}(x) \equiv \pm m_\beta$, with $m_\beta > 0$ solving the mean field equation

$$m_\beta = \tanh\{\beta m_\beta\}, \quad (\beta > 1) \tag{1.5}$$

$m^{(\pm)}$ are in fact spatially homogeneous solutions of (1.3) and $F_L(m^{(\pm)}) = 0$. $F_L(m) > 0$ for any other m as it follows from (1.4) recalling that J is a smooth probability kernel. The choice $\beta > 1$ is responsible for the non uniqueness of minimizers, which, in the mesoscopic theory, means that there is a phase transition. Each minimizer is in fact interpreted as a "pure phase" and our model has a plus phase, $m^{(+)}$, and a minus phase, $m^{(-)}$. At $\beta \le 1$ instead the minimizer is unique and given by $m^{(0)} \equiv 0$; $m^{(0)}$ remains a critical point also at $\beta > 1$, but it is no longer a minimizer (or a pure phase). If L is large enough there are also space dependent critical points, in particular $\hat{m}_L$, which is an antisymmetric increasing solution of (1.3) the existence and properties of which have been studied in [4, 7, 8, 11]. There are several seemingly different reasons to study $\hat{m}_L$ that we outline below.

• Energy. For L large enough $\hat{m}_L$ is the first excited state of F_L: in [2] it is proved that there is $\epsilon > 0$ so that

$$\text{if } m = \tanh\{\beta J * m\} \text{ and } F_L(m) < F_L(\hat{m}_L) + \epsilon \text{ then } m \in \{m^{(+)}, m^{(-)}, \hat{m}_L\}. \tag{1.6}$$

• Finite volume interfaces. There is a stationary solution $\bar{m}(x)$ of (1.1) in the whole of $\mathbb{R}$, i.e.

$$\bar{m}(x) = \tanh\{\beta(J * \bar{m})(x)\}, \qquad x \in \mathbb{R}, \tag{1.7}$$

$$\lim_{x\to\pm\infty} \bar{m}(x) = \pm m_\beta.$$

Thus $\bar{m}$ (called instanton) interpolates between the two equilibria states $m^{(\pm)}$ at $\pm\infty$ and it has the interpretation of the interface at phase coexistence. In [4], [11] it has been proved that

$$\lim_{L\to\pm\infty} \sup_{|x|\le L/2} |\bar{m}(x) - \hat{m}_L(x)| = 0 \tag{1.8}$$

Thus $\hat{m}_L$ has the meaning of the interface at finite volumes.
• The Wulff problem. According to thermodynamics, the minimizers of the free energy are the equilibrium states. If the total magnetization is a conserved quantity, then equilibrium at 0 magnetization is described by minimizing $F_L(m)$ with the constraint $\int m = 0$, Wulff problem. In [20] it is proved that for L large enough the infimum of F_L with the 0 magnetization constraint is attained at a unique point, $\hat{m}_L$.
• Tunnelling. While the instanton $\bar{m}$ and its finite volume version $\hat{m}_L$ describe the "optimal" spatial pattern to connect $m^{(\pm)}$, $\hat{m}_L$ is also the saddle point of optimal orbits connecting $m^{(\pm)}$ in time, as proved in [1].

In this paper we will investigate the latter issue establishing first the existence of a dynamical connection between $\hat{m}_L$ and $m^{(\pm)}$. We will prove that there are two one-dimensional, invariant manifolds, $\mathcal{W}_\pm = \{v^{(\pm)}(x, s) : |x| \le L/2,\ s \in \mathbb{R}\}$, such that, denoting by $S_t(m)$ the semigroup generated by (1.1), see Section 2 for a precise definition,

$$\lim_{t\to-\infty} \|v^{(\pm)}(\cdot, s) - \hat{m}_L\|_\infty = 0, \quad \lim_{t\to\infty} \|v^{(\pm)}(\cdot, s) - m^{(\pm)}\|_\infty = 0 \tag{1.9}$$

and moreover, for any $t \ge 0$ and any $s \in \mathbb{R}$,

$$S_t(v^{(\pm)}(\cdot, s)) = v^{(\pm)}(\cdot, t + s). \tag{1.10}$$

The existence of such manifolds has been much studied in the context of the Allen-Cahn equation $u_t = u_{xx} + u - u^3$, starting from the works of [17] and [6]. It has also been studied, [4]–[2], for the non local "Glauber" evolution equation

$$u_t = -u + \tanh\{\beta J^{\text{neum}} * u\} \tag{1.11}$$

which is similar to (1.1) and with the same critical points. But unfortunately the orbits of (1.11) and (1.1) do not coincide: therefore a new analysis is needed, and this is what we carry out in the present paper. Our main motivation for doing it is related to the tunnelling problems mentioned above. A characterization of the optimal orbit connecting $m^{(\pm)}$ requires a proof of existence of the two manifolds $v^{(\pm)}$ and its local stability, we refer to [1] for a discussion on this point. We also mention that the question appears also in tunnelling in $d = 2$ dimensions. Indeed a key estimate in the proof in [3] (namely that any optimizing orbit from $m^{(-)}$ to $m^{(+)}$ in $d = 2$ dimensions is planar and follows first $v^{(-)}$ with time reversed and then $v^{(+)}$) is based on the analysis we carry out here. The present paper deals only with the $d = 1$ case, but the extension to $d = 2$ is immediate using the spectral analysis in [3], and is discussed in Section 7.

2 Definitions and results

2.1 Existence of dynamics

The velocity field $f_L(m)$ is Lipschitz when restricted to sets of the form $\{\|m\|_\infty \le b\}$, $b < 1$. Then, by classical arguments, the Cauchy problem for (1.1) with initial datum $m \in L^\infty([-L/2, L/2], (-1, 1))$ has a unique, local [in time] solution, denoted by $S_t(m)$. See Appendix A for details on this and the other statements in the present section.
To derive global existence we use a priori bounds, namely that if $\|m\|_\infty < 1$, then there is $b < 1$ such that for all t, $\|S_t(m)\|_\infty < b$, a statement which follows from the comparison theorem. Recall that a smooth function $v(x, t)$ is a sub-solution (respectively super-solution) of (1.1) if

$$v_t \le f_L(v) \qquad \text{(respectively } v_t \ge f_L(v)). \tag{2.1}$$

Proposition 2.1. *Let $m \in L^\infty([-L/2, L/2], (-1, 1))$. If v is a sub-solution (respectively super-solution) of (1.1) and $v(\cdot, 0) \le m$ (respectively $v(\cdot, 0) \ge m$) then*

$$v(\cdot, t) \le S_t(m) \qquad \textit{(respectively } v(\cdot, t) \ge S_t(m)) \tag{2.2}$$

In this way we will prove that $S_t(m)$ is well defined for all $m \in L^\infty([-L/2, L/2], (-1, 1))$ and all $t > 0$. Moreover

Proposition 2.2. *If m and $\tilde{m}$ are both in $L^\infty([-L/2, L/2], (-1, 1))$, then for any $t > 0$*

$$\|S_t(m) - S_t(\tilde{m})\|_\infty \le e^t \|m - \tilde{m}\|_\infty \tag{2.3}$$

By continuity S_t extends to $L^\infty([-L/2, L/2], [-1, 1])$.

Proposition 2.3. *Let $\|m\|_\infty \le 1$. For any $t > 0$, $\|S_t(m)\|_\infty < 1$, $S_t(m)$ solves (1.1) for $t > 0$, and $S_t(m) \to m$ in L^∞ as $t \to 0$.*

Using sub and super solutions, existence, uniqueness and regularity theorems extend to the case of a bounded, smooth external force as considered in Section 5, see (5.1).

2.2 Instantons manifold and the finite volume instanton

In [13],[14] it is proved that there exists a solution $\bar{m}$ of (1.7), called instanton, which is a $\mathcal{C}^\infty$, strictly increasing, antisymmetric function. Moreover, with $\alpha > 0$ such that

$$\beta(1-m_\beta^2)\int J(0,z)e^{\alpha z} = 1, \tag{2.4}$$

there are $a > 0$, $\alpha_0 > \alpha$, and $c > 0$ so that, for all $x \geq 0$,

$$\left|\bar{m}(x) - (m_\beta - ae^{-\alpha x})\right| + \left|\bar{m}'(x) - \alpha a e^{-\alpha x}\right| + \left|\bar{m}''(x) + \alpha^2 a e^{-\alpha x}\right| \leq ce^{-\alpha_0 x}, \tag{2.5}$$

where $\bar{m}'$ and $\bar{m}''$ are respectively the first and second derivatives of $\bar{m}$.
Given any $\xi \in \mathbb{R}$ we denote by

$$\bar{m}_\xi(x) = \bar{m}(x-\xi), \qquad x \in \mathbb{R}$$

the instanton with center ξ and $\{\bar{m}_\xi : \xi \in \mathbb{R}\}$ the instantons manifold. Any solution of (1.7) which is definitively strictly negative (respectively positive) as $x \to -\infty$ (respectively as $x \to \infty$) is an element of the instantons manifold.
In [4, 7, 8, 11] it has been proved that a finite volume analogue of the instanton does exist. If L is large enough there is in fact a C^∞ solution $\hat{m}_L$ of (1.3), called the finite volume instanton, which is antisymmetric and strictly increasing; morevover $\hat{m}_L$ converges to $\bar{m}$ as $L \to \infty$ in the sense of (1.8). For finite L however there is no analogue of the instantons manifold, but we will prove that there are invariant manifolds starting from $\hat{m}_L$ where the motion is "extremely slow" as L becomes large. Uniqueness of finite volume instantons is proved (in the above quoted references) in the following sense. Given $\epsilon > 0$ and $r \in (0,1)$ let

$$B_{\epsilon,r} := \left\{m \in L^\infty([-L/2, L/2], (-1,1)) : \inf_{|\xi| \leq rL/2} \|m - \bar{m}_\xi \mathbf{1}_{|x|\leq L}\|_\infty \leq \epsilon\right\} \tag{2.6}$$

Then for any $r \in (0,1)$ and $\epsilon > 0$ small enough, there is $L_{\epsilon,r}$ such that $\hat{m}_L$ is the only solution of (1.3) in $B_{\epsilon,r}$ for any $L \geq L_{\epsilon,r}$.

2.3 Spectral properties of linearized operators

The content of this section is based on the papers [7, 8, 10], in Appendix B we fill in the missing details.
Let $m \in L^\infty([-L/2, L/2], (-1,1))$ and define

$$\Omega_m u := -u + p_m J * u, \qquad p_m(x) := \beta\left[1 - m(x)^2\right] \tag{2.7}$$

If m is a stationary solution of (1.11) then Ω_m is obtained by linearizing (1.11) around m. Ω_m is self-adjoint in $L^2([-L/2, L/2], p_m^{-1}dx)$. We will denote by $\langle\cdot,\cdot\rangle$, $\|\cdot\|_2$ and $\|\cdot\|_\infty$ the

scalar product, the L^2 and the L^∞ norm in $L^2([-L/2,L/2],dx)$ and add a subscript p_m when the scalar product is with weight p_m^{-1}.
In [7, 8, 10] it is proved that there are positive constants $\hat c_\pm$, ω and $\epsilon(L)$ so that for any L large enough if $\|m-\hat m_L\|_\infty \le \epsilon(L)$, then Ω_m has a maximal eigenvalue λ (dependence on m is not made explicit)

$$\hat c_- e^{-2\alpha L} \le \lambda \le \hat c_+ e^{-2\alpha L} \tag{2.8}$$

with eigenfunction e smooth and strictly positive. To the left of λ the spectrum (regarding Ω_m as an operator on $L^2([-L/2,L/2],p_m^{-1}dx)$) has a gap, namely

$$\langle u,\Omega_m u\rangle_{p_m} \le -\omega\langle u,u\rangle_{p_m}, \qquad \langle u,e\rangle_{p_m}=0 \tag{2.9}$$

The semigroup $e^{\Omega_m t}$ has a spectral gap also in L^∞, in fact it has been proved in [7, 8, 10] that there are positive constants λ', ω' and c' so that

$$\|e^{\Omega_m t}u\|_\infty \le c' e^{\lambda' t}\|u\|_\infty, \qquad \|e^{\Omega_m t}u\|_\infty \le c' e^{-\omega' t}\|u\|_\infty,\ \langle u,e\rangle_{p_m}=0 \tag{2.10}$$

We prove in Appendix B that the L^2 analysis extends to the operator $\mathcal{L}_m = p_m^{-1}\Omega_m$, explicitly

$$\mathcal{L}_m u = -\frac{1}{p_m}u + J^{\rm neum} * u \tag{2.11}$$

$\mathcal{L}_m$ is obtained by linearizing (1.1) around m. Referring to Appendix B for details of the proof which is based on the previous analysis on Ω_m, we have

Theorem 2.4. *There are positive constants $c_\pm$, ω and given any $\delta>0$ there exists $\epsilon(L,\delta)$ so that for L large enough if $\|m-\hat m_L\|_2 \le \epsilon(L,\delta)$ and $\|m\|_\infty < 1-\delta$ then the L^2 norm of the operator $\mathcal{L}_m$ is bounded by $1+\beta^{-1}\mathrm{arctanh}''((1+m_\beta)/2)$ and:*

(i) $\mathcal{L}_m$ has a positive eigenvalue λ_m

$$c_- e^{-2\alpha L} \le \lambda_m \le c_+ e^{-2\alpha L} \tag{2.12}$$

with eigenfunction e_m which is smooth and strictly positive;

(ii)

$$\langle u,\mathcal{L}_m u\rangle \le -\omega\langle u,u\rangle, \qquad u\in L^2([-L/2,L/2];[-1,1]), \quad \langle u,e_m\rangle = 0 \tag{2.13}$$

(iii) let $\dfrac{\delta e_m}{\delta m}$ be the linear operator such that $\dfrac{de_{m(s)}}{ds}\Big|_{s=0} = \dfrac{\delta e_m}{\delta m}\dfrac{dm(s)}{ds}\Big|_{s=0}$ for any smooth curve $m(s)$, $m(0)=m$. Then there is $c_1>0$ so that

$$\|\frac{\delta e_m}{\delta m}\|_2 \le c_1 \tag{2.14}$$

Notice that for L large enough, $\lambda_m - \omega < 0$ and there is a constant c so that

$$\|\mathcal{L}_m^{-1}u\|_2 \le c, \qquad \langle u,e_m\rangle = 0 \tag{2.15}$$

2.4 Invariant manifolds: existence

The first of the two main results in this paper is the following theorem which extends to the present case results proved in [4] for the evolution (1.11).

Theorem 2.5. *There is $L_0 > 0$ so that the following holds. For any $L > L_0$, there are two distinct one-dimensional manifolds*

$$\mathcal{W}_\pm = \left\{ v^\pm(x,s) : |x| \le L/2, s \in \mathbb{R} \right\},$$

for which (1.9)–(1.10) *hold.*

The proof is reported in Section 4.

2.5 Invariant manifolds: stability

There is a L^2 neighborhood of $\mathcal{W}_+$ which is attracted by $m^{(+)}$ (for $\mathcal{W}_-$ the analogous statements hold). As discussed in the beginning of Section 5 the statement is almost evident due to the continuity of S_t and the stability of $m^{(+)}$. Less trivial is the property that for any $\epsilon > 0$ there is $\delta > 0$ so that if $\|m - \hat{m}_L\|_2 < \delta$ and $\lim\limits_{t\to\infty} \|S_t(m) - m^{(+)}\|_2 = 0$, then

$$\inf_{s\in\mathbb{R}} \|S_t(m) - v^{(+)}(s)\|_2 < \epsilon \qquad \forall t > 0. \tag{2.16}$$

A stronger statement is actually proved in Theorems 5.5–5.8 (the other main results in this paper) where also sufficiently small external forces are added to (1.1). In Section 6 we prove that the initial condition can be controlled in terms of the energy, as we will prove a lower bound for $F_L(m) - F_L(\hat{m}_L)$, $m \in \Sigma = \{u : \lim_{t\to\infty} \|S_t(u) - \hat{m}_L\|_2 = 0\}$, proportional to $\|m - \hat{m}_L\|_2^2$. The result is needed in the applications to tunnelling.

2.6 Tunnelling.

Let

$$\begin{aligned}\mathcal{U}_\tau[m^{(-)}, m^{(+)}] = \Big\{ u \in C^\infty\big((-L/2, L/2) \times (0,\tau); (-1,1)\big) :& \\ \lim_{t\to 0^+} u(\cdot,t) = m^{(-)},\ \lim_{t\to\tau^-} u(\cdot,t) = m^{(+)} \Big\}.&\end{aligned} \tag{2.17}$$

the set of tunnelling events in a time τ;

$$I_\tau(u) = \frac{1}{4}\int_0^\tau \int_{-L}^{L} K(x,t)^2\, dx\, dt, \quad K = u_t - f_L(u) \tag{2.18}$$

the "cost" of an orbit u and

$$P_{[m^{(-)},m^{(+)}]} := \inf_{\tau>0} \inf_{u\in\mathcal{U}_\tau[m^{(-)},m^{(+)}]} I_\tau(u) \tag{2.19}$$

the tunnelling penalty. Then in [1] it is proved that

Theorem 2.6. *For any L large enough,*

$$P_{[m^{(-)},m^{(+)}]} = F_L(\hat{m}_L) \tag{2.20}$$

The proof of Theorem 2.6 in [1] shows that any minimizing sequence in (2.19) passes arbitrarily close to $\hat{m}_L$. The validity of Theorem 2.6 extends to $d = 2$ dimensions for the model defined in the square $Q_L = [-L/2, L/2]^2$. In [3] it is proved that the penalty $P_{[m^{(-)},m^{(+)}]}$ is again given by the $d = 1$ energy $F_L(\hat{m}_L)$ multiplied by the factor $2L$. Indeed the function $\hat{m}^e_L$ on Q_L which only depends on the x-coordinate and as a function of x is equal to $\hat{m}_L$ is stationary and again any minimizing sequence passes arbitrarily close to $\hat{m}^e_L$. In [3] using our Theorems 5.5–5.8 (whose validity extends to $d = 2$, as briefly discussed in Section 7), it is shown that any minimizing sequence is an orbit which follows backward in time $v^{(-)}$ from $m^{(-)}$ to $\hat{m}^e_L$ and then forward in time $v^{(+)}$ from $\hat{m}^e_L$ to $m^{(+)}$ (the analogous statement holds in $d = 1$ as well).
While Theorem 2.6 answers the first question about tunnelling, namely the minimal cost for the tunnelling to occur, the above result specifies also the way the tunnelling occurs. While it is well established that a minimizing sequence can be obtained by following the reversed flow on the invariant manifolds, see [15], our statements in Theorems 5.5–5.8 completes the picture by saying that "this is in fact the only possible way", as any other pattern would lead to a larger penalty.

3 Approximants of the invariant manifolds

In this section we study manifolds which approximate the invariant manifolds $\mathcal{W}_\pm$ of Theorem 2.4. In agreement with Section 2.2 we denote by $\hat{m}_L$ the finite volume instanton and write $\hat{\lambda}$ and $\hat{e}$ for the maximal eigenvalue and corresponding eigenfunction of

$$\hat{\mathcal{L}} := \mathcal{L}_{\hat{m}_L},$$

recalling from Theorem 2.4 that $\hat{\lambda} > 0$ and $\hat{e} := e_{\hat{m}_L}$ is strictly positive and smooth, and we normalize $\hat{e}$ so that $\langle \hat{e}, \hat{e} \rangle = 1$. Finally we denote by $\hat{\omega}$ the L^2 spectral gap of $\hat{\mathcal{L}}$ and by ω the spectral gap of $\mathcal{L}_m$ when m is in a suitable neighborhood of $\hat{m}_L$, see (2.13). Recall that the remaining part of the spectrum is strictly negative.
The proof of Theorem 2.5 is based on constructing portions of the invariant manifolds $\mathcal{W}_\pm$ in a small neighborhood of $\hat{m}_L$ and then prove that such manifolds can be continued reaching respectively $m^{(+)}$ and $m^{(-)}$. By the symmetry under change of sign it suffices to consider the former case to which we restrict in the sequel. The proof is in spirit close to the one in [4] relative to the evolution $u_t = -u + \tanh\{\beta J^{\mathrm{neum}} * u\}$, but the absence in our case of a L^∞ spectral gap for the linearized evolution around $\hat{m}_L$ complicates the proofs.
In a small neighborhood of $\hat{m}_L$, the manifold $\mathcal{W}_+$ is to a linear order approximation given by

$$\mathcal{M}_+ = \Big\{ v_a = \hat{m}_L + a\,\hat{e} \Big\} \tag{3.1}$$

(we will only consider small positive values of a). However if non linear effects are taken into account,

$$v_a(t) = S_t(v_a) \tag{3.2}$$

leaves $\mathcal{M}_+$ as soon as t is positive, while, on the contrary, the solution of the linearized evolution around $\hat{m}_L$ is $v_{e^{\hat\lambda t}a} \in \mathcal{M}_+$ (for t not too large). We will prove that in a suitably small neighborhood of $\hat{m}_L$, the orbits $v_a(\cdot)$ are close to $\mathcal{M}_+$ and to the linearized evolution on $\mathcal{M}_+$ converging, as $a \to 0$ to a limit manifold which in the end will be identified with $\mathcal{W}_+$. We thus have two natural ways to parameterize points in a small neighborhood of $\mathcal{M}_+$, by using orthogonal projections either onto $\mathcal{M}_+$ or onto $v_a(\cdot)$.

Definition 3.1. *The a-coordinate of m is*

$$a(m) = \Big\langle m - \hat{m}_L, \hat{e} \Big\rangle. \tag{3.3}$$

We denote by $a(t;b)$ the a-coordinate of $v_b(t)$, so that we can write

$$v_b(t) = v_{a(t;b)} + \psi(t;b), \qquad \langle \psi(t;b), \hat{e} \rangle = 0. \tag{3.4}$$

Given $b > 0$, the t-coordinate of m relative to the orbit $\{v_b(\cdot)\}$ is a non negative number $\tau_{m;b}$ such that

$$\Big\langle m - v_b(\tau_{m;b}), e_{v_b(\tau_{m;b})} \Big\rangle = 0 \tag{3.5}$$

We will prove later that the t-coordinate relative to the orbit $\{v_b(\cdot)\}$, $b > 0$, is well defined provided m is sufficiently close to $\mathcal{M}_+$ and b is sufficiently smaller than the a-coordinate of m (see Proposition 3.4 (ii)). We will also establish relations between a and t-coordinates.
We start with some a-priori estimates on the orbit $S_t(\hat{m}_L + u_0)$. We suppose u_0 a small, smooth function and study the orbit up to times t which grow proportionally to $|\log(\|u_0\|_2)|$. We linearize around $\hat{m}_L$ and control the non linear terms using smallness and smoothness of the initial datum. We prove that the orbit follows $\mathcal{M}_+$ with orthogonal deviations which are much smaller than the displacement along $\mathcal{M}_+$.

Lemma 3.1. *There exists a constant $c_0' > 0$ so that the following holds. Let $u_0 \in \mathcal{C}^1([-L/2, L/2])$ be such that*

$$\|u_0\|_2^{2/3} < \frac{\hat\lambda}{c_0'}, \quad \|u_0\|_\infty \le \frac{1+m_\beta}{2}. \tag{3.6}$$

Define

$$\sigma = \sigma_{u_0} := \frac{1}{\hat\lambda} \log\Big(\frac{\hat\lambda}{c_0' \|u_0\|_2^{2/3}} \Big) > 0, \tag{3.7}$$

$$u(t) := S_t(\hat{m}_L + u_0) - \hat{m}_L.$$

Then

$$\left\| u(t) - e^{\hat{\mathcal{L}}t} u_0 \right\|_2 < \frac{c_0'}{\hat{\lambda}} \left(e^{\hat{\lambda}t} \| u_0 \|_2 \right)^{5/3} \qquad \forall t \in [0, \sigma], \tag{3.8}$$

$$\| u(t) \|_2 \le 2 e^{\hat{\lambda}t} \| u_0 \|_2 \qquad \forall t \in [0, \sigma]. \tag{3.9}$$

Proof. We will prove the lemma with $c_0' = 2^6 C C_2$, $C = C_{t=0}$ as in (C.14), $C_2 = (2\beta)^{-1} \text{arctanh}''(x)$, $x = (m_\beta + 1)/2$, see (C.16).
Since $S_t(\hat{m}_L) = \hat{m}_L$ and $f_L(\hat{m}_L) = 0$ we have

$$u_t = \hat{\mathcal{L}} u + \left[f_L(\hat{m}_L + u) - f_L(\hat{m}_L) - \hat{\mathcal{L}} u \right] \tag{3.10}$$

which implies

$$u = e^{\hat{\mathcal{L}}t} u_0 + \int_0^t ds \, e^{\hat{\mathcal{L}}(t-s)} \left[f_L(\hat{m}_L + u) - f_L(\hat{m}_L) - \hat{\mathcal{L}} u \right] \tag{3.11}$$

Then by (C.16), (C.14), recalling that $\hat{\lambda}$ is the maximal eigenvalue of $\hat{\mathcal{L}}$, and using Jensen's inequality (with respect to the measure $\frac{e^{\hat{\lambda}(t-s)} ds}{\hat{\lambda}^{-1}}$)

$$\left\| u(t) - e^{\hat{\mathcal{L}}t} u_0 \right\|_2^2 \le C C_2 \int_0^t e^{\hat{\lambda}(t-s)} \| u(s) \|_2^{10/3} \, ds. \tag{3.12}$$

We prove (3.8) by contradiction. We suppose, without loss of generality, that $\|u_0\|_2 > 0$. Let $T \le \sigma$ be the first time when the inequality (3.8) becomes an equality. Note that for any s such that (3.8) holds

$$\| u(s) \|_2 \le e^{\hat{\lambda}s} \| u_0 \|_2 + \frac{c_0'}{\hat{\lambda}} \left(e^{\hat{\lambda}s} \| u_0 \|_2 \right)^{5/3}. \tag{3.13}$$

Define

$$\rho_t := [e^{\hat{\lambda}t} \| u_0 \|_2]^{2/3}.$$

We use (3.13) in (3.12) with $t = T$. We estimate the integral on the right hand side of (3.12) as follows

$$\begin{aligned} & \int_0^T e^{\hat{\lambda}(T-s)} \left[e^{\hat{\lambda}s} \| u_0 \|_2 + \frac{c_0'}{\hat{\lambda}} \left(e^{\hat{\lambda}s} \| u_0 \|_2 \right)^{5/3} \right]^{10/3} ds \\ \le & \int_0^T e^{\hat{\lambda}(T-s) + (10/3)\hat{\lambda}s} \, \| u_0 \|_2^{10/3} \left[1 + \frac{c_0'}{\hat{\lambda}} \rho_s \right]^{10/3} ds \\ \le & \left(1 + \frac{c_0' \rho_\sigma}{\hat{\lambda}} \right)^{10/3} \| u_0 \|_2^{10/3} e^{\hat{\lambda}T} \int_0^T e^{\frac{7}{3}\hat{\lambda}s} ds \le \frac{6}{7\hat{\lambda}} \left(e^{\hat{\lambda}T} \| u_0 \|_2 \right)^{10/3} \end{aligned} \tag{3.14}$$

In the last inequality we have used that $c_0'\rho_\sigma < \hat{\lambda}$, which follows from (3.6). Observing that $\big(2CC_2\frac{3}{7\hat{\lambda}}\big)^{1/2} \le \frac{c_0'}{\hat{\lambda}}$ for L sufficiently large (see (2.12)), from (3.14) and (3.12) we get

$$\left\|u(T) - e^{\hat{\mathcal{L}}T}u_0\right\|_2 < \frac{c_0'}{\hat{\lambda}}\left(e^{\hat{\lambda}T}\|u_0\|_2\right)^{5/3} \tag{3.15}$$

We have thus reached a contradiction and (3.8) is proved.
By (3.8) and (3.13) it follows that

$$\|u(t)\|_2 \quad \le \quad \left(1 + \frac{c_0'}{\hat{\lambda}}\rho_\sigma\right)e^{\hat{\lambda}t}\|u_0\|_2 \le 2e^{\hat{\lambda}t}\|u_0\|_2 \tag{3.16}$$

for all $t \le \sigma$, and (3.9) is proved. □

In the next lemmas we study the orbits $S_t(\hat{m}_L + b\hat{e}) = v_b(t)$ with $b > 0$ small, first for short times and then, by an iterative procedure, for longer times.

Lemma 3.2. *Let c_0' be as in Lemma 3.1, and define*

$$\sigma_b := \frac{1}{\hat{\lambda}}\log\Big(\frac{\hat{\lambda}}{c_0' b^{2/3}}\Big), \qquad b \in \Big(0, \big(\frac{\hat{\lambda}}{c_0'}\big)^{3/2}\Big). \tag{3.17}$$

Let $\epsilon_0 \in \Big(0, \frac{1-m_\beta}{4\|\hat{e}\|_\infty}\Big)$. Then

$$\left\|v_b(t) - v_{e^{\hat{\lambda}t}b}\right\|_2 \le \frac{c_0'}{\hat{\lambda}}\big(e^{\hat{\lambda}t}b\big)^{5/3} \qquad \forall b \in (0,\epsilon_0), \quad \forall t \in [0,\sigma_b]. \tag{3.18}$$

Moreover

$$\|\psi(t;b)\|_2 \le \frac{c_0'}{\hat{\lambda}}\big(e^{\hat{\lambda}t}b\big)^{5/3}, \qquad |a(t;b) - e^{\hat{\lambda}t}b| \le \frac{c_0'}{\hat{\lambda}}\big(e^{\hat{\lambda}t}b\big)^{5/3} \qquad \forall t \in [0,\sigma_b]. \tag{3.19}$$

Proof. We have $\|v_b - \hat{m}_L\|_\infty \le \|\hat{e}\|_\infty b < (1-m_\beta)/2$. Then (3.18) follows from Lemma 3.1 observing that

$$e^{\hat{\mathcal{L}}t}b\hat{e} = e^{\hat{\lambda}t}b\hat{e}.$$

By (3.4) we have $\|\psi(t;b)\|_2^2 = \inf_{a\in\mathbb{R}}\|v_b(t) - v_a\|_2^2 \le \|v_b(t) - v_{e^{\hat{\lambda}t}b}\|_2^2 \le \{\frac{c_0'}{\hat{\lambda}}[e^{\hat{\lambda}t}b]^{5/3}\}^2$, where the last inequality follows from (3.18). Then the first inequality in (3.19) follows. We write

$$v_b(t) = v_{e^{\hat{\lambda}t}b} + \phi, \qquad \phi := v_b(t) - v_{e^{\hat{\lambda}t}b}$$

so that $a(t;b) = e^{\hat{\lambda}t}b + \langle\phi,\hat{e}\rangle$ and the second inequality in (3.19) then follows from (3.18). □

Lemma 3.3. *Let* $\epsilon_0 \in \Big(0, \dfrac{1-m_\beta}{4\|\hat{e}\|_\infty}\Big)$*. There exists* $\epsilon_1 \in \big(0, \min(\epsilon_0, \frac{\hat{\lambda}}{8c_0'})\big)$ *so that the following holds. If at time* $t^* > 0$ *equality* (3.4) *holds with* $b \in (0, \epsilon_1)$ *and*

$$a(t^*; b) \le \epsilon_1, \quad \|\psi(t^*; b)\|_2 \le \frac{4c_0'}{\hat{\lambda}} a(t^*; b)^{5/3}, \tag{3.20}$$

then (3.4) *holds at time* $t^* + \sigma_{a(t^*;b)}$ *and*

$$\|\psi(t^* + \sigma_{a(t^*;b)}; b)\|_2 \le \frac{4c_0'}{\hat{\lambda}} a(t^* + \sigma_{a(t^*;b)}; b)^{5/3}. \tag{3.21}$$

Proof. Set for simplicity $a_* := a(t^*; b)$. We observe that from (3.4), (3.1) and (2.13),

$$\|e^{\hat{\mathcal{L}}s}(v_b(t^*) - \hat{m}_L) - e^{\hat{\lambda}s} a_* \hat{e}\|_2 \le e^{-\hat{\omega}s} \|\psi(t^*; b)\|_2, \qquad s \ge 0.$$

Then (3.8) and (3.18) imply that for $s \in (0, \sigma_{a_*}]$,

$$\begin{aligned}
\|v_b(t^* + s) - v_{e^{\hat{\lambda}s} a_*}\|_2 = \quad & \|S_s(v_b(t^*)) - \hat{m}_L - e^{\hat{\lambda}s} a_* \hat{e}\|_2 \\
\le \quad & \|u(s) - e^{\hat{\mathcal{L}}s}(v_b(t^*) - \hat{m}_L)\|_2 + \|e^{\hat{\mathcal{L}}s}(v_b(t^*) - \hat{m}_L) - e^{\hat{\lambda}s} a_* \hat{e}\|_2 \\
\le \quad & e^{-\hat{\omega}s} \|\psi(t^*; b)\|_2 + \frac{c_0'}{\hat{\lambda}} \Big(e^{\hat{\lambda}s}[a_* + \|\psi(t^*; b)\|_2]\Big)^{5/3} \\
\le \quad & e^{-\hat{\omega}s} \frac{4c_0'}{\hat{\lambda}} a_*^{5/3} + \frac{c_0'}{\hat{\lambda}} \big(e^{\hat{\lambda}s}[a_* + \frac{4c_0'}{\hat{\lambda}} a_*^{5/3}]\big)^{5/3} \\
\le \quad & \frac{c_0'}{\hat{\lambda}} (e^{\hat{\lambda}s} a_*)^{5/3} \Big(4e^{-\hat{\omega}s} e^{-\frac{5}{3}\hat{\lambda}s} + \big[1 + \frac{4c_0'}{\hat{\lambda}} \epsilon_1^{2/3}\big]^{5/3}\Big)
\end{aligned} \tag{3.22}$$

Recalling the choice of a_*, the definition (3.17) of σ_{a_*} and by taking $\epsilon_1 > 0$ sufficiently small, from (3.22) we get

$$\|v_b(t^* + \sigma_{a_*}) - v_{e^{\hat{\lambda}\sigma_{a_*}} a_*}\|_2 \le \frac{2c_0'}{\hat{\lambda}} (e^{\hat{\lambda}\sigma_{a_*}} a_*)^{5/3}. \tag{3.23}$$

From the second inequality in (3.19) we get that

$$|a(t^* + \sigma_{a_*}; b) - e^{\hat{\lambda}\sigma_{a_*}} a_*| \le \frac{c_0'}{\hat{\lambda}} \big(e^{\hat{\lambda}\sigma_{a_*}} a_*\big)^{\frac{5}{3}}.$$

Recalling again the definition of σ_{a_*}, for ϵ_1 small enough, we get

$$a(t^* + \sigma_{a_*}; b) \ge e^{\hat{\lambda}\sigma_{a_*}} a_* \Big(1 - \frac{c_0'}{\hat{\lambda}} \big(\frac{\hat{\lambda}}{c_0'} \epsilon_1^{1/3}\big)^{2/3}\Big) \ge e^{\hat{\lambda}\sigma_{a_*}} a_* 2^{-3/5}$$

Thus

$$e^{\hat{\lambda}\sigma_{a_*}} a_* \le a(t^* + \sigma_{a_*}; b) 2^{3/5} \tag{3.24}$$

Then from (3.23) and (3.24) we get

$$\|v_b(t^* + \sigma_{a_*}) - v_{e^{\hat\lambda \sigma_{a*}} a_*}\|_2 \le 2\frac{c_0'}{\hat\lambda}\{e^{\hat\lambda \sigma_{a_*}} a_*\}^{5/3} \le 4\frac{c_0'}{\hat\lambda} a(t^* + \sigma_{a_*}; b)^{5/3}$$

Since $\|\psi(t^* + \sigma_{a_*}; b)\|_2 = \inf_{a' \in \mathbb{R}} \|v_b(t^* + \sigma_{a_*}) - v_{a'}\|_2$, (3.21) follows.

$\square$

Remark 3.4. Note that the constant $\frac{4c_0'}{\hat\lambda}$ in conclusion (3.21) is the same as in (3.20).

We set

$\|\hat{\mathcal{L}}\|$ the (L^2) norm of $\hat{\mathcal{L}}$,

c_0' is as in Lemma 3.1,

C_2 defined as in (C.16), with $\mathtt{a}$ replaced by $1-\sigma$, $\sigma > 0$ small enough,

$C = C_{t=0}$ as in (C.14), where m is of the form $\hat m_L + b\hat e$,

$\tilde c_2 := \|\hat{\mathcal{L}}\| \frac{4c_0'}{\hat\lambda} + 2C_2 C$,

c_1 the constant appearing in (2.14), for m of the form $\hat m_L + b\hat e$,

ϵ_1 as in Lemma 3.3 and also such that

$$4\hat\lambda c_1\, \epsilon_1^{1/3} < 1, \qquad \frac{4c_0' + \tilde c_2}{\hat\lambda}\, \epsilon_1^{2/3} < \frac{1}{4}, \tag{3.25}$$

$T(\epsilon_1; b)$ the time for which $a(T(\epsilon_1; b); b) = \epsilon_1$, for a given $b \in (0, \epsilon_1)$.

In the next proposition we prove that if b is small enough then (3.21) holds for all times $t \in (0, T(\epsilon_1; b))$, the function $a(t; b)$ is strictly positive and increasing and $f_L\big(v_b(t)\big)$ is essentially parallel to $\hat e$.

Proposition 3.2. *There exists $\epsilon_2 \in (0, \epsilon_1]$ so that for all $b \in (0, \epsilon_2)$ and all $t \in (0, T(\epsilon_1; b))$:*

$$\|v_b(t) - v_{a(t;b)}\|_2 \leq \frac{4c_0'}{\hat{\lambda}} a(t;b)^{5/3}, \tag{3.26}$$

$t \to a(t;b)$ is strictly positive, differentiable and strictly increasing with respect to t,

$$\Big|\frac{da(t;b)}{dt} - \hat{\lambda}\, a(t;b)\Big| \leq \tilde{c}_2\, a(t;b)^{5/3}, \tag{3.27}$$

and

$$\|f_L\big(v_b(t)\big) - \hat{\lambda}\, a(t;b)\, \hat{e}\|_2 \leq \tilde{c}_2\, a(t;b)^{5/3} \tag{3.28}$$

$$\big|\langle f_L\big(v_b(t)\big), e_{v_b(t)}\rangle - \hat{\lambda}\, a(t;b)\big| \leq 2\tilde{c}_2 a(t;b)^{5/3} \tag{3.29}$$

$$|\langle f_L\big(v_b(t)\big), e_{v_b(t)}\rangle| \geq \frac{\hat{\lambda}}{2}\, a(t;b) \geq \frac{\|f_L\big(v_b(t)\big)\|_2}{4} \tag{3.30}$$

Finally for all $t_1 < t_2 \leq T(\epsilon_1; b)$ we have

$$e^{(t_2-t_1)(\hat{\lambda} - \tilde{c}_2\, a(t_2;b)^{2/3})} \leq \frac{a(t_2;b)}{a(t_1;b)} \leq e^{(t_2-t_1)(\hat{\lambda} + \tilde{c}_2\, a(t_2;b)^{2/3})} \tag{3.31}$$

$$\|v_b(t_2) - v_b(t_1)\|_2 \leq a(t_2;b)\Big(1 - e^{-(t_2-t_1)(\hat{\lambda} + \tilde{c}_2\, a(t_2;b)^{2/3})}\Big) + 2\frac{4c_0'}{\hat{\lambda}} a(t_2;b)^{5/3} \tag{3.32}$$

Proof. We fix $b \in (0, \epsilon_2)$ with $\epsilon_2 \in (0, \epsilon_1)$ small enough. We shorthand $a = a(t;b)$ with $t < T(\epsilon_1; b)$. Inequality (3.26) follows by an iterative procedure using Lemmas 3.2 and 3.3. Let us prove (3.28). Recalling the definition of $R(t)$ in (C.15) we have

$$f_L\big(v_b(t)\big) \quad = \quad \hat{\mathcal{L}}\big(v_b(t) - \hat{m}_L\big) + R(t) = \hat{\mathcal{L}}\big(v_b(t) - v_a + v_a - \hat{m}_L\big) + R(t),$$

so that

$$f_L\big(v_b(t)\big) - \hat{\lambda} a \hat{e} = \hat{\mathcal{L}}\big(v_b(t) - v_a\big) + R(t). \tag{3.33}$$

From (3.26) it follows that $\|\hat{\mathcal{L}}\big(v_b(t) - v_a\big)\|_2 \leq \|\hat{\mathcal{L}}\| \frac{4c_0'}{\hat{\lambda}} a^{5/3}$. From (C.16) and (C.14), we have $\|R(t)\|_2 \leq C_2 C \|v_b(t) - \hat{m}_L\|_2^{5/3}$. Moreover

$$\|v_b(t) - \hat{m}_L\|_2 \leq \|v_b(t) - v_a\|_2 + a \leq \frac{4c_0'}{\hat{\lambda}} a^{5/3} + a \leq a(1 + \frac{4c_0'}{\hat{\lambda}} \epsilon_1^{2/3})$$

Since by (3.25), $\frac{4c_0'}{\hat{\lambda}} \epsilon_1^{2/3} \leq 1$, collecting together the above estimates we have

$$\|f_L\big(v_b(t)\big) - \hat{\lambda}\, a\, \hat{e}\|_2 \leq \|\hat{\mathcal{L}}\| \{\frac{4c_0'}{\hat{\lambda}} a^{5/3}\} + 2C_2 C a^{5/3} \leq \tilde{c}_2 a^{5/3} \tag{3.34}$$

and (3.28) is proved.

We now show (3.27). From (3.28) it follows that

$$\big|\langle f_L\big(v_b(t)\big), \hat e\rangle - \hat\lambda\, a(t;b)\big| \le \tilde c_2 a(t;b)^{5/3} \tag{3.35}$$

Then (3.27) follows from (3.3) and (3.35), and the fact that $a(t;b)$ is positive because $a(0,b) = b > 0$ and $a(t;b)$ is a continuous function of t. Observe that from (3.27) and (3.25) it follows that

$$\frac{da(t;b)}{dt} \ge \hat\lambda\, a(t;b)[1 - \frac{\tilde c_2}{\hat\lambda}\,\epsilon_1^{2/3}] \ge \frac{\hat\lambda}{2} a(t;b) > 0, \qquad \text{for all } t \le T(\epsilon_1;b)$$

Let us now prove (3.29) and (3.30). By (3.35) and since $\epsilon_2 < \epsilon_1$, and ϵ_1 satisfies (3.25),

$$|\langle f_L\big(v_b(t)\big), \hat e\rangle| \ge \hat\lambda\, a(t;b) - \tilde c_2 a(t;b)^{5/3} \ge \frac{\hat\lambda}{2}\, a(t;b) \tag{3.36}$$

By (2.14),

$$\|e_{v_{a(t;b)}} - \hat e\|_2 \le c_1 a(t;b), \qquad \|e_{v_b(t)} - e_{v_{a(t;b)}}\|_2 \le c_1 \frac{4c_0'}{\lambda} a(t;b)^{5/3} \tag{3.37}$$

We then have

$$\|e_{v_b(t)} - \hat e\|_2 \le 2c_1 a(t;b) \tag{3.38}$$

By (3.28) $\|f_L\big(v_b(t)\big)\|_2 \le \hat\lambda\, a(t;b)\big(1 + \frac{\tilde c_2}{\hat\lambda}\epsilon_1^{2/3}\big)$ so that from (3.25) we get

$$\|f_L\big(v_b(t)\big)\|_2 \le\; 2\hat\lambda\, a(t;b) \tag{3.39}$$

From (3.35), (3.39), (3.38), (3.25) we get

$$\begin{aligned}\Big|\langle f_L\big(v_b(t)\big), e_{v_b(t)}\rangle - \hat\lambda\, a(t;b)\Big| &\le \tilde c_2 a(t;b)^{5/3} + \Big|\langle f_L\big(v_b(t)\big), e_{v_b(t)} - \hat e\rangle\Big| \\ &\le \tilde c_2 a(t;b)^{5/3} + 2\hat\lambda\, a(t;b)(2c_1 a(t;b)) \le 2\tilde c_2 a(t;b)^{5/3}\end{aligned}$$

and (3.29) is proved.

Finally, from (3.29) using (3.25) we get

$$|\langle f_L\big(v_b(t)\big), e_{v_b(t)}\rangle| \ge \hat\lambda\, a(t;b)[1 - \frac{2\tilde c_2}{\hat\lambda} a(t;b)^{2/3}] \ge \frac{\|f_L\big(v_b(t)\|_2}{4}$$

and this concludes the proof of (3.30).

It remains to prove (3.31) and (3.32). From (3.27), by using that $a(\cdot;b)$ is increasing, we get

$$a(t_1;b) + \int_{t_1}^{t_2} a(s;b)[\hat\lambda - \tilde c_2 a(t_2;b)]ds \le a(t_2;b) \le a(t_1;b) + \int_{t_1}^{t_2} a(s;b)[\hat\lambda + \tilde c_2 a(t_2;b)]ds$$

that gives (3.31).

Recalling that $v_b(t_i) = v_{a(t_i;b)} + \psi(t_i,b)$ for $i = 1,2$, from (3.26) we get

$$\|v_b(t_2) - v_b(t_1)\|_2 \le [a(t_2,b) - a(t_1,b)] + \|\psi(t_2,b)\|_2 + \|\psi(t_1,b)\|_2 \tag{3.40}$$

By using (3.31) and the monotonicity of a we get (3.32).

□

We will next describe functions in terms of their t-coordinates relative to orbits $v_b(\cdot)$, see Definition 3.1. To this end it is convenient to give the following definition.

Definition 3.3. *We define the function $\theta(a_1;b)$ as the inverse of $t \to a(t;b)$:*

$$a\big(\theta(a_1;b);b\big) = a_1 \qquad \forall a_1 \in [b,\epsilon_1]. \tag{3.41}$$

Namely, $\theta(a_1;b)$ is the time when the orbit $v_b(\cdot)$ has its a-coordinate equal to a_1. The notion is well defined, since by Proposition 3.2, $a(\cdot;b)$ is strictly increasing and $a(0;b) = b$.

For later applications we establish conditions for the existence of t-coordinates for functions m not necessarily of the form $v_a(t)$ (to which we may restrict in the proof of existence of $\mathcal{W}_+$). We will need m close to $\mathcal{M}_+$, the condition will involve the quantity $\|m - v_{a(m)}\|_2$ (see (3.42) below), $a(m)$ the a-coordinate of m defined in (3.3).

Proposition 3.4. *Let ϵ_1 be as in Lemma 3.3 and $\epsilon_2 \in (0,\epsilon_1)$ be as in Proposition 3.2. Then there exists $\epsilon_3 \in (0,\epsilon_2)$ such that, if we suppose*

$$\|m - v_{a(m)}\|_2 \le \epsilon_3\, a(m), \qquad a(m) \le \epsilon_2, \qquad\qquad b \in \big(0, \frac{1}{2}a(m)\big) \tag{3.42}$$

and define

$$S := \frac{16}{\hat\lambda a(m)} \Big\{ \|m - v_{a(m)}\|_2 + \frac{4c_0'}{\hat\lambda} a(m)^{5/3} \Big\} \tag{3.43}$$

then

(i) $\theta(a(m);b) + S < T(\epsilon_1;b)$ *and for all* $t \in [\theta(a(m);b) - S, \theta(a(m);b) + S]$,

$$\frac{1}{2}a(m) < a(\theta(a(m);b) - S;b) \le a(t;b) \le a(\theta(a(m);b) + S;b) < 2a(m) \tag{3.44}$$

(ii) m *has a unique* t*-coordinate* $\tau_{m,b}$ *relative to* $\{v_b(\cdot)\}$ *in* $[\theta(a(m);b) - S, \theta(a(m);b) + S]$

(iii) $\|m - v_b(t)\|_2 \le 81\|m - v_{a(m)}\|_2 + 360\frac{c_0'}{\hat\lambda} a(m)^{5/3}$ *for all* $t \in [\theta(a(m);b) - S, \theta(a(m);b) + S]$

(iv)

$$\|v_b(\theta(a(m);b)) - v_b(\tau_{m,b})\|_2 \le 80\|m - v_{a(m)}\|_2 + 576\frac{4c_0'}{\hat\lambda} a(m)^{5/3} \tag{3.45}$$

$$|a(m) - a\big(\tau_{m,b}\big)| \le 80\|m - v_{a(m)}\|_2 + 80\frac{4c_0'}{\hat\lambda} a(m)^{5/3} \tag{3.46}$$

(v) If in particular m *has the form* $m = v_{a_1}(t^*)$ *then we take* $S = \dfrac{64c_0'}{\hat\lambda^2} a(m)^{2/3}$ *and*

$$\|m - v_b(\tau_{m,b})\|_2 \le 360\frac{c_0'}{\hat\lambda}\, a(m)^{5/3} \tag{3.47}$$

Proof. Set for simplicity $\theta = \theta(a(m); b)$. Recall that by (3.25) it follows that

$$\frac{\tilde{c}_2}{\hat{\lambda}} \epsilon_1^{2/3} < \frac{1}{4} \tag{3.48}$$

Necessarily $T := T(\epsilon_1; b) > \theta$ because $a(\theta; b) = a(\theta(a(m); b); b) = a(m) < \epsilon_2 < \epsilon_1$. We assume by contradiction that $T \leq \theta + S$; then from (3.27) we get

$$\epsilon_1 = a(T; b) \leq a(\theta) + \int_\theta^T \hat{\lambda} a(t; b)[1 + \frac{\tilde{c}_2}{\hat{\lambda}} \epsilon_1^{2/3}] \leq \epsilon_2 + \frac{5}{4} \hat{\lambda} \epsilon_1 S$$

By taking ϵ_2 small enough the right hand side of the above inequality is less than ϵ_1 which gives the desired contradiction.
From (3.32), (3.48) for ϵ_3 and ϵ_2 small enough we get that

$$\frac{a(\theta; b)}{a(\theta - S; b)} \leq e^{\frac{5}{4}\hat{\lambda} S} \leq 2, \qquad \frac{a(\theta + S; b)}{a(\theta; b)} \leq e^{\frac{5}{4}\hat{\lambda} S} \leq 2$$

which implies (3.44).

Let us prove (iii) and (v). For $t \in [\theta - S, \theta + S]$ write

$$m - v_b(t) = [m - v_{a(m)}] + [v_{a(m)} - v_b(\theta)] + [v_b(\theta) - v_b(t)]. \tag{3.49}$$

From (3.26) (recall that $v_{a(m)} = v_{a(\theta;b)}$) we get

$$\|v_{a(m)} - v_b(\theta)\|_2 \leq \frac{4c_0'}{\hat{\lambda}} a(m)^{5/3}. \tag{3.50}$$

Recalling the monotonicity of $a(\cdot; b)$, we get

$$\|v_b(\theta) - v_b(t)\|_2 \leq a(\theta + S; b)\Big\{(1 - e^{-5\hat{\lambda} S/4}) + 2\frac{4c_0'}{\hat{\lambda}} a(\theta + S; b)^{2/3}\Big\}, \tag{3.51}$$

where the last inequality is a consequence of (3.32) (applied with $t_1 = t$ and $t_2 = \theta$), since

$$e^{(t_2 - t_1)(\hat{\lambda} + \tilde{c}_2 a(\theta; b)^{2/3})} \leq e^{S(\hat{\lambda} + \tilde{c}_2 a(\theta; b)^{2/3})},$$

and $\hat{\lambda} + \tilde{c}_2 a(\theta; b)^{2/3} \leq 5\hat{\lambda}/4$ from (3.48).
Hence, from (3.49), (3.50), (3.51) and (3.44) we obtain

$$\begin{aligned} \|m - v_b(t)\|_2 \leq \quad & \|m - v_{a(m)}\|_2 + \frac{4c_0'}{\hat{\lambda}} a(m)^{5/3} \\ & + a(\theta + S; b)\Big\{(1 - e^{-5\hat{\lambda} S/4}) + 2\frac{4c_0'}{\hat{\lambda}} a(\theta + S; b)^{2/3}\Big\} \\ \leq \quad & \|m - v_{a(m)}\|_2 + \frac{4c_0'}{\hat{\lambda}} a(m)^{5/3} + 2a(m)\Big\{\frac{5\hat{\lambda} S}{2} + \frac{8c_0'}{\hat{\lambda}} [2a(m)]^{2/3}\Big\} \end{aligned} \tag{3.52}$$

Therefore, recalling the definition (3.43) of S we get

$$\|m - v_b(t)\|_2 \leq 81\|m - v_{a(m)}\|_2 + \frac{4c_0'}{\hat{\lambda}} a(m)^{5/3}[1 + 80 + 4 \cdot 2^{2/3}]$$

Observe that for ϵ_2 small enough the above inequality implies the following inequality that we are going to use later.

$$\sup_{|t-\theta|\le S} \|m - v_b(t)\|_2 \le \frac{1}{8c_1}, \qquad c_1 \text{ defined in (2.14)} \tag{3.53}$$

Notice that if m has the form $m = v_{a_1}(t^*)$ then by letting $S = \frac{64c_0'}{\hat\lambda^2} a(m)^{2/3}$ in (3.52), for ϵ_2 small enough, we get (3.47).

We now show (ii). For $t \in [\theta - S, \theta + S]$ we define $\xi(t) := \langle m - v_b(t), e_{v_b(t)}\rangle$. We have to show that ξ vanishes at only one point of $[\theta - S, \theta + S]$.
We compute

$$\xi'(t) = -\langle f(v_b(t)), e_{v_b(t)}\rangle + \langle m - v_b(t), \frac{\delta e_v}{\delta v} f(v_b(t))\rangle, \tag{3.54}$$

where $\frac{\delta e_v}{\delta v}$ is the functional derivative at $v_b(t)$. We claim that $\xi'(t) < 0$.
From (3.29) and (3.48) it follows

$$\langle f(v_b(t)), e_{v_b(t)}\rangle \ge \hat\lambda a(t;b)[1 - 2\frac{\tilde c_2}{\hat\lambda}\epsilon_1^{2/3}] \ge \frac{1}{2}\,\hat\lambda a(t;b), \tag{3.55}$$

while from (2.14), (3.53), (3.28) and (3.48) we obtain

$$\big|\langle m - v_b(t), \frac{\delta e_v}{\delta v} f(v_b(t))\rangle\big| \le c_1\|m - v_b(t)\|_2 \|f(v_b(t))\|_2 \le c_1 \frac{1}{8c_1} 2\hat\lambda a(t;b). \tag{3.56}$$

Therefore, from (3.54), (3.55) and (3.56), we deduce

$$\xi'(t) \le -\frac{1}{2}\hat\lambda a(t;b) + \frac{1}{4}\hat\lambda a(t;b) \le -\frac{\hat\lambda}{4} a(t;b) < 0, \tag{3.57}$$

and the claim is proved. As a consequence, there exist at most one t-coordinate of m in $[\theta - S, \theta + S]$. To conclude the proof of (ii) it remains to show that ξ changes sign in $[\theta - S, \theta + S]$. We will show that if $\xi(\theta) > 0$ then $\xi(\theta + S) < 0$, and if $\xi(\theta) < 0$ then $\xi(\theta - S) > 0$.
We write $\psi = [m - v_{a(m)}] - [v_b(\theta) - v_{a(\theta,b)}]$, where we recall once more that $v_{a(m)} = v_{a(\theta,b)}$.
From (3.26) and (3.50) we get

$$|\xi(\theta)| = |\langle \psi, e_{v_b(\theta)}\rangle| \le \|m - v_{a(m)}\|_2 + \frac{4c_0'}{\hat\lambda}\, a(m)^{5/3} \tag{3.58}$$

Assume that $\xi(\theta) > 0$; from (3.57) and the fact that $a(\cdot;b)$ is increasing, we get

$$\xi(\theta + S) \le \xi(\theta) - \frac{\hat\lambda}{4}\int_\theta^{\theta+S} a(t;b)dt \le \xi(\theta) - \frac{\hat\lambda}{4} a(m)S \tag{3.59}$$

The definition (3.43) of S and (3.58) imply

$$\frac{\hat\lambda}{4} a(m)S = 4\|m - v_{a(m)}\|_2 + 4\frac{4c_0'}{\hat\lambda}\, a(m)^{5/3} > 4\xi(\theta). \tag{3.60}$$

Inequality (3.60) together with (3.59) imply that $\xi(\theta+S)<0$.
If instead $\xi(\theta)<0$ we have from (3.57) and (3.44)

$$\xi(\theta-S)\geq\xi(\theta)+\frac{\hat\lambda}{4}\int_{\theta-S}^{\theta}a(t;b)dt\geq\xi(\theta)+\frac{\hat\lambda}{4}a(\theta-S)S\geq\xi(\theta)+\frac{\hat\lambda}{8}a(m)S$$

The definition (3.43) of S and (3.58) implies that $\frac{\hat\lambda}{8}a(m)S>-2\xi(\theta)$ that, in turn, implies $\xi(\theta-S)>0$.
Thus $\xi(t)$ must change sign in $[\theta-S,\theta+S]$, and the existence of $\tau_{m;b}$ in $[\theta-S,\theta+S]$ follows.

It remains to prove (iv). From (3.32), (3.48), (3.43) and (3.44) we get

$$\begin{aligned}\|v_b(\theta)-v_b(\tau_{m;b})\|_2\leq&\quad a(\theta+S)[1-e^{2S\hat\lambda\frac{5}{4}}]+2\frac{4c_0'}{\hat\lambda}a(\theta+S)^{5/3}\\ \leq&\quad 2a(m)2S\hat\lambda\frac{5}{2}+2\frac{4c_0'}{\hat\lambda}[2a(m)]^{5/3}\\ \leq&\quad 80\|m-v_{a(m)}\|_2+(80+64)\frac{4c_0'}{\hat\lambda}a(m)^{5/3}\end{aligned}$$

which gives (3.45).
From (3.27), (3.43) and (3.44) we get

$$\begin{aligned}|a(m)-a(\tau_{m;b}|\leq&\quad\frac{5}{4}\hat\lambda\Big|\int_{\theta-S}^{\theta+S}a(t;b)dt\Big|\leq\frac{5}{4}\hat\lambda a(\theta+S;b)2S\\ \leq&\quad 80\|m-v_{a(m)}\|_2+80\frac{4c_0'}{\hat\lambda}a(m)^{5/3}\end{aligned}$$

Thus (3.46) is proved and the proof of the proposition is concluded. □

Remark 3.5. From (ii) and (iii) of Proposition 3.4 it follows in particular

$$\|m-v_b(\tau_{m,b})\|_2\leq 81\|m-v_{a(m)}\|_2+360\frac{c_0'}{\hat\lambda}a(m)^{5/3}. \tag{3.61}$$

Let us suppose that m, $b>0$ and $t^*>0$ are such that $S_t(m)$ satisfies the assumptions of Proposition 3.4 for any $t\in[0,t^*]$, so that the t-coordinate $\tau_{S_t(m),b}$ of $S_t(m)$ relative to $\{v_b(\cdot)\}$ is well defined for all $t\in[0,t^*]$.
We set

$$\tau=\tau(t)=\tau_{S_t(m),b},$$

$$\mathcal{L}_{b,\tau}=\mathcal{L}_{v_b(\tau)}.$$

Moreover, we denote derivative with respect to t either by a superscript dot or by a suffix t. Our purpose is to study the evolution of $\tau(t)$ and of $\mathtt{u}(t)$ which is the component of $S_t(m)$ orthogonal to $e_{v_b(\tau(t))}$, namely

$$S_t(m)=v_b(\tau(t))+\mathtt{u}(t), \tag{3.62}$$

$$\langle\mathtt{u}(t),e_{v_b(\tau(t))}\rangle=0 \tag{3.63}$$

We have

$$\mathtt{u}_t = f_L(S_t(m)) - \dot\tau f_L(v_b(\tau)) = (1-\dot\tau) f_L(v_b) + \mathcal{L}_{b,\tau}\mathtt{u} + R(S_t(m))$$

and

$$R(S_t(m)) := \beta^{-1}[\operatorname{arctanh} S_t(m) - \operatorname{arctanh} v_b(\tau) - \operatorname{arctanh}'(v_b(\tau))\mathtt{u}] \tag{3.64}$$

By differentiating (3.63) and using (3.64) we get

$$\begin{aligned} 0 &= \langle \mathtt{u}_t, e_{v_b}\rangle + \dot\tau\, \langle \mathtt{u}, \frac{\delta e_{v_b}}{\delta v_b} f_L(v_b)\rangle \\ &= (1-\dot\tau)\langle f_L(v_b), e_{v_b}\rangle + \langle R(S_t(m)), e_{v_b}\rangle + \dot\tau\, \langle \mathtt{u}, \frac{\delta e_{v_b}}{\delta v_b} f_L(v_b)\rangle, \end{aligned}$$

so that

$$\dot\tau = \frac{\langle f_L(v_b), e_{v_b}\rangle + \langle R(S_t(m)), e_{v_b}\rangle}{\langle f_L(v_b), e_{v_b}\rangle - \langle \mathtt{u}, \frac{\delta e_{v_b}}{\delta v_b} f_L(v_b)\rangle}$$

provided the denominator is non zero. In such a case we can also write

$$\dot\tau - 1 \quad = \quad A(S_t(m); b) \tag{3.65}$$

where the "force field" $A(m;b)$ is defined as

$$A(m;b) \quad := \quad \frac{\langle R(m), e_{v_b(\tau)}\rangle + \langle \mathtt{u}, \frac{\delta e_{v_b(\tau)}}{\delta v_b(\tau)} f_L(v_b(\tau))\rangle}{\langle f_L(v_b(\tau)), e_{v_b(\tau)}\rangle - \langle \mathtt{u}, \frac{\delta e_{v_b(\tau)}}{\delta v_b(\tau)} f_L(v_b(\tau))\rangle} \tag{3.66}$$

where $v_b(\tau) = v_b(\tau_{m;b})$, $\tau_{m;b}$ the t-coordinate of m. The definition applies only if m has a t-coordinate and the denominator in (3.66) is non zero.
By taking the scalar product with $\mathtt{u}$ of both sides of (3.64) we get

$$\frac{1}{2}\frac{d}{dt}\langle \mathtt{u}, \mathtt{u}\rangle \quad = \quad B(S_t(m); b) \tag{3.67}$$

where

$$B(m;b) \quad := \quad -A(m;b)\langle \mathtt{u}, f_L(v_b(\tau))\rangle + \langle \mathtt{u}, \mathcal{L}_{b,\tau}\mathtt{u}\rangle + \langle \mathtt{u}, R(m)\rangle \tag{3.68}$$

with same meaning of symbols and same restrictions on m.
The analysis of the system of equations (3.65)–(3.67) is based on the following bounds on the "forces" $A(m;b)$ and $B(m;b)$.
We recall that

ϵ_3 is given by Proposition 3.4,

ω is a bound for the spectral gap uniform in a neighborhood of $\hat m_L$, see Theorem 2.4,

c_1 is defined in (2.14),

$\|e_{v_b(\tau)}\|_\infty$, see Theorem 2.4,

C_2 is defined at the beginning of the proof of Lemma 3.1.

Proposition 3.6. *There exists $\epsilon_4 \in (0, \epsilon_3)$ so that if we suppose*

$$\|m - v_{a(m)}\|_2 \le \epsilon_4 a(m), \qquad a(m) \le \epsilon_4, \qquad b > 0 \text{ small enough}$$

and define $\mathtt{u} := m - v_b(\tau)$, $v_b(\tau) = v_b(\tau_{m,b})$ (recall Proposition 3.4), then

(i) $\|\mathtt{u}\|_2 \le \min\left\{\frac{1}{8c_1}\ ,\ \frac{\omega}{80C_2\tilde{c}_+}\right\}$, *where* $\tilde{c}_+ \ge \|e_{v_b(\tau)}\|_\infty$,

(iii) $A(m; b)$ *and* $B(m; b)$ *are well defined and*

$$|A(m;b)| \le 2\Big\{4c_1\|\mathtt{u}\|_2 + \frac{C_2\|e_{v_b(\tau)}\|_\infty\|\mathtt{u}\|_2^2}{|\langle f_L(v_b(\tau)), e_{v_b(\tau)}\rangle|}\Big\} \tag{3.69}$$

(iv) If in addition

$$\mathtt{u} = u_1 + u_2, \quad |\mathrm{supp}(u_2)| \le \zeta, \quad \|u_1\|_\infty \le \frac{\omega}{10C_2} \tag{3.70}$$

then

$$B(m;b) \le -\frac{\omega}{2}\|\mathtt{u}\|_2^2 + 8C_2\zeta \tag{3.71}$$

Proof. By taking ϵ_4 small enough we get (i) from (3.61). By (2.14) and (3.30), shorthanding $v = v_b(\tau)$,

$$\begin{aligned} |\langle f_L(v), e_v\rangle - \langle \mathtt{u}, \frac{\delta e_v}{\delta v} f_L(v)\rangle| &\ge \Big|\langle f_L(v), e_v\rangle\Big|\Big|1 - c_1\|\mathtt{u}\|_2\frac{\|f_L(v)\|_2}{|\langle f_L(v), e_v\rangle|}\Big| \\ &\ge \Big|\langle f_L(v), e_v\rangle\Big|\Big|1 - 4c_1\|\mathtt{u}\|_2\Big| \end{aligned} \tag{3.72}$$

and by (i),

$$|\langle f_L(v), e_v\rangle - \langle \mathtt{u}, \frac{\delta e_v}{\delta v} f_L(v)\rangle| \ge \frac{1}{2}\Big|\langle f_L(v), e_v\rangle\Big| \tag{3.73}$$

Then (3.69) follows from (C.16) and (3.30).

Let us now show (3.71). By (2.13),

$$B(m;b) \le |A(m;b)\langle \mathtt{u}, f_L(v)\rangle| - \omega\|\mathtt{u}\|_2^2 + |\langle \mathtt{u}, R\rangle| \tag{3.74}$$

and by (3.69) and (3.30) we have

$$|A(m;b)\langle \mathtt{u}, f_L(v)\rangle| \le 8c_1\|\mathtt{u}\|_2^2\|f_L(v)\|_2 + 8C_2\|\mathtt{u}\|_2^3\|e_v\|_\infty \tag{3.75}$$

By (3.28), $\|f_L(v)\|_2 \le 2\hat{\lambda}\epsilon_4$ and supposing ϵ_4 small enough, $8c_1\|f_L(v)\|_2 \le \omega/10$.
By (i), $8C_2\|\mathtt{u}\|_2\|e_v\|_\infty \le \omega/10$ so that

$$|A(m;b)\langle \mathtt{u}, f_L(v)\rangle| \le \frac{\omega}{5}\|\mathtt{u}\|_2^2 \tag{3.76}$$

Finally from (3.70) we get

$$|(\mathtt{u}, R)| \le C_2\|\mathtt{u}\|_2^2\|u_1\|_\infty + 8C_2\zeta \le \frac{\omega}{10}\|\mathtt{u}\|_2^2 + 8C_2\zeta \tag{3.77}$$

Inserting the estimates (3.76) and (3.77) in (3.74) we get (3.71).

□

4 Construction of the invariant manifolds

In this section we prove Theorem 2.5 by constructing the invariant manifolds $\mathcal{W}_\pm$ as a suitable limit of the manifolds $\mathcal{M}_\pm$.
Given ϵ_4 as in Proposition 3.6, we fix $a_0 \in (0, \epsilon_4)$ and such that

$$\frac{4c_0'}{\hat{\lambda}} a_0^{2/3} < \epsilon_4 \tag{4.1}$$

(the ratio $4c_0'/\hat{\lambda}$ appears in Lemma 3.3).
We call $\theta(a_0; b)$, $b \in (0, a_0)$, the time when the a-coordinate of $v_b(t)$ is equal to a_0.

Theorem 4.1. *There exists a family of functions $v^{(+)}(s) \in L^\infty([-L/2, L/2]; (-m_\beta, m_\beta))$, $s \le 0$, such that*

$$\lim_{b\to 0} \sup_{0\ge s'\ge s} \|v_b(\theta(a_0;b)+s') - v^{(+)}(s')\|_2 = 0 \qquad \forall s \le 0. \tag{4.2}$$

Proof. We will prove that $\{v_b(\theta(a_0;b)+s), b \in (0,a_0)\}$ is a Cauchy sequence for any $s < 0$, namely for any $\zeta > 0$ there is $a_1 \in (0, a_0)$ so that for any $b \in (0, a_1)$

$$\|v_b(\theta(a_0;b)+s) - v_{a_1}(\theta(a_0;a_1)+s)\|_2 < \zeta \tag{4.3}$$

By the continuity of S_t, (4.2) then follows, where

$$v^{(+)}(s) := \lim_{b\to 0^+} v_b(\theta(a_0;b)+s),$$

(the limit taken in L^2).

Step 1. For any $s<0$ and $\delta>0$ there is $a_1\in(0,a_0)$ so that for any $b<a_1$ and any t such that $a(t;a_1)\le\epsilon_4$, the pair $m=v_{a_1}(t)$ and $v_b(\cdot)$ satisfy the assumptions in Proposition 3.6. Since $m=v_{a_1}(t)$ and t is such that $a(m)\le\epsilon_4$, by (3.26) we find $\|m-v_{a(m)}\|_2\le\dfrac{4c_0'}{\hat\lambda}a(m)^{5/3}$. By the second inequality in (3.27) and since $a(m)\le\epsilon_4\le\epsilon_2$, we then have $\|m-v_{a(m)}\|_2\le\epsilon_4 a(m)$.
To verify (3.70) we set $u_2=0$. Since the derivative with respect to x of $\mathtt{u}=v_{a_1}(t)-v_b(\tau(t))$ is bounded, by Lemma C.3 $\|\mathtt{u}\|_\infty\le c\|\mathtt{u}\|_2^{2/3}$. In order to bound $\|\mathtt{u}\|_2$, we use (3.47) and obtain

$$\|v_{a_1}(t)-v_b(\tau(t))\|_2\le\frac{360c_0'}{\hat\lambda}\,a(v_{a_1}(t))^{5/3},\quad a_1<\epsilon_4,\tag{4.4}$$

and the inequality in (3.70) follows, using also (i) of Proposition 3.6.

Step 2. For any $s<0$ and $\delta>0$ there is $a_1\in(0,a_0)$ so that if we indicate by $\theta(a_0;a_1)$ the time when $a(t;a_1)=a_0$, then the t-coordinate of $v_{a_1}(t)$ relative to $\{v_b(\cdot)\}$, denoted by $\tau(t)$, is well defined for $t\le\theta(a_0;a_1)$ and

$$\|v_b\big(\tau(\theta(a_0;a_1)+s)\big)-v_{a_1}(\theta(a_0;a_1)+s))\|_2\le\delta\tag{4.5}$$

By (4.4), $\|\mathtt{u}(0)\|_2\le ca_1^{5/3}$, c a suitable constant. By (3.67) and (3.71),

$$\frac{d\|\mathtt{u}\|_2^2}{dt}\le-\frac{\omega}{4}\|\mathtt{u}\|_2^2$$

so that (4.5) holds with $\delta=ca_1^{5/3}$.

Let us conclude the proof of the theorem. Shorthand $m=v_{a_1}(\theta(a_0;a_1)+s)$ and $m^*=v_b\Big(\tau\big(\theta(a_0;a_1)+s\big)\Big)$. Since $S_{|s|}(m)=v_{a_1}(\theta(a_0;a_1))$, $a(S_{|s|}(m))=a_0$. On the other hand

$$\|S_t(m)-S_t(m^*)\|_2\le e^{ct}\|m-m^*\|_2,\quad c=1+\frac{1}{\beta}\max_{|x|\le(1+m_\beta)/2}\operatorname{arctanh}'(x)\tag{4.6}$$

so that $\|S_{|s|}(m^*)-v_{a_1}(\theta(a_0;a_1))\|_2\le e^{c|s|}\delta$ and there is a constant c' so that $|a(S_{|s|}(m^*))-a_0|\le c'e^{c|s|}\delta$. For a in the interval $[a_0-c'e^{c|s|}\delta,a_0+c'e^{c|s|}\delta]$, $\dfrac{da}{dt}\ge\dfrac{\hat\lambda}{2}a\ge\dfrac{\hat\lambda}{4}a_0$, see (3.27). Letting t^* be such that $\dfrac{\hat\lambda a_0}{4}t^*=c'e^{c|s|}\delta$, there is $|t|\le t^*$ so that $a(S_{|s|+t}(m^*))=a_0$. Thus, letting $c''=\dfrac{4}{a_0\hat\lambda}c'$, we have

$$\theta(a_0,b)=\tau\big(\theta(a_0;a_1)+s\big)+|s|+t,\quad |t|\le c''e^{c|s|}\delta\tag{4.7}$$

hence, using (3.27) and (4.5) to derive the second inequality below,

$$\begin{aligned}
&\|v_b\big(\theta(a_0;b)+s)\big)-v_{a_1}(\theta(a_0;a_1)+s))\|_2\\
\le\;&\|v_b\big(\tau(\theta(a_0;a_1)+s)+t\big)-v_b\big(\tau(\theta(a_0;a_1)+s)\big)\|_2\\
&+\|v_b\big(\tau(\theta(a_0;a_1)+s)\big)-v_{a_1}(\theta(a_0;a_1)+s))\|_2\\
\le\;&2\hat\lambda a_0 t+\delta\le(8c'e^{c|s|}+1)\delta
\end{aligned}$$

which proves (4.3) if δ is small enough.

$\square$

We extend the definition of $v^{(+)}(t)$ to all $t \in \mathbb{R}$, by setting

$$v^{(+)}(t) := S_t\big(v^{(+)}(0)\big), \qquad t > 0. \tag{4.8}$$

Theorem 4.2. *The family $\{v^{(+)}(s) : s \in \mathbb{R}\}$ is time-invariant, i.e.,*

$$S_t(v^{(+)}(s)) = v^{(+)}(t+s), \quad s \in \mathbb{R},\ t \geq 0, \tag{4.9}$$

the a-coordinate $a(v^{(+)}(t))$ of $v^{(+)}(t)$ is increasing and satisfies

$$|\frac{d}{dt}a\big(v^{(+)}(t)\big) - \hat{\lambda}\, a\big(v^{(+)}(t)\big)| \leq \tilde{c}_2\, a\big(v^{(+)}(t)\big)^{5/3} \tag{4.10}$$

Furthermore

$$\lim_{t\to-\infty} \|v^{(+)}(t) - \hat{m}_L\|_2 = 0 \tag{4.11}$$

Proof. If $s \geq 0$ equality (4.9) follows directly from (4.8) Let then $s < 0$ and suppose first $s + t \leq 0$. By (4.2),

$$S_t(v^{(+)}(s)) = S_t\Big(\lim_{b\to 0} v_b(\theta(a_0; b) + s)\Big) = \lim_{b\to 0} v_b(\theta(a_0; b) + s + t) = v^{(+)}(s+t)$$

If $s + t > 0$, we write $S_t(v^{(+)}(s)) = S_{t-|s|}\big(S_{|s|}(v^{(+)}(s))\big)$ and since we have already proved that $S_{|s|}(v^{(+)}(s)) = v^{(+)}(0)$, (4.9) is proved.

Let us prove (4.10). By (3.3) and since $v^{(+)}(t)$ satisfies the equations of motion, i.e.,

$$\frac{d}{dt}a\big(v^{(+)}(t)\big) = \Big\langle f_L\big(v^{(+)}(t)\big), \hat{e}\Big\rangle \tag{4.12}$$

it follows

$$\frac{d}{dt}a\big(v^{(+)}(t)\big) = \lim_{b\to 0}\Big\langle f_L\big(v_b(\theta(a_0; b) + t)\big), \hat{e}\Big\rangle \tag{4.13}$$

which extends the validity of (3.27) to $a\big(v^{(+)}(t)\big)$, namely (4.10) holds. Observe that from (4.10) it follows that

$$\lim_{t\to-\infty} a\big(v^{(+)}(t)\big) = 0 \tag{4.14}$$

It remains to show (4.11). Let $s < 0$ and $a^*_s := a(v^{(+)}(s))$. Since from Theorem 4.1 we know that $a(v^{(+)}(0) = a_0$, from the fact that a^*_s is an increasing function, we get that $a^*_s < a_0$.

From (3.40) and (3.27) we get

$$\begin{aligned}\|v_b(\theta(a_0;b)+s)-v_b(\theta(a_s^*;b))\|_2 &\le e^{-\hat\lambda|s|}a_0\big[1+(\tilde c_2+\frac{4c_0'}{\hat\lambda})a_0^{2/3}\big]+a_s^*\big[1+\frac{4c_0'}{\hat\lambda}(a_s^*)^{2/3}\big]\\ &\le 2e^{-\hat\lambda|s|}a_0+2a_s^* \qquad (4.15)\end{aligned}$$

and from (3.26) we get that

$$\|v_b(\theta(a_s^*;b))-\hat m_L\|_2\le a_s^*+\frac{4c_0'}{\hat\lambda}(a_s^*)^{5/3} \qquad (4.16)$$

Hence

$$\|v^{(+)}(s)-\hat m_L\|_2=\lim_{b\to 0}\|v_b(\theta(a_0;b)+s)-\hat m_L\|_2\le 2e^{-\hat\lambda|s|}a_0+3a_s^*$$

Thus (4.11) follows from (4.14).

□

Theorem 4.3. *The function $v^{(+)}(t)$ verifies*

$$\lim_{t\to\infty}\|v^{(+)}(t)-m^{(+)}\|_2=0 \qquad (4.17)$$

so that the manifold $\mathcal{W}_+$ can be identified with $\{v^{(+)}(t)\}_{t\in\mathbb{R}}$.

Proof. Since $s\to F_L(v^{(+)}(s))$ is strictly decreasing and F_L is continuous in L^2, $F_L(v^{(+)}(s))<F_L(\hat m_L))$. For any m, $S_t(m)$ converges by subsequences as $t\to\infty$ and any limit point is stationary. Thus $v^{(+)}(t)$ converges by subsequences and any limit point m^* is a stationary solution of (1.1). By (1.6), $m^*\in\{m^{(+)},\hat m_L,m^{(-)}\}$. On the other hand $m^*\neq\hat m_L$ because its free energy is strictly smaller. Since $v_b\ge\hat m_L$, $v_b(t)\ge\hat m_L$ hence $v^{(+)}(t)\ge\hat m_L$. Thus $m^*\ge\hat m_L$ excluding the possibility that $m^*=m^{(-)}$. □

5 Stability of the invariant manifolds

The manifolds $\mathcal{W}_\pm$ are asymptotically stable: referring for instance to $\mathcal{W}_+$, since $m^{(+)}$ is stable, its basin of attraction $\mathcal{B}^+$ is an open set (in the L^2 topology) and each element $v^{(+)}(\cdot)\in\mathcal{W}_+$, being in $\mathcal{B}^+$, has a neighborhood which is attracted by $m^{(+)}$; in this sense therefore $\mathcal{W}_+$ is asymptotically stable. The property is indeed a consequence of the existence of $\mathcal{W}_+$ and of the stability of $m^{(+)}$.
We discuss here a different stability property of $\mathcal{W}_+$, namely that given any neighborhood I of $\mathcal{W}_+$ there is a neighborhood U of $\hat m_L$ so that for any $m\in U\cap\mathcal{B}^+$, $S_t(m)$ converges to $m^{(+)}$ being at all times in I and "following closely" the orbit $v^{(+)}(\cdot)$. For applications to

tunnelling and the characterization of its optimal orbits we generalize the context by adding a time dependent force $K(x,t)$ and thus considering the evolution equation

$$m_t = f_L(m) + K \tag{5.1}$$

supposing $K(x,t)$ smooth and that

$$\|K\|^2 = \int_0^\infty k(t)^2\, dt < \infty, \qquad k(t)^2 := \int_{-\frac{L}{2}}^{\frac{L}{2}} K(x,t)^2 dx \tag{5.2}$$

We will denote by $S_t^K(m)$ the orbit which solves (5.1) starting from $m \in L^2([-L/2, L/2], (-1,1))$ and with K satisfying (5.2). As noticed in Subsection 2.1, $|S_t^K(m)| < 1$ for all t.
The presence of the additional force may have the effect to stabilize $\hat{m}_L$ and we need assumptions to avoid such a case which can be dropped if $K = 0$, as in Theorem 5.8 below. When $K \neq 0$ we thus split our results in two theorems, which will be proved, together with Theorem 5.8, in the remaining of the section. Theorem 5.5 below describes the initial part of the orbit which stays close to $\hat{m}_L$. Theorem 5.6 describes what happens when the orbit leaves such a neighborhood.
Before stating and proving the above Theorems we give some definitions and preliminary lemmas.

Recalling Definition 3.1, we write any m as

$$m = \hat{m}_L + a(m)\hat{e} + \phi, \qquad \langle \phi, \hat{e} \rangle = 0.$$

Definition 5.1. *For any $a^* > 0$ and m such that $|a(m)| < a^*$ we define*

$$t_{a^*}(m) := \inf\Big\{ t \geq 0 : |a\big(S_t^K(m)\big)| = a^* \Big\}, \tag{5.3}$$

so that $t_{a^*}(m) = \infty$ if $|a\big(S_t^K(m)\big)| < a^*$ for all t.

Definition 5.2. *We define $\phi(t)$ as*

$$S_t^K(m) = \hat{m}_L + a(t)\hat{e} + \phi(t), \quad \langle \phi(t), \hat{e} \rangle = 0, \quad a(t) := a\big(S_t^K(m)\big). \tag{5.4}$$

In the next lemma we give a first estimate of $\phi(t)$ for m_0 such that $a(m_0) < \epsilon_0$.

Lemma 5.3. *Let $\epsilon_0 \in (0, \frac{1-m_\beta}{4\|\hat{e}\|_\infty})$ be as in Lemma 3.2. There exists a constant $c_M > 0$ such that if $a(m_0) \in (0, \epsilon_0)$ then*

$$\|\phi(t)\|_2^2 \leq e^{8c_M t}\Big(\|\phi(0)\|_2^2 + \frac{2}{\sqrt{c_M}}\|K\| + 8c_M \int_0^t e^{-8c_M s} a(s)^2\, ds \Big) \qquad \forall t \in [0, t_{\epsilon_0}(m_0)]. \tag{5.5}$$

Proof. We set $m(t) := S_t^K(m_0) = m(\cdot,t)$ and define

$$\mathcal{C}_t := \big\{x \in [-L/2, L/2] : |m(x,t)| \ge \frac{m_\beta + 1}{2}\big\} \qquad \forall t \in [0, t_{\epsilon_0}(m_0)].$$

We claim that there is $c > 0$ so that for all $t \le t_{\epsilon_0}(m_0)$

$$\phi(t)\,[\operatorname{arctanh} m(t) - \operatorname{arctanh} \hat{m}_L] \ge \phi(t) \operatorname{arctanh}'(\hat{m}_L)[m(t) - \hat{m}_L] - c|\phi(t)|(m(t) - \hat{m}_L)^2 \quad \text{in } \mathcal{C}_t. \tag{5.6}$$

We prove the claim assuming $m \ge (1 + m_\beta)/2$, the case $m \le -(1 + m_\beta)/2$ is analogous and omitted. Write for simplicity ϕ in place of $\phi(t)$. Since $\phi = m - \hat{m}_L - a(t)\hat{e}$ and $m - \hat{m}_L > (1 - m_\beta)/2$, by using that $\|a(t)\hat{e}\|_\infty \le \epsilon_0 \|\hat{e}\|_\infty \le (1 - m_\beta)/4$ (recall $t \le t_{\epsilon_0}(m_0)$) we get that $\phi > 0$ so that ϕ drops from (5.6).
If $\hat{m}_L \ge 0$, then $[\operatorname{arctanh} m - \operatorname{arctanh} \hat{m}_L] \ge \operatorname{arctanh}'(\hat{m}_L)[m - \hat{m}_L]$ because $x \to \operatorname{arctanh}' x$ is increasing when $x > 0$. If instead $\hat{m}_L < 0$, $[m - \hat{m}_L] \ge (1 + m_\beta)/2$ and the r.h.s. of (5.6) becomes negative when $c(1 + m_\beta)/2 > 1/(1 - m_\beta^2)$. Therefore (5.6) is proved.

Denoting by $\mathcal{C}_t^c = [-L/2, L/2] \setminus \mathcal{C}_t$, from (5.6) and (C.16) of Lemma C.3 (proved with $K = 0$ but true also for a bounded K, possibly modifying the constant C_2 in (C.16) by replacing $\operatorname{arctanh}''(\mathtt{a})$ with $\operatorname{arctanh}''(1 - \sigma)$ for a suitable $\sigma > 0$ small enough) we get

$$\begin{aligned}
\langle \phi, \operatorname{arctanh} m - \operatorname{arctanh} \hat{m}_L \rangle \ge \quad & \int_{\mathcal{C}_t^c} \phi[\operatorname{arctanh} m - \operatorname{arctanh} \hat{m}_L] \\
& + \int_{\mathcal{C}_t} \phi \operatorname{arctanh}'(\hat{m}_L)[m - \hat{m}_L] - c \int |\phi|(m - \hat{m}_L)^2 \\
\ge \quad & \int \phi \operatorname{arctanh}'(\hat{m}_L)[m - \hat{m}_L] - [C_2 + c] \int |\phi|(m - \hat{m}_L)^2.
\end{aligned}$$

By using that $f_L(\hat{m}_L) = 0$ and $\langle \phi, \hat{e} \rangle = 0$, from (5.1) we get

$$\begin{aligned}
\langle \phi, \phi_t \rangle \le \quad & \langle \phi, J^{\text{neum}} * [m - \hat{m}_L] \rangle - \langle \phi, \operatorname{arctanh}'(\hat{m}_L)[m - \hat{m}_L] \rangle \\
& + \langle \phi, K \rangle + [C_2 + c] \langle |\phi|(m - \hat{m}_L)^2 \rangle
\end{aligned}$$

Recalling the definition of $\hat{\mathcal{L}}$ and using again that $\langle \phi, \hat{e} \rangle = 0$, we then get that there is $c_M > 0$ so that

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\langle \phi, \phi \rangle \quad & \le \langle \phi, \hat{\mathcal{L}}\phi \rangle + \langle |\phi|, c_M[|\phi|^2 + (a(t)\hat{e})^2] \rangle + |\langle \phi, K \rangle| \\
& \le -\hat{\omega}\langle \phi, \phi \rangle + \langle |\phi|, c_M[|\phi|^2 + (a(t)\hat{e})^2] \rangle + |\langle \phi, K \rangle| \qquad \forall t \in (0, t_{\epsilon_0}(m_0)),
\end{aligned} \tag{5.7}$$

where the last inequality follows from (2.13). Since $|\phi| < 4$ (recall that $|m| < 1$, $|\hat{m}_L| < 1$, the assumption $a(m_0) \in (0, m_0)$ and the choice of ϵ_0), from (5.7) we have

$$\frac{1}{2}\frac{d}{dt}\langle \phi, \phi \rangle \le -\hat{\omega}\langle \phi, \phi \rangle + 4c_M\Big(\langle \phi, \phi \rangle + a(t)^2\Big) + |\langle \phi, K \rangle| \tag{5.8}$$

We bound

$$|\langle \phi(t), K \rangle| \le 4k(t) \tag{5.9}$$

and estimate

$$\langle \phi(t), \phi(t)\rangle \le e^{8c_M t}\langle \phi(0), \phi(0)\rangle + 8\int_0^t e^{8c_M(t-s)}\{c_M a(s)^2 + k(s)\}.$$

By Cauchy-Schwartz we then have

$$8\int_0^t e^{-8c_M s} k(s) \le \frac{2}{\sqrt{c_M}}\|K\|$$

and (5.5) follows. □

To have an estimate better than (5.5), we must improve the trivial bound $|\phi| < 4$ used in the proof of Lemma 5.3. The idea is to bound the L^∞ norm of ϕ in terms of its L^2 norm. This requires some regularity properties of ϕ which will follow from regularizing properties of the evolution (see Appendix C, Lemmas C.1, C.2 and C.3). There is however a very first time interval when the evolution has not yet produced the desired regularity, and in such an interval we cannot do better than (5.5).

We will use the parameters listed below:
$\delta > 0$ is the parameter which specifies the initial datum and the strength of the external force:

$$\|m - \hat{m}_L\|_2 \le \delta, \quad a(0)^2 + \|\phi(0)\|_2^2 \le \delta^2; \qquad \|K\| \le \delta \tag{5.10}$$

$a_1 > 0$ is a parameter which controls $a(t)$. By the continuity of the motion for any m verifying (5.10), $t_{a_1}(m)$ can be made as large as desired by letting $\delta \ll a_1$. One condition on a_1 is that $a_1 < \epsilon_0$ so that we can apply Lemma 5.3.
$\tau^* > 0$ denotes the initial time layer, after which regularity properties of time evolution apply (see Lemma C.2).
$\mu_0 \in (0,1)$ is a control parameter for $\|\phi(t)\|_2$.
We choose $\tau^* > \tau_\beta$ (τ_β as in (C.25)) so large that together with μ_0 they satisfy

$$2e^{-\tau^*/\beta} + c_M(2\mu_0^2)^{1/3} < \min\{\frac{\hat{\omega}}{4c_M}, \frac{\omega}{10^2 C_2}\} \tag{5.11}$$

(the ratio $\omega/(10C_2)$ appears in Proposition 3.6).
We also assume that $\tau^* < t_{\epsilon_0}(m)$ for all m satisfying (5.10). This because $t_{\epsilon_0}(m) > t_{a_1}(m)$ can be made as large as desired by letting ϵ_0 and therefore δ sufficiently small.

To control strength and location of the external force fluctuations we introduce for any $t > \tau^*$ and $\delta > 0$,

$$A_{t,\delta} = \Big\{x \in \Big[-\frac{L}{2}, \frac{L}{2}\Big] : \int_{t-\tau^*}^t K(x,s)^2 < \delta\Big\} \tag{5.12}$$

By the Chebyshev inequality,

$$|A_{t,\delta}^c| \le \frac{1}{\delta}\int_{A_{t,\delta}^c}\int_{t-\tau^*}^t K(x,s)^2 \, ds \, dx \le \frac{\delta^2}{\delta} \tag{5.13}$$

Let finally

$$\alpha^* = c^* e^{\|J\|_\infty \tau^*} \big(\sqrt{\delta} + \delta\big)^{1/2} \tag{5.14}$$

with c^* the constant in (C.21).

Before stating smallness conditions on δ and a_1, we prove the following lemma.

Lemma 5.4. *There exists $c'_M > 0$ so that if K and m satisfy* (5.10) *then*

$$\sup_{x \in A_{t,\delta}} |\phi(x,t)| \le \alpha^* + 2e^{-\frac{\tau^*}{\beta}} + c'_M \Big(2\|\phi(t)\|_2^2 + 2L(\alpha^*)^2 + 8\delta\Big)^{1/3} \qquad \forall t \in (\tau^*, t_{\epsilon_0}(m)) \tag{5.15}$$

with

$$\|\phi(t)\|_2^2 \le e^{8c_M\tau^*}\big(\|\phi(t-\tau^*)\|_2^2 + 2\delta + a_t^{+\,2}\big) \qquad \forall t \in (\tau^*, t_{\epsilon_0}(m)) \tag{5.16}$$

where $a_t^+ := \max\big\{|a(S_s^K(m))| : s \in [t-\tau^*, t]\big\}$.

Proof. (5.16) follows from Lemma 5.3, so that we only need to prove (5.15). By (C.21),

$$\sup_{x \in A_{t,\delta}} |S_t^K(m)(x) - S_{\tau^*}(m(t-\tau^*))(x)| \le c^* e^{\|J\|_\infty \tau^*} \Big(\sqrt{\delta} + \delta\Big)^{1/2} = \alpha^* \tag{5.17}$$

Calling $\psi(t) := S_{\tau^*}(m(t-\tau^*)) - [\hat{m}_L + a(t)\hat{e}]$,

$$\phi(t) = [S_t^K(m) - S_{\tau^*}(m(t-\tau^*))] + \psi(t) \tag{5.18}$$

so that

$$\sup_{x \in A_{t,\delta}} |\phi(x,t)| \le \alpha^* + \sup_{x \in A_{t,\delta}} |\psi(x,t)| \tag{5.19}$$

By Lemma C.2 and C.1,

$$\|\psi(t)\|_\infty \le 2e^{-\tau^*/\beta} + c\|\psi(t)\|_2^{2/3} \tag{5.20}$$

By (5.18),

$$\begin{aligned} \frac{1}{2}\|\psi\|_2^2 \le \quad & \|\phi\|_2^2 + \|S_t^K(m) - S_{\tau^*}(m(t-\tau^*))\|_2^2 \\ \le \quad & \|\phi\|_2^2 + (\alpha^*)^2 L + 4|A_{t,\delta}^c| \end{aligned} \tag{5.21}$$

so that

$$\|\psi(t)\|_\infty \le 2e^{-\tau^*/\beta} + c\Big(2\|\phi(t)\|_2^2 + 2L(\alpha^*)^2 + 8|A_{t,\delta}^c|\Big)^{1/3} \tag{5.22}$$

which together with (5.19) and (5.13) proves (5.15).

□

A first set of conditions on δ and a_1

Given $a_1 \in (0, \epsilon_0)$ we will require δ so small that for any m satisfying (5.10), $t_{a_1}(m) > \tau^*$. Another set of requirements (to which others will be added later) is the following one:

$$e^{8c_M\tau^*}\Big(\delta^2 + \frac{2}{\sqrt{c_M}}\delta + a_1^2\Big) + \frac{2c_M}{\hat{\omega}}\Big(4^3\delta + \tilde{c}_+\mu_0 a_1^2\Big) + \frac{4\delta}{\sqrt{\hat{\omega}}} < \mu_0^2 \tag{5.23}$$

$$\alpha^* + 2e^{-\tau^*/\beta} + c'_M\Big(2\mu_0^2 + 2L(\alpha^*)^2 + 8\delta\Big)^{1/3} < \min\{\frac{\hat{\omega}}{4c_M}, \frac{\omega}{10^2 C_2}\} \tag{5.24}$$

with α^* as in (5.14).

Theorem 5.5. *There exists $c > 0$ so that for any $a_1 \in (0, \mu_0)$ small enough and any $\delta > 0$ small enough the following holds. If K and m satisfy* (5.10)*, then*

$$\sup_{t \le t_{a_1}(m)} \|\phi(t)\|_2 \le c a_1 \tag{5.25}$$

Moreover $t_{a_1}(m) > \tau^*$ and

$$\frac{1}{2}\frac{d}{dt}\langle\phi,\phi\rangle \le -\frac{\hat{\omega}}{2}\langle\phi,\phi\rangle + 4^3 c_M\delta + c_M\tilde{c}_+\|\phi\|_2 a(t)^2 + |\langle\phi, K\rangle| \qquad \forall t \in [\tau^*, t_{a_1}(m)) \tag{5.26}$$

$$\sup_{x \in A_{t,\delta}} |\phi(x,t)| \le \frac{\omega}{10^2 C_2}, \qquad \left|A^c_{t,\delta}\right| \le \delta \qquad \forall t \in [\tau^*, t_{a_1}(m)) \tag{5.27}$$

Finally

$$t_{a_1}(m) < \infty \Rightarrow \|\phi(t_{a_1})\|_2 \le \frac{\epsilon_4^2}{10^2}\, a_1 \tag{5.28}$$

(ϵ_4 as in Proposition 3.6).

Proof. For $t \ge \tau^*$ (5.7) yields (with $\tilde{c}_+ \ge \|\hat{e}^2\|_\infty$),

$$\frac{1}{2}\frac{d}{dt}\|\phi\|_2^2 \le -\hat{\omega}\|\phi\|_2^2 + 4^3 c_M\delta + c_M[\langle|\phi|, \mathbf{1}_{A_{t,\delta}}|\phi|^2\rangle + \tilde{c}_+\|\phi\|_2 a(t)^2] + |\langle\phi, K\rangle| \tag{5.29}$$

Call S the first time when $\|\phi(s)\|_2 = \mu_0$ (we will show later that $S > t_{a_1}(m)$). Then by (5.15) and (5.24), for all $t \le S$,

$$c_M\langle|\phi|, \mathbf{1}_{A_{t,\delta}}|\phi|^2\rangle \le \frac{\hat{\omega}}{4}\|\phi\|_2^2$$

so that from (5.29) we get

$$\frac{1}{2}\frac{d}{dt}\langle\phi,\phi\rangle \le -\frac{\hat{\omega}}{2}\langle\phi,\phi\rangle + 4^3 c_M\delta + c_M\tilde{c}_+\mu_0 a(t)^2 + |\langle\phi, K\rangle| \tag{5.30}$$

Thus for $t \in [\tau^*, \min\{t_{a_1}, S\}]$ we get an improvement of (5.16), namely recalling that $a_t^+ = \max\{|a(S_s^K(m))|, s \in [t-\tau^*, t]\}$,

$$\|\phi(t)\|_2^2 \le e^{-\hat{\omega}(t-\tau^*)}\|\phi(\tau^*)\|_2^2 + \frac{2c_M}{\hat{\omega}}\Big(4^3\delta + \tilde{c}_+\mu_0 a_t^{+\,2}\Big) + \frac{4\delta}{\sqrt{\hat{\omega}}} \tag{5.31}$$

having used (5.9) to bound

$$\int_{\tau^*}^t e^{-\hat{\omega}(t-s)}|\langle\phi, K\rangle| \le 4\int_{\tau^*}^t e^{-\hat{\omega}(t-s)}k(s) \le 4(2\hat{\omega})^{-1/2}\delta$$

Since $a_t^+ \le a_1$

$$\|\phi(t)\|_2^2 \le e^{-\hat{\omega}(t-\tau^*)}\|\phi(\tau^*)\|_2^2 + \frac{2c_M}{\hat{\omega}}\Big(4^3\delta + \tilde{c}_+\mu_0 a_1^2\Big) + \frac{4\delta}{\sqrt{\hat{\omega}}} \tag{5.32}$$

By (5.5) and (5.10)

$$\|\phi(\tau^*)\|_2^2 \le e^{8c_M\tau^*}\Big(\delta^2 + \frac{2}{\sqrt{c_M}}\delta + a_1^2\Big) \tag{5.33}$$

and by (5.23) and (5.32), $\|\phi(t)\|_2^2 < \mu_0^2$. Hence for all $t \le t_{a_1}(m)$, $\|\phi(t)\|_2^2 < \mu_0^2$ and (5.30) and (5.31) hold for all $t \in [\tau^*, t_{a_1}(m)]$. Thus (5.26) is proved.
From (5.32) it follows that for $\delta < a_1^2$ and sufficiently small, there is c so that

$$\|\phi(t)\|_2^2 \le c a_1^2, \qquad \forall t \in [0, t_{a_1}(m)]$$

and (5.25) is proved. (5.27) follows from (5.15), $\|\phi(t)\|_2 \le \mu_0$ and (5.24).
Thus it only remains to prove (5.28). Going back to (5.26) we use (5.25) to bound $\|\phi\|_2$ in the term containing $a(t)^2$. We then have the following improvement of (5.31):

$$\|\phi(t)\|_2^2 \le e^{-\hat{\omega}(t-\tau^*)}\|\phi(\tau^*)\|_2^2 + \frac{2c_M}{\hat{\omega}}\Big(4^3\delta + \tilde{c}_+(ca_1)a_1^2\Big) + \frac{4\delta}{\hat{\omega}}$$

By (5.33) if $t-\tau^*$ is large enough, the first term becomes smaller than $\epsilon_4^2 a_1/10^3$. This is possible if $t_{a_1}(m)$ is larger than such a value of t, a condition which can be achieved by supposing a_1 small enough and then δ consequently small. By taking δ small enough also the second term can be made smaller than $\epsilon_4^2 a_1/10^3$, and we then get (5.28).

□

As already remarked the external force even if small may win if the initial condition is too close to $\hat{m}_L$ thus determining the future outcome of the orbit, which may either stay always close to $\hat{m}_L$ or leave it. The next theorem shows that if we know that $a(S_t^K(m))$ reaches a (still very small) positive or negative value then the orbit goes to $m^{(+)}$ or $m^{(-)}$ according to the sign of $a(S_t^K(m))$, following closely along the way, $\mathcal{W}_+$ or, respectively, $\mathcal{W}_-$.

We will use the following properties of $v^{(\pm)}$ that are consequences of (1.9), (1.10) and of the results of Section 4. We state them only for $v^{(+)}$, for $v^{(-)}$ the statements are completely analogous.

We have chosen the parametrization of $\mathcal{W}_+$ in such a way that

$$a\big(v^{(+)}(0)\big) = a_0 < \epsilon_4 \tag{5.34}$$

with ϵ_4 as in Proposition 3.6.
For $\zeta > 0$ let $s^-(\zeta)$ be the (unique) number such that

$$a\big(v^{(+)}(s^-(\zeta))\big) = \zeta. \tag{5.35}$$

From (5.34) it follows that if $\zeta < a_0$ then $s^-(\zeta) < 0$. By (1.9) there exists a number $s^+(\zeta) > 0$ such that

$$\sup_{t \geq s^+(\zeta)} \|v^{(+)}(t) - m^{(+)}\|_2 \leq \zeta \tag{5.36}$$

In order to obtain uniqueness, one can choose the smallest number that satisfies (5.36), but this will not be important later on.

Definition 5.1. *Given $a_1 > 0$ and $\delta \in (0, a_1)$ we define*

$$\mathcal{C}^+_{a_1,\delta} := \Big\{ m \in L^2([-L/2, L/2]) : \|m - \hat{m}_L\|_2 < \delta,\ t_{a_1}(m) < \infty,\ a(S^K_{t_{a_1}}(m)) > 0 \Big\} \tag{5.37}$$

The set $\mathcal{C}^-_{a_1,\delta}$ is defined in a similar way with $a(S^K_{t_{a_1}}(m)) < 0$.

Theorem 5.6. *Given $\zeta \in (0, a_0)$ there exist $a^* > 0$, a positive function $\delta^*(\cdot)$ such that for any $a_1 \in (0, a^*)$ and $\delta \in (0, \delta^*(a_1))$, if $\|K\| < \delta$ then*

$$t_\zeta(m) < \infty \qquad \forall m \in \mathcal{C}^+_{a_1,\delta},$$

and

$$\begin{aligned} &\sup_{0 \leq t \leq s^+(\zeta) + |s^-(\zeta)|} \|S^K_{t_\zeta(m)+t}(m) - v^{(+)}(s^-(\zeta) + t)\|_2 \leq \zeta, \\ &\sup_{t > s^+(\zeta) + |s^-(\zeta)|} \|S^K_{t_\zeta(m)+t}(m) - m^{(+)}\|_2 \leq \zeta \end{aligned} \tag{5.38}$$

If $m \in \mathcal{C}^-_{a_1,\delta}$, then (5.38) holds with $v^{(-)}$ in place of $v^{(+)}$ and $m^{(-)}$ in place of $m^{(+)}$.

Remark 5.7. From Theorem 5.5 it follows that $\|S^K_t(m) - \hat{m}_L\|_2 \leq c\zeta$ for all $t \leq t_\zeta(m)$ if $\|K\|$ is small enough.

If instead the external force is absent, $K = 0$ (and of course the two previous theorems still apply) we can prove that $\mathcal{B}^+$, the basin of attraction of $m^{(+)}$ in a neighborhood of $\hat{m}_L$ contains a triangular shape with vertex $\hat{m}_L$:

Theorem 5.8. *Given $c > 0$, for any $\epsilon > 0$ small enough and any m such that $a(m) \in (0, \epsilon)$ and $\|m - v_{a(m)}\|_2 \le ca(m)$, we have*

$$\lim_{t\to\infty} \|S_t(m) - m^{(+)}\|_2 = 0 \tag{5.39}$$

Proof of Theorem 5.6. Without loss of generality we will suppose ζ small enough and such that

$$\zeta^{2/3} < \frac{\hat{\lambda}}{32c_0'} \frac{\epsilon_4}{10^6} \tag{5.40}$$

(the ratio $\frac{\hat{\lambda}}{c_0'}$ appears for the first time in Lemma 3.1).

By using the stability of $m^{(+)}$, there are ζ, $t_0 > s^+(\zeta)$, $\zeta_0 \in (0, \zeta)$ and δ so that if $\|K\| < \delta$

$$\sup_{t\ge 0} \|S_t^K(u) - m^{(+)}\|_2 \le \frac{\zeta}{10}, \quad \text{for all } u : \|u - v^{(+)}(t_0)\|_2 \le \frac{\zeta}{10^2} \tag{5.41}$$

$$\sup_{0\le t\le t_0+|s^-(\zeta)|} \|S_t^K(u) - v^{(+)}(s^-(\zeta) + t)\|_2 \le \frac{\zeta}{10^2}, \text{ for all } u : \|u - v^{(+)}(s^-(\zeta))\|_2 \le \zeta_0 \tag{5.42}$$

We finally choose $a_1 \le \zeta_0^2$ (other conditions will be given later) and then any $b > 0$ so small that

$$\sup_{s^-(\frac{1}{2}a_1)\le t\le s^-(2\zeta)} \|v_b(\theta(a_0; b) + t) - v^{(+)}(t)\|_2 \le \frac{\zeta_0}{10^2} \tag{5.43}$$

(this is possible because of Theorem 4.1, see (4.2)).

For $a_1 < \zeta_0^2 < \epsilon_4$, we now consider an $m \in \mathcal{C}^+_{a_1,\delta}$ and we prove the following. We call

$$m_1 = S^K_{t_{a_1}}(m), \qquad m_1 = \hat{m}_L + a_1\hat{e} + \phi(0) = v_{a_1} + \phi(0), \quad \langle \phi(0), \hat{e}\rangle = 0 \tag{5.44}$$

and we observe that by Theorem 5.5,

$$\|\phi(0)\|_2 \le \frac{\epsilon_4^2}{10^2} a_1 \tag{5.45}$$

Recalling Definition 3.1, we denote by $a(t) = a(S_t^K(m_1))$, the a-coordinate of $S_t^K(m_1)$, letting

$$S_t^K(m_1) = v_{a(t)} + \phi(t), \quad \langle \phi(t), \hat{e}\rangle = 0 \tag{5.46}$$

and we denote by $\tau(t)$ the t-coordinate of $S_t^K(m_1)$ relative to $\{v_b(\cdot)\}$.
We observe that since $\epsilon_4 \le \epsilon_3$, by Proposition 3.4 $\tau(0)$ is well defined and we call s^* the largest time such

$$\|S_t^K(m_1) - v_{a(t)}\|_2 = \|\phi(t)\|_2 < \epsilon_4 a(t), \qquad \text{for all } t < s^* \tag{5.47}$$

so that $\tau(t)$ is well defined for all $t \le s^*$.

We also denote by $a(\tau(t); b) := \langle v_b(\tau(t)), \hat{m}_L \rangle$ the a-coordinate of $v_b(\tau(t))$. We finally call

$$t^* = \inf\{t : a(\tau(t); b) = 2\zeta\}, \qquad t^* = \infty \text{ if } a(\tau(t); b) < 2\zeta \text{ for all } t \tag{5.48}$$

Observe that from (3.44) it follows that

$$\frac{1}{2} a(t) < a(\tau(t); b) < 2a(t), \qquad \forall t \in [0, s^*] \tag{5.49}$$

We are going to prove that

$$t^* < \infty, \qquad s^* \ge t^* \tag{5.50}$$

and for δ small enough

$$\|S_t^K(m_1) - v_b(\tau(t))\|_2 \le \frac{\epsilon_4}{8}\zeta_0^2 + C^*\delta \le \frac{\epsilon_4}{4}\zeta_0^2, \qquad \forall t \le t^* \tag{5.51}$$

We conclude the proof of the theorem by using (5.50) and (5.51) that we will prove afterwards.

Proof that $t_\zeta(m) < t^* < \infty$. From (5.49) and (5.50) it follows that $a(t^*) \in [\zeta, 4\zeta]$ and since $a(0) = a_1 < \zeta$ by continuity of $a(t)$ we get that $t_\zeta(m) \le t^* < \infty$.

Proof of (5.38). From (5.41) and (5.42) it follows that we only need to prove that

$$\|S^K_{t_\zeta(m)}(m) - v^{(+)}(s^-(\zeta))\|_2 \le \zeta_0 \tag{5.52}$$

We first observe that the a-coordinate of a L^2-function u denoted by $a(u)$ is a continuous function, more precisely there is c^* so that

$$|a(u) - a(v)| \le c^* \|u - v\|_2 \tag{5.53}$$

Thus, from (5.43) we get that for $t \in \big(s^-(\frac{1}{2}a_1), s^-(2\zeta)\big)$

$$a\big(v_b(\theta(a_0; b) + t)\big) \in \Big(\frac{a_1}{2} - c^*\frac{\zeta_0}{10^2}, 2\zeta + c^*\frac{\zeta_0}{10^2}\Big) \tag{5.54}$$

and from (5.51) we get that

$$a\big(\tau(t_\zeta(m) : b)\big) \in \Big(\zeta - c^*\frac{\epsilon_4}{4}\zeta_0^2, \zeta + c^*\frac{\epsilon_4}{4}\zeta_0^2\Big) \tag{5.55}$$

In Proposition 3.2 it has been proved that $a(t; b)$ is a strictly increasing function of t so from (5.54) and (5.55) we get that (for ϵ_4 and ζ_0 small enough)

$$t_\zeta^* := \tau(t_\zeta(m)) - \theta(a_0; b) \in \Big(s^-(a_1/2), s^-(2\zeta)\Big) \tag{5.56}$$

From (5.43) and (5.56) we get that

$$\|v_b(\tau(t_\zeta(m))) - v^{(+)}(t^*_\zeta)\|_2 \le \frac{\zeta_0}{10^2} \tag{5.57}$$

$$a(v^{(+)}(t^*_\zeta)) \in \Big(\zeta - c^*\frac{\zeta_0}{10^2}, \zeta + c^*\frac{\zeta_0}{10^2}\Big) \tag{5.58}$$

Since $a(v^+(t))$ is a strictly increasing function of t (see Theorem 4.2) from (5.58) we get that for a suitable constant c,

$$\|v^{(+)}(t^*_\zeta) - v^{(+)}(s^-(\zeta))\|_2 \le c\frac{\zeta_0}{10^2} \tag{5.59}$$

Thus

$$\begin{aligned}
\|S^K_{t_\zeta(m)}(m) - v^{(+)}(s^-(\zeta))\|_2 &\le \|S^K_{t_\zeta(m)}(m) - v_b(\tau(t_\zeta(m)))\|_2 \\
&\quad + \|v_b(\tau(t_\zeta(m))) - v^{(+)}(t^*_\zeta)\| + \|v^{(+)}(t^*_\zeta) - v^{(+)}(s^-(\zeta))\|_2 \\
&\le \frac{\epsilon_4}{4}\zeta_0^2 + \frac{\zeta_0}{10^2} + c\frac{\zeta_0}{10^2}
\end{aligned}$$

which gives (5.52) for ζ_0 and ϵ_4 small enough.

Proof of (5.51). We denote by $u(t) = S^K_t(m_1) - v_b(\tau(t))$ and with computations similar to (3.65) and (3.67) we get

$$\dot\tau - 1 = A^K(S^K_t(m_1); b) \tag{5.60}$$

$$A^K(m;b) = A(m;b) + \frac{\langle K, e_{v_b(\tau)}\rangle}{\langle f_L(v_b(\tau)), e_{v_b(\tau)}\rangle - \langle u, \dfrac{\delta e_{v_b(\tau)}}{\delta v_b(\tau)} f(v_b(\tau))\rangle} \tag{5.61}$$

$$\frac{1}{2}\frac{d}{dt}\|u\|_2^2 = B^K(S^K_t(m_1); b) \tag{5.62}$$

$$B^K(m;b) = B(m;b) - [A^K(m;b) - A(m;b)]\langle u, f_L(v_b(\tau))\rangle + \langle u, K\rangle \tag{5.63}$$

where $A(m;b)$ is defined in (3.66) and $B(m;b)$ in (3.68).
To bound $A(S^K_t(m_1);b)$ and $B(S^K_t(m_1);b)$ we use Proposition 3.6. We thus have to split $u(t) = S^K_t(m_1) - v_b(\tau(t))$ into $u(t) = u_1(t) + u_2(t)$. We choose $u_2(t) = \phi(x,t)\mathbf{1}_{x\in A^c_{t,\delta}}$, recalling that $\phi(t) = S^K_t(m_1) - v_{a(t)}$, $a(t) = a(S^K_t(m_1))$, and the parameter ζ of Proposition 3.6 equal to δ (this ζ is obviously not the ζ of Theorem 5.6). Thus

$$u_1(t) = \phi(t)\mathbf{1}_{x\in A_{t,\delta}} + v_{a(t)} - v_b(\tau(t))$$

We then use (5.15) bounding on its r.h.s. the term $\|\phi(t)\|_2^2 \le [\epsilon_4 a(t)]^2$ because $t \le s^*$, getting

$$\|u_1(t)\|_\infty \le \alpha^* + 2e^{-\tau^*/\beta} + c'_M\Big(2[\epsilon_4 a(t)]^2 + 2L(\alpha^*)^2 + 8\delta\Big)^{1/3}$$
$$+|a(t) - a(\tau(t);b)|\ \|\hat{e}\|_\infty + \frac{4c'_0}{\hat{\lambda}} a(\tau(t);b)^{5/3} \tag{5.64}$$

where the last term comes from (3.26).
By (3.46) and (5.11) we then have $\|u_1(t)\|_\infty \le \dfrac{\omega}{10C_2}$ as required in (3.70).
Denoting by $k(t)^2 = \int K(x,t)^2 dx$ and using (3.69) and (3.73), we then have

$$|A^K(S^K_t(m);b)| \le 2\Big\{4c_1\|u(t)\|_2 + \frac{C_2\|e_{v_b(\tau)}\|_\infty \|u(t)\|_2^2}{|\langle f_L(v_b(\tau)), e_{v_b(\tau)}\rangle|}\Big\} + \frac{4k(t)}{\|f_L(v_b(\tau))\|_2} \tag{5.65}$$

and from (3.71) we have

$$B^K(S^K_t(m);b) \le -\frac{\omega}{2}\|u(t)\|_2^2 + 8C_2\delta + \frac{4k(t)}{\|f_L(v_b(\tau))\|_2}\|u(t)\|_2 \|f_L(v_b(\tau))\|_2 + \|u(t)\|_2 k(t)$$

Thus, since $\|u(t)\|_2 \le 2$,

$$\begin{aligned} B^K(S^K_t(m);b) &\le -\frac{\omega}{2}\|u(t)\|_2^2 + 8C_2\delta + 5\|u(t)\|_2 k(t) \\ &\le -\frac{\omega}{2}\|u(t)\|_2^2 + 8C_2\delta + 10k(t) \end{aligned} \tag{5.66}$$

that, together with (5.62) implies that

$$\|u(t)\|_2^2 \le e^{-\omega t}\|u(0)\|_2^2 + C^*\delta, \qquad C^* := \frac{1}{\omega}\Big(16C_2 + 10\Big) \tag{5.67}$$

By (3.61) and (5.45) we have that

$$\|u(0)\|_2 \le 360\frac{c'_0}{\hat{\lambda}} a_1^{5/3} + 81\|m - v_{a_1}\|_2 \le 360\frac{c'_0}{\hat{\lambda}} a_1^{5/3} + 81\frac{\epsilon_4^2}{10^2} a_1 \le \frac{\epsilon_4}{8} a_1$$

Therefore from our choice of a_1 we get that

$$\|u(t)\|_2^2 \le \frac{\epsilon_4}{8}\zeta_0^2 + C^*\delta \tag{5.68}$$

which proves (5.51).

Proof that $t^* < \infty$. We need to exclude the fact that $a(\tau(t);b)$ decreases reaching a value less than a_1. To this purpose we call $t^{**} \le \infty$ the first time when $a(\tau(t^{**});b) = a_1/2$ and $\sigma^* = \min\{t^*, t^{**}\}$ so that

$$\frac{a_1}{2} \le a(\tau(t)) \le 2\zeta, \qquad \text{for all } t \le \sigma^* \tag{5.69}$$

For $t \le \sigma^*$, using Proposition 3.2, see (3.30), we get from (5.65)

$$|A^K(S_t^K(m);b)| \le 2\Big\{4c_1\|u(t)\|_2 + \frac{C_2\|e_{v_b(\tau)}\|_\infty\|u(t)\|_2^2}{\hat\lambda a_1/2}\Big\} + \frac{4k(t)}{\hat\lambda a_1/2} \tag{5.70}$$

By (5.68) supposing ζ_0 and δ small enough,

$$|A^K(S_t^K(m);b)| \le \frac{1}{4} + \frac{8k(t)}{\hat\lambda a_1} \tag{5.71}$$

From (3.27), (3.44), (5.60), (5.69) and (5.71), assuming that $\hat\lambda - \tilde c_2(2\zeta)^{2/3} \ge \frac{1}{2}$ and using Cauchy Schwartz, we get for all $t \le \sigma^*$

$$\begin{aligned} a(\tau(t);b) - a_1 \;&\ge\; a(\tau(0);b) - a_1 + \int_0^t a(\tau(s);b)[\hat\lambda - \tilde c_2(2\zeta)^{2/3}]\dot\tau(s)ds \\ &\ge\; -\frac{a_1}{2} + \frac{a_1}{4}\frac{3t}{4} - \frac{16\delta\sqrt{t}}{\hat\lambda} \end{aligned} \tag{5.72}$$

Let h be a small positive number. Given h, suppose $\delta > 0$ so small that $16\delta < \hat\lambda\sqrt{h}\frac{a_1}{16}$. Then

$$a(\tau(h);b) - a_1 \ge -\frac{a_1}{2} + \frac{a_1}{8}h \tag{5.73}$$

which shows that $h < t^{**}$. Moreover $a(\tau(h);b) \ge \frac{a_1}{2} + \frac{a_1}{8}h$. By using the same estimates we get

$$a(\tau(2h);b) - a(\tau(h);b) \ge \frac{a_1}{8}h$$

Thus $a(\tau(2h);b) \ge \frac{a_1}{2} + \frac{a_1}{4}h$. By iteration we then get that $t^* < t^{**}$ is finite.

Proof of the inequality $s^* > t^*$.
Since

$$\|\phi(t)\|_2^2 = \inf_a \|S_t^K(m_1) - v_a\|^2,$$

from (3.26) and (5.40) we get for $t \le t^*$

$$\begin{aligned} \|\phi(t)\| \;&\le\; \|u(t)\|_2 + \|v_b(\tau(t)) - v_{a(\tau(t);b)}\|_2 \le \|u(t)\|_2 + \frac{4c_0'}{\hat\lambda}a(\tau(t);b)^{5/3} \\ &\le\; \|u(t)\|_2 + \frac{4c_0'}{\hat\lambda}(2\zeta)^{2/3}a(\tau(t);b) \le \|u(t)\|_2 + \frac{\epsilon_4}{10^6}a(\tau(t);b) \end{aligned}$$

From (3.46), (5.49) and (5.40) we have that for $t \le t^*$

$$\begin{aligned} |a(\tau(t);b) - a(t)| \;&\le\; 80\|S_t^K(m_1) - v_{a(t)}\|_2 + 80\frac{4c_0'}{\hat\lambda}a(t)^{5/3} \le 80\|\phi(t)\| + 80\frac{4c_0'}{\hat\lambda}(4\zeta)^{2/3}a(t) \\ &\le\; 80\|\phi(t)\| + 80\frac{\epsilon_4}{10^6}a(t) \end{aligned}$$

Thus since $1 - 80\frac{\epsilon_4}{10^6} < \frac{1}{2}$, from (5.51) we get

$$\|\phi(t)\|_2 \le 2\|u(t)\|_2 + 2\frac{\epsilon_4}{10^6}[1 + 80\frac{\epsilon_4}{10^6}]a(t) \le 2\frac{\epsilon_4}{4}\zeta_0^2 + \frac{\epsilon_4}{2}a(t) \tag{5.74}$$

Thus for ζ_0 small enough we get that $\|\phi(t)\|_2 < \epsilon_4 a(t)$ for all $t < t^*$ namely $s^* > t^*$. Theorem 5.6 is thus proved. □

Proof of Theorem 5.8. Let m be as in the statement of the Theorem and denote by $a(0) := a(m) > 0$. We call $a(t) := a(S_t(m))$, the a-coordinate of $S_t(m)$, thus

$$S_t(m) = \hat{m}_L + a(t)\hat{e} + \phi(t), \qquad \langle \phi(t), \hat{e} \rangle = 0$$

We then compute the time derivative of $a(t)$ getting

$$\dot{a} = \hat{\lambda} a + \langle R, \hat{e} \rangle, \qquad R = f_L(S_t(m)) - f_L(\hat{m}_L) - \hat{\mathcal{L}}(S_t(m) - \hat{m}_L)$$

Let c_M be as in the proof of Lemma 5.3 and let $a_1 > a(0)$ be as in Theorem 5.6 small enough (this is possible by taking $\epsilon > a(0)$ small enough). Then the time derivative $\dot{a}(t)$ satisfies for all $t \le t_{a_1}(m)$

$$\dot{a} \ge \hat{\lambda} a - c_M(a^2 + \|\phi\|_2^2) \tag{5.75}$$

and we want to prove that $\dot{a} > 0$. Let t^* be the largest time such that $\dot{a}(t) > 0$ for $t \le t^*$ and let $t^{**} = \min\big(t^*, t_{a_1}(m)\big)$. Recalling the definition of τ^* given in (5.11), we first consider $t \le \min\{\tau^*, t^*\}$ and use (5.5) to bound

$$\|\phi(t)\|_2^2 \le e^{8c_M\tau^*}(c+1)a(t)^2 \tag{5.76}$$

having bounded $\|\phi(0)\|_2^2 \le ca(0)^2 \le ca(t)^2$. By (5.75) and (5.76),

$$\dot{a} \ge \hat{\lambda} a - c_M[1 + e^{8c_M\tau^*}(c+1)]a^2 \ge \frac{\hat{\lambda}}{2}\, a \tag{5.77}$$

for a_1 small enough. Thus $t^* > \tau^*$. For $t \ge \tau^*$ and $t \le t^{**}$, we use (5.31) to bound

$$\begin{aligned} \|\phi(t)\|_2^2 \;&\le\; \|\phi(\tau^*)\|_2^2 + \frac{2c_M}{\hat{\omega}}\tilde{c}_+\mu_0 a(t)^2 & (5.78)\\ &\le\; \{e^{8c_M\tau^*}(c+1) + \frac{2c_M}{\hat{\omega}}\tilde{c}_+\mu_0\}a(t)^2 & (5.79) \end{aligned}$$

For a_1 small enough we then get as in (5.77) the bound $\dot{a} \ge \hat{\lambda}a/2$. We thus conclude that till when $a(t) < a_1$, $\dot{a} \ge \hat{\lambda}a/2$ so that the time when $a(t) = a_1$ is finite and Theorem 5.8 follows from Theorem 5.6. □

6 Energy bounds

Let Σ be the basin of attraction of $\hat{m}_L$, namely

$$\Sigma = \Big\{ m \in L^\infty\big([-L/2, L/2], [-1,1]\big) : \lim_{t\to\infty} \|S_t(m) - \hat{m}_L\|_2 = 0 \Big\} \tag{6.1}$$

Theorem 6.1. *There is $c_\Sigma > 0$ so that for any $m \in \Sigma$*

$$F_L(m) - F_L(\hat{m}_L) \geq c_\Sigma \|m - \hat{m}_L\|_2^2 \tag{6.2}$$

We will first reduce the proof to the case $\|m\|_\infty < 1$.
With v^0 the solution of (C.24), let t_a, $a \in (m_\beta, 1)$, be such that $v^0(t_a) = a$ (since $v^0(t)$ is strictly decreasing t_a is well defined).

Lemma 6.1. *For any a sufficiently close to 1 and any $m \in L^\infty([-L/2, L/2], [-1,1])$,*

$$\sup_{t \geq t_a} \|S_t(m)\|_\infty \leq a, \qquad \inf_{t \leq t_a} \|S_t(m) - \hat{m}_L\|_2 \geq \frac{1 - m_\beta}{30} \|m - \hat{m}_L\|_2 \tag{6.3}$$

Proof. Given any $m \in L^\infty([-L/2, L/2], [-1,1])$, we shorthand $D_+ := \{x : m(x) \geq a\}$, $D_- := \{x : m(x) \leq -a\}$ and $D = D_+ \cup D_-$. Call

$$\psi(t) = S_t(m) - \hat{m}_L, \quad \psi_1(t) = \mathbf{1}_{D^c}\, \psi(t), \quad \psi_2(t) = \mathbf{1}_D\, \psi(x,t) \tag{6.4}$$

Since v^0 is a super-solution and $-v^0$ a sub-solution of (1.1), $\|S_{t_a}(m)\|_\infty \leq a$. Since $a > m_\beta$, a is a super-solution and $-a$ a sub-solution of (1.1), so that $\|S_t(S_{t_a}(m))\|_\infty \leq a$ for all $t \geq 0$, and the first inequality in (6.3) is proved.

- Proof of the second inequality in (6.3) under the assumption that

$$\|\psi_2(0)\|_2 > \frac{1}{2} \|\psi_1(0)\|_2 \tag{6.5}$$

We postpone the proof that if a is sufficiently close to 1 then

$$|S_{t_a}(m)| \geq 1 - 3(1-a) \quad \text{in } D \tag{6.6}$$

By (6.6), $\|\psi_2(t_a)\|_2^2 \geq [1 - 3(1-a) - m_\beta]^2 |D|$ and since $|D| \geq \dfrac{1}{4}\displaystyle\int_D \psi_2(0)^2$,

$$\|\psi_2(t_a)\|_2^2 \geq [1 - 3(1-a) - m_\beta]^2 \frac{1}{4} \|\psi_2(0)\|_2^2$$

By (6.5), $\|\psi_2(0)\|_2 \geq \|\psi(0)\|_2/3$. We choose a so close to 1 that $[1-3(1-a)-m_\beta] \geq (1-m_\beta)/2$ thus deriving the second inequality in (6.3) when (6.5) holds and pending the validity of (6.6), which we will prove next.

The solution $w(t)$ of

$$w_t = -1 - \frac{1}{\beta}\text{arctanh}(w), \quad w(0) = a \tag{6.7}$$

is a sub-solution of the equation satisfied by $S_t(m)$ restricted to D_+, considering $S_t(m)\mathbf{1}_{D_+^c}$ as a known term. Obviously,

$$w(t_a) \geq a - \{1 + \frac{1}{\beta}\text{arctanh}(a)\}t_a$$

By (C.24),

$$t_a \leq \frac{1-a}{\beta^{-1}\text{arctanh}(a) - 1} \tag{6.8}$$

Let a be so close to 1 that $\dfrac{\text{arctanh}(a) + \beta}{\text{arctanh}(a) - \beta} \leq 2$, then $w(t_a) \geq a - 2(1-a) = 1 - 3(1-a)$. The same argument applies in D_- and (6.6) is proved.

- Proof of the second inequality in (6.3) when (6.5) does not hold.

Suppose a so close to 1 that $\text{arctanh}\,(a) > \beta$, then a is a super-solution of the equation $u_t = 1 - \beta^{-1}\text{arctanh}\,(u)$ and since $m\mathbf{1}_{D^c} \leq a$, $S_t(m)\mathbf{1}_{D^c} \leq a$. Analogously, $S_t(m)\mathbf{1}_{D^c} \geq -a$, hence

$$\|S_t(m)\mathbf{1}_{D^c}\|_\infty \leq a \tag{6.9}$$

By (6.9),

$$\frac{1}{2}\frac{d\|\psi_1(t)\|_2^2}{dt} \geq -|\langle\psi_1(t), J^{\text{neum}} * [\psi_1(t) + \psi_2(t)]\rangle| - \frac{\|\psi_1(t)\|_2^2}{\beta(1-a^2)}$$

hence

$$\frac{1}{2}\frac{d\|\psi_1(t)\|_2^2}{dt} \geq -\|\psi_1(t)\|_2^2 - \|\psi_1(t)\|_2\|\psi_2(t)\|_2 - \frac{\|\psi_1(t)\|_2^2}{\beta(1-a^2)} \tag{6.10}$$

Let t_a^* be the maximal $\tau \leq t_a$ such that

$$\|\psi_2(t)\|_2 \leq \|\psi_1(t)\|_2, \quad \text{for all } t \leq \tau \tag{6.11}$$

$t_a^* > 0$, because (6.5) does not hold. (6.10) then gives

$$\frac{d\|\psi_1(t)\|_2^2}{dt} \geq -2\{2 + \frac{1}{\beta(1-a^2)}\}\|\psi_1(t)\|_2^2, \quad t \leq t_a^*$$

hence

$$\|\psi_1(t)\|_2^2 \geq e^{-2\{2+\frac{1}{\beta(1-a^2)}\}t}\|\psi_1(0)\|_2^2, \quad t \leq t_a^* \tag{6.12}$$

For a sufficiently close to 1, $\operatorname{arctanh}(a) - \beta \geq |\log(1-a)|/4$ and by (6.8),

$$t_a \leq \frac{4\beta(1-a)}{|\log(1-a)|} \tag{6.13}$$

so that the exponent in (6.12) is bounded by

$$-2\{2 + \frac{1}{\beta(1-a^2)}\}t_a \geq -8\frac{2\beta(1-a^2)+1}{(1+a)|\log(1-a)|}$$

Hence, for a sufficiently close to 1, (6.12) yields

$$\|\psi_1(t)\|_2^2 \geq \{1 - \frac{5}{|\log(1-a)|}\}\|\psi_1(0)\|_2^2, \quad t \leq t_a^* \tag{6.14}$$

We will next show that $t_a^* = t_a$. Indeed, in D_+, $m_t(x,t) < 0$ for all $t \leq t_a$: in fact

$$m(x,t) \geq w(t), \quad w \text{ as in (6.7)}$$

and for a sufficiently close to 1,

$$\operatorname{arctanh}(w(t_a)) > \beta$$

Hence $m_t(x,t) < 0$. An analogous argument shows that $m_t(x,t) > 0$ in D_-. Since $w(t_a) \geq m_\beta \geq |\hat{m}|$, $|\psi_2(x,t)| \leq |\psi_2(x,0)|$. The condition (6.11) is then satisfied with $t = t_a$. Suppose in fact by contradiction that $t_a^* < t_a$, then, by (6.14) and with a sufficiently close to 1,

$$\|\psi_1(t_a^*)\|_2 > \frac{\|\psi_1(0)\|_2}{2} \geq \|\psi_2(0)\|_2 \geq \|\psi_2(t_a^*)\|_2$$

so that, by continuity, (6.11) is satisfied also for $t > t_a^*$ against the assumption of maximality of the latter.
Thus (6.14) holds for all $t \leq t_a$. Writing $\epsilon = 5/|\log(1-a)|$ in (6.14), (6.3) follows from

$$\|\psi(t)\|_2 \geq \|\psi_1(t)\|_2 \geq (1-\epsilon)^{1/2}\|\psi_1(0)\|_2 \geq \frac{2(1-\epsilon)^{1/2}}{3}\|\psi(0)\|_2$$

having used that

$$\|\psi(0)\|_2 \leq \|\psi_1(0)\|_2 + \|\psi_2(0)\|_2 \leq \frac{3}{2}\|\psi_1(0)\|_2$$

Lemma 6.1 is proved. $\square$

Proof of Theorem 6.1. We will prove equivalently that for any $\delta > 0$ small enough,

$$\inf_{m \in \Sigma : \|m - \hat{m}_L\|_2 \geq \delta} F_L(m) \;\geq\; F_L(\hat{m}_L) + c\delta^2 \tag{6.15}$$

Given m as above, let t^* be the first time t when $\|S_t(m)-\hat{m}_L\|_2 = \delta/30$. By Lemma 6.1, $t^* \geq t_a$, so that $m^* := S_{t^*}(m)$ verifies the first inequality in (6.3). Calling $v(\cdot,t) = S_t(m^*) - \hat{m}_L$, we observe that by its definition

$$\|v(\cdot,0)\|_2 = \frac{\delta}{30} \tag{6.16}$$

Moreover, there is $c_1 = c_1(a)$ so that

$$\frac{1}{2}\left|\frac{d}{dt}\|v(t)\|_2^2\right| \leq c_1\|v(t)\|_2^2$$

which yields, for any $t > 0$,

$$e^{-c_1 t}\|v(0)\|_2 \leq \|v(t)\|_2 \leq e^{c_1 t}\|v(0)\|_2 \tag{6.17}$$

Since $F_L(m) \geq F_L(S_t(m^*))$,

$$F_L(m) - F_L(\hat{m}_L) \geq -\frac{1}{2}\,\langle v(t), \hat{\mathcal{L}} v(t)\rangle - c_2\big(\epsilon + C\|v(t)\|_2^{2/3}\big)\|v(t)\|_2^2 \tag{6.18}$$

where c_2 comes from the Taylor expansion of arctanh, while ϵ and C from Lemma C.1. Denoting by $v_\perp$ and $v_{//}$ the components of $v(\cdot,t)$ perpendicular and parallel to $\hat{e}$, we have, using (6.17) and (2.13),

$$\begin{aligned} F_L(m) - F_L(\hat{m}_L) \quad &\geq \quad \Big\{\frac{\hat{\omega}}{2} - c_2\big(\epsilon + Ce^{(2/3)c_1 t}\|v(0)\|_2^{2/3}\Big\}\,\|v_\perp(t)\|_2^2 \\ &- \quad \Big\{\frac{\lambda}{2} + c_2\big(\epsilon + Ce^{(2/3)c_1 t}\|v(0)\|_2^{2/3}\Big\}\,\|v_{//}(t)\|_2^2 \end{aligned}$$

By Lemma C.2

$$\epsilon \leq 2e^{-t/\beta}, \quad C \leq \frac{3}{\sqrt{8}}\,\big(\beta\|J'\|_\infty\big)^{1/3}$$

We then choose $t = \tau$ and τ such that

$$c_2\,(2e^{-\tau/\beta}) \leq \frac{\hat{\omega}}{16} \tag{6.19}$$

and recalling that by (6.16), $\|v(0)\|_2 = \delta/30$, we choose δ in (6.15) so small that

$$c_2 C e^{(2/3)c_1\tau}\delta^{2/3} \leq \frac{\hat{\omega}}{16} \tag{6.20}$$

Thus

$$F_L(m) - F_L(\hat{m}_L) \quad \geq \quad \frac{\hat{\omega}}{4}\|v_\perp(\tau)\|_2^2 - \{\frac{\lambda}{2} + \frac{\hat{\omega}}{8}\}\,\|v_{//}(\tau)\|_2^2$$

By Theorem 5.8 for δ small enough

$$\frac{\hat{\omega}}{4}\|v_\perp(\tau)\|_2^2 \geq 2\{\frac{\lambda}{2} + \frac{\hat{\omega}}{8}\}\,\|v_{//}(\tau)\|_2^2$$

otherwise m would be attracted by $m^{(+)}$ or by $m^{(-)}$, against the assumption that $m \in \Sigma$. Hence

$$F_L(m) - F_L(\hat{m}_L) \quad \geq \quad \frac{\hat{\omega}}{8}\|v_\perp(\tau)\|_2^2 \geq \frac{\kappa^2\hat{\omega}}{8(1+\kappa^2)}v(\tau)^2$$

where

$$\kappa^2 = 2\{\frac{\lambda}{2} + \frac{\hat{\omega}}{8}\}\frac{4}{\hat{\omega}}$$

which, using (6.17), proves Theorem 6.1. □

7 Extensions

In this section we discuss several extensions of the above results mainly in view of applications to tunnelling. They involve a repetition of the proofs we have done so far, just an adaptation to the new contexts without any really new ideas, for this reason we just state the results without proofs.

Glauber dynamics.

By this we refer to the evolution (1.11) (indeed the scaling limit of Glauber dynamics in Ising systems with Kac potentials gives rise to (1.11)) and denote by T_t the corresponding flow. Stationary solutions are the same for T_t and S_t but invariant sets may differ. Indeed the invariant manifolds $v^{(\pm)}$ are not invariant for T_t, however there are invariant manifolds that we denote by $v_{\rm gl}^{(\pm)}$ for T_t as well satisfying (1.9)–(1.10). Actually their existence has been proved prior to the present paper, see [4]. The proof in [4] is similar but not exactly as in the present paper, as it exploits the existence of a spectral gap in L^∞. The stability properties in Section 5 require some additional properties on the external force as one should also impose that the solution has values in $[-1,1]$; such a condition is automatically satisfied for the equation (5.1). To avoid complications we thus just state that Theorems 5.5, 5.6 and 5.8 retain their validity for S_t^K replaced by T_t, i.e. without the additional force K, and with $v_{\rm gl}^{(\pm)}$. We just state here from [2] that

Theorem 7.1. *There is $\epsilon > 0$ and for any $r \in (0,1)$ there is $L^* > 0$ so that if $L \geq L^*$ and $\|m - \bar{m}_\xi \mathbf{1}_{|x|\leq L}\|_\infty < \epsilon$, $|\xi| > rL$ then*

$$\lim_{t\to\infty} \|T_t(m) - m^{(\sigma)}\|_\infty = 0 \tag{7.1}$$

where σ is equal to the sign of ξ.

The same proofs of Section 6 apply to T_t showing that:

Theorem 7.2. *There is $c > 0$ so that if $\lim_{t\to\infty} \|T_t(m) - \hat{m}_L\|_2 = 0$, then*

$$\|m - \hat{m}_L\|_2^2 \le c[F_L(m) - F_L(\hat{m}_L)] \tag{7.2}$$

The two-dimensional case

The stability properties of Section 5 and the lower bound on the energy of Section 6 extend to the $d = 2$ case as we explain below.
Consider the spatial domain $Q_L = [-L/2, L/2]^2$ and the evolution equation with a kernel $j^{\text{neum}}(r, r')$ defined with reflections at the boundaries and supposing that the probability kernel $j(r, r')$ is smooth, depends on $|r - r'|$ and vanishes for $|r - r'| \ge 1$. The solution to the corresponding equation on Q_L with an initial datum $m(r)$ which depends only on $r \cdot e_1$ (e_1 the unit vector along the x-axis) preserves the planar symmetry and verifies (1.1) with

$$J(x, x') = \int_{\mathbb{R}} j(0, (x' - x)e_1 + ye_2)\, dy \tag{7.3}$$

(the role of j and J are interchanged in [3]). As a consequence the invariant manifolds $v^{(\pm)}$ are still invariant for S_t in $d = 2$. The stability properties given in Section 5 use the structure of the equation which is the same in $d = 2$ and are based on spectral properties of the linearized operators.

Spectral analysis. The analogue of Theorem 2.4 holds also in $d = 2$ as we explain next. Given any function $m \in L^2([-L/2, L/2], [-1, 1])$ we denote by $m^{(e)}$ its extension to Q_L namely

$$m^{(e)}(r) := m(r \cdot e_1), \qquad r \in Q_L$$

and we call planar a function $m \in L^2(Q_L, [-1, 1])$ that depends only on the x-coordinate. We also denote by $\mathcal{M}_L \subset L^2(Q_L, [-1, 1])$ the set that contains the instanton $\hat{m}_L^{(e)}$ and all the planar functions m such that

$$\|m - \hat{m}_L^{(e)}\|_\infty \le \epsilon(L)$$

where $\epsilon(L)$ is a small number as required by Theorem 7.3 below.
Given any $m \in \mathcal{M}_L$ we denote by $\mathcal{L}_m$ the linearized operator in Q_L namely,

$$\mathcal{L}_m u(r) := -\frac{1}{p_m(r \cdot e_1)} u(r) + j^{\text{neum}} \star u(r), \qquad p_m(x) := \beta[1 - m(x)^2], \qquad r \in Q_L$$

Theorem 7.3. *There are constants $c_\pm$, $\hat{c}$ and a function $\epsilon(L)$ all positive so that for L large enough if $m \in \mathcal{M}_L$ then the L^2 norm of the operator $\mathcal{L}_m$ is bounded by $1+\beta^{-1}\mathrm{arctanh}''((1+m_\beta)/2)$ and:*

(i) $\mathcal{L}_m$ has a positive eigenvalue λ_m

$$c_- e^{-2\alpha L} \le \lambda_m \le c_+ e^{-2\alpha L} \tag{7.4}$$

with eigenfunction e_m which is planar, smooth and strictly positive;

(ii) $\mathcal{L}_m$ has a gap

$$\langle u, \mathcal{L}_m u\rangle \le -\frac{\hat{c}}{L^2}\langle u, u\rangle, \qquad u \in L^2(Q_L;[-1,1]), \quad \langle u, e_m\rangle = 0 \tag{7.5}$$

(iii) let $\dfrac{\delta e_m}{\delta m}$ be the linear operator such that $\dfrac{de_{m(s)}}{ds}\Big|_{s=0} = \dfrac{\delta e_m}{\delta m}\dfrac{dm(s)}{ds}\Big|_{s=0}$ for any smooth curve $m(s)$, $m(0)=m$. Then there is $c_1>0$ so that

$$\|\frac{\delta e_m}{\delta m}\|_2 \le c_1 \tag{7.6}$$

Proof. The proof is the same as the one of Theorem 2.4 given in Appendix B since it is a consequence of the following remark. Let Ω_m, $m \in \mathcal{M}_L$ be the linearized operator of the glauber semigroup T_t, namely

$$\Omega_m u(r) := -u(r) + p_m(r\cdot e_1) j^{\mathrm{neum}} \star u(r), \qquad r \in Q_L$$

As explained in Appendix B, by using that $\mathcal{L}_m = \frac{1}{p_m}\Omega_m$ the spectral analysis of $\mathcal{L}_m$ in $L^2(Q_L; dxdy)$ is a consequences of the spectral properties of Ω_m in $L^2(Q_L, p_m^{-1}(x)dxdy)$. Since the statements (i)–(iii) are proven in [3] for the operator Ω_m in $L^2(Q_L, p_m^{-1}dxdy)$, we get that the same holds for $\mathcal{L}_m$ as well. We omit the details. $\square$

Stability under a small additive force.
Here we briefly comment on the proof in $d=2$ of Theorem 5.5, 5.6 and 5.8.
Consider the evolution (5.1) for $r \in Q_L$ and with the time depending force $K(r,t)$ smooth and such that

$$\|K\|^2 = \int_0^\infty k(t)^2 < \infty, \quad k(t)^2 := \int_{Q_L} K(r,t)^2 dr$$

As in (5.4), for $\|m - \hat{m}_L^{(e)}\|_\infty$ small enough, we write for any $r=(x,y)\in Q_L$,

$$S_t^K(m)(r) = v_{a(t)}(x) + \phi(r,t), \quad \langle \phi(t), \hat{e}\rangle = 0. \tag{7.7}$$

where $a(t)$ is the a-coordinate of $S_t^K(m)$. Observing that the first function on the right hand side of (7.7) is planar, it is possible to repeat line by line the proofs given in Section 5 based on the spectral-gap property (7.5). We omit the details.

Application to tunnelling.
From Theorem 5.5 and Theorem 5.6 in $d = 1, 2$ we get Theorem 7.4 below, used in [1] for $d = 1$ and in [3] for $d = 2$.

Theorem 7.4. *For any $\tau > 0$ and for any $\zeta > 0$ there is $\delta > 0$ so that if $\|K\| < \delta$ and $\|m - \hat{m}_L^e\|_2 < \delta$ only the following two alternatives hold:*

- *(i) For all times $t \geq 0$, $\|S_t^K(m) - \hat{m}_L^e\|_2 < \zeta$*
- *(ii) There are $t^* > 0$ and $\sigma \in \{-, +\}$ so that $\|S_t^K(m) - \hat{m}_L^e\|_2 < 2\|v_L^{(\sigma)}(\cdot, -\tau) - \hat{m}_L^e\|_2$ for all $t \leq t^*$ while $\|S_t^K(m) - v_L^{(\sigma)}(\cdot, -\tau + (t - t^*)\|_2 < \zeta$ for all $t \geq t^*$.*

Proof. Let m be such that $\|m - \hat{m}_L^e\|_2 < \delta$ with δ small enough so that $a_1 \equiv a(m)$ satisfies the hypothesis of Theorems 5.5 and 5.6. Then if $t_{a_1}(m) = \infty$ from Theorem 5.5 we get that (i) holds. If instead $t_{a_1}(m) < \infty$, Theorem 5.6 implies that (ii) holds. $\square$

We finally state, again without proofs, that Theorem 7.2 retains its validity also in $d = 2$ with Q_L.

A Existence of dynamics

Let $m \in \mathcal{C}([-L/2, L/2], (-1, 1))$, $b := \|m\|_\infty$, $s \in (0, \infty]$ and

$$\mathcal{X}(m; s) = \Big\{u \in \mathcal{C}([-L/2, L/2] \times [0, s]) : u(\cdot, 0) = m, \sup_{t \leq s} \|u(\cdot, t)\|_\infty < 1\Big\} \tag{A.1}$$

Define $\Psi(u)(t)$, $u \in \mathcal{X}(m; s)$, $t \leq s$, by setting

$$\Psi(u)(t) = m + \int_0^t f_L(u) \tag{A.2}$$

There is $a > 0$ so that for all $m \in \mathcal{C}([-L/2, L/2], (-1, 1))$ $f_L(m)(x) \leq a$ if $m(x) \geq 0$ and $f_L(m)(x) \geq -a$ if $m(x) \leq 0$. Hence, given m, there is $s(m) > 0$ so that Ψ maps $\mathcal{X}(m; s(m))$ into itself. Moreover, Ψ is a contraction on $\mathcal{X}(m; s^*)$ if $s^* \leq s(m)$ is small enough, then Ψ has a fixed point that we may call $S_t(m)$ because it solves (1.1) for $t \leq s^*$ with initial datum m.

Ψ preserves the order, so that if $b(t)$ solves the ordinary differential equation $b_t = -b + \beta^{-1}\text{arctanh}(b)$, $b(0) = b$, then $\sup_{t\geq 0} b(t) < 1$ and for any $s > 0$ the subset

$$\mathcal{X}^0(m;s) = \Big\{u \in \mathcal{X}(m;\infty) : u(\cdot,0) = m, \|u(\cdot,t)\|_\infty \leq b(t), t \leq s\Big\} \tag{A.3}$$

is well defined and left invariant by Ψ. Thus $S_t(m) \in \mathcal{X}^0(m;s^*)$ and by setting $S_{t+s^*}m = S_tS_{s^*}(m)$ we can extend the evolution past s^* and by iteration to all times, thus concluding that (1.1) has a unique global solution for any initial datum $m \in \mathcal{C}([-L/2, L/2], (-1,1))$.

Proof of Proposition 2.1

Let v be a super-solution of (1.1) with $v(\cdot,0) \geq m$. Since $v(\cdot,t) \geq \Psi(v)(t)$, the set $\{u \in \mathcal{X}(m;s(m)) : u(\cdot,t) \leq v(\cdot,t), t \leq s(m)\}$ is left invariant by Ψ and consequently $S_t(m)$, $t \leq s(m)$, is in such a set, thus proving that $S_t(m) \leq v(\cdot,t)$ for $t \leq s(m)$. The argument can be repeated starting from $s(m)$ and we then conclude that $S_t(m) \leq v(\cdot,t)$ for all $t \geq 0$ and since the analogous argument applies to sub-solutions Proposition 2.1 is proved.

Proof of Proposition 2.2

Calling $v(x,t) = S_t(m)(x) - S_t(\tilde{m})(x)$. If $v(x,t) > 0$ (< 0) then $-[\text{arctanh}(S_t(m)(x)) - \text{arctanh}(S_t(\tilde{m})(x))] < 0$ (respectively > 0). Therefore,

$$|v(x,t)| \leq |v(x,0)| + \int_0^t \int_{-L}^{L} J^{\text{neum}}(x,y)|v(y,s)| \tag{A.4}$$

so that $\|v(\cdot,t)\|_\infty \leq e^t\|v(\cdot,0)\|_\infty$.

Proof of Proposition 2.3

Define $K_t^+(b)$ and $K_t^-(b)$ as the solutions of the ordinary differential equation $b_t = \pm 1 + \beta^{-1}\text{arctanh}(b)$, $b(0) = b$, $|b| < 1$. Since $|J^{\text{neum}} * m| \leq 1$, for any $m \in L^\infty([-L/2, L/2], (-1,1))$, $K_t^-(-b) \leq S_t(m) \leq K_t^+(b)$ where $b := \|m\|_\infty$. For any $t > 0$, $\lim_{b\to\pm 1} K_t^\pm(b)$ exists and is in $(-1,1)$ hence Proposition 2.3.

B Proof of Theorem 2.4

To bound the L^2 norm of $\mathcal{L}_m$, we write

$$\langle f, Jf\rangle \leq \frac{1}{2}\int J^{\text{neum}}(x,y)\{f^2(x) + f^2(y)\}dxdy \leq \langle f, f\rangle$$

while $\text{arctanh}''(m) \leq \text{arctanh}''((1+m_\beta)/2)$ if $\epsilon(L)$ is small enough, hence the desired bound on the norm of $\mathcal{L}_m$.

There are positive constants $a_\pm$ so that for any m as in the hypothesis of the Theorem,

$$\|p_m\|_\infty < a_+, \qquad \|p_m^{-1}\|_\infty < a_-$$

Proof of the upper bound in (2.12). We denote by v the strictly positive eigenfunction of Ω_m (see (2.7)) corresponding to the positive eigenvalue λ. Then, using (2.8) we get the following upper bound for the maximal eigenvalue λ_m:

$$\lambda_m := \sup_u \frac{\langle u, \mathcal{L}_m u\rangle}{\langle u, u\rangle} = \sup_u \frac{\langle u, \Omega_m u\rangle_p}{\langle u, u\rangle_p} \frac{\langle u, u\rangle_p}{\langle u, u\rangle} \le a_- \lambda \le a_- \hat{c}_+ e^{-2\alpha L} \tag{B.1}$$

Proof of the lower bound in (2.12). Using that $\langle v, \Omega_m v\rangle = \lambda \langle v, v\rangle$ and (2.8) we get

$$\sup_u \frac{\langle u, \mathcal{L}_m u\rangle}{\langle u, u\rangle} \ge \frac{\langle v, \mathcal{L}_m v\rangle}{\langle v, v\rangle} = \lambda \frac{\langle v, v\rangle_p}{\langle v, v\rangle} = \lambda \frac{\langle v, v\rangle_p}{\langle v, pv\rangle_p} \ge \frac{\lambda}{a_+} \ge \frac{\hat{c}_-}{a_+} e^{-2\alpha L} \tag{B.2}$$

Conclusion of the proof of (i). Let e_m be such that $\langle e_m, e_m\rangle = 1$ and $\lambda_m = \langle e_m, \mathcal{L}_m e_m\rangle$ then, if $|e_m| \neq e_m$ we get that since $J \ge 0$

$$\langle |e_m|, \mathcal{L}_m |e_m|\rangle - \langle e_m, \mathcal{L}_m e_m\rangle = \int J(x,y)\big\{|e_m(x)e_m(y)| - e_m(x)e_m(y)\big\} > 0$$

which is a contradiction. Let x_0 be such that $e_m(x_0) = 0$, since $J * e_m(x_0) > 0$, we get that $\mathcal{L}_m e_m(x_0) \neq \lambda_m e_m(x_0)$.

Proof of (ii)
The second largest spectral point is

$$\inf_w \sup_{u:\langle u, w\rangle = 0} \frac{\langle u, \mathcal{L}_m u\rangle}{\langle u, u\rangle}$$

Let $C > 0$ be the number defined in (2.9),

$$-C = \sup_{u:\langle u, v_m\rangle_p = 0} \frac{\langle u, \Omega_m u\rangle_p}{\langle u, u\rangle_p}$$

choosing $w = p_m^{-1} v$

$$\begin{aligned} \inf_w \sup_{u:\langle u, w\rangle = 0} \frac{\langle u, \mathcal{L}_m u\rangle}{\langle u, u\rangle} &\le \sup_{u:\langle u, p^{-1}v\rangle = 0} \frac{\langle u, \mathcal{L}_m u\rangle}{\langle u, u\rangle} = \sup_{u:\langle u, v\rangle_p = 0} \frac{\langle u, \Omega_m u\rangle_p}{(u, u)} \\ &\le -C \frac{(u, u)_p}{\langle u, u\rangle} \le -C a_- \end{aligned} \tag{B.3}$$

Proof of (2.14). Recalling that $\dfrac{\delta e_m}{\delta m}$ is given by

$$\frac{de_{m(\cdot,s)}}{ds}\Big|_{s=0} = \frac{\delta e_m}{\delta m} \frac{dm(\cdot, s)}{ds}\Big|_{s=0}$$

we differentiate the following identity:

$$\mathcal{L}_m e_m = \lambda_m e_m, \quad \langle e_m, e_m\rangle = 1 \tag{B.4}$$

Thus, denoting by $\dot{u}$ derivative w.r.t. s at $s = 0$, we get

$$(\mathcal{L}_m - \lambda_m)\dot{e}_m = [\dot{\mathcal{L}}_m - \dot{\lambda}_m]e_m, \quad \dot{\lambda}_m = \langle e_m, \dot{\mathcal{L}}_m e_m \rangle \tag{B.5}$$

$$\dot{\mathcal{L}}_m e_m = -\frac{2m\dot{m}}{\beta(1-m^2)^2}\, e_m \tag{B.6}$$

so that

$$\frac{\delta e_m}{\delta m}\psi = -(\mathcal{L}_m - \lambda_m)^{-1}[\frac{2me_m\psi}{\beta(1-m^2)^2} - \langle e_m, \frac{2me_m\psi}{\beta(1-m^2)^2}\rangle] \tag{B.7}$$

From (B.7) and (2.15), (2.14) easily follows. □

C Estimates on the evolution

In the following lemmas m is a measurable function on $[-L/2, L/2]$ with $\|m\|_\infty \le 1$.

Lemma C.1. *Suppose there are $\rho \ge 0$ and $c > 0$ so that*

$$|m(x) - m(y)| \le \rho + c\,|x-y| \qquad \forall x, y \in [-L/2, L/2]. \tag{C.1}$$

If

$$c \ge \frac{\sqrt{2}\|m\|_2}{L^{3/2}} \tag{C.2}$$

then

$$\|m\|_\infty \le \rho + C\|m\|_2^{2/3}, \qquad C := (2^{-1/2} + 2^{-5/3})c^{1/3}. \tag{C.3}$$

Proof. Let u be the function on $[-L/2, L/2]$ obtained from m by reflections around $\pm L/2$. For $\delta \in (0, L/2]$ and $|x| \le L/2$,

$$2\delta m(x) = \int_{x-\delta}^{x+\delta} u(y)\,dy + \int_{x-\delta}^{x+\delta} (u(x) - u(y))\,dy$$

By Cauchy-Schwartz and (C.1), $2\delta|m(x)| \le \sqrt{2\delta}\|m\|_2 + 2\delta\rho + \delta^2 c/2$, hence

$$|m(x)| \le \frac{\|m\|_2}{\sqrt{2\delta}} + \rho + \frac{\delta c}{4} \tag{C.4}$$

Let $\delta = \min\{(\frac{\sqrt{2}\|m\|_2}{c})^{2/3}, L\}$. By (C.2) we have $\delta = (\frac{\sqrt{2}\|m\|_2}{c})^{2/3}$. Then (C.3) follows from (C.4).

□

We next prove that for $t > 0$ the solution $S_t(m)$ satisfies the hypothesis of Lemma C.1.

Lemma C.2. *Let $\|m\|_\infty < 1$ and $m(x,t) = S_t(m)(x)$. Then for any $x, y \in [-L/2, L/2]$ and any $t > 0$,*

$$|m(x,t) - m(y,t)| \le 2e^{-t/\beta} + \beta \|J'\|_\infty \, |x-y| \tag{C.5}$$

Proof. Fix arbitrarily two points $x, y \in [-L/2, L/2]$ and write

$$m_t(x,t) = -\frac{\operatorname{arctanh}\{m(x,t)\}}{\beta} + g_0(t), \quad m_t(y,t) = -\frac{\operatorname{arctanh}\{m(y,t)\}}{\beta} + g_1(t), \tag{C.6}$$

where $g_0(t) = J * m(x,t)$ and $g_1(t) = J * m(y,t)$ are regarded as "known" terms. By the smoothness of J,

$$\|g_1 - g_0\|_\infty \le \|J'\|_\infty |x-y|. \tag{C.7}$$

Let $\lambda \in [0,1]$, define $g(t;\lambda) = \lambda g_1(t) + (1-\lambda) g_0(t)$ and $u(t;\lambda)$, $t \ge 0$, as the solution of

$$u_t(t;\lambda) = -\frac{\operatorname{arctanh}\{u(t;\lambda)\}}{\beta} + g(t;\lambda) \tag{C.8}$$

with initial datum

$$u(0;\lambda) = \lambda m(y,0) + (1-\lambda) m(x,0). \tag{C.9}$$

Note that $u(0;\lambda)$ is a constant in $[-L/2, L/2]$, since x and y are fixed. Calling $v(t;\lambda) := \frac{d}{d\lambda} u(t;\lambda)$, we have

$$v_t = -\frac{v}{\beta(1-u^2)} + (g_1 - g_0) \tag{C.10}$$

whose solution is

$$v(t;\lambda) = e^{-a(t,0;\lambda)} v(0;\lambda) + \int_0^t e^{-a(t,s;\lambda)} [g_1(s) - g_0(s)] \tag{C.11}$$

where $a(t,s;\lambda) := \displaystyle\int_s^t \frac{1}{\beta(1-u^2)}$. The lemma is concluded by recalling that

$$|m(y,t) - m(x,t)| \le \int_0^1 |v(t;\lambda)| d\lambda, \quad |v(0,\lambda)| \le 2, \quad a(t,s;\lambda) \ge \frac{t-s}{\beta} \tag{C.12}$$

□

Lemma C.3. *Assume that $m([-L/2, L/2])$ is compactly contained in $[-1,1]$. Then the following assertions hold:*

(i) let $\|m\|_\infty \le \mathtt{a}$ and $\mathtt{a} \in (m_\beta, 1)$. Then $\|S_t(m)\|_\infty \le \mathtt{a}$ for any $t > 0$;

(ii) suppose $\|m_x\|_\infty < \infty$. *Then*

$$\|\frac{\partial}{\partial x} S_t(m)\|_\infty < \infty \qquad \forall t \geq 0 \tag{C.13}$$

and for L *sufficiently large*

$$\|S_t(m)\|_\infty \leq C_t \|S_t(m)\|_2^{2/3}, \quad C_t = \frac{3}{\sqrt{8}}\Big(e^{-t/\beta}\|m_x\|_\infty + \beta\|J'\|_\infty\Big)^{1/3} \tag{C.14}$$

(iii) let $u(t) = S_t(m) - \hat{m}_L$ *with* m *as in (i), and let*

$$R(t) := \beta^{-1}\Big\{\text{arctanh}\{S_t(m)\} - \text{arctanh}\{\hat{m}_L\} - \text{arctanh}'\{\hat{m}_L\}u(t)\Big\} \tag{C.15}$$

where $'$ *denotes the derivative. Then*

$$|R(t)| \leq C_2(u(t))^2, \qquad C_2 := \frac{1}{2\beta}\text{arctanh}''(\mathtt{a}). \tag{C.16}$$

Proof. Statement (i) follows from the maximum principle.
Proof of (ii). Calling $v(\cdot, t) := S_t(m)(\cdot)$, by differentiating (1.1) with respect to x we get

$$v_{xt} = -\frac{1}{\beta(1-v^2)}v_x + J_x^{\text{neum}} * v \tag{C.17}$$

The solution of (C.17) is

$$v_x(x,t) = \exp\Big\{\int_0^t a(x,s)ds\Big\}v_x(x,0) + \int_0^t \exp\Big\{\int_s^t a(x,s')ds'\Big\}b(x,s)ds \tag{C.18}$$

where $a = -[\beta(1-v^2)]^{-1}$, $b = J_x^{\text{neum}} * v$. Recalling that $|v| < 1$, we get

$$\|v_x(\cdot,t)\|_\infty \leq e^{-t/\beta}\|v_x\|_\infty + \beta\|J'\|_\infty \tag{C.19}$$

which proves (C.13). To prove (C.14) we use that from (C.19) it follows that we can apply Lemma C.1 to v with $\rho = 0$ and $c = e^{-t/\beta}|v_x|_\infty + \beta|J'|_\infty$. Indeed, for L sufficiently large we have that (C.2) is satisfied, since

$$\frac{\sqrt{2}\|S_t(m)\|_2}{L^{3/2}} \leq e^{-t/\beta}|v_x|_\infty + \beta\|J'\|_\infty,$$

where we have used the fact that $\|v\|_2 \leq 2L$.
Finally (C.16) follows from (i). $\square$

In the next Lemma we give an estimate of the difference between $S_t(m)$ and the solution u_K of the equation

$$u_t = f_L(u) + K, \qquad u(\cdot,0) = m, \tag{C.20}$$

where $K \in L^\infty([-L/2, L/2] \times (0, +\infty))$.

Lemma C.4. *There exists a constant $c^* > 0$ so that the following holds. Let $u(\cdot,t) = S_t(m)$ and $u_K(\cdot,t)$ solve* (C.20)*. Define*

$$A_{t,\delta} := \Big\{x \in \big(-\frac{L}{2},\frac{L}{2}\big) : \int_0^t |K(x,s)|^2\, ds < \delta\Big\}, \qquad t>0,\ \delta>0$$

Then

$$\sup_{x\in A_{t,\delta}} |u(x,t) - u_K(x,t)| \le c^* e^{\|J\|_\infty t}\Big(\sqrt{\delta} + |A^c_{t,\delta}|\Big)^{1/2}, \tag{C.21}$$

where $A^c_{t,\delta} := [-L/2, L/2] \setminus A^c_{t,\delta}$.

Proof. Set $\phi := u - u^0$, $w(s) := \sup_{x\in A_{t,\delta}} |\phi(x,s)|$. Pick $x \in A_{t,\delta}$. Then, using also that $|\phi| \le 2$ and that $\phi(x,t)[\operatorname{arctanh}(u(x,t)) - \operatorname{arctanh}(u_K(x,t))] \ge 0$, we have

$$\frac{1}{2}\frac{d}{ds}\left(\phi(x,s))^2\right) \le |\phi(x,s)|\, \|J\|_\infty \Big(w(s) + 2|A^c_{t,\delta}|\Big) + 2|K(x,s)|. \tag{C.22}$$

Let us integrate (C.22) over $s \in [0,t]$, and then take the supremum over $x \in A_{t,\delta}$. We get

$$w(t)^2 \le 2\|J\|_\infty \int_0^t w(s)^2 + 8t\|J\|_\infty |A^c_{t,\delta}| + 4t^{1/2}\delta$$

hence (C.21). □

We now give estimates on the evolution (C.20) in order to reduce to the case when the solution has support compactly inside the interval $[-1,1]$.
Let

$$m^+_\beta = \tanh\beta; \qquad 1 > m^+_\beta > m_\beta \tag{C.23}$$

and $v^0(t)$, $t \ge 0$, be the solution of

$$v^0_t = 1 - \frac{1}{\beta}\operatorname{arctanh}(v^0), \qquad v^0(0) = 1 \tag{C.24}$$

Then $v^0(t)$ decreases to m^+_β as $t \to \infty$, so that we can define τ_β as

$$v^0(\tau_\beta) = m^+_\beta + \frac{1}{2}(1 - m^+_\beta) \tag{C.25}$$

Define

$$A_t := \Big\{x : \int_0^t |K(x,s)|\, ds < \frac{1}{6}(1 - m^+_\beta)\Big\}, \quad B := \Big\{x : u(x,0) < m^+_\beta + \frac{1}{2}(1 - m^+_\beta)\Big\}$$

Lemma C.5. *If u_K solves* (C.20)*, then for any $t > 0$ and $x \in A_t \cap B$, and for any $t \ge \tau_\beta$ and $x \in A_t$, $u_K(x,s) < m^+_\beta + \frac{2}{3}(1 - m^+_\beta)$, $s \le t$.*

Proof. We start from the proof of the last statement. Call $K^+ = \max\{K, 0\}$ and $v(x,t)$ the solution of

$$v_t = 1 - \frac{1}{\beta}\,\text{arctanh}\,(v) + K^+, \qquad v(x,0) = 1 \tag{C.26}$$

We claim that $v(x,t)$ is a super-solution of (C.20). Indeed, since $v \le 1$ (as K is bounded),

$$\frac{dv}{dt} = 1 - \frac{1}{\beta}\,\text{arctanh}\,(v) + K^+ \ge J^{\text{neum}} * v - \frac{1}{\beta}\,\text{arctanh}\,(v) + K$$

Thus $u_K(x,t) \le v(x,t)$. In turns, $v(x,t) \le w(x,t) := w(x,t) = v^0(t) + \int_0^t K^+ ds$, $v^0(t)$ the solution of (C.24), because $w(x,t)$ is a super-solution of (C.26). Indeed, since $w \le v^0$,

$$\frac{dw}{dt} = 1 - \frac{1}{\beta}\,\text{arctanh}\,(v^0) + K^+ \ge 1 - \frac{1}{\beta}\,\text{arctanh}\,(w) + K^+$$

We have thus proved that $u_K(x,t) \le w(x,t)$, hence for $t \ge \tau_\beta$, and recalling that $v^0(t)$ is a decreasing function of t,

$$\begin{aligned} u_K(x,t) &\le v^0(\tau_\beta) + \int_0^t K^+\,ds < v^0(\tau_\beta) + \frac{1}{6}(1 - m_\beta^+) \\ &= m_\beta^+ + \frac{1}{2}(1 - m_\beta^+) + \frac{1}{6}(1 - m_\beta^+) = m_\beta^+ + \frac{2}{3}(1 - m_\beta^+) \end{aligned}$$

which concludes the proof of the last statement in the lemma. The first one is proved in the same way after defining $v(t)$ as the solution of (C.26) with $v(0) = m_\beta^+ + \frac{1}{2}(1 - m_\beta^+)$, and redefining $w(x,t) = v^0(t + \tau_\beta) + \int_0^t K^+ ds$. $\square$

References

[1] G. Bellettini, A. De Masi, E. Presutti: Tunnelling for nonlocal evolution equations *J. Nonlin. Math. Phys.*, **12** Suppl. 1 (2005), 50-63.

[2] G. Bellettini, A. De Masi, E. Presutti: Energy levels of critical points of a non local evolution equation, *J. Math. Phys.*, **46** (2005).

[3] G. Bellettini, A. De Masi, N. Dirr, E. Presutti: *Tunnelling in two dimensions*, preprint (2005).

[4] P. Buttà, A. De Masi, E. Rosatelli: Slow motion and metastability for a nolocal evolution equation. *J. Stat. Phys.* **112** (2003), 709–764.

[5] J. Carr, R. Pego: Metastable patterns in solutions of $u_t = \epsilon^2 u_{xx} - f(u)$. *Proc. Roy. Soc. Edinburgh Sect. A* **116** (1990), 133–160.

[6] J. Carr, R. Pego: Invariant manifolds for metastable patterns in $u_t = \epsilon^2 u_{xx} - f(u)$. *Comm. Pure Appl. Math.* **42** (1989), 523–576.

[7] Chen X.: Existence, uniqueness and asymptotic stability of traveling waves in non local evolution equations. *Adv. Differential Equations* **2** (1997), 125–160.

[8] Chmaj A., Ren X.: Homoclinic solutions of an integral equation: existence and stability. *J. Differential Equations* **155** (1999), 17–43.

[9] A. De Masi, T. Gobron, E. Presutti: Traveling fronts in non-local evolution equations. *Arch. Rational Mech. Anal.* **132** (1995), 143–205.

[10] A. De Masi, E. Olivieri, E. Presutti: Spectral properties of integral operators in problems of interface dynamics and metastability. *Markov Process. Related Fields* **4** (1998), 27–112.

[11] A. De Masi, E. Olivieri, E. Presutti: Critical droplet for a non local mean field equation. *Markov Process. Related Fields* **6** (2000), 439–471.

[12] A. De Masi, E. Orlandi, E. Presutti, L. Triolo: Glauber evolution with Kac potentials. I. Mesoscopic and macroscopic limits, interface dynamics. *Nonlinearity* **7** (1994), 1–67.

[13] A. De Masi, E. Orlandi, E. Presutti, L. Triolo: Stability of the interface in a model of phase separation. *Proc. Roy. Soc. Edinburgh Sect. A* **124** (1994), 1013–1022.

[14] A. De Masi, E. Orlandi, E. Presutti, L. Triolo: Uniqueness and global stability of the instanton in non local evolution equations. *Rend. Math. Appl.* (7) **14** (1994), 693–723.

[15] W. G. Faris, G. Jona Lasinio: Large fluctuations for nonlinear heat equation with noise. *J. Phys. A.: Math. Gen.* **15** (1982), 3025–3055.

[16] M. I. Freidlin, A.D. Wentzell: Random Perturbations of Dynamical Systems. *Springer-Verlag Vol. 260*, Berlin-New York 1984.

[17] G. Fusco, J.K. Hale: Slow motion manifolds, dormant instability and singular perturbations. *J. Dynamics Differential Equations* **1** (1989), 75–94.

[18] R. V. Kohn, M.G. Reznikoff, Y. Tonegawa: The sharp interface limit of the action functional for Allen-Cahn in one space dimension, preprint (2005).

[19] R. V. Kohn, Felix Otto, M.G. Reznikoff, E. Vanden-Eijnden: Action minimization and sharp interface limits for the stochastic Allen-Cahn Equation, preprint (2005).

[20] G. Manzi: The Wulff problem for diffuse interfaces, preprint (2005).

[21] E. Presutti: From Statistical Mechanics towards Continuum Mechanics. *Course given at Max-Plank Institute, Leipzig* (1999).