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Kai Chen, Sergio Albeverio, and Shao-Ming Fei

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Two-Setting Bell Inequalities for Many Qubits

Kai Chen[1], Sergio Albeverio[1], and Shao-Ming Fei[2,3]

[1]*Institut für Angewandte Mathematik,*

Universität Bonn, D-53115, Germany

[2]*Department of Mathematics, Capital Normal University, Beijing 100037, China*

[3]*Max Planck Institute for Mathematics in the Sciences, D-04103 Leipzig, Germany*

Abstract

We present a family of Bell inequalities involving *only two* measurement settings of each party for $N > 2$ qubits. Our inequalities include all the standard ones with less than N qubits and thus gives a natural generalization. It is shown that all the Greenberger-Horne-Zeilinger states violate the inequalities maximally, with an amount that grows exponentially as $2^{(N-2)/2}$. The inequalities are also violated by some states which do satisfy *all* the standard Bell inequalities. Remarkably, our results yield in an efficient and simple way a new implementation of non-locality tests of many qubits favorably within reach of the well-established technology of linear optics.

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Quantum states can exhibit one of the most striking features of quantum mechanics, producing remarkable correlations which are impossible within a local realistic description based on the notion of Einstein, Podolsky, and Rosen [1]. Constraints on statistical correlations imposed by local realism are termed as Bell inequalities after the pioneering work of Bell [2]. Derivations of new and stronger Bell inequalities are one of the most important and challenging subject in quantum information processing. It is an essentially conceptional problem to find out to what extent a state can rule out any possibly local realistic description, and thus certify its quantum origin and true non-locality. Violation of the inequalities is very closely related to the extraordinary power of realizing certain tasks in quantum information processing, outperforming its classical counterpart, such as building quantum protocols to decrease communication complexity [3] and making secure quantum communication [4].

Since Bell's work there have appeared many important generalizations, including Clauser-Horne-Shimony-Holt (CHSH) inequality [5] and Mermin-Ardehali-Belinskii-Klyshko (MABK) inequalities [6]. A set of multipartite Bell inequalities has been elegantly derived by Werner and Wolf and by Żukowski and Brukner (WWZB), by using two dichotomic observables per site [7, 8]. One usually refers to such inequalities as "standard" ones. Tailored for high dimensional systems, Bell inequalities are constructed in such a way that each measurement can bear more than two outcomes [9]. This further motivated successive experimental verification of non-locality [10, 11]. Moreover, the inequalities can lead to a detailed classification of multipartite entanglement [12], while the GHZ states are shown to be the only states that violate maximally the MABK inequalities [13]. We refer to [14] and references therein for recent nice reviews.

However, Scarani and Gisin, and Żukowski *et al.* find that there exists a family of pure $N > 2$ qubit states which *escape* violation of the "complete" set of Bell inequalities [15, 16] when considering a restricted setup (two dichotomic measurements and full correlation functions among all the parties). Note that such a restricted setup is sufficient to detect entanglement of any pure *bipartite* state and is known as Gisin's theorem [17]. A notable work in [18] shows that, for a fully-entangled N-partite pure state there exist some projective measurements for $N-2$ parties such that one can still observe a violation of the CHSH inequality for the remaining two particles. The insights can be further proved to lead to a violation of two-setting Bell inequalities and thus implies Gisin's theorem for any number of qubits [19]. However, such a construction substantially relies on a localized entanglement only between two parties with the help of all the other parties. Furthermore, an amount with exponentially increasing violation (which is a key character mainly coming from true multipartite entanglement) is totally lost, as the maximal violation is less or equal to $\sqrt{2}$. The family escaping violation is a subset of the generalized Greenberger-Horne-Zeilinger (GHZ) states given by $|\psi\rangle = \cos\alpha|0,...,0\rangle + \sin\alpha|1,...,1\rangle$ with $0 \leq \alpha \leq \pi/4$. It describes GHZ states [20] for $\alpha = \pi/4$. For $\sin 2\alpha \leq 1/\sqrt{2^{N-1}}$ and N *odd*, these states are proved to satisfy all the standard inequalities [16]. This is rather surprising as they are a generalization of the GHZ states which maximally violate the MABK inequalities.

There has been also notable progress in deriving stronger Bell inequalities by employing more measurement settings [21, 22], which can be violated by a larger class of states including the generalized GHZ states. Chen *et al.* recently obtained a Bell inequality for 3 qubits involving two dichotomic observables per site, that can be seen numerically to be violated by any pure entangled state [23]. Can one find any Bell inequalities satisfying the conditions that: i) they recover the standard Bell inequalities as a special case; ii) they provide an exponentially increasing violation for GHZ states; iii) they essentially involve *only two* measurement settings per observer; iv) they yield violation for the generalized GHZ states in the whole region of α for *any number* of qubits? This is highly desirable, as such Bell inequalities will lead to a much easier and more efficient way to test non-locality, and contribute to the development of novel *multi-party* quantum protocols and cryptographic schemes by exploiting *much less entangled resources and experimental efforts.*

In this Letter, we present the first family of two-settings Bell inequalities with all these advantages. We then show that it leads to a natural generalization of the standard Bell inequalities. The GHZ states are demonstrated to violate the inequalities maximally, with an amount that grows exponentially as $2^{(N-2)/2}$. Finally, we provide practical settings to test experimentally the non-locality of any generalized GHZ entangled states.

The scenario is as follows. We consider N parties and allow each of them to choose independently between two dichotomic observables A_j, A'_j for the jth observer, specified by some local parameters, each measurement having two possible outcomes -1 and 1. We define

$$\mathcal{B} = \mathcal{B}_{N-1} \otimes \frac{1}{2}(A_N + A'_N) + \mathbb{1}_{N-1} \otimes \frac{1}{2}(A_N - A'_N), \quad (1)$$

$$\mathcal{B}_{N-1} = \frac{1}{2^{N-1}} \sum_{s_1,...,s_{N-1}=-1,1} S(s_1,...,s_{N-1}) \sum_{k_1,...,k_{N-1}=1,2} s_1^{k_1-1}...s_{N-1}^{k_{N-1}-1} \otimes_{j=1}^{N-1} O_j(k_j), \quad (2)$$

where $\mathcal{B}_{N-1}$ is the quantum mechanical Bell operator of WWZB inequalities [7, 8], $S(s_1,...,s_{N-1})$ is an arbitrary function taking only values ± 1. Here $O_j(1) = A_j$ and $O_j(2) = A'_j$ with $k_j = 1, 2$. The notation $\mathbb{1}_{N-1}$ represents an identity matrix of dimension 2^{N-1}, with the meaning of "not measuring" the $N-1$ parties [24].

From the classical view of local realism, the values of A_j, A'_j are predetermined by a local hidden variable λ before measurement, and independent of any measurements, orientations or actions performed on other parties at spacelike separation. The correlation among all N observations is then a statistical average over many runs of the experiment

$$E_{\text{LHV}}(k_1,...,k_N) = \int d\lambda\rho(\lambda) \prod_{j=1}^{N} O_j(k_j, \lambda), \quad (3)$$

where $\rho(\lambda)$ is a statistical distribution of λ satisfying $\rho(\lambda) \geq 0$ and $\int d\lambda\rho(\lambda) = 1$. Noting that local realism requires that $|\langle\mathcal{B}_{N-1}\rangle_{\text{LHV}}| \leq 1$ shown in [8], we obtain

$$|\langle\mathcal{B}\rangle_{\text{LHV}}| = \frac{1}{2} |\langle\mathcal{B}_{N-1}(A_N + A'_N) + (A_N - A'_N)\rangle_{\text{LHV}}| \leq 1. \quad (4)$$

In fact $A_N = \pm 1$ and $A'_N = \pm 1$ for the observer N, and one has either $|A_N + A'_N| = 2$ and $|A_N - A'_N| = 0$, or vice versa. This implies that (4) holds. For a given function of $S(s_1,...,s_{N-1})$, one can generate the full set of members of a family by simply permuting different locations, or the measurement orientations A_i and A'_i.

Let us now see the quantum mechanical representation of the Bell inequalities given by Eq. (4) tailored for qubits. Since any quantum observable A_i that describes a measurement with ± 1 as possible outcomes can be represented by $\vec{a}_i \cdot \vec{\sigma} \equiv \sigma_{a_i}$, with $\vec{a}_i$ a unit vector and $\vec{\sigma}$ the Pauli matrices ($A'_i = \vec{a'}_i \cdot \vec{\sigma} = \sigma_{a'_i}$ respectively), the Bell operator of Eq. (1) can be parameterized by all these σ_a. A violation by the quantum state with density matrix ρ reads $|\langle\mathcal{B}\rangle| = |Tr(\rho\mathcal{B})| > 1$. Moreover, every unit vector $\vec{a}_i$ can be parameterized completely by its polar angle θ_i and azimuthal angle ϕ_i in the Bloch sphere as $\vec{a}_i = (\sin\theta_i \cos\phi_i, \sin\theta_i \sin\phi_i, \cos\theta_i)$.

For the convenience of later use, we first derive an alternative form of MABK inequalities, different from the usual one through a recursive definition [6]. It is shown in [8] that one can recover the MABK inequalities by taking $S(s_1,...,s_N) = \sqrt{2}\cos[(s_1+...+s_N-N+1)\pi/4]$ in Eq. (2). Thus Eq. (2) is symmetric with respect to s_i, and one concludes that the coefficients c_m for the correlation function $\otimes_{j=1}^{N} O_j(k_j)$ will be the same if the number of items for $k_i = 2$ is fixed to be m. Without loss of generality, supposing $k_i = 2$ only for $i = 1,\ldots,m$, we have

$$\begin{aligned} c_m &= 2^{\frac{1}{2}-N} \sum_{\substack{s_1,\ldots,s_N\\=-1,1}} \cos[(s_1+...+s_N-N+1)\frac{\pi}{4}]s_1\cdots s_m \\ &= 2^{\frac{1}{2}-N}\mathrm{Re}\Big(e^{i(2m-N+1)\pi/4}(e^{i\pi/4}+e^{-i\pi/4})^N\Big) \\ &= 2^{(1-N)/2}\cos[(2m-N+1)\pi/4], \end{aligned} \tag{5}$$

where we have used the fact that $s_i = \exp[i(1-s_i)\pi/2]$ holds for any $s_i = \pm 1$.

With the above notation and preparation, we can state several main results.

Theorem 1: The generalized Bell inequalities Eq. (4) include the standard Bell inequalities as a special case.

Proof: If one takes $A_N = A'_N$, the inequalities reduce to $|\langle \mathcal{B}_{N-1}\rangle_{\mathrm{LHV}}| \le 1$, which are precisely the WWZB inequalities for $N-1$ parties. Furthermore, our inequalities inherit the property that all the standard inequalities for less than $N-1$ parties will be recovered, as it is valid for the standard inequalities [7]. ■

One may wonder whether all the GHZ states violate the inequalities maximally, similarly in the case of MABK inequalities. We provide the following answer.

Theorem 2: All the GHZ states violate the Bell inequality Eq. (4) maximally.

Proof: Squaring Eq. (1) gives

$$\begin{aligned} \mathcal{B}^2 &= \mathcal{B}_{N-1}^2 \otimes \frac{1}{2}(1+\vec{a}_N\cdot\vec{a'}_N)\mathbb{1}_1 \\ &\quad +\mathbb{1}_{N-1}\otimes\frac{1}{2}(1-\vec{a}_N\cdot\vec{a'}_N)\mathbb{1}_1 \end{aligned} \tag{6}$$

Noting that $2^{N-2}\mathbb{1}_{N-1} \ge \mathcal{B}_{N-1}^2$ as proved in [7], one has $2^{N-2}\mathbb{1}_N \ge \mathcal{B}^2$. Here by $A \ge B$ we mean that $A-B$ is semi-positive definite. Thus a possible maximal violation is $2^{(N-2)/2}$ due to the observation that the maximally possible eigenvalue for $\mathcal{B}$ is $2^{(N-2)/2}$. This can indeed be saturated by the GHZ states as seen from Eq. (10) with $\alpha = \pi/4$, as will be shown in an example later. All the other GHZ states up to a local unitary transformation will lead to the same violation, since the case corresponds to a local unitary transformation of local observables. ■

As is well known the GHZ states are major resources for many quantum information tasks, while the generalized GHZ state are crucial for distributed quantum computing [25]. Moreover, in any real experiments for preparing the GHZ states, one usually gets the generalized GHZ states due to unavoidable imperfections in the devices. In the following we will find practical experimental settings to detect this distinctive class of states. The correlation function is of the form

$$\langle \otimes_{j=1}^{N} O_j(k_j) \rangle = (\cos^2\alpha + (-1)^N \sin^2\alpha) \prod_{i=1}^{N} \cos\theta_i + \sin 2\alpha \prod_{i=1}^{N} \sin\theta_i \cos(\sum_{j=1}^{N} \phi_j), \tag{7}$$

as shown in [21]. Let us take $\theta_i^1 = \theta_i^2 = \pi/2$, $\phi_i^1 = 0$, and $\phi_i^2 = \pi/2$ for all $i = 1, \ldots, N-1$, and set $\phi_N^1 = \phi_N^2 = \phi_N$ and $\theta_N^1 = \pi - \theta_N^2 = \theta_N$. These special choices of the angles correspond to measuring the first $N-1$ parties along σ_x (corresponding to A_i) and σ_y basis (A_i'). Taking $\mathcal{B}_{N-1}$ as the MABK polynomial [6] and using Eqs. (5) and (7), one arrives at

$$\begin{aligned}\left\langle \mathcal{B}_{N-1} \otimes \frac{1}{2}(A_N + A_N') \right\rangle &= 2^{(2-N)/2} \sin 2\alpha \sin\theta_N \\ &\times \sum_{m=0}^{N-1} \binom{N-1}{m} \cos[(2m-N+2)\frac{\pi}{4}] \cos[\frac{m\pi}{2} + \phi_N] \\ &= 2^{(N-2)/2} \sin 2\alpha \sin\theta_N \sin[\frac{N\pi}{4} + \phi_N] \\ &= 2^{(N-2)/2} \sin 2\alpha \sin\theta_N, \end{aligned} \tag{8}$$

which can be easily derived by using exponential representations of trigonometric functions, and noting that the sum corresponds to the binomial expansion of a certain function. Here m is the number of the parties that are measured along the σ_y basis. In the last formula of Eq. (8), we have further set $\phi_N = (2-N)\pi/4$.

It is easy to see that $\langle \mathbb{1}_{N-1} \otimes (A_N - A_N')/2 \rangle = \cos 2\alpha \cos\theta_N$, one thus has

$$\langle \mathcal{B} \rangle = 2^{(N-2)/2} \sin 2\alpha \sin\theta_N + \cos 2\alpha \cos\theta_N. \tag{9}$$

Since $\max(x\sin\theta + y\cos\theta) = \sqrt{x^2+y^2}$, we get

$$\langle \mathcal{B} \rangle = (2^{N-2} \sin^2 2\alpha + \cos^2 2\alpha)^{1/2} > 1, \quad \text{for } \alpha \neq k\pi/2 \ \ (k \in \text{integer}, N \geq 3), \tag{10}$$

where we have taken $\theta_N = \tan^{-1}[2^{(N-2)/2}\tan 2\alpha]$ if $0 \leq \alpha \leq \pi/4$, and $\theta_N = \tan^{-1}[2^{(N-2)/2}\tan 2\alpha]+\pi$ if $\pi/4 \leq \alpha \leq \pi/2$ in Eq. (9). Therefore, we see that the whole class of generalized GHZ states can violate the Bell inequalities Eq. (4) except for the product states ($\alpha = 0$ or $\pi/2$).

By suitable choices of the two observables in each site, one thus can reveal hidden nonlocality for any generalized GHZ state in a very subtle way through our inequalities. Note that the result leads to *the same violation factor* as the one obtained by the *many settings* approach [22], where however the required experimental effort is *exponentially larger* than ours. In addition, for a given α the violation will increase exponentially, with the maximal violation achieved by GHZ states.

Let us highlight the significance of our inequalities. First, the implementations involve only two measurement settings per site, and should be immediately feasible due to rapidly developing technology for generation and manipulation of multipartite entangled states in linear optical, atomic or trapped ions systems [10, 26]. Second, the standard Bell inequalities are recovered as a special case of our inequalities as shown in Theorem 1. Third, for N even our inequalities demand asymptotically only half of the experimental efforts. In such a case the MABK inequalities are combinations of all the correlation functions with 2^N terms [6]. Our inequalities require only $2^{N-1}+2$ terms, as seen from Eq. (1) ($\mathcal{B}_{N-1}$ is a combination of 2^{N-2} correlation functions in this case). Fourth, they fill the well known gap for the states which the standard Bell inequalities fail to detect, and keep the exponentially increasing violation in the mean time.

From Eq. (4), one can see that our inequalities not only include the full correlations, but also account for less than N particle contributions. This novel construction goes beyond the restricted set classified in [7, 8], and exhibits superior power by admitting a wider class of LHV description to include all possible correlations, as done before for 3 qubits in [23], and for 4 qubit cluster states in [27] with a linear optical demonstration in [11].

We remark that our inequalities apply as well to arbitrary dimensional multipartite systems. Moreover, they can be violated by a $|W\rangle$ state of the form $1/\sqrt{N}(|100...0\rangle + |010...0\rangle + ... + |000...1\rangle)$, and by cluster states that are effective resource for one-way universal quantum computation [28]. For example, taking $\mathcal{B}_{N-1}$ as the MABK polynomial in Eq. (1), the $|W\rangle$ state can be violated with maximal violation factors of $1.202, 1.316, 1.382$ for $N = 3, 4, 5$, respectively, while for the cluster states $|\psi_3\rangle = 1/\sqrt{2}(|000\rangle + |111\rangle)$ (GHZ state), $|\psi_4\rangle = 1/2(|0000\rangle + |0011\rangle + |1100\rangle - |1111\rangle)$ [11, 27] a factor of $\sqrt{2}$ for both. Considering a practical noise admixture to the N-particle GHZ state $|GHZ\rangle$ of the form $\rho = (1-V)\rho_{\text{noise}} + V|GHZ\rangle\langle GHZ|$, with $\rho_{\text{noise}} = \mathbb{1}/2^N$, one has a threshold visibility of $V_{\text{thr}} = 2^{(2-N)/2}$ above which a local realism is impossible. This suggests that our inequalities are rather efficient, as it is only a slightly bigger threshold visibility than that required for the MABK inequalities with $V_{\text{thr}} = 2^{(1-N)/2}$. In addition, they share the same behavior with $V_{\text{thr}} \sim 2^{-N/2}$ that is exponentially decreasing to 0 in the asymptotics $N \to \infty$. For $N \geq 4$, they are also significantly better than the one derived from [18] which requires the very strict condition $V_{\text{thr}} \geq 0.7071$ for any N.

Summarizing our results, we have proposed a novel family of Bell inequalities for many qubits. They are entirely compatible with the simplicity requirements of current linear optical experiments for non-locality tests, i.e., involving only two measurement settings per location. The inequalities recover the standard Bell's inequalities as a special case and can be maximally violated by GHZ states. In addition, practical experimental settings are derived for revealing violation of local realism for some class of states which the standard Bell's inequalities fail to detect. This permits to reduce significantly experimental efforts comparing with those which utilize many settings, and at the same time, can be achieved without compromise of an exponentially increasing amount of violation. Complementary to the standard inequalities and a number of existing results, our inequalities offer a new prospective tool for much *stronger* non-locality tests and a *more economic* way of performing experiments.

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