

Max-Planck-Institut  
für Mathematik  
in den Naturwissenschaften  
Leipzig

Remarks on the variational nature of the heat  
equation and of mean curvature flow

by

*Giovanni Bellettini, and Luca Mugnai*

Preprint no.: 34

2006





# Remarks on the variational nature of the heat equation and of mean curvature flow

G. Bellettini\*      L. Mugnai†

## Abstract

We show that the classical solution of the heat equation can be seen as the minimizer of a suitable functional defined in space-time. Using similar ideas, we introduce a functional  $\mathcal{F}$  on the class of space-time tracks of moving hypersurfaces, and we study suitable minimization problems related  $\mathcal{F}$ . We show some connections between minimizers of  $\mathcal{F}$  and mean curvature flow.

*Key words: heat equation, space-time energy minimizers, mean curvature flow.*

*AMS (MOS) subject classification: 49J45, 53C44, 35K10*

## 1 Introduction

In this paper we show that solutions to the classical heat equation can be viewed as minimizers of a suitable space-time functional  $F$  of second order. The argument leading to the expression of  $F$  (defined in (2.1), or more generally in (2.9)) is based on a elementary and well known integration by parts formula, see Section 2.1, which is used together with the least squares method. The functional  $F$  consists of two parts, which equally contribute to the energy of the minimizers, as it happens in the theory of curves of maximal slope in the sense of De Giorgi, see [14], [3]. Other approaches in the direction of looking at solutions of gradient flows of convex functionals as minimizers of suitable energies were proposed in [12] and developed further in [5] and in [21], [20].

More interestingly, an idea similar to the one leading to the expression of  $F$  can be applied to the more difficult situation of geometric evolutions, for instance concerning the mean curvature flow of an embedded hypersurface in  $\mathbb{R}^n$ . In this case it turns out that, assuming enough regularity, the functional  $\mathcal{F}$  reads as

$$\mathcal{F}(\Sigma) = \int_{\Sigma} \frac{(\nu_{n+1})^2}{\sqrt{1 - (\nu_{n+1})^2}} d\mathcal{H}^n + \int_0^{+\infty} \int_{\Sigma(t)} (H_{\Sigma(t)})^2 d\mathcal{H}^{n-1} dt, \quad (1.1)$$

where

$$\Sigma = \cup_{t \geq 0} (\Sigma(t) \times \{t\}) \subset \mathbb{R}^{n+1}$$

---

\*Dipartimento di Matematica, Università di Roma “Tor Vergata”, via della Ricerca Scientifica 00133 Roma, Italy, and INFN Laboratori Nazionali di Frascati, Frascati, Italy. E-mail: belletti@mat.uniroma2.it

†Max Planck Institute for Mathematics in the Sciences, Inselstr. 22, D-04103 Leipzig, Germany. E-mail: mugnai@mis.mpg.de

is the space-time track of the evolution,  $\nu = (\nu_1, \dots, \nu_{n+1}) \in \mathbb{R}^{n+1}$  is a unit normal vector field to  $\Sigma$ , and  $H_{\Sigma(t)}$  is the mean curvature of the projection  $\Sigma(t)$  on  $\mathbb{R}^n$  of the  $t$ -time slice of  $\Sigma$ . The energy  $\mathcal{F}$  is therefore a functional on a suitable class of hypersurfaces (in space-time), and consists of two competing terms: one of first order depending on the unit normal vector field  $\nu$  to  $\Sigma$  (singular at those points of  $\Sigma$  where  $\nu_{n+1} = 1$ ) and the other one (of second order) which is the Willmore functional of the time-slices of  $\Sigma$  integrated further in time. The important role of the Willmore functional in motion by mean curvature is well known, see for instance [29], [15]; such a role is displayed also in our variational approach. Observe that being  $\mathcal{F}$  (and also  $F$ ) of second order, the variational analysis (i.e., the direct method and the regularity theory) is much more difficult than the one required for functionals depending only on gradients such as those leading to elliptic PDEs. In particular, the study of the lower semicontinuity of  $\mathcal{F}$  with respect to suitable topologies deserves further investigation. In this respect we remark that the possible formation of singularities in mean curvature flow is one of the difficulties that one has to face in setting the functional (1.1) in a weak form. Another general remark concerning both the heat equation and the mean curvature flow is that, because of their structure, minimizing the functionals  $F$  and  $\mathcal{F}$  could lead to backward solutions (or to combinations of forward and backward solutions). This problem is avoided for the heat equation using the convexity of  $F$  and some properties of its domain, while for the geometric functional  $\mathcal{F}$  the situation is much less clear.

The plan of the paper is the following. In Section 2 we derive the functional  $F$ , and we define what we mean by a variational solution to the heat equation starting from an initial datum  $g$ . For simplicity of presentation, we distinguish the case of the whole space and the case of a bounded domain with Dirichlet boundary conditions. The existence and uniqueness of a variational solution to the heat equation is proved in Theorem 2.5. Also it is not difficult to check that a classical solution is a variational solution, see Remark 2.4. In Section 3 we derive the functional  $\mathcal{F}$  in (1.1), and we define what we mean by a strong variational mean curvature flow starting from an initial solid set  $E$  with smooth compact boundary (see Definitions 3.6). Definition 3.6 requires some care; indeed, without assuming condition (ii) it is possible (combining forward and backward curvature flows) to construct an initial smooth set  $E \subset \mathbb{R}^2 \times \{0\}$  and a smooth space-time track  $\Sigma \subset \mathbb{R}^2 \times (0, +\infty)$  with  $\partial\Sigma = \partial E$  in such a way that  $\mathcal{F}(\Sigma) < \mathcal{F}(\Sigma_c)$ , where  $\Sigma_c$  denotes the space-time track of the classical curvature flow of  $\partial E$  (Example 3.5). Similar counterexamples cannot be constructed in the previous case of the heat equation, due to the regularity properties enjoyed by functions belonging to the competitor space  $X(\Omega)$ . In Proposition 3.10 we show that classical mean curvature flow is a stationary point of  $\mathcal{F}$ .

The expression of  $\mathcal{F}$  on space-time tracks which are graphs of a function  $f$  on the initial set  $E$  is

$$\mathcal{F}(\Sigma) = \int_E \frac{1}{|\nabla f|} dx + \int_E \left( \operatorname{div} \frac{\nabla f}{|\nabla f|} \right)^2 |\nabla f| dx =: \mathcal{F}(f), \quad (1.2)$$

see Remark 3.3 (this situation happens, as it is well known [18], in mean curvature flow of convex sets). In Section 4 we study our minimization problem when the initial set  $E$  is a convex set in the plane and the minimization is restricted to graphs of concave functions: under these assumptions, we show that the minimum problem has a solution (Theorem 4.1). The proof inspects lower semicontinuity and coercivity properties of the functional (1.2). The minimum problem as stated in Definition 3.6 has not, in general, a solution. A possible

weak form of the minimum problem (3.12) is proposed in Section 5: in Definition 5.1 we introduce the weak variational mean curvature flow starting from  $E$ , and the corresponding weak minimum problem is given in (5.1). In Proposition 3.9 we show that if a weak variational solution is sufficiently smooth, then there is a sort of equipartition of energy in (1.1) in an integral sense. Finally, in Section 6 we point out some preliminary relations between the asymptotic properties (as  $\epsilon \rightarrow 0$ ) of the sequence of second order functionals

$$\mathcal{F}_\epsilon(v) := \int_0^{+\infty} \int_{\mathbb{R}^n} \left[ \epsilon(v_t)^2 + \frac{1}{\epsilon} \left( -\epsilon \Delta v + \frac{1}{\epsilon} W'(v) \right)^2 \right] dx dt \quad (1.3)$$

and the functional  $\mathcal{F}$ , see the papers [27], [26] for related results. This result relates parabolic problems with  $\Gamma$ -convergence (see also [32]), and is a further motivation for an independent study of the functional  $\mathcal{F}$  in (1.1).

We conclude this introduction by mentioning that the approach of the present paper may have some applications to large deviation problems in statistical mechanics, see for instance [8].

## 2 The heat equation: the functional $F$

If  $\Omega$  is an open set, and  $w \in L^2(\Omega \times (0, +\infty))$  we denote by  $w(t) = w(t)(\cdot)$  the map  $w(\cdot, t)$ . In order to illustrate the method in the simplest case, in the following section we show how to find the functional  $F$  in the ideal situation of the heat equation in the whole of  $\mathbb{R}^n$ .

### 2.1 The heat equation in $\mathbb{R}^n$

Let  $I$  be the usual Dirichlet functional

$$I(f) := \frac{1}{2} \int_{\mathbb{R}^n} |\nabla f|^2 dx, \quad f \in H^1(\mathbb{R}^n).$$

Set  $X(\mathbb{R}^n) := L^2(0, +\infty; H^2(\mathbb{R}^n)) \cap H^1(0, +\infty; L^2(\mathbb{R}^n))$  and define the functional  $F : X(\mathbb{R}^n) \rightarrow [0, +\infty]$  as

$$F(v) = \int_0^{+\infty} \int_{\mathbb{R}^n} [(v_t)^2 + (\Delta v)^2] dx dt, \quad v \in X(\mathbb{R}^n). \quad (2.1)$$

Let  $g : \mathbb{R}^n \rightarrow \mathbb{R}$  be a given function with  $I(g) < +\infty$ , and denote by  $u^c : \mathbb{R}^n \times [0, +\infty) \rightarrow \mathbb{R}$  the solution to the Cauchy problem

$$\begin{cases} u_t = \Delta u & \text{in } \mathbb{R}^n \times (0, +\infty), \\ u(0) = g & \text{in } \mathbb{R}^n \times \{t = 0\}, \\ u \in X(\mathbb{R}^n). \end{cases} \quad (2.2)$$

In view of well known regularity and decay properties of  $u := u^c$  we can compute

$$\begin{aligned}
0 &= \int_0^{+\infty} \int_{\mathbb{R}^n} (u_t - \Delta u)^2 dxdt = \int_0^{+\infty} \int_{\mathbb{R}^n} [(u_t)^2 + (\Delta u)^2 - 2u_t \Delta u] dxdt \\
&= F(u) - 2 \int_0^{+\infty} \int_{\mathbb{R}^n} u_t \Delta u dxdt = F(u) + 2 \int_0^{+\infty} \int_{\mathbb{R}^n} \nabla u_t \cdot \nabla u dxdt \\
&= F(u) + \int_{\mathbb{R}^n} \int_0^{+\infty} \frac{d}{dt} |\nabla u|^2 dt dx = F(u^c) - 2I(g).
\end{aligned}$$

Forgetting now that  $u^c$  is the solution to the heat equation, the above computation leads naturally to the following definition.

**Definition 2.1.** *We say that a function  $u^v \in X(\mathbb{R}^n)$  is a variational solution to (2.2) if  $u^v$  solves the minimum problem*

$$\inf \left\{ F(v) : v \in X(\mathbb{R}^n), v(0) = g \right\}. \quad (2.3)$$

The general strategy behind Definition 2.1 is the following: try to prove that (2.3) has a solution  $u^v$ ; in positive case, prove that no other minimizer exists; *then* show that

$$F(u^v) = 2I(g). \quad (2.4)$$

Finally, prove that  $u^v = u^c$ . This strategy will be partially pursued, for the Dirichlet problem in a bounded domain  $\Omega$  (see Remark 2.4 and Theorem 2.5 below) and can be adapted to the case when  $\Omega = \mathbb{R}^n$ . Here we only mention that, assuming the existence and sufficient regularity of  $u^v$ , the equality  $u^v = u^c$  can be seen as follows. We denote by  $u_t^v$  the function  $(u^v)_t$  and  $\Delta u^v$  the function  $\Delta(u^v)$ . Since  $u^v$  is a minimum point of  $F$  it is a weak solution of the Euler-Lagrange equation of  $F$ , i.e.,

$$\int_0^{+\infty} \int_{\mathbb{R}^n} u_t^v \varphi_t + \Delta u^v \Delta \varphi dxdt = 0 \quad \forall \varphi \in C_c^\infty(\mathbb{R}^{n+1}). \quad (2.5)$$

Denoting by  $\widehat{u^v}$  the Fourier transform of  $u^v$  with respect to  $x$ , from (2.5) we get

$$\int_0^{+\infty} \int_{\mathbb{R}^n} \widehat{u_t^v} \widehat{\varphi_t} + |\xi|^4 \widehat{u^v} \widehat{\varphi} d\xi dt = 0 \quad \forall \varphi \in C_c^\infty(\mathbb{R}^{n+1}). \quad (2.6)$$

Under sufficient regularity assumptions on  $u^v$  we have  $\widehat{u_t^v} = (\widehat{u^v})_t$ , therefore  $\widehat{u^v}$  is a weak (hence a strong) solution of the ordinary differential equation

$$\frac{d^2}{dt^2} \widehat{u^v} = |\xi|^4 \widehat{u^v}, \quad \widehat{u^v}(0) = \widehat{g}. \quad (2.7)$$

It follows that  $\widehat{u^v}(\xi, t) = \exp(-|\xi|^2 t) h(\xi) + \exp(|\xi|^2 t) (\widehat{g}(\xi) - h(\xi))$ , where  $h \in L^2(\mathbb{R}^n)$  is not explicit, as (2.7) is under-determined. For  $t \in (0, +\infty)$  we have

$$\begin{aligned}
\int_{\mathbb{R}^n} |\nabla u^v(t)|^2 dx &= \int_{\mathbb{R}^n} |\xi|^2 \widehat{u^v} \overline{\widehat{u^v}} d\xi \\
&= \int_{\mathbb{R}^n} |\xi|^2 (e^{-2|\xi|^2 t} |h|^2 + e^{2|\xi|^2 t} |\widehat{g} - h|^2 + h \overline{(\widehat{g} - h)} + \overline{h} (\widehat{g} - h)) d\xi.
\end{aligned} \quad (2.8)$$

If  $(\widehat{g} - h)$  is not zero, from (2.8) we deduce  $\lim_{t \rightarrow +\infty} \int_{\mathbb{R}^n} |\nabla u^v(t)|^2 dx = +\infty$ , which contradicts  $\int_{\mathbb{R}^n} |\nabla u^v(t)|^2 dx \leq \int_{\mathbb{R}^n} |\nabla g|^2 dx + \frac{1}{2} F(u^v)$ . Therefore  $\widehat{u^v}(\xi, t) = \exp(-|\xi|^2 t) \widehat{g}(\xi)$  and eventually  $u^v = u^c$ .

## 2.2 The heat equation in $\Omega$ with Dirichlet boundary conditions

Several different methods are known to lead to the solution of the heat equation on a given domain (with some kind of boundary and initial conditions), see for instance [30], [11], [16]. It is not the aim of the present paper to make another theory of existence, uniqueness and regularity of parabolic PDEs. Nevertheless, in this section we briefly point out some facts related to Definition 2.1.

Let  $\Omega$  be a bounded open set of class  $C^2$  and let  $g \in H_0^1(\Omega)$  be a given function. Set

$$I(g) = I(g, \Omega) := \frac{1}{2} \int_{\Omega} |\nabla g|^2 dx.$$

Define

$$X(\Omega) := L^2(0, +\infty; H_0^1(\Omega) \cap H^2(\Omega)) \cap H^1(0, +\infty; L^2(\Omega)),$$

and

$$F(v) = F(v, \Omega \times (0, +\infty)) := \int_0^{+\infty} \int_{\Omega} [(v_t)^2 + (\Delta v)^2] dx dt, \quad v \in X(\Omega). \quad (2.9)$$

**Definition 2.2.** We say that a function  $u^v$  is a variational solution to

$$\begin{cases} u_t = \Delta u & \text{in } \Omega \times (0, +\infty), \\ u(0) = g \in H_0^1(\Omega), & \text{in } \Omega \times \{t = 0\}, \\ u \in X(\Omega), \end{cases} \quad (2.10)$$

if  $u^v$  solves the minimum problem

$$\inf \left\{ F(v) : v \in X(\Omega), v(0) = g \right\}. \quad (2.11)$$

Let us recall some crucial properties shared by functions belonging to  $X(\Omega)$ .

**Remark 2.3.** We recall (see for instance [30]) that

- (a) if  $v \in X(\Omega)$  then  $v \in C^0([0, +\infty); H_0^1(\Omega))$ , the map  $t \in (0, +\infty) \rightarrow \int_{\Omega} |\nabla v(t)|^2 dx$  has first derivative in  $L^1(0, +\infty)$ , and

$$-\frac{d}{dt} \frac{1}{2} \int_{\Omega} |\nabla v(t)|^2 dx = \int_{\Omega} v_t(t) \Delta v(t) dx \quad \text{for a.e. } t \in (0, +\infty). \quad (2.12)$$

In particular  $\lim_{t \rightarrow +\infty} \int_{\Omega} |\nabla v(t)|^2 dx = 0$ ;

- (b) the set of all  $v \in X(\Omega)$  such that  $v(0) = g$  is non empty, and if  $v$  is one of these functions then  $\lim_{t \rightarrow 0^+} \int_{\Omega} |\nabla v(t)|^2 dx = \int_{\Omega} |\nabla g|^2 dx$ ;
- (c) on the space  $H_0^1(\Omega) \cap H^2(\Omega)$ , the  $H^2(\Omega)$ -norm of  $f$  is equivalent to  $(\int_{\Omega} (\Delta f)^2 dx)^{1/2}$  (see for instance [16], 6.3.2).

As in the case of the whole space, the classical solution  $u^c$  to (2.10) satisfies  $u^c \in X(\Omega)$  and  $F(u^c) = 2I(g)$ .

**Remark 2.4.**

- (a) Since  $\alpha^2 + \beta^2 \geq 2\alpha\beta$  for  $\alpha, \beta \in \mathbb{R}$ , it follows from (a) and (b) of Remark 2.3 that

$$F(v) \geq 2I(g) \quad \forall v \in X(\Omega) \text{ with } v(0) = g.$$

Hence  $\inf \{F(v) : v \in X(\Omega), v(0) = g\} \geq 2I(g) = F(u^c)$ , and therefore  $u^c$  is a variational solution to (2.2).

- (b) The functional  $F$  is convex on the linear space  $X(\Omega)$ . Hence, if  $u$  and  $v$  are two different solutions to (2.11), it follows that  $u_t = v_t$  (and  $\Delta u = \Delta v$ ) almost everywhere on  $\Omega \times (0, +\infty)$ ; since  $u(0) = v(0)$  it follows that  $u = v$ . This fact, together with (a), shows that  $u^v = u^c$  is the unique solution to (2.11). In particular, it is not possible to have minimizers of  $F$  solving the backward heat equation in some space-time region.
- (c) Simple modifications of the previous arguments allow to find energy functionals for a large class of parabolic equations in divergence form. This approach is strongly related to the theory of the curves of maximal slope in the sense of De Giorgi [14], [3]. Moreover also in our case the functional  $F$  consists of two parts, which equally contribute to the energy of the minimizers.
- (d) Assume for simplicity  $n = 1$ ; if the solution of (2.11) is known to be smooth enough we can prove that  $u^v$  is a classical solution of the heat equation only using its minimality as follows. Let  $I \subset\subset \Omega$  be an interval and denote by  $u_I^v$  the  $I$ -periodic extension of the restriction of  $u^v$  to  $I$ . Let  $\{C_k(t)\}_{k \in \mathbb{Z}}$  be the coefficients of the Fourier series of  $u_I^v$ .  
By the smoothness assumptions made on  $u_I^v$  and the fact that (being a minimizer)  $u^v$  is a solution of the Euler Lagrange equation of  $F$  it follows

$$C_k''(t) = C_k(t)|k|^4 \quad \forall k \in \mathbb{Z}, \quad C_k(0) = c_k^{g_I},$$

where  $\{c_k^{g_I}\}_{k \in \mathbb{Z}}$  are the coefficient of the Fourier series of the  $I$ -periodic extension of  $g|_I$ . Repeating the same argument above using Fourier series in place of Fourier transform we will eventually obtain that  $u^v$  is a classical solution of the heat equation on  $I$ .

We now want to consider the minimum problem (2.11) *independently* of the knowledge of  $u^c$ . The following result shows in particular the existence and uniqueness of  $u^v$ .

**Theorem 2.5.** *Let  $g \in H_0^1(\Omega)$ . Then there exists a unique solution  $u^v$  of (2.11) and  $F(u^v) \geq 2I(g)$ . Moreover*

$$\int_{\Omega} (u_t^v(t))^2 dx = \int_{\Omega} (\Delta u^v(t))^2 dx \quad \text{for a.e. } t \in (0, +\infty). \quad (2.13)$$

*Proof.* The space  $X(\Omega)$  is a Hilbert space if endowed with the norm given by  $F$ . Define  $K := \{v \in X(\Omega) : v(0) = g\}$ . Then  $K$  is non empty (see [30, pag. 21]), and is a convex subset of  $X(\Omega)$ . Moreover,  $K$  is closed in  $X(\Omega)$  (see [30, Rem. 3.5]). Therefore the minimum



problem (2.11) has a unique solution  $u^v$  by [11, Th. V.6]. Moreover,  $F(u^v) \geq 2I(g)$  as a consequence of Remark 2.4 (a).

Let  $s > 0$  and set  $u := u^v$ . Let us show that

$$\int_s^{+\infty} \int_{\Omega} (u_t)^2 dxdt = \int_s^{+\infty} \int_{\Omega} (\Delta u)^2 dxdt. \quad (2.14)$$

For any  $\lambda > 0$  let  $v_{\lambda}(x, t) := u(x, s)$  if  $t \in [0, s]$  and  $v_{\lambda}(x, t) := u(x, s + \lambda(t - s))$  if  $t \geq s$ . Then  $v_{\lambda} \in X(\Omega)$  and  $v_{\lambda}(0) = g$ . The minimality of  $u$  then implies

$$\begin{aligned} F(u) \leq F(v_{\lambda}) &= \int_0^s \int_{\Omega} [(u_t)^2 + (\Delta u)^2] dxdt \\ &+ \int_s^{+\infty} \int_{\Omega} [\lambda^2 (u_t(x, s + \lambda(t - s)))^2 + (\Delta u(x, s + \lambda(t - s)))^2] dxdt \\ &= \int_0^s \int_{\Omega} [(u_t)^2 + (\Delta u)^2] dxdt + \int_s^{+\infty} \int_{\Omega} \left[ \lambda (u_t)^2 + \frac{1}{\lambda} (\Delta u)^2 \right] dxdt =: f(\lambda), \end{aligned}$$

hence  $f$  has a minimum for  $\lambda = 1$ . The equation  $f'(1) = 0$  yields (2.14). Let now  $\delta > 0$ . From (2.14) it follows

$$(2\delta)^{-1} \int_{s-\delta}^{s+\delta} \int_{\Omega} (u_t)^2 dxdt = (2\delta)^{-1} \int_{s-\delta}^{s+\delta} \int_{\Omega} (\Delta u)^2 dxdt,$$

and (2.13) is a consequence of Lebesgue differentiation theorem.  $\square$

**Remark 2.6.** Deviation of a function  $w \in X(\Omega)$  with  $w(0) = g \in H_0^1(\Omega)$  from being a classical solution to the heat equation with zero Dirichlet boundary condition could be measured by the following geodesic-type minimum problem:

$$\inf \{ F(w) : w \in X(\Omega), w(0) = g, w(T) = h \},$$

where  $h \in H^1(\Omega)$  and  $T \in (0, +\infty]$ . This kind of problems could be related to large deviations theory, see for instance [8].

**Remark 2.7.** As already mentioned in the Introduction, in the paper [12] solutions of gradient flows of proper convex lower semicontinuous energies  $\Phi$  in a Hilbert space  $H$  are obtained through the minimization of functionals of the form

$$\int_0^T [\Phi(v(t)) + \Phi^*(-\dot{v}(t))] dt + \frac{1}{2} \|v(T)\|_H^2,$$

where  $\Phi^*$  denotes the Legendre transform of  $\Phi$  and  $T > 0$  is fixed. The minimization is made on the space  $\{v \in C([0, T]; H); \phi^* (-\frac{dv}{dt}) \in L^1([0, T]), v(0) = g\}$ . Such a functional differs from  $F$  when  $\Phi$  is the Dirichlet integral.

As in our case, once the existence of a minimizer  $u$  is proved, in order to show that  $u$  is weak solution of the evolution equation up to the time  $T$ , one needs to prove that the minimum value equals twice the energy of the initial datum  $g$ . More recently, in [21], [20] a modified version of the functional proposed in [12] was studied. The modification is such that it makes possible to deduce directly from the minimality that  $u$  is also a weak solution of the evolution equation up to time  $T$ .

We observe that the functionals considered in [12] and [21], [20] are defined by duality using the Legendre transform; extending such an approach to geometric evolution seems not to be immediate.

### 3 Motion by mean curvature: the functional $\mathcal{F}$

Given a Borel set  $B \subseteq \mathbb{R}^n$  we denote by  $P(B)$  the perimeter of  $B$  in  $\mathbb{R}^n$ , and by  $\mathcal{H}^{n-1}$  the  $(n-1)$ -dimensional Hausdorff measure [2]. We denote by  $\{e_1, \dots, e_{n+1}\}$  the canonical basis of  $\mathbb{R}^n \times (0, +\infty)$ . We indicate by  $\nabla$  (resp.  $\Delta$ ) the gradient (resp. the laplacian) with respect to the space variable  $x$ .  $B_r$  denotes the ball of  $\mathbb{R}^n$  centered at the origin with radius  $r > 0$ . Let us recall the definition of classical mean curvature flow of a hypersurface, see for instance [18]. Let  $E \subset \mathbb{R}^n$  be an open set with smooth compact boundary. We say that  $(E(t))_{t \in [0, T]}$  is a smooth mean curvature flow on  $[0, T]$  starting from  $E = E(0)$  if, setting  $d(x, t) := \text{dist}(x, E(t)) - \text{dist}(x, \mathbb{R}^n \setminus E(t))$ ,

- (i) there exists an open set  $A$  such that  $A \supset \partial E(t)$  for any  $t \in [0, T]$  and  $d \in C^\infty(A \times [0, T])$ ;
- (ii) for all  $(x, t) \in A \times [0, T]$  such that  $d(x, t) = 0$  we have

$$\frac{\partial d}{\partial t}(x, t) = \Delta d(x, t). \quad (3.1)$$

Set  $\Sigma(t) := \partial E(t)$  and  $\Sigma := \cup_{t \in (0, T]} (\Sigma(t) \times \{t\})$ . We recall that, if  $x \in \partial E(t)$ , then  $\Delta d(x, t) \nabla d(x, t)$  is the mean curvature vector of  $\partial E(t)$  at  $x$  and that  $H_{\Sigma(t)}(x) = \Delta d(x, t)$ . Assume that  $E(t)$  is smooth up to the extinction time  $T \in (0, +\infty)$ . Define

$$\mathcal{F}(\Sigma) := \int_0^{+\infty} \int_{\partial E(t)} [(d_t)^2 + (\Delta d)^2] d\mathcal{H}^{n-1} dt. \quad (3.2)$$

Using (3.1) let us compute

$$0 = \int_0^{+\infty} \int_{\partial E(t)} (d_t - \Delta d)^2 d\mathcal{H}^{n-1} dt = \int_0^{+\infty} \int_{\partial E(t)} [(d_t)^2 + (\Delta d)^2 - 2d_t \Delta d] d\mathcal{H}^{n-1} dt \quad (3.3)$$

$$= \mathcal{F}(\Sigma) - 2 \int_0^{+\infty} \int_{\partial E(t)} d_t \Delta d d\mathcal{H}^{n-1} dt. \quad (3.4)$$

We now recall [1] that

$$\frac{d}{dt} P(E(t)) = - \int_{\partial E(t)} d_t \nabla d \cdot \nabla d \Delta d d\mathcal{H}^{n-1} = - \int_{\partial E(t)} d_t \Delta d d\mathcal{H}^{n-1},$$

where we have used  $|\nabla d|^2 = 1$  in a neighbourhood of  $\partial E(t)$ . Therefore from (3.3) we derive

$$0 = \int_0^{+\infty} \int_{\partial E(t)} (d_t - \Delta d)^2 d\mathcal{H}^{n-1} dt = \mathcal{F}(\Sigma) + 2 \int_0^{+\infty} \frac{d}{dt} P(E(t)) dt = \mathcal{F}(\Sigma) - 2P(E).$$

The above computation leads naturally to try to define the variational mean curvature flow  $\Sigma \subset \mathbb{R}^{n+1}$  starting from the set  $E \subset \mathbb{R}^n \times \{0\}$  as a minimizer of the functional  $\mathcal{F}$  in (3.2).

Before doing that, in order to have a better flavour of the meaning of the functional  $\mathcal{F}$ , we rewrite it on the space-time track  $\Sigma$ . Denote by  $\nu = (\nu_1, \dots, \nu_{n+1}) \in \{n \in \mathbb{R}^{n+1} : |n| = 1\}$  the unit normal to  $\Sigma$  outer to the set  $A := \cup_{t \in (0, T]} (E(t) \times \{t\})$ . If we indicate by  $\pi : \mathbb{R}^n \times (0, +\infty) \rightarrow (0, +\infty)$  the projection function  $\pi(x, t) := t$  on the last coordinate, the coarea formula on manifolds [19, pag. 103] guarantees, for  $B \subseteq \Sigma$ ,

$$\int_B |\nabla_{\Sigma} \pi| d\mathcal{H}^n = \int_{I_B} \int_{\partial E(t)} d\mathcal{H}^{n-1} dt, \quad (3.5)$$

where  $I_B := \{t \in (0, +\infty) : (\partial E(t) \times \{t\}) \cap B \neq \emptyset\}$  and  $\nabla_{\Sigma}$  denotes the tangential gradient on  $\Sigma$ . Therefore  $\nabla_{\Sigma} \pi = e_{n+1} - e_{n+1} \cdot \nu \nu = e_{n+1} - \nu_{n+1} \nu$ , so that  $|\nabla_{\Sigma} \pi|^2 = (\nu_{n+1})^2 ((\nu_1)^2 + \dots + (\nu_n)^2) + (1 - (\nu_{n+1})^2)^2 = 1 - (\nu_{n+1})^2$ . Hence from (3.5) we get

$$\int_B \sqrt{1 - (\nu_{n+1})^2} d\mathcal{H}^n = \int_{I_B} \int_{\partial E(t)} d\mathcal{H}^{n-1} dt. \quad (3.6)$$

In addition observe that  $\Sigma = \{(x, t) : d(x, t) = 0\}$ , so that the equation  $|\nabla d(x, t)| = 1$  implies  $\nu = \frac{1}{\sqrt{1+(d_t)^2}} (\nabla d, d_t)$  on  $\Sigma$ , and therefore

$$\frac{(\nu_{n+1})^2}{1 - (\nu_{n+1})^2} = (d_t)^2 \quad \text{on } \Sigma. \quad (3.7)$$

From (3.5) and (3.7) we deduce that the first term on the right hand side of (3.2) can be rewritten as  $\int_{\Sigma} \frac{(\nu_{n+1})^2}{\sqrt{1 - (\nu_{n+1})^2}} d\mathcal{H}^n$ . We conclude that the functional  $\mathcal{F}$  in (3.2) has also the expression (1.1), i.e.,

$$\mathcal{F}(\Sigma) = \int_{\Sigma} \frac{(\nu_{n+1})^2}{\sqrt{1 - (\nu_{n+1})^2}} d\mathcal{H}^n + \int_0^{+\infty} \int_{\Sigma(t)} (H_{\Sigma(t)})^2 d\mathcal{H}^{n-1} dt. \quad (3.8)$$

Set

$$\mathcal{F}_v(\Sigma) := \int_{\Sigma} \frac{(\nu_{n+1})^2}{\sqrt{1 - (\nu_{n+1})^2}} d\mathcal{H}^n, \quad \mathcal{F}_{\kappa}(\Sigma) := \int_0^{+\infty} \int_{\Sigma(t)} (H_{\Sigma(t)})^2 d\mathcal{H}^{n-1} dt, \quad (3.9)$$

in such a way that

$$\mathcal{F}(\Sigma) = \mathcal{F}_v(\Sigma) + \mathcal{F}_{\kappa}(\Sigma).$$

**Remark 3.1.**  $\mathcal{F}_v(\Sigma)$  penalizes regions of  $\Sigma$  where  $\nu = e_{n+1}$  (corresponding, to fat regions in the level set formulation of mean curvature flow). On the other hand, regions of  $\Sigma$  where  $\nu_{n+1} = 0$  do not contribute to  $\mathcal{F}_v(\Sigma)$ .

**Remark 3.2.** Assume that there exists the classical mean curvature flow  $E(t)$  of  $E$  for all times up to the (finite) extinction time. Define  $\Sigma_c := \cup_{t \in (0, +\infty)} (\partial E(t) \times \{t\})$  for any  $t$ . Then  $\mathcal{F}(\Sigma_c) = 2P(E)$ .

**Remark 3.3.** Assume that  $\Sigma = \{(x, t) \in E \times (0, +\infty) : t = f(x)\}$  for a function  $f \in \mathcal{C}^2(E)$  with nonvanishing gradient and  $f = 0$  on  $\partial E$ . Then

$$\int_{\Sigma} \frac{(\nu_{n+1})^2}{\sqrt{1 - (\nu_{n+1})^2}} d\mathcal{H}^n = \int_E \frac{1}{|\nabla f|} dx,$$

so that  $\mathcal{F}(\Sigma)$  takes the form (1.2).

**Remark 3.4.** Let

$$\phi(\xi) := \frac{(\xi_{n+1})^2}{\sqrt{1 - (\xi_{n+1})^2}}, \quad \xi \in \mathbb{R}^n \times (0, +\infty), \quad |\xi| = 1. \quad (3.10)$$

Denote by  $\phi^e$  the one-homogeneous extension of  $\phi$  to the whole of  $\mathbb{R}^{n+1}$ , i.e.,

$$\phi^e(\xi) = \frac{(\xi_{n+1})^2}{\sqrt{|\xi|^2 - (\xi_{n+1})^2}}, \quad \xi \in \mathbb{R}^{n+1} \setminus \{\xi_1 = \dots = \xi_n = 0\}. \quad (3.11)$$

If  $n = 1$  then  $\phi^e$  is convex. If  $n > 1$  then  $\phi^e$  is not convex.

Before giving the definition of strong variational solution of the mean curvature flow, let us examine one example which shows that not all smooth  $\Sigma$  (with finite energy) must be included in the domain of  $\mathcal{F}$ , since it is possible to construct smooth  $\Sigma$  with  $\mathcal{F}(\Sigma) < \mathcal{F}(\Sigma_c)$ ,  $\Sigma_c$  the space-time track of the classical mean curvature flow. The construction is based on the fact that, if we look at  $\Sigma$  as a graph with respect to the  $n$ -space variables, such a graph may contain regions of local maxima (or local minima) with dimension  $n - 1$ .

**Example 3.5.** Let  $R_0 > 0$  and  $0 < \delta < R_0/4$ . Let us consider the circular annulus  $E = B_{R_0} \setminus \overline{B_{R_0-\delta}} \subset \mathbb{R}^2$  as initial set, see (as in Figure 1. The space-time track  $\Sigma_c$  of the classical curvature flow starting from  $E$  is given by

$$\Sigma_c := \left( \bigcup_{t \in [0, t_{\text{ext}}^\dagger]} \partial B^{\text{ext}}(t) \times \{t\} \right) \cup \left( \bigcup_{t \in [0, t_{\text{int}}^\dagger]} \partial B^{\text{int}}(t) \times \{t\} \right),$$

where  $t_{\text{ext}}^\dagger = \frac{R_0^2}{2}$ ,  $t_{\text{int}}^\dagger = \frac{(R_0-\delta)^2}{2}$ ,  $B^{\text{ext}}(t) = B_{\sqrt{R_0^2-2t}}$ ,  $B^{\text{int}}(t) = B_{\sqrt{(R_0-\delta)^2-2t}}$ , and

$$\mathcal{F}(\Sigma_c) = 2P(E) = 4\pi(2R_0 - \delta).$$

Let us evolve now  $B_{R_0-\delta}$  by the *backward* curvature flow up to time  $t^* := \frac{1}{2}(t_{\text{ext}}^\dagger - t_{\text{int}}^\dagger)$ , and denote such an evolution by  $B_{\text{rev}}^{\text{int}}(t) = B_{\sqrt{(R_0-\delta)^2+2t}}$ ,  $t \in [0, t^*]$ . Define  $\Sigma^\delta$  as

$$\Sigma^\delta := \bigcup_{t \in [0, \frac{t^*}{2}]} (\partial B^{\text{ext}}(t) \cup \partial B_{\text{rev}}^{\text{int}}(t)) \times \{t\},$$

see Figure 1 (b).

Observe that  $\Sigma^\delta$  is not smooth (only along the time-slice circle  $\Sigma^\delta \cap \{t = \frac{t^*}{2}\}$ ) and that

$$P(B_{R_0}) - P(B^{\text{ext}}(t^*/2)) + P(B_{\text{rev}}^{\text{int}}(t^*/2)) - P(B_{R_0-\delta}) = 2\pi\delta = \mathcal{F}(\Sigma^\delta)/2.$$

We now smoothen the singularity of  $\Sigma^\delta$  with a sequence of radially symmetric  $\Sigma_k^\delta$  of smooth surfaces such that

$$\lim_{k \rightarrow \infty} \mathcal{F}(\Sigma_k^\delta) = \mathcal{F}(\Sigma^\delta) = 4\pi\delta < \mathcal{F}(\Sigma_c).$$

In view of the choice of  $\delta$ , it follows that  $\mathcal{F}(\Sigma_k^\delta) < \mathcal{F}(\Sigma_c)$  for  $k$  large enough.

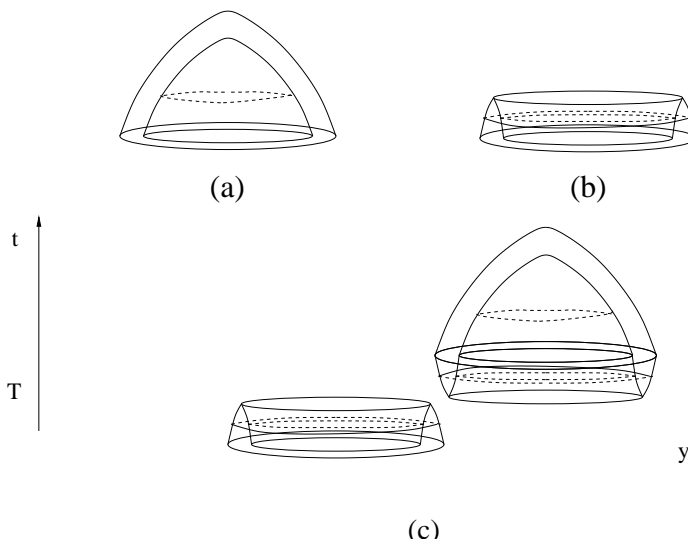


Figure 1: Fig.(a) represents the space-time track  $\Sigma_c$  of the curvature flow starting from an annulus. Fig.(b) shows  $\Sigma^\delta$ , constructed in Example 3.5. In Fig.(c) we depict an extinction and a nucleation (at time  $T$ ) of the same perimeter. This explains the local nature of condition (ii) in Definition 3.6, needed to rule out such examples.

Example 3.5 forces to add condition (ii) in the next definition, in order to rule out the surfaces  $\Sigma_k^\delta$  in the domain of  $\mathcal{F}$ , see also in analogy (a) of Remark 2.3.

**Definition 3.6.** Denote by  $\mathcal{S}$  the class of all hypersurfaces  $\Sigma \subset \mathbb{R}^n \times (0, +\infty)$  of class  $\mathcal{C}^2$  such that  $\partial\Sigma \subset \mathbb{R}^n \times \{0\}$ ,  $\mathcal{F}(\Sigma) < +\infty$  and satisfying the following assumptions:

- (i) there exists a bounded open set  $A \subset \mathbb{R}^n \times (0, +\infty)$  such that  $\partial A = \Sigma$ ;
- (ii) if we set  $\Sigma(t) := \{x \in \mathbb{R}^n : (x, t) \in \Sigma\}$ , for any Borel set  $B \subseteq \mathbb{R}^n$  the function  $\mathcal{H}^{n-1} \llcorner B(\Sigma(\cdot))$  is absolutely continuous in  $(0, +\infty)$

Let  $E \subset \mathbb{R}^n \times \{0\}$  be an open set with smooth compact boundary. We say that  $\Sigma_s$  is a strong variational solution of the mean curvature flow starting from  $E$  if  $\Sigma_s$  solves

$$\inf \left\{ \mathcal{F}(\Sigma) : \Sigma \in \mathcal{S}, \partial\Sigma = \partial E \right\}. \quad (3.12)$$

**Remark 3.7.** Unlike the previous case of the heat equation, proving that  $\mathcal{S}$  is nonempty is not immediate (it is not difficult to prove that  $\mathcal{S}$  is nonempty if  $E$  is convex or is smoothly diffeomorphic to a sphere). In addition, the existence of a solution to (3.12) is not guaranteed; therefore, it seems natural to study limits of minimizing sequences, and to relax the space  $\mathcal{S}$  of competitors and the definition of  $\mathcal{F}$  (see Section 5 below). This requires some care: indeed, weakening the regularity assumptions on  $\Sigma$  could lead to introduce other minimizers beside the one, for instance, given by the level set flow of  $E$ .

**Remark 3.8.** Condition (ii) in Definition 3.6 is satisfied in case  $\Sigma$  is the space-time track of a smooth flow in the sense of [1, Def. 5.1]. Moreover such an assumption ensures that sudden

loss of mass or nucleation of sets of  $\mathcal{H}^{n-1}$  positive measure are not allowed. These phenomena are related to suitable subsets of “horizontal” portions  $\{(x, t) \in \Sigma : \nu(x, t) = \pm e_{n+1}\}$  of  $\Sigma$ . In fact, let  $\Sigma \in \mathcal{S}$  and suppose that for a certain  $t$  we have  $\nu(x, t) = e_{n+1}$  for every  $x \in \Xi \subset \Sigma(t)$ , where  $\Xi$  is a relatively open subset of  $\Sigma(t)$ . It is then possible to find an open neighborhood  $U \subset \mathbb{R}^{n+1}$  of  $\Xi$  such that  $\Sigma \cap U$  can be written as a graph of a function defined on an open subset of  $\mathbb{R}^n$ . Condition (ii) implies that  $\Xi$  cannot be a subset of a level set of local maxima (which corresponds to a loss of mass) or local minima (which corresponds to a nucleation).

In the case of the heat equation, note that the fundamental equality (2.12) is a consequence of the function  $v$  to belong to  $X(\Omega)$ ; the analog of (2.12) would be to require in Definition 3.6

$$\frac{d}{dt} \int_{\Sigma(t)} \varphi \, d\mathcal{H}^{n-1} = \int_{\Sigma(t)} \frac{\nu_{n+1}}{\sqrt{1 - (\nu_{n+1})^2}} H_{\Sigma(t)} \varphi \, d\mathcal{H}^{n-1} + \int_{\Sigma(t)} \frac{\nu_{n+1}}{\sqrt{1 - (\nu_{n+1})^2}} \nu_{\Sigma(t)} \cdot \nabla \varphi \, d\mathcal{H}^{n-1}$$

for a.e.  $t \in (0, +\infty)$  and for any  $\varphi \in \mathcal{C}_c^\infty(\mathbb{R}^n)$ .

The following result is the analog of (2.13) in the geometric context.

**Proposition 3.9.** *Let  $\Sigma \in \mathcal{S}$  be a solution of (5.1). Then*

$$\int_{\Sigma(t)} (H_{\Sigma(t)})^2 \, d\mathcal{H}^{n-1} = \int_{\Sigma(t)} \frac{(\nu_{n+1})^2}{1 - (\nu_{n+1})^2} \, d\mathcal{H}^{n-1} \quad \text{for a.e. } t \in (0, +\infty). \quad (3.13)$$

*Proof.* Let  $(x, t) \in \Sigma$  and  $\{v_1, \dots, v_{n-1}\} \subset \mathbb{R}^n \times \{0\}$  be vectors spanning the tangent space  $T_{(x,t)}(\Sigma^t)$  to  $\Sigma^t$  at  $(x, t)$ . Let  $\gamma \in \mathcal{C}^1((-1, 1); \Sigma)$  be such that  $\gamma(0) = (x, t)$  and  $\gamma'_{n+1} \neq 0$  in  $(-1, 1)$ . In particular  $\text{span}\{v_1, \dots, v_{n-1}, \dot{\gamma}(0)\} = T_{(x,t)}(\Sigma)$ . Using Cramer’s formula for the solution of linear systems we have  $\text{span}\{\nu(x, t)\} = \text{span}(\sum_{j=1}^{n+1} (-1)^j \det A_j(x, t) e_j)$ , and  $A_j(x, t)$  is the  $(n \times n)$ -matrix obtained removing the  $j$ -th column from the matrix

$$\begin{pmatrix} v_{1,1} & \dots & v_{1,n} & 0 \\ \vdots & \dots & \vdots & 0 \\ v_{n-1,1} & \dots & v_{n-1,n} & 0 \\ \gamma'_1(0) & \dots & \gamma'_n(0) & \gamma'_{n+1}(0) \end{pmatrix}. \quad (3.14)$$

Fix  $s > 0$ . For every  $\lambda > 0$  we define  $\Sigma(t) := \{x : (x, t) \in \Sigma^t\}$ ,  $\Sigma_\lambda := \bigcup_{t \in (0, +\infty)} (\Sigma(t) \times \{\psi_\lambda(t)\})$ , where  $\psi_\lambda(t) := t$  if  $t \leq s$  and  $\psi_\lambda(t) := s + \lambda(t - s)$  if  $t \geq s$ . Set also  $\text{spt}(V_\lambda) := \Sigma_\lambda$ . Then  $(\Sigma_\lambda, V_\lambda) \in \mathcal{X}$  and  $\partial \Sigma_\lambda = \partial E$ . Moreover, since  $\Sigma_\lambda(\tau) = \Sigma(\psi_\lambda^{-1}(\tau))$ , we have

$$T_{(x, \psi_\lambda(t))} \Sigma_\lambda(\psi_\lambda(t)) = T_{(x,t)} \Sigma(t), \quad H_{\Sigma_\lambda(\psi_\lambda(t))}(x) = H_{\Sigma(t)}(x) \quad \forall x \in \Sigma(t).$$

Define  $\gamma_\lambda \in \mathcal{C}^1((-1, 1); \mathbb{R}^{n+1})$  as  $\sigma \mapsto (\gamma_1(\sigma), \dots, \gamma_n(\sigma), \psi_\lambda(\gamma_{n+1}(\sigma)))$ . Then

$$\gamma_\lambda(\sigma) \in \Sigma(\gamma_{n+1}(\sigma)) \times \{\psi_\lambda(\gamma_{n+1}(\sigma))\} \subset \Sigma_\lambda, \quad \gamma_\lambda(0) = (x, \psi_\lambda(t)), \quad \dot{\gamma}_\lambda \neq 0 \text{ in } (-1, 1),$$

and  $T_{(x, \psi_\lambda(t))} \Sigma_\lambda = \text{span}(v_1, \dots, v_{n-1}, \dot{\gamma}_\lambda(0))$ . Moreover,

$$\text{span}(\nu_{\Sigma_\lambda}(x, \psi_\lambda(t))) = \text{span}\left(\sum_{j=1}^{n+1} (-1)^j \det A_j^\lambda(x, t) e_j\right),$$

where  $A_j^\lambda(x, t)$  is the  $(n \times n)$ -matrix obtained removing the  $j$ -th column from the matrix (3.14) with  $\gamma'_{n+1}(0)$  replaced by  $\psi'_\lambda(\gamma_{n+1}(0))$ . Hence

$$\frac{(\nu_{\Sigma_\lambda, n+1}(x, \psi_\lambda(t)))^2}{1 - \nu_{\Sigma_\lambda, n+1}^2(x, \psi_\lambda(t))} = \lambda^{-2} \frac{(\nu_{n+1}(x, t))^2}{1 - (\nu_{n+1}(x, t))^2}. \quad (3.15)$$

By the minimality of  $\Sigma$  and (3.15), we have

$$\begin{aligned} \mathcal{F}^w(\Sigma) &\leq \mathcal{F}^w(\Sigma_\lambda) \\ &= \int_0^{+\infty} \int_{\Sigma_\lambda(\tau)} \frac{(\nu_{\Sigma_\lambda, n+1})^2}{1 - (\nu_{\Sigma_\lambda, n+1})^2} d\mathcal{H}^{n-1} d\tau + \int_0^{+\infty} \int_{\Sigma_\lambda(\tau)} (H_{\Sigma(\psi_\lambda^{-1}(\tau))})^2 d\mathcal{H}^{n-1} d\tau \\ &= \lambda^{-1} \int_0^{+\infty} \int_{\Sigma(t)} \frac{(\nu_{n+1})^2}{1 - (\nu_{n+1})^2} d\mathcal{H}^{n-1} dt + \lambda \int_0^{+\infty} \int_{\Sigma(t)} (H_{\Sigma(t)})^2 d\mathcal{H}^{n-1} dt =: f(\lambda) \end{aligned}$$

The proof then proceeds exactly as in the proof of (2.13).  $\square$

Classical mean curvature flow is a stationary point of  $\mathcal{F}$ , as shown by the following result.

**Proposition 3.10.** *Let  $E \subset \mathbb{R}^n$  be a smooth bounded open set, let  $\Sigma_c \subset \mathbb{R}^n \times (0, t^\dagger)$  be the time track of the mean curvature flow starting from  $\partial E$ , which is assumed to be smooth on  $(0, t^\dagger)$ , so that  $\Sigma_c \in \mathcal{S}$ . Let  $\Psi \in C^\infty(\mathbb{R}^{n+2}; \mathbb{R}^{n+1})$  and set  $\Psi_\lambda(z) := \Psi(\lambda, z)$  for  $\lambda \in \mathbb{R}$  and  $z \in \mathbb{R}^{n+1}$ . Assume that  $\Psi_0 = \text{Id}$  on  $\mathbb{R}^{n+1}$ ,  $\Psi_\lambda = \text{Id}$  out of a compact set of  $\mathbb{R}^{n+1}$  for  $|\lambda|$  small enough, and that there exists  $\psi \in C_0^\infty(\mathbb{R}^n \times (0, t^\dagger))$  such that*

$$\frac{d}{d\lambda} \Psi_\lambda(x, t) \Big|_{\lambda=0} = \psi(x, t) (\nu_{\Sigma_c(t)}(x), 0), \quad \forall (x, t) \in \Sigma, \quad (3.16)$$

where  $\nu_{\Sigma_c(t)}$  is the outer unit normal vector to  $\Sigma_c(t)$ . Define  $\Sigma_\lambda := \Psi_\lambda(\Sigma_c)$  and suppose that  $\Sigma_\lambda \in \mathcal{S}$ . Then

$$\frac{d}{d\lambda} \mathcal{F}(\Sigma_\lambda) \Big|_{\lambda=0} = 0. \quad (3.17)$$

*Proof.* By (3.7) and (3.1) we have, for  $x \in \Sigma_c(t)$ ,

$$(\nu_{n+1}(x, t))^2 = \frac{|H_{\Sigma_c(t)}(x)|^2}{1 + |H_{\Sigma_c(t)}(x)|^2}, \quad (\nu(x, t) \cdot (\nu_{\Sigma_c(t)}(x), 0))^2 = \frac{1}{1 + |H_{\Sigma_c(t)}(x)|^2}. \quad (3.18)$$

By the regularity of  $\Sigma_c$ , (3.16) we have (see for instance [35])

$$\begin{aligned} \frac{d}{d\lambda} \mathcal{F}_\kappa(\Sigma_\lambda) \Big|_{\lambda=0} &= \int_0^{+\infty} \frac{d}{d\lambda} \left( \int_{\Sigma_c(t)} |H_{\Sigma_c(t)}|^2 d\mathcal{H}^{n-1} \right) dt \Big|_{\lambda=0} \\ &= \int_0^{+\infty} \int_{\Sigma_c(t)} \psi(x, t) (2\Delta_{\Sigma_c(t)} H_{\Sigma_c(t)} + 2H_{\Sigma_c(t)} |A_{\Sigma_c(t)}|^2 - (H_{\Sigma_c(t)})^3) d\mathcal{H}^{n-1} dt, \end{aligned} \quad (3.19)$$

where  $\Delta_{\Sigma_c(t)}$  is the Laplace-Beltrami operator on  $\Sigma_c(t)$ , and  $|A_{\Sigma_c(t)}|^2$  is the square of the (euclidean) norm of the second fundamental form of  $\Sigma_c(t)$ . In addition, by [7, Theorem 5.1] we have

$$\frac{d}{d\lambda} \mathcal{F}_v(\Sigma_\lambda) \Big|_{\lambda=0} = - \int_{\Sigma} (\psi \kappa_\phi) \nu \cdot (\nu_{\Sigma_c(t)}, 0) d\mathcal{H}^n,$$

where  $\kappa_\phi := \operatorname{div} [(\nabla \phi^e)(\nu^e)]|_{\xi=\nu}$ ,  $\phi^e$  is defined in (3.11) and  $\nu^e$  is a smooth extension of  $\nu$  in a neighbourhood of  $\Sigma$  keeping the constraint  $|\nu^e| = 1$ . Using (3.18) we have

$$\begin{aligned} \nabla \phi^e(\nu^e)|_{\xi=\nu(x,t)} &= \left( -\frac{(\nu_{n+1})^2 \nu_1}{(1 - (\nu_{n+1})^2)^{3/2}}, \dots, -\frac{(\nu_{n+1})^2 \nu_n}{(1 - (\nu_{n+1})^2)^{3/2}}, 2 \frac{\nu_{n+1}}{\sqrt{1 - (\nu_{n+1})^2}} \right) \\ &= (-|H_{\Sigma_c(t)}|^2 \nu_{\Sigma_c(t),1}, \dots, -|H_{\Sigma_c(t)}|^2 \nu_{\Sigma_c(t),n}, 2H_{\Sigma_c(t)}), \end{aligned}$$

where the right hand sides are evaluated at  $(x, t) \in \Sigma$ , and hence

$$\kappa_\phi = -\sum_{j=1}^n \partial_{x_j} (|H_{\Sigma_c(t)}|^2 \nu_{\Sigma_c(t),j}) + 2\partial_t(H_{\Sigma_c(t)}) = -(H_{\Sigma_c(t)})^3 + 2\Delta_{\Sigma_c(t)} H_{\Sigma_c(t)} + 2H_{\Sigma_c(t)} |A_{\Sigma_c(t)}|^2,$$

where in the last equality we used  $\nu \nabla \nu = 0$  and (see e.g. [15, Appendix B]) that

$$\partial_t H_{\Sigma_c(t)} = \Delta_{\Sigma_c(t)} H_{\Sigma_c(t)} + H_{\Sigma_c(t)} |A_{\Sigma_c(t)}|^2.$$

Eventually, from (3.18), (3.5) and  $\sqrt{1 - (\nu_{n+1}(x, t))^2} = 1/\sqrt{1 + |H_{\Sigma_c(t)}(x)|^2}$ , we deduce

$$\begin{aligned} \frac{d}{d\lambda} \mathcal{F}_v(\Sigma_\lambda)|_{\lambda=0} &= -\int_{\Sigma_c} (\psi \kappa_\phi) \nu \cdot (\nu_{\Sigma_c(t)}, 0) d\mathcal{H}^n \\ &= \int_{\Sigma_c} \psi \left( (H_{\Sigma_c(t)})^3 - 2\Delta_{\Sigma_c(t)} H_{\Sigma_c(t)} - 2H_{\Sigma_c(t)} |A_{\Sigma_c(t)}|^2 \right) \frac{1}{\sqrt{1 + |H_{\Sigma_c(t)}|^2}} d\mathcal{H}^n \\ &= \int_0^{+\infty} \int_{\Sigma_c(t)} \psi \left( (H_{\Sigma_c(t)})^3 - 2\Delta_{\Sigma_c(t)} H_{\Sigma_c(t)} - 2H_{\Sigma_c(t)} |A_{\Sigma_c(t)}|^2 \right) d\mathcal{H}^{n-1} dt. \end{aligned} \tag{3.20}$$

Then (3.17) follows from (3.19) and (3.20).  $\square$

We conclude this section by observing that the idea leading to the functional  $\mathcal{F}$  could be applied to other systems of geometric PDEs. As an example, let us mention mean curvature flow of a smooth compact embedded manifold  $\Gamma(t)$  of dimension  $h \in \{1, \dots, n-2\}$  without boundary in  $\mathbb{R}^n$ , see [4]. In this case, the same arguments in Section 3 (cfr. Figure 2) lead to study the following functional:

$$\mathcal{G}(\Sigma) := \int_{\Sigma} \frac{1 - |\nabla_{\Sigma} \pi|^2}{|\nabla_{\Sigma} \pi|} d\mathcal{H}^{h+1} + \int_0^{+\infty} \int_{\Gamma(t)} |H_{\Gamma(t)}|^2 d\mathcal{H}^h dt,$$

where now  $\Sigma$  is a  $(h+1)$ -dimensional manifold in  $\mathbb{R}^n \times (0, +\infty)$  and  $H_{\Gamma(t)}$  still denotes the mean curvature vector of  $\Gamma(t)$ .

## 4 The geometric minimum problem for convex sets in $\mathbb{R}^2$

In this section we study problem (3.12) when  $E$  is a bounded open smooth *convex* set of  $\mathbb{R}^2$ . We also restrict our minimization problem to those  $\Sigma$  which are subgraphs of concave functions defined on  $E$ .



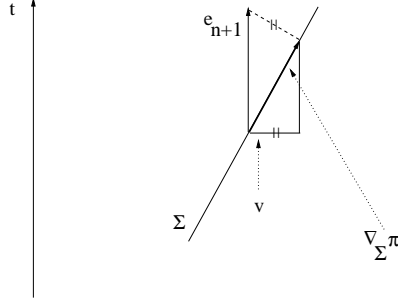


Figure 2: Computation of the velocity of a flowing manifold of arbitrary codimension. Note that  $|v|^2 = |\nabla_\Sigma \pi - (\nabla_\Sigma \pi \cdot e_{n+1}) e_{n+1}|^2 = 1 - |\nabla_\Sigma \pi|^2$ .

We recall that the existence of a unique smooth mean curvature flow starting from a smooth compact convex set  $E$ , until its extinction time, was proved by Huisken in [22].

Let us define

$$\mathcal{C} := \{ \Sigma : \Sigma = \text{graph}(f), f : E \rightarrow [0, +\infty) \text{ concave,} \quad (4.1)$$

$$\{f = t\} := \Sigma(t) \text{ has curvature } H_{\Sigma(t)} \text{ in } L^2 \text{ for a.e. } t \in (0, +\infty) \}.$$

It is possible to show that the class  $\mathcal{C}$  is not empty and that there exists  $\Sigma \in \mathcal{C}$  with  $\mathcal{F}(\Sigma) < +\infty$  (where the curvature part  $\mathcal{F}_\kappa(\Sigma)$  of  $\mathcal{F}(\Sigma)$  is obviously defined as  $\int_0^{+\infty} \|H_{\Sigma(t)}\|_{L^2_{\mathcal{H}^1}}^2 dt$ ).

The main result of this section is the following.

**Theorem 4.1.** *Let  $n = 2$ . The problem*

$$\inf \{ \mathcal{F}(\text{graph}(f)) : \text{graph}(f) \in \mathcal{C}, f = 0 \text{ on } \partial E \} \quad (4.2)$$

*admits a solution.*

*Proof.* Write  $\text{graph}(f) = \Sigma_f$  for a concave function  $f$  as in (4.2) with  $\mathcal{F}(\Sigma_f) < +\infty$ . We divide the proof into five steps.

*Step 1.* Let  $\Sigma_f \in \mathcal{C}$ . Then

$$0 \leq s \leq \sigma \Rightarrow P(\{f \geq s\}) \geq P(\{f \geq \sigma\}). \quad (4.3)$$

Since  $f$  is concave, if  $0 \leq s \leq \sigma$  the sets  $\{f \geq s\}$  and  $\{f \geq \sigma\}$  are convex and  $\{f \geq s\} \supseteq \{f \geq \sigma\}$ . The assertion then follows from [13, Lemma 2.4].

Observe that  $\mathcal{F}_v(\Sigma_f) < +\infty$  implies that  $|\nabla f| \neq 0$  almost everywhere in  $E$ , namely the level  $\{f = t^\dagger\}$ ,  $t^\dagger := \sup\{t \in (0, +\infty) : \{f \geq t\} \neq \emptyset\}$ , has zero Lebesgue measure.

The following step implies that if the velocity contribution  $\mathcal{F}_v(\Sigma_{f_h})$  to the energy is uniformly bounded by a constant  $c$  along a sequence  $\{\Sigma_{f_h}\} \subset \mathcal{C}$ , then all  $\{f_h \geq t\}$  are not empty for  $t$  smaller than some  $T_*$  depending only on  $c$ .

*Step 2.* Let  $c > 0$  and define

$$V_c := \{\Sigma_f \in \mathcal{C} : \mathcal{F}_v(\Sigma_f) \leq c\}.$$

Then

$$\sup_{x \in E} f(x) \geq T_* \quad \forall f \in V_c, \quad T_* := \frac{|E|^2}{cP(E)}.$$

By the Schwartz inequality we have

$$|E| = \int_E \frac{1}{|\nabla f|^{1/2}} |\nabla f|^{1/2} dx \leq \left( \int_E \frac{1}{|\nabla f|} dx \right)^{1/2} \left( \int_E |\nabla f| dx \right)^{1/2} \leq c^{1/2} \left( \int_E |\nabla f| dx \right)^{1/2}. \quad (4.4)$$

We can assume that  $f$  is bounded above by some constant  $\lambda$ , otherwise there is nothing to prove. Using (4.4), the coarea formula and *step 1* we then get

$$|E| \leq c^{1/2} \left( \int_0^\lambda P(\{f \geq t\}) dt \right)^{1/2} \leq c^{1/2} \lambda^{1/2} P(E)^{1/2}.$$

It follows  $\lambda \geq \frac{|E|^2}{cP(E)}$ .

The following step implies that if the curvature contribution  $\mathcal{F}_\kappa(\Sigma_{f_h})$  to the energy is uniformly bounded by a constant  $c$  along a sequence  $\{\Sigma_{f_h}\} \subset \mathcal{C}$  then all  $\{f_h \geq t\}$  are empty for  $t$  larger than some  $T$  depending only on  $c$ , provided  $n = 2$ .

*Step 3.* Let  $c > 0$  and define

$$K_c := \{\Sigma_f \in \mathcal{C} : \mathcal{F}_\kappa(\Sigma_f) \leq c\}.$$

Then

$$\sup_{x \in E} f(x) \leq T^* \quad \forall f \in K_c,$$

where  $T^* := \frac{cP(E)}{4\pi}$ .

Let  $f \in K_c$ . For any  $t \in [0, +\infty)$  such that  $\{f = t\}$  is non empty and has curvature  $H_t$  in  $L^2$ , using the Schwartz inequality and *step 1* we have

$$4\pi = \left( \int_{\Sigma_f(t)} H_{\Sigma_f(t)} d\mathcal{H}^1 \right)^2 \leq P(\Sigma_f(t)) \int_{\Sigma_f(t)} (H_{\Sigma_f(t)})^2 d\mathcal{H}^1 \leq P(E) \int_{\Sigma_f(t)} (H_{\Sigma_f(t)})^2 d\mathcal{H}^1.$$

Integrating over  $[0, \tau]$  we get

$$\tau \leq \frac{P(E)}{4\pi} \int_0^\tau \int_{\Sigma_f(t)} (H_{\Sigma_f(t)})^2 d\mathcal{H}^1 dt \leq c \frac{P(E)}{4\pi},$$

and the assertion follows.

*Step 4.* Let  $\{\Sigma_{f_h}\} \subset \mathcal{C}$  be a sequence with

$$\mathcal{F}(\Sigma_{f_h}) \leq c \quad \forall h \in \mathbb{N}. \quad (4.5)$$

Then there exist a subsequence  $\{\Sigma_{f_{h_k}}\} \subset \mathcal{C}$  and  $\Sigma_f \in \mathcal{C}$  such that  $\{f_{h_k}\}$  converges to  $f$  uniformly on compact subsets of  $E$ , and  $\{\nabla f_{h_k}\}$  converges to  $\nabla f$  almost everywhere in  $E$ . Thanks to (4.5), we can apply *step 3* to deduce that there exists a constant  $T^*$  such that

$$f_h(x) \leq T^* \quad \forall x \in E, \forall h \in \mathbb{N}.$$

Therefore  $\{f_h\}$  is equi-Lipschitz on compact subsets of  $E$ , so that there exists a subsequence  $\{f_{h_k}\}$  uniformly converging, on compact subsets of  $E$ , to a function  $f : E \rightarrow [0, +\infty)$ . It follows that  $f$  is concave,  $f : E \rightarrow [0, T^*]$  and  $f = 0$  on  $\partial E$ . Additionally, reasoning as in [13, Lemma 2.2], we have also  $\nabla f_{h_k} \rightarrow \nabla f$  almost everywhere on  $E$ .

Note that since  $f_{h_k} \rightarrow f$  uniformly on compact subsets of  $E$ , then  $\{f_{h_k} \geq t\} \rightarrow \{f \geq t\}$  in  $L^1$  for any  $t \in [0, T^*]$ , hence  $\{f_{h_k} \geq t\} \rightarrow \{f \geq t\}$  in the Hausdorff distance (see [13, Lemma 4.4]), and  $\mathcal{H}^1(\Sigma_{f_{h_k}}) \rightarrow \mathcal{H}^1(\Sigma_f(t))$ .

Using again (4.5) and the Fatou's lemma, we have that for almost every  $t \in [0, T^*]$

$$\liminf_{k \rightarrow +\infty} \int_{\Sigma_{f_{h_k}}(t)} (H_{\Sigma_{f_{h_k}}(t)})^2 d\mathcal{H}^1 < +\infty. \quad (4.6)$$

It follows that for almost every  $t \in [0, T^*]$  there exists a subsequence  $\{h_{k'}\}$  (depending on  $t$ ) such that the  $\liminf$  in (4.6) is a limit along  $\{h_{k'}\}$ . Since  $\{f_{h_{k'}} \geq t\} \rightarrow \{f \geq t\}$  in  $L^1$  and  $\mathcal{H}^1(\Sigma_{f_{h_{k'}}}(t)) \rightarrow \mathcal{H}^1(\Sigma_f(t))$ , and (4.6) holds, it follows that  $\Sigma_f(t)$  has curvature in  $L^2$ , since

$$\int_{\Sigma_f(t)} (H_{\Sigma_f(t)})^2 d\mathcal{H}^1 \leq \liminf_{k' \rightarrow \infty} \int_{\Sigma_{f_{h_{k'}}}(t)} (H_{\Sigma_{f_{h_{k'}}}(t)})^2 d\mathcal{H}^1. \quad (4.7)$$

holds (see for instance [6]), so that  $\Sigma_f \in \mathcal{C}$ .

The proof of the next step is a consequence of Fatou's Lemma.

*Step 5.* Let  $f_h, f : E \rightarrow [0, +\infty[$  be concave functions such that  $\{\nabla f_h\}$  converges to  $\nabla f$  almost everywhere on  $E$  as  $h \rightarrow +\infty$ . Then

$$\int_E \frac{1}{|\nabla f|} dx \leq \liminf_h \int_E \frac{1}{|\nabla f_h|} dx.$$

The proof of the next step is a consequence of Fatou's Lemma and the results in [6].

*Step 6.* Let  $\{\Sigma_{f_h}\}$  be a sequence of elements of  $\mathcal{C}$ , and let  $\Sigma_f \in \mathcal{C}$ . Assume that  $\{f_h \geq t\} \rightarrow \{f \geq t\}$  in  $L^1(\mathbb{R}^2)$  as  $h \rightarrow +\infty$  for almost every  $t \in \mathbb{R}^+$ . Then

$$\mathcal{F}_\kappa(\Sigma_f) \leq \liminf_h \mathcal{F}_\kappa(\Sigma_{f_h}). \quad (4.8)$$

The assertions of the theorem then follows by standard arguments.  $\square$

**Remark 4.2.** We expect that Theorem 4.1 holds with the same proof in  $n = 3$  space dimensions, using the results of [23], [24] [25].

## 5 Weak form of the geometric minimum problem

Finding the domain of the lower semicontinuous envelope of  $\mathcal{F}$  (with respect to a suitable topology) and studying its compactness properties is not easy. In particular, one has to keep in mind that, in general, mean curvature flow admits singularities at finite time, and that these singularities have not been completely classified. In the following definition we propose a possible domain  $\mathcal{X}$  for the relaxed functional of  $\mathcal{F}$ , and a possible expression for an extension of  $\mathcal{F}$  (originally defined on  $\mathcal{S}$ ) on  $\mathcal{X}$ . Our definition is partially inspired by the definition of enhanced-flow given by Ilmanen in [29]. The main idea is that the classical evolution  $\partial E(t) \times \{t\}$  is replaced by the time slices of a varifold, slices having mean curvature in  $L^2$ , and containing the interior of the time slices  $\Sigma^t$  of a current  $\Sigma$ ; such an interior makes possible to devise what, in the smooth case, makes the role of  $E(t) \times \{t\}$ . Concerning the theory of currents and of varifolds we refer to [19] and [34]. If  $V$  is a varifold, its projection on the base space is denoted by  $\|V\|$ .

**Definition 5.1.** *Let  $\mathcal{X}$  be the class of all pairs  $(\Sigma, V)$  having the following properties:  $\Sigma$  is a  $n$ -rectifiable integer current in  $\mathbb{R}^n \times (0, +\infty)$  with  $\partial\Sigma \subset \mathbb{R}^n \times \{0\}$  and  $V$  is a rectifiable unoriented integer varifold in  $\mathbb{R}^n \times (0, +\infty)$  with compact support such that:*

- (i) *for  $\mathcal{H}^1$ -almost every  $t \in (0, +\infty)$  the support of the restriction  $v(t) := \|V\| \llcorner \{x_{n+1} = t\}$  of  $\|V\|$  to  $\{x_{n+1} = t\}$  contains the slice  $\Sigma^t := \Sigma \cap \{x_{n+1} = t\}$ ;*
- (ii) *for  $\mathcal{H}^1$ -almost every  $t \in (0, +\infty)$ ,  $v(t)$  is a rectifiable measure [29] with mean curvature  $H_{v(t)}$  in  $L^2_{v(t)}$ , and*

$$\mathcal{F}_\kappa^w(V) := \int_0^{+\infty} \int_{\text{spt}(v(t))} (H_{v(t)})^2 dv(t) dt < +\infty;$$

- (iii) *if  $\nu$  denotes a ( $\mathcal{H}^n$ -almost everywhere defined) unit normal vector field to  $\text{spt}(\|V\|)$  then*

$$\mathcal{F}_\nu^w(V) := \int_{\mathbb{R}^n \times (0, +\infty)} \phi(\nu) d\|V\| < +\infty,$$

where  $\phi$  is defined in (3.10);

- (iv) *for any Borel set  $B \subseteq \mathbb{R}^n$  the function  $v(\cdot)(B)$  is absolutely continuous in  $(0, +\infty)$ .*

**Definition 5.2.** *Let  $E \subset \mathbb{R}^n$  be an open set with smooth compact boundary. We say that  $(\Sigma, V) \in \mathcal{X}$  is a weak variational solution to the mean curvature flow starting from  $E$  if  $(\Sigma, V)$  solves*

$$\inf \left\{ \mathcal{F}^w(V) : (\Sigma, V) \in \mathcal{X}, \partial\Sigma = \partial E \right\} \quad (5.1)$$

where  $\mathcal{F}^w(V) := \mathcal{F}_\nu^w(V) + \mathcal{F}_\kappa^w(V)$ .

Some comments are in order concerning Definition 5.2.

**Remark 5.3.**

- (a) If  $n = 1$  the minimum problem (5.1) has the unique solution  $\Sigma(t) = \text{spt}(v(t)) = \partial E$  for any  $t$ .
- (b) If  $(\Sigma, V)$  is a solution of (5.1) and  $\text{spt}(v(t)) = \emptyset$  for some  $t > 0$ , then  $\text{spt}(v(s)) = \emptyset$  for all  $s \geq t$ .
- (c) If  $\Sigma_s$  is a strong variational solution of the mean curvature flow starting from  $E$  then, setting  $v(t) := \Sigma_s \cap \{x_{n+1} = t\}$ , we have that  $(\Sigma_s, V)$  is a weak variational solution of the mean curvature flow starting from  $E$ .

**Remark 5.4.** If  $n = 2$  and  $\partial E$  is connected, it should be possible to prove that there exists  $(\Sigma, V) \in \mathcal{X}$  with  $\partial \Sigma = \partial E$ , namely to find a smooth map between a regular parametrization of the curve  $\partial E$  and a parametrization of the circle within a time interval  $[0, T]$  and then letting flow the circle by curvature starting at time  $T$ , without violating the smoothness requirements in Definition 5.1.

**Definition 5.5.** We say that a sequence  $\{(\Sigma_n, V_n)\} \subset \mathcal{X}$  converges to  $(\Sigma, V) \in \mathcal{X}$  as  $n \rightarrow +\infty$  if  $\Sigma_n$  converges to  $\Sigma$  in the flat norm, and  $v_n(t)$  converges to  $v(t)$  as a varifold, for almost every  $t \in (0, +\infty)$ .

The above definition allows to consider the lower semicontinuous envelope  $\overline{\mathcal{F}^w}$  of the functional  $\mathcal{F}^w$ , and to try to prove that  $\overline{\mathcal{F}^w} = \mathcal{F}^w$  on some subset of  $\mathcal{X}$ . In this respect it becomes useful to find the weakest topology for which  $\mathcal{F}^w$  (or even  $\mathcal{F}$ ) is lower semicontinuous; we refer to [9] (for  $n = 2$ ) and especially [33] (for  $n \geq 3$ ) for results concerning the lower semicontinuity of the Willmore functional.

## 6 Application to reaction-diffusion equations and $\Gamma$ -convergence

Given a function  $v \in H_{\text{loc}}^1(\mathbb{R}^n)$  and  $\epsilon > 0$  let

$$m_\epsilon(v) := \frac{\epsilon}{2} |\nabla v|^2 + \frac{1}{\epsilon} W(v), \quad M_\epsilon(v) := \int_{\mathbb{R}^n} m_\epsilon(v) \, dx, \quad (6.1)$$

where  $W(s) = \frac{1}{4}(1 - s^2)^2$ . Define  $c_0 := \frac{2\sqrt{2}}{3}$ . If  $v \in H_{\text{loc}}^2(\mathbb{R}^n)$ , set

$$\text{eul}_\epsilon(v) := -\epsilon \Delta v + \frac{1}{\epsilon} W'(v). \quad (6.2)$$

Let  $R > 0$  and  $Y_R := L^\infty([0, +\infty); H^2(B_R)) \cap H_{\text{loc}}^1([0, +\infty); L^2(B_R))$ ,  $\mathcal{Y} := \cap_R Y_R$ . Define the sequence  $\mathcal{F}_\epsilon : \mathcal{Y} \rightarrow [0, +\infty]$  of functionals as

$$\mathcal{F}_\epsilon(v) := \begin{cases} \int_0^{+\infty} \int_{\mathbb{R}^n} \left[ \epsilon (v_t)^2 + \frac{1}{\epsilon} (\text{eul}_\epsilon(v))^2 \right] \, dx dt & \text{if } v \in \mathcal{Y}, \\ +\infty & \text{if } v \in L_{\text{loc}}^2(\mathbb{R}^n \times (0, +\infty)) \setminus \mathcal{Y}. \end{cases} \quad (6.3)$$

Let  $g \in H_{\text{loc}}^1(\mathbb{R}^n)$  be a given function with  $M_\epsilon(g) < +\infty$ . Denote by  $u_\epsilon \in \mathcal{Y}$  the solution to the Cauchy problem

$$\begin{cases} u_{\epsilon t} = \frac{1}{\epsilon} \text{eul}(u_\epsilon) & \text{in } \mathbb{R}^n \times (0, +\infty), \\ u_\epsilon(0) = g & \text{in } \mathbb{R}^n \times \{0\}. \end{cases} \quad (6.4)$$

Following the same integration by parts in Section 2.1, let us compute

$$\begin{aligned} 0 &= \int_0^{+\infty} \int_{\mathbb{R}^n} (u_{\epsilon t} - \frac{1}{\epsilon} \text{eul}_\epsilon(u_\epsilon))^2 dxdt \\ &= \int_0^{+\infty} \int_{\mathbb{R}^n} \left[ (u_{\epsilon t})^2 + \frac{1}{\epsilon^2} (\text{eul}_\epsilon(u_\epsilon))^2 - \frac{2}{\epsilon} u_{\epsilon t} \text{eul}_\epsilon(u_\epsilon) \right] dxdt, \end{aligned}$$

so that

$$\begin{aligned} 0 &= \mathcal{F}_\epsilon(u_\epsilon) - 2 \int_0^{+\infty} \int_{\mathbb{R}^n} u_{\epsilon t} \text{eul}(u_\epsilon) dxdt = \mathcal{F}_\epsilon(u) \\ &\quad + 2 \int_0^{+\infty} \int_{\mathbb{R}^n} \frac{d}{dt} m_\epsilon(u_\epsilon) dt dx = \mathcal{F}_\epsilon(u_\epsilon) - 2M_\epsilon(g). \end{aligned}$$

Given  $B \subseteq \mathbb{R}^n$ , we let  $\chi_B(x) := 1$  if  $x \in B$  and  $\chi_B(x) := -1$  if  $x \notin B$ .

**Theorem 6.1.** *The following assertions hold.*

- (i) *Let  $A$  and  $\Sigma$  be connected and as in Definition 3.6. Assume that  $\Sigma(T) \times \{T\} = \{(x_0, T)\} = \{(x, t) \in \Sigma : \nu(x, t) = e_{n+1}\}$  and for every  $0 < \delta < T/2$  the manifold  $\Sigma \cap [0, T - \delta] \times \mathbb{R}^n$  represents the space-time track of a smooth evolution starting from  $\partial\Sigma$ . Then there exists a sequence  $\{u_\epsilon\} \subset \mathcal{Y}$  converging to  $\Sigma$  in  $L^1(\mathbb{R}^n \times (0, +\infty))$  such that*

$$\lim_{\epsilon \rightarrow 0} \mathcal{F}_\epsilon(u_\epsilon) = c_0 \mathcal{F}(\Sigma). \quad (6.5)$$

- (ii) *Let  $n \in \{2, 3\}$  and  $\{v_\epsilon\} \subset \mathcal{Y}$  be such that  $\sup_\epsilon \mathcal{F}_\epsilon(v_\epsilon) < +\infty$ . Suppose that  $v_\epsilon \rightarrow 1_B$  in  $L^1(\mathbb{R}^n \times (0, +\infty))$ , where  $B \subset \mathbb{R}^n \times (0, +\infty)$  is a smooth bounded open set, and that  $\lim_{\epsilon \rightarrow 0} \int_0^{+\infty} \int_{\mathbb{R}^n} m_\epsilon(v_\epsilon) dxdt = c_0 \mathcal{H}^{n-1}(\Sigma)$ , where  $\Sigma := \partial B \cap (\mathbb{R}^n \times (0, +\infty))$ . Then*

$$\liminf_{\epsilon \rightarrow 0} \mathcal{F}_\epsilon(v_\epsilon) \geq c_0 \mathcal{F}(\Sigma), \quad (6.6)$$

*Proof.* The sequence  $\{u_\epsilon\}$  is defined through a straightforward adaptation of the construction made in [10]. Let  $d(\cdot, t)$  be the signed distance function from  $\Sigma(t)$ , negative inside. For any  $0 < \epsilon < 1$  and  $s \in \mathbb{R}$ , let  $\gamma_\epsilon(s) := \gamma(s/\epsilon)$  and  $\tilde{\gamma}_\epsilon$  be defined as follows:  $\tilde{\gamma}_\epsilon := \gamma_\epsilon$  in  $(0, \epsilon|\log \epsilon|)$ ,  $\tilde{\gamma}_\epsilon := p_\epsilon$  in  $(\epsilon|\log \epsilon|, s_\epsilon^0)$ ,  $\tilde{\gamma}_\epsilon := +1$  in  $(s_\epsilon^0, +\infty)$ , and  $\tilde{\gamma}_\epsilon(s) := -\tilde{\gamma}_\epsilon(-s)$  if  $s < 0$ . Here,  $p_\epsilon$  is an arc of parabola on  $(\epsilon|\log \epsilon|, s_\epsilon^0)$  connecting the points  $(\epsilon|\log \epsilon|, \gamma_\epsilon(\epsilon|\log \epsilon|))$  and  $(s_\epsilon^0, 1)$ , that is  $p_\epsilon(s) := -a_\epsilon(s - s_\epsilon^0)^2 + 1$ ,  $a_\epsilon > 0$ . To find  $a_\epsilon$  and  $t_\epsilon^0$ , we impose the condition  $\tilde{\gamma}_\epsilon \in H^2(\mathbb{R})$ , that gives  $s_\epsilon^0 = \epsilon + \epsilon^3 + \epsilon|\log \epsilon|$  and  $a_\epsilon = \frac{2}{(1+\epsilon^2)^3}$ .

We define

$$u_\epsilon := \begin{cases} \tilde{\gamma}_\epsilon(-d), & \text{in } \mathbb{R}^n \times [0, T) \\ -1 & \text{in } \mathbb{R}^n \times [T, +\infty) \end{cases} \quad (6.7)$$

Then  $\lim_{\epsilon \rightarrow 0} u_\epsilon = \chi_A$  in  $L^1(\mathbb{R}^n \times (0, +\infty))$ , and

$$\lim_{\epsilon \rightarrow 0} \int_0^{+\infty} \int_{\mathbb{R}^n} \frac{1}{\epsilon} (\text{eul}_\epsilon(u))^2 dx dt = c_0 \mathcal{F}_\kappa(\Sigma),$$

see [10]. Therefore to show (6.5) it remains to prove that

$$\lim_{\epsilon} \int_0^{+\infty} \int_{\mathbb{R}^n} \epsilon (u_{\epsilon t})^2 dx dt = c_0 \mathcal{F}_\kappa(\Sigma). \quad (6.8)$$

Applying the coarea formula and (6.7) we have

$$\begin{aligned} & \int_0^{+\infty} \int_{\mathbb{R}^n} \epsilon (u_{\epsilon t})^2 dx dt = \int_0^{+\infty} \int_{\mathbb{R}^n} \epsilon |\tilde{\gamma}_\epsilon(d/\epsilon) d_t|^2 |\nabla d| dx dt \\ &= \int_0^{+\infty} \int_{-\epsilon |\ln \epsilon|}^{\epsilon |\ln \epsilon|} \epsilon^{-1} |\gamma'(s/\epsilon)|^2 \left( \int_{\{d=s\}} (d_t)^2 d\mathcal{H}^{n-1} \right) ds dt \\ &+ \int_0^{+\infty} \int_{\epsilon |\ln \epsilon|}^{\epsilon + \epsilon^3 + \epsilon |\ln \epsilon|} \epsilon |p'_\epsilon(s)|^2 \left( \int_{\{d=s\}} (d_t)^2 d\mathcal{H}^{n-1} \right) ds dt \\ &+ \int_0^{+\infty} \int_{-\epsilon - \epsilon^3 - \epsilon |\ln \epsilon|}^{-\epsilon |\ln \epsilon|} \epsilon |p'_\epsilon(s)|^2 \left( \int_{\{d=s\}} (d_t)^2 d\mathcal{H}^{n-1} \right) ds dt =: I_\epsilon + II_\epsilon + III_\epsilon. \end{aligned}$$

By assumption we have that (3.7) holds for every  $(x, t) \in \Sigma(t)$  with  $t \in [0, T]$ . Moreover letting  $\nu^s(x, t)$  be the outward normal to  $\partial\{d \leq s\}$  in  $(x, t)$ , from [1, Theorem 4.3] we have  $\nu^s(x, t) = \nu(\pi_{\Sigma(t)}(x), t) + o(1)$ . Therefore using the change of variables  $s/\epsilon = z$ , we have

$$\begin{aligned} \lim_{\epsilon \rightarrow 0} I_\epsilon &= \lim_{\epsilon \rightarrow 0} \int_{(0, +\infty)} \int_{-\epsilon |\ln \epsilon|}^{\epsilon |\ln \epsilon|} (\gamma'(z))^2 \left( \int \frac{(\nu_{n+1}^s)^2}{1 - (\nu_{n+1}^s)^2} d\mathcal{H}^{n-1} \right) dz dt \\ &= \lim_{\epsilon \rightarrow 0} \int_{-\epsilon |\ln \epsilon|}^{\epsilon |\ln \epsilon|} (\gamma'(z))^2 \left( \int_{(0, +\infty)} \int \frac{(\nu_{n+1})^2}{1 - (\nu_{n+1})^2} d\mathcal{H}^{n-1} dt + o(1) \right) dz = c_0 \mathcal{F}_v(\Sigma) \end{aligned} \quad (6.9)$$

Since  $\epsilon |p'_\epsilon(s)|^2 = \frac{8\epsilon(s - \epsilon - \epsilon^3 - \epsilon |\log \epsilon|)^2}{(1 + \epsilon^2)^6}$ , making the change of variable  $\sigma = s - \epsilon |\log \epsilon|$ , it follows

$$\int_{\epsilon |\log \epsilon|}^{\epsilon + \epsilon^3 + \epsilon |\log \epsilon|} \epsilon |p'_\epsilon(s)|^2 ds = \frac{32\epsilon}{(1 + \epsilon^2)^6} \int_0^{\epsilon + \epsilon^3} (\tau - \epsilon - \epsilon^3)^2 d\tau = O(\epsilon^4),$$

as  $\epsilon \rightarrow 0$ , hence

$$\begin{aligned} & \lim_{\epsilon \rightarrow 0} II_\epsilon + III_\epsilon \\ &= \lim_{\epsilon \rightarrow 0} \int_{\epsilon |\log \epsilon|}^{\epsilon + \epsilon^3 + \epsilon |\log \epsilon|} \epsilon |p'_\epsilon(s)|^2 \left( \int_{(0, +\infty)} \int_{\partial\{d \leq s\}} \frac{\nu_{n+1}^2}{1 - (\nu_{n+1})^2} d\mathcal{H}^{n-1} dt + o(1) \right) ds \\ &+ \int_{-\epsilon - \epsilon^3 - \epsilon |\log \epsilon|}^{-\epsilon |\log \epsilon|} \epsilon |p'_\epsilon(s)|^2 \left( \int_{(0, +\infty)} \int_{\partial\{d \leq s\}} \frac{\nu_{n+1}^2}{1 - (\nu_{n+1})^2} d\mathcal{H}^{n-1} dt + o(1) \right) ds = 0 \end{aligned} \quad (6.10)$$

Summing up (6.9) and (6.10) we obtain (6.8)

Finally, assertion (ii) is a direct consequence of [26, Section 4.3] and [31].  $\square$

We conclude the paper by observing that a full  $\Gamma$ -convergence and coerciveness result could be used to prove the convergence of the minimizers of  $\mathcal{F}_\varepsilon$  (that reasonably can be supposed to be solutions of (6.4)) to a minimizer of  $\mathcal{F}$ , and obtain as a by-product [28] that the Brakke-flow is a minimizer of the latter functional.

## References

- [1] L. Ambrosio: Lectures Notes on Geometric Evolution Problems, Distance Function and Viscosity Solutions, (Pisa, 1996), 5–93, Springer, Berlin, 2000.
- [2] L. Ambrosio, N. Fusco, D. Pallara: *Special Functions of Bounded Variation and Free Discontinuity Problems*, Clarendon Press (Oxford), 2000.
- [3] L. Ambrosio, N. Gigli, G. Savaré: *Gradient Flows in Metric Spaces and in the Space of Probability Measures*, Lectures in Mathematics, ETH Zürich, Birkhäuser, 2005.
- [4] L. Ambrosio, H.-M. Soner: A level set approach to the evolution of surfaces of any codimension, *J. Differential Geom.* **43** (1996), 693–737.
- [5] G. Auchmuty: Saddle points and existence-uniqueness for evolution equations. *Diff. Integral Eqs.* **6** (1993), 1161–1171
- [6] G. Bellettini, G. Dal Maso, M. Paolini: Semicontinuity and relaxation properties of a curvature depending functional in 2d, *Ann. Scuola Norm. Pisa Cl. Sc.* (4) **20** (1993), 247–297.
- [7] G. Bellettini, M. Paolini: Anisotropic motion by mean curvature in the context of Finsler geometry. *hokkaido Math. J.* **25** (1996), 537–566
- [8] G. Bellettini, A. De Masi, N. Dirr, E. Presutti: Tunnelling in two dimensions, submitted
- [9] G. Bellettini, L. Mugnai: A varifolds representation of the relaxed elastica functional. *Preprint*.
- [10] G. Bellettini, M. Paolini: Approssimazione variazionale di funzionali con curvatura. *Seminario di Analisi Matematica Univ. Bologna, Tecnoprint Bologna* (1993), 87–97.
- [11] H. Brezis: Analyse Fonctionnelle. Théorie et Applications. *Masson* (1987)
- [12] H. Brezis, I. Ekeland: Un principe variationnel associe á certaines equations paraboliques. Le cas dependant du temps *C.R. Acad. Sci. Paris Ser. A* **282** (1976), 1197–1198
- [13] G. Buttazzo, V. Ferone, B. Kawhol: Minimum problems over sets of concave functions and related questions. *Math. Nachr.* **173** (1995), 71–89
- [14] E. De Giorgi, A. Marino, M. Tosques: Evolution equations for a class of nonlinear operators. *Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur.* (8) 75 (1983), no. 1-2, 1–8 (1984).
- [15] K. Ecker: *Regularity theory for mean curvature flow*. Progr. in Nonlinear Diff. Equations and their Applications, vol. 57, Birkhuser Boston 2004.



- [16] L.C. Evans: *Partial Differential Equations*, Graduate Studies in Mathematics, vol. 19, Am. Math. Soc. 1998
- [17] L.C. Evans and J. Spruck: Motion of level sets by mean curvature I, *J. Differential Geom.*, **33** (1991), 635–681.
- [18] L.C. Evans and J. Spruck: Motion of level sets by mean curvature II, *Trans. Amer. Math. Soc.*, **330** (1992), 321–332.
- [19] M. Giaquinta, G. Modica, J. Soucek: *Cartesian Currents in the Calculus of Variations I*. *Ergebnisse der Mathematik und ihrer Grenzgebiete* vol. 37, Springer-Verlag, Berlin 1998.
- [20] N. Ghoussoub: A theory of anti-selfdual Lagrangians: dynamical case. *C. R. Math. Acad. Sci. Paris* **340** (2005), 325–330
- [21] N. Ghoussoub, L. Tzou: A variational principle for gradient flows *Math. Ann.* **330** (2004), 519–549.
- [22] G. Huisken: Flow by mean curvature of convex surfaces into spheres, *J. Differential Geom.* **20** (1984), 237–266.
- [23] D. Hug: Absolute continuity for curvature measures of convex sets I. *Math. Nach.* **195** (1998), 139–158.
- [24] D. Hug: Absolute continuity for curvature measures of convex sets II. *Math. Z.* **232** (1999), 437–485.
- [25] D. Hug: Absolute continuity for curvature measures of convex sets III. *Adv. Math.* **169** (2002), 92–117.
- [26] R.V. Kohn, F. Otto, M.G. Reznikoff, E. Vanden-Eijnden : Action Minimization and Sharp-Interface Limitz for the Stochastic Allen-Cahn Equation. *Comm. Pure Appl. Math.*, in press.
- [27] R.V. Kohn, M.G. Reznikoff, Y. Tonegawa: The Sharp-Interface Limit of the Action Functional for Allen-Cahn in One Space Dimension. *Calc. Var. Part. Diff. Eq.* , in press
- [28] T. Ilmanen: *Convergence of the Allen-Cahn equation to Brakke’s motion by mean curvature*, *J. Differential Geom.*, **38** (1993), 417–461.
- [29] T. Ilmanen: *Elliptic Regularization and Partial Regularity for Motion by Mean Curvature*, *Mem. Amer. Math. Soc.*, **108** (1994).
- [30] J.L. Lions, E. Magenes: *Non-Homogeneous Boundary Value Problems and Applications* vol. I, Springer-Verlag, Berlin (1972).
- [31] M. Roeger, R. Schaetzle: On a modified conjecture of De Giorgi. *to appear on Mathematische Zeitschrift*
- [32] E. Sandier, S. Serfaty: Gamma-convergence of gradient flows with applications to Ginzburg-Landau *Comm. Pure Appl. Math.* **57** (2004), 1627–1672

- [33] R. Schaetzle: Lower semicontinuity of the Willmore functional for currents, *Preprint n. 167/2004*
- [34] L. Simon: *Lectures on geometric measure theory*, Proceedings of the Centre for Mathematical Analysis, Australian National University, Canberra, (1983)
- [35] T. J. Willmore: *An Introduction to Riemannian Geometry*, Clarendon Press (Oxford), 1993.