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NORMAL MODES AND NONLINEAR STABILITY BEHAVIOUR OF DYNAMIC PHASE BOUNDARIES IN ELASTIC MATERIALS

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ABSTRACT. This paper considers an ideal non-thermal elastic medium described by a stored-energy function W. It studies time-dependent configurations with subsonically moving phase boundaries across which, in addition to the jump relations (of Rankine-Hugoniot type) expressing conservation, some kinetic rule g acts as a two-sided boundary condition. The paper establishes a concise version of a normal-modes determinant that characterizes the local-in-time linear and nonlinear (in)stability of such patterns. Specific attention is given to the case where W has two local minimizers U^A, U^B which can coexist via a static planar phase boundary. Dynamic perturbations of such configurations being of particular interest, the paper shows that the stability behaviour of corresponding almost-static phase boundaries is uniformly controlled by an explicit expression that can be determined from derivatives of W and g at U^A and U^B.

1. INTRODUCTION

In this paper we consider the equations

$$\begin{aligned} U_t - \nabla_x V &= 0, \\ V_t - \mathrm{div}_x\, \sigma(U) &= 0, \end{aligned} \tag{1}$$

with

$$\mathrm{curl}_x\, U = 0 \tag{2}$$

of non-thermal elasticity, in which $t \in \mathbb{R}_+$, $x \in \mathbb{R}^d$, $U \in \mathbb{R}_+^{d\times d}$, $V \in \mathbb{R}^d$ $(d \geq 2)$, denote time, space, local deformation gradient and local velocity, respectively. The stress $\sigma(U)$ is supposed to derive as

$$\sigma(U) = \frac{\partial W}{\partial U},$$

from a stored-energy density function $W : \mathbb{R}_+^{d\times d} \to \mathbb{R}$. For $U \in \mathbb{R}_+^{d\times d}$ and $\xi \in \mathbb{R}^d$, let $\kappa_{min}(\xi, U) \in \mathbb{R}$ be the smallest eigenvalue of the acoustic tensor

$$\mathcal{N}(\xi, U) = D^2W(U)(\xi, \xi).$$

We study subsonic phase boundaries, i. e., weak solutions of (1) of form

$$(U, V)(x, t) = \begin{cases} (U^-, V^-), & x \cdot N < st, \\ (U^+, V^+), & x \cdot N > st \end{cases} \tag{3}$$

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with $N \in S^{d-1}$, and

$$s^2 < \min\{\kappa_{min}(N, U^-), \kappa_{min}(N, U^+)\}. \tag{4}$$

Besides the classical Rankine-Hugoniot type jump relations

$$\begin{aligned} -s[U] - [V] \otimes N = 0, \\ -s[V] - [\sigma(U)]N = 0 \end{aligned} \tag{5}$$

and the jump conditions

$$[U] \ \times \ N = 0 \tag{6}$$

associated with (2), solutions (3) are required to satisfy an additional *kinetic rule*

$$g((U^-, V^-), (U^+, V^+), s, N) = 0, \tag{7}$$

where g is a real-valued function on $\Omega = (\mathbb{R}_+^{d\times d} \times \mathbb{R}^d) \times (\mathbb{R}_+^{d\times d} \times \mathbb{R}^d) \times \mathbb{R} \times S^{d-1}$.

The purpose of this paper is to characterize stability properties of general subsonically moving phase boundaries (3). Particular attention is given to the case of small dynamic perturbations of a static configuration

$$(U^*, V^*)(x,t) = \begin{cases} (U^A, 0), & x \cdot N^* < 0, \\ (U^B, 0), & x \cdot N^* > 0 \end{cases} \tag{8}$$

For $U \in \mathbb{R}_+^{d\times d}$ consider the hypotheses

(H1) W is rank-one convex at U (local hyperbolicity).

(H2) For all $\tilde{U}$ near U and all directions of propagation $\xi \in \mathbb{R}^d$, $\xi \neq 0$, the eigenvalues of $\mathcal{N}(\xi, \tilde{U})$ are all semi-simple and their multiplicity is independent of $\tilde{U}$ and ξ (constant multiplicity).

For quadruples $((U^-, V^-), (U^+, V^+), s, N) \in \Omega$, summarize (5) and (7) as

(H3) $h((U^-, V^-), (U^+, V^+), s, N) = 0,$

and formulate

(H4) The $(d^2 + d + 1) \times 2(d^2 + d)$ matrix

$$\big(d_{(U^+,V^+)}h\,, \quad d_{(U^-,V^-)}h\big)\big|_{((U^-,V^-),(U^+,V^+),s,N)}$$

has full rank.

Finally consider the possible assumptions on an equilibrium configuration

(E1) There exist two states $U^A \neq U^B$ in $\mathbb{R}_+^{d\times d}$, local minima of W, and $U^A - U^B$ is rank one. W is rank-one convex both at U^A and U^B.

(E2) Hypothesis (H2) is satisfied both with $U = U^A$ and $U = U^B$. Hypotheses (H3), (H4) hold with $((U^-, V^-), (U^+, V^+), s, N) = ((U^A, 0), (U^B, 0), 0, N^*)$, where $N^* \in S^{d-1}$ such that with some $\upsilon \in \mathbb{R}^d$, $U^B - U^A = \upsilon \otimes N^*$.

The paper shows the following.

Theorem 1. *For every $U \in \mathbb{R}_+^{d\times d}$ satisfying (H1) and any $(s,N) \in \mathbb{R}\times S^{d-1}$ with $s^2 < \kappa_{min}(N,U)$, there exist continuous mappings (analytic for $\operatorname{Re}\lambda > 0$)*

$$\hat{R}^s_{s,N}(U) : \Gamma_N \to \mathbb{C}^{2d\times d}, \quad \hat{R}^u_{s,N}(U) : \ \Gamma_N \to \mathbb{C}^{2d\times d},$$
$$\mathbb{M}_{s,N}(U) : \Gamma_N \to \mathbb{C}^{2d\times 2d}, \quad \mathcal{K}_{s,N}(U) : \ \Gamma_N \to \mathbb{C}^{(d^2+d)\times 2d},$$

on $\Gamma_N := \{(\lambda,\xi) \in \mathbb{C}\times\mathbb{R}^{d-1} : \operatorname{Re}\lambda \geq 0, \xi\cdot N = 0, |\lambda|^2 + |\xi|^2 = 1\}$ with which the following holds:

(i) For any subsonic phase boundary (3) satisfying hypotheses (H3), (H4), and (H1), (H2) for $U = U^-$ as well as for $U = U^+$, the stability behaviour is controlled by the Lopatinski function

$$\hat{\Delta}(U^-,U^+) = \det\begin{pmatrix} \hat{R}^s_{s,N}(U^-) & \hat{Q}(U^-,U^+) & \hat{R}^u_{s,N}(U^+)) \\ \hat{p}^-(U^-,U^+) & \hat{q}(U^-,U^+) & \hat{p}^+(U^-,U^+) \end{pmatrix} : \Gamma_N \to \mathbb{C}, \tag{9}$$

in which

$$\hat{Q}(U^-,U^+)(\lambda,\xi) := \begin{pmatrix} [U]N \\ -(\lambda s[U]N + i[\sigma(U)]\xi) \end{pmatrix},$$
$$\hat{q}(U^-,U^+)(\lambda,\xi) := -\lambda(d_s g) + i(\xi\cdot d_N)g,$$
$$\hat{p}^-(U^-,U^+)(\lambda,\xi) := -(d_{(U^-,V^-)}g)\mathcal{K}_{s,N}(U^-)\hat{R}^s_{s,N}(U^-),$$
$$\hat{p}^+(U^-,U^+)(\lambda,\xi) := (d_{(U^+,V^+)}g)\mathcal{K}_{s,N}(U^+)\hat{R}^u_{s,N}(U^+)\ .$$

More precisely:
(i)$_1$ If $\hat{\Delta}(U^-,U^+)$ has no zero on Γ_N, then (3) is nonlinearly stable.
(i)$_2$ If $\hat{\Delta}(U^-,U^+)$ vanishes for some $(\lambda,\xi)\in\Gamma_N$ with $\operatorname{Re}\lambda > 0$, then (3) is strongly unstable.

(ii) $\mathbb{M}$ and $\mathcal{K}$ are given by simple explicit formulae in terms of first and second derivatives of W. $\hat{R}^s$ and $\hat{R}^u$ represent the right stable and unstable spaces of $\mathbb{M}$. In their whole domain of definition, given by

$$-\kappa_{min}(N,U) < s < \kappa_{min}(N,U),$$

$\mathbb{M}_{s,N}(U), \mathcal{K}_{s,N}(U), \hat{R}^s_{s,N}(U), \hat{R}^u_{s,N}(U)$, depend continuously on (U,s,N).

Corollary 1. *If W satisfies hypotheses (E1) and (E2), then the dynamic stability of the static phase boundary* (3) *is uniformly controlled by the* static-case Lopatinski function

$$\hat{\Delta}(U^A,U^B) : \Gamma_{N^*} \to \mathbb{C},$$

in the sense that if $\hat{\Delta}(U^A,U^B)$ has no zero on Γ_{N^*}, then any phase boundary (3) *with (H3) and (U^-,U^+) sufficiently close to (U^A,U^B) is nonlinearly stable, while if $\hat{\Delta}(U^A,U^B)$ vanishes for some $(\lambda,\xi)\in\Gamma_{N^*}$ with $\operatorname{Re}\lambda > 0$, then any such phase boundary is strongly unstable.*

Theorem 2. *(i) Under the assumptions of Theorem 1, the left stable and the left unstable spaces of* $\mathbb{M}_{s,N}(U)$ *are represented by mappings*

$$\hat{L}^s_{s,N}(U):\ \Gamma_N \to \mathbb{C}^{d\times 2d}, \qquad \hat{L}^u_{s,N}(U):\ \Gamma_N \to \mathbb{C}^{d\times 2d},$$

with the same regularity properties as the $\hat{R}^s_{s,N}(U)$, $\hat{R}^u_{s,N}(U)$.
(ii) The $(d+1)\times(d+1)$ *determinants*

$$\hat{\Delta}^u(U^-,U^+) := \det\begin{pmatrix} \hat{L}^u_{s,N}(U^-)\hat{Q}(U^-,U^+) & \hat{L}^u_{s,N}(U^-)\hat{R}^u_{s,N}(U^+) \\ \hat{q}^u(U^-,U^+) & \hat{p}^u(U^-,U^+) \end{pmatrix}, \tag{10}$$

where

$$\begin{aligned} \hat{p}^u &:= ((d_{(U^+,V^+)}g)\mathcal{K}_{s,N}(U^+) + (d_{(U^-,V^-)}g)\mathcal{K}_{s,N}(U^-))\hat{R}^u_{s,N}(U^+) \\ \hat{q}^u &:= \hat{q}(U^+,U^-) + (d_{(U^-,V^-)}g)\mathcal{K}_{s,N}(U^-)\hat{Q}(U^-,U^+), \end{aligned}$$

and

$$\hat{\Delta}^s(U^-,U^+) = \det\begin{pmatrix} \hat{L}^s_{s,N}(U^+)\hat{R}^s_{s,N}(U^-) & \hat{L}^s_{s,N}(U^+)\hat{Q}(U^-,U^+) \\ \hat{p}^s(U^-,U^+) & \hat{q}^s(U^-,U^+) \end{pmatrix}, \tag{11}$$

where

$$\begin{aligned} \hat{p}^s &:= -((d_{(U^+,V^+)}g)\mathcal{K}_{s,N}(U^+) + (d_{(U^-,V^-)}g)\mathcal{K}_{s,N}(U^-))\hat{R}^s_{s,N}(U^-), \\ \hat{q}^s &:= \hat{q}(U^+,U^-) - (d_{(U^+,V^+)}g)\mathcal{K}_{s,N}(U^+)\hat{Q}(U^-,U^+), \end{aligned}$$

are equivalent to $\hat{\Delta}(U^-,U^+)$,

$$\hat{\Delta}(U^-,U^+) \sim \hat{\Delta}^u(U^-,U^+) \sim \hat{\Delta}^s(U^-,U^+),$$

in the sense that the three differ from each other only by non-vanishing factors.

Remark 1.1. Hypothesis (H1) is both the Legendre-Hadamard ellipticity condition for the static problem and the natural well-posedness criterion for the dynamic problem (cf., e.g. [8, 10]), at some constant state U. Hypothesis (H2) means that the system is symmetrizable hyperbolic with constant multiplicity (cf. notably [24]). (H3) just summarizes the Rankine-Hugoniot relations and the kinetic rule, and (H4) constitutes a non-degeneracy condition the need for which, in general contexts, was pointed out in [9]. The reference configuration described by condition (E1) is standard in steady-state two-phase elasticity (cf., e.g. [26]).

Remark 1.2. An interesting alternative for characterizing stability properties of moving phase boundaries (3) is [15] via the second-order system

$$X_{tt} - \mathrm{div}_x \sigma(\nabla_x X) = 0 \tag{12}$$

and Sakamoto's theory [27, 28]. In contrast to the situation for (12), the static case $s = 0$ is characteristic for (1), which may make it seem difficult at first sight; in fact, however, the constraint (2) prevents the 0-speed mode from being active, so that $s = 0$ poses no problem. One (certainly temporary) advantage of the first-order framework consists in the fact that the theory for *nonlinear non-constant-coefficients* settings is readily available for it in the literature [22, 23, 9].

Remark 1.3. The literature offers significant approaches towards the issues of (i) whether simple kinetic rules like (7) are at all capable to capture at least some of the complexities of phase-boundary dynamics in real solid materials, and (ii) how such rules may be derived from considerations, of deterministic or stochastic nature, at microscopic and mesoscopic levels; cf., e. g., [2, 33, 11]. We indeed view our results as a critical contribution to this modeling problem. To say it negatively: a kinetic rule that passes not even the test of a multidimensional stability analysis can hardly be accepted for a mathematically satisfactory description of stably moving phase boundaries!

Plan of the paper. In Section 2 we gather basic facts about moving interfaces in conservative systems and show how to reduce the order of Lopatinski determinants for general undercompressive or Lax shock fronts. Section 3 describes the objects of our study, namely subsonic phase boundaries for hyperelastic materials. We justify assumptions (H1)-(H4) and (E1), (E2) by discussing the model and explaining its principal features. The central Section 4 contains a careful investigation into the peculiarities of the normal-mode analysis in the specific situation of this model. Section 5 combines the previous findings into proofs of Theorems 1 and 2.

2. Lopatinski determinants and undercompressive shock waves

2.1. **Conservation laws and shock fronts.** Consider a system of n conservation laws in d spatial variables of form

$$u_t + \sum_{j=1}^{d} f_j(u)_{x_j} = 0. \tag{13}$$

where $u \in \mathcal{U} \subset \mathbb{R}^n$, $\mathcal{U}$ open and convex, $f_j \in C^\infty(\mathcal{U};\mathbb{R}^n)$, $j = 1,\dots,d$. We assume that system (13) is *hyperbolic*, i. e., for any $u \in \mathcal{U}$ and all $\xi \in \mathbb{R}^d$, the matrix

$$A(\xi,u) := \sum_{j=1}^{d} \xi_j A_j(u), \quad A_j(u) := Df_j(u),$$

is diagonalizable over $\mathbb{R}$ with C^∞ real eigenvalues $a_1(u;\xi) \le \cdots \le a_n(u;\xi)$ (called characteristic speeds) of fixed algebraic multiplicities $\alpha_1,\dots,\alpha_n$. System (13) supports planar discontinuity fronts

$$u(x,t) = \begin{cases} u^+, & \text{if } x \cdot N > st, \\ u^-, & \text{if } x \cdot N < st, \end{cases} \tag{14}$$

where $u^\pm$ are constant states in $\mathcal{U}$, $u^+ \ne u^-$, $N \in S^{d-1}$ is the direction of propagation and $s \in \mathbb{R}$ is the speed of the discontinuity. The classical Rankine-Hugoniot type jump relations

$$-s[u] + [f(u)]N = 0, \tag{15}$$

where $f := (f_1,\dots,f_d) \in \mathbb{R}^{n\times d}$, are necessary for (14) being a weak solution to (13). We assume that the discontinuity is non-characteristic, that is, there exist integers $o_-, o_+ \in \{1,\dots,n\}$ (the "numbers of outgoing modes") such that

$$a_j(N,u^-) < s < a_k(N,u^-) \quad \text{for all} \quad j \le o_-, \qquad k > o_-, \tag{16}$$

$$a_j(N,u^+) < s < a_k(N,u^+) \quad \text{for all} \quad j \le n - o_+, k > n - o_+, \tag{17}$$

and define a "degree of undercompressivity" as

$$l = o_- + o_+ + 1 - n.$$

Obviously, l counts the amount by which the total number $o = o_- + o_+$ of outgoing modes exceeds $n-1$. The case $l = 0$ corresponds to the classical "Lax type" shock wave [21], while discontinuity waves with $l > 0$ are often called *undercompressive shock waves.* For undercompressive shock waves one augments (15) to

$$0 = h(u^+, u^-, s, N) := \begin{pmatrix} -s[u] + [f(u)]N \\ g(u^+, u^-, s, N) \end{pmatrix}, \tag{18}$$

with the last l *kinetic conditions* given by a "kinetic function" [31, 32, 30, 1, 35, 13, 14]

$$g : \mathcal{U} \times \mathcal{U} \times \mathbb{R} \times S^{d-1} \to \mathbb{R}^l.$$

2.2. **Lopatinski determinants.** Due to the fundamental work of Majda and Métivier [22, 12, 23], the nonlinear stability behaviour of shock fronts is known to be controlled by so called Lopatinski conditions, as they were introduced for hyperbolic problems by Kreiss [20] and Sakamoto [27, 28]. The Majda-Métivier theory has been extended to general undercompressive shocks [14, 4, 9].

The starting point of these analyses is a Fourier decomposition of the constant coefficients linearized problem associated with (13) and (18) at (14). Introducing a level set function ($\phi = x \cdot N - st$ at the reference configuration), we write (18) as

$$h(u^-, u^+, -\phi_t, \nabla_x \phi) = 0.$$

The linearized problem reads

$$\begin{aligned} w_t^\pm + \sum_{j=1}^d A^j(u^\pm) w_{x_j}^\pm &= 0, \quad \text{for } \ x \cdot N - st \gtrless 0, \\ (d_{u^+}h)w^+ + (d_{u^-}h)w^- - (d_s h)\psi_t + (d_N h) \cdot \nabla \psi &= 0, \quad \text{ at } x \cdot N - st = 0. \end{aligned}$$

Considering a single Fourier mode

$$\begin{aligned} w^\pm(x,t) &= \hat{w}^\pm(x \cdot N - st) e^{i\xi \cdot x + \lambda t}, \quad x \cdot N - st \gtrless 0, \\ \psi(x,t) &= \hat{\psi} e^{i\xi \cdot x + \lambda t}, \end{aligned}$$

with $\lambda \in \mathbb{C}$, $\xi \cdot N = 0$, we obtain

$$\begin{aligned} \lambda \hat{w}^\pm + (A_N^\pm - sI)(\hat{w}^\pm)' + iA_\xi^\pm \hat{w}^\pm &= 0, \\ (d_{u^+}h)\hat{w}^+(0) + (d_{u^-}h)\hat{w}^-(0) - \hat{\psi}(\lambda(d_s h) + i(\xi \cdot d_N)h) &= 0, \end{aligned} \tag{19}$$

where $A_\nu^\pm$ is a short-cut for $A(\nu, u^\pm)$, for every $\nu \in \mathbb{R}^d$. The bounded solutions $\hat{w}^+ : [0, +\infty) \to \mathbb{C}^n$, $\hat{w}^- : (-\infty, 0] \to \mathbb{C}^n$ correspond to initial values

$$\hat{w}^+ = \hat{w}(0) \in \text{span } \tilde{R}_+^u, \quad \hat{w}^- = \hat{w}^-(0) \in \text{span } \tilde{R}_-^s,$$

with matrices $\tilde{R}_+^s, \tilde{R}_-^u$ whose columns span the stable and unstable spaces of

$$(A_N^\pm - sI)^{-1}(\lambda I + iA_\xi^\pm),$$

respectively. The basic stability requirement of Lopatinski, Kreiss, Majda and successors is that for $\operatorname{Re}\lambda \geq 0$, no pair $(\hat{w}^-, \hat{w}^+) \in \tilde{R}^s_- \times \tilde{R}^u_+$ allow a solution $\hat{\psi}$ of (19). This yields the *uniform Lopatinski condition* that

$$\Delta(\lambda,\xi) = \det\big((d_{u_-}h)\tilde{R}^s_-(\lambda,\xi), \quad -\lambda(d_s h) + i(d_N h)\xi, \quad (d_{u_+}h)\tilde{R}^u_+(\lambda,\xi)\big)$$

have no zero on

$$\Gamma_N := \{(\lambda,\xi) \in \mathbb{C}\times\mathbb{R}^d \,:\, \operatorname{Re}\lambda \geq 0, \xi\cdot N = 0, |\lambda|^2 + |\xi|^2 = 1\}.$$

In the Lax case ($l = 0$), this *Lopatinski determinant* reads

$$\Delta = \det\begin{pmatrix} R^s_- & Q & R^u_+ \end{pmatrix}$$

with

$$R^{s,u}_\pm(\lambda,\xi) \text{ spanning the stable/unstable space of } (\lambda I + iA^\pm_\xi)(A^\pm_N - sI)^{-1} \tag{20}$$

and

$$Q = Q(\lambda,\xi) = \lambda[u] + i[f(u)]\xi. \tag{21}$$

For undercompressive shocks, one obtains

$$\Delta = \det\begin{pmatrix} R^s_- & Q & R^u_+ \\ -(d_{u^-}g)(A^-_N - sI)^{-1}R^s_- & q & (d_{u^+}g)(A^+_N - sI)^{-1}R^u_+ \end{pmatrix} \tag{22}$$

with (20),(21), and

$$q = q(\lambda,\xi) = -\lambda(d_s g) + i(d_N g)\xi.$$

2.3. **A reduction.** In this subsection we indicate a systematic way of decreasing the order of Lopatinski determinants. The Lopatinski determinant (22) of an undercompressive or Lax shock can be reduced as follows. First, if $l > 0$, multiplying the upper $n\times(n+l)$ block of the matrix on the right hand side of (22) from the left by $(d_{u^-}g)(A^-_N - sI)^{-1}$ and subtracting the result from the lower $l\times(n+l)$ block, we get a matrix of the form

$$\begin{pmatrix} R^s_- & Q & R^u_+ \\ 0 & q^u & p^u \end{pmatrix}.$$

We let $L^u_-(\lambda,\xi)$ denote an $(n - o_-)\times n$ matrix whose rows represent the left unstable space of $(\lambda I + iA^\pm_\xi)(A^\pm_N - sI)^{-1}$. Necessarily, $L^u_- R^s_- = 0$. We multiply

$$\begin{pmatrix} (R^s_-)^t & 0 \\ L^u_- & 0 \\ 0 & I \end{pmatrix}\begin{pmatrix} R^s_- & Q & R^u_+ \\ 0 & q^u & p^u \end{pmatrix} = \begin{pmatrix} (R^s_-)^t R^s_- & * & * \\ 0 & L^u_- Q & L^u_- R^u_+ \\ 0 & q^u & p^u \end{pmatrix}. \tag{23}$$

The matrix on the far left of (23) and the matrix $(R^s_-)^t R^s_-$ are not singular. Thus, the Lopatinski determinant reduces, up to a non-vanishing factor, to the $(o_+ + l)\times(o_+ + l)$ determinant

$$\Delta^u := \det\begin{pmatrix} L^u_- Q & L^u_- R^u_+ \\ q^u & p^u \end{pmatrix}, \tag{24}$$

where

$$p^u := \left((d_{u^+}g)(A^+_N - sI)^{-1} + (d_{u^-}g)(A^-_N - sI)^{-1}\right)R^u_+,$$
$$q^u := q + d_{u^-}g\,(A^-_N - sI)^{-1}Q.$$

Obviously, the reduction can equally well be performed on the right column. Multiplying the upper block by $(d_{u^+}g)(A_N^+ - sI)^{-1}$, subtracting the result from the lower block, and multiplying by a suitable non-singular matrix on the left, we obtain

$$\Delta^s = \det \begin{pmatrix} L_+^s R_-^s & L_+^s Q \\ p^s & q^s \end{pmatrix}, \tag{25}$$

where

$$\begin{aligned} p^s &:= -\big((d_{u^-}g)(A_N^- - sI)^{-1} + (d_{u^+}g)(A_N^+ - sI)^{-1}\big) R_-^s, \\ q^s &:= q - (d_{u^+}g)(A_N^+ - sI)^{-1} Q. \end{aligned}$$

Equation (25) is an $(o_- + l) \times (o_- + l)$ determinant.

Lemma 2.1 (Reduced Lopatinski determinant). *Suppose (14) is a Lax or undercompressive planar shock front of degree $l \geq 0$, satisfying Rankine-Hugoniot jump conditions plus l transition conditions of form $g = 0$. Then the associated Lopatinski determinant and the reduced versions (24), (25) are equivalent to each other,*

$$\Delta \sim \Delta^u \sim \Delta^s,$$

in the sense that they differ only by a non-vanishing factor.

Remark 2.2. For extreme Lax k-shocks, $k = n$ ($o_+ = 0$) or $k = 1$ ($o_- = 0$), the reduced Lopatinski determinants are just products of a left Lopatinski vector with the jump vector,

$$\Delta^u = l_-^u Q \;\text{ if }\; k = n, \qquad \text{and} \qquad \Delta^s = l_+^s Q \;\text{ if }\; k = 1;$$

these expressions are familiar from, e. g. [29, 17]. The general expressions

$$\begin{aligned} \Delta^u &= \det \begin{pmatrix} L_-^u Q & L_-^u R_+^u \end{pmatrix} \\ \Delta^s &= \det \begin{pmatrix} L_+^s R_-^s & L_+^s Q \end{pmatrix} \end{aligned}$$

may be useful for investigations on non-extreme Lax shocks.

3. Elastodynamics and moving phase boundaries

3.1. **Modeling.** We consider an elastic body identified at rest by a reference configuration, which is an open set $\Xi \subset \mathbb{R}^d$, $d \geq 2$, and describe its motion by a mapping $(x,t) \mapsto X$, $\Xi \times [0, +\infty) \to \mathbb{R}^d$, where X is the position at instant t of the particle that was situated in $x \in \Xi$ at rest. We assume that, (i) no thermal effects play a role, (ii) the forces in the medium derive from a stored-energy function $W(\nabla_x X)$, and (iii) there are no external forces. Then basic principles of continuum mechanics show that $X(t,x)$ satisfies the second-order PDE system [8]

$$X_{tt} - \mathrm{div}_x((DW)(\nabla_x X)) = 0. \tag{26}$$

We define the velocity $V : \Xi \times [0, +\infty) \to \mathbb{R}^d$ and the deformation gradient $U : \Xi \times [0, +\infty) \to \mathbb{R}^{d\times d}$ by

$$V := X_t, \quad U := \nabla_x X$$

or, component-wise, by $V_j = \partial X_j / \partial t$, $U_{ij} = \partial X_i / \partial x_j$, $i, j = 1, \dots, d$. Eqs. (26) and various equalities of mixed partial derivatives yield the $d^2 + d$ first-order equations of motion

$$\partial_t U_{ij} - \partial_j V_i = 0, \qquad i,j = 1,\dots,d,$$

$$\partial_t V_i - \sum_{j=1}^{d} \partial_j \left(\frac{\partial W(U)}{\partial U_{ij}} \right) = 0, \qquad i = 1,\dots,d,$$

and the constraints

$$\partial_k U_{ij} = \partial_j U_{ik}, \quad i,j,k = 1,\dots,d.$$

The equations of motion account for conservation of mass, momentum, and more [10]. The stored-energy density W is defined (at most) for $U \in \mathbb{R}_+^{d\times d}$, the set of $d\times d$-matrices with positive determinant (the material does not change orientation), and is fundamentally nonlinear. A basic restriction on W is the principle of *frame indifference*,

$$W(U) = W(OU) \qquad \text{for all } O \in \mathbf{SO}_d(\mathbb{R}),$$

where $\mathbf{SO}_d(\mathbb{R})$ denotes the set of $d \times d$ proper orthogonal real matrices (rotations). This restriction has important consequences [8] for the possible shapes of W; we do not enter any details since they do not matter for the considerations in this paper.

From now on, we assume that $\Xi = \mathbb{R}^d$; due to finite speed of propagation and the fact that we are interested in the local-in-time, local-in-space evolution near the phase boundary, this means no loss of generality.

Notation. In the sequel, we shall adopt the following notation. We write the *stress tensor* as

$$\sigma(U) := \frac{\partial W}{\partial U},$$

and denote U_j and σ_j as the j-th columns of U and σ, respectively; those are,

$$U_j = \begin{pmatrix} U_{1j} \\ \vdots \\ U_{dj} \end{pmatrix}, \quad \text{and} \quad \sigma(U)_j = W_{U_j} = \begin{pmatrix} W_{U_{1j}} \\ \vdots \\ W_{U_{dj}} \end{pmatrix}.$$

Without confusion we occasionally write V_j as the j-th scalar component of the velocity. To express the second derivatives of W, we define for each pair $1 \le i,j \le d$, the $d \times d$ matrices

$$B_i^j(U) := \frac{\partial \sigma_j}{\partial U_i} = \begin{pmatrix} W_{U_{1j}U_{1i}} & \cdots & W_{U_{1j}U_{di}} \\ \vdots & & \vdots \\ W_{U_{dj}U_{1i}} & \cdots & W_{U_{dj}U_{di}} \end{pmatrix} \in \mathbb{R}^{d\times d}.$$

Clearly, each B_i^i is symmetric, and $(B_j^i)^t = B_i^j$.

3.2. **Rank-one convexity and hyperbolicity.** Equations (27) constitute a system of conservation laws of form (13), where $u := (U_1^t, \dots, U_d^t, V^t) \in \mathbb{R}^n$, $n = d^2 + d$. We write $u = (U,V)^t$ for short. The fluxes in (13) are given by

$$f_j(U,V) := - \begin{pmatrix} 0 \\ \vdots \\ V \\ \vdots \\ 0 \\ \sigma(U)_j \end{pmatrix} \in \mathbb{R}^{d^2+d}, \quad j = 1,\dots,d, \tag{27}$$

where the vector V appears in the j-th position. In our notation, the Jacobians are, correspondingly,

$$A_j(U) = -\begin{pmatrix} & & & 0 \\ & & & \vdots \\ & 0 & & I \\ & & & \vdots \\ & & & 0 \\ B_1^j(U) & \cdots & B_d^j(U) & 0 \end{pmatrix} \in \mathbb{R}^{(d^2+d)\times(d^2+d)}, \tag{28}$$

where the 0 matrix in the upper left is the $d^2 \times d^2$ null matrix, and the matrix I on the last column appears in the j-th $d \times d$ block from top to bottom. Notice that A_j is a matrix-valued function of the deformation gradient alone.

Definition 3.1. [10] We define the $d \times d$ *acoustic tensor* $\mathcal{N}(\xi, U)$ as

$$\mathcal{N}(\xi, U) := \sum_{i,j=1}^{d} \xi_i \xi_j B_i^j(U), \tag{29}$$

for $U \in \mathbb{R}_+^{d\times d}$ and for all $\xi \in \mathbb{R}^d$.

Definition 3.2. [8, 10] We say the energy density function W is *rank-one convex* at U if it satisfies Legendre-Hadamard condition,

$$\nu^t \mathcal{N}(\xi, U)\nu > 0, \qquad \text{for all } \nu \text{ and } \xi \text{ in } \mathbb{R}^d, \tag{30}$$

that is, if W is convex along any direction $\xi \otimes \nu$ with rank one (or equivalently, if the acoustic tensor is positive definite for any $\xi \in \mathbb{R}^d$).

Lemma 3.3. *At any state U, if W is rank-one convex, then system* (27) *is hyperbolic.*

Proof. From the expression of the Jacobians we note that $a = 0$ is an eigenvalue with algebraic multiplicity bigger than or equal to d^2. For $a \neq 0$, the eigenvalue problem

$$A(\xi, U)(\tilde{U}, \tilde{V})^t = a(\tilde{U}, \tilde{V})^t$$

with $A_j(\xi, U) := \sum_j \xi_j A_j(U)$ can be written as

$$\xi_i \tilde{V} + a\tilde{U}_i = 0, \quad i = 1, \ldots, d,$$
$$\sum_{i,j} \xi_j B_i^j(U)\tilde{U}_i + a\tilde{V} = 0.$$

Upon substitution,

$$a^2 \tilde{V} = \sum_{i,j} \xi_j \xi_i B_i^j(U)\tilde{V} = \mathcal{N}(\xi, U)\tilde{V}.$$

Since Legendre-Hadamard condition (30) holds, then $a^2 \in \mathbb{R}^+$ for $\xi \neq 0$, and the eigenvalues a of $A(\xi, U)$ are all real for every $\xi \in \mathbb{R}^d$. □

Note that the characteristic speeds of $A(\xi, U), \xi \neq 0$, are the $2d$ square roots of the d positive eigenvalues of the acoustic tensor, with the same multiplicity, and $a = 0$ with algebraic multiplicity $d^2 - d$. More precisely, since by assumptions (H1), (H2) and by continuity, the eigenvalues of $\mathcal{N}(\xi, U), \xi \neq 0$ are all semi-simple and positive with multiplicity depending neither on ξ nor on U near U^A or U^B, we obtain

Corollary 3.4. *Under assumptions (H1) and (H2), for any $\xi \in \mathbb{R}^d \setminus \{0\}$, the characteristic speeds of $A(\xi, U)$ are (with numbering slightly different from section 2.1)*

1. $a_0(\xi, U) = 0$ *with constant algebraic multiplicity* $\alpha_0 = d^2 - d$, *and*
2. $a_j^{\pm}(\xi, U) = \pm\sqrt{\kappa_j(\xi, U)}$, $j = 1, \dots, m$, *where* κ_j *are the* m *distinct semi-simple eigenvalues of* $\mathcal{N}$, $m \leq d$, *with constant multiplicities* α_j, *and with* $\sum \alpha_j = d$.

3.3. **Static rank-one connections and spinodality.** Two constant-state phases coexist in a static configuration when there is a piecewise linear deformation $X(x)$ with

$$\nabla_x X = \begin{cases} U^A, & \text{if } x \cdot N^* < 0, \\ U^B, & \text{if } x \cdot N^* > 0. \end{cases}$$

for some unit vector $N^* \in \mathbb{R}^d$. Continuity of the tangential derivatives of X across the boundary—formally a consequence of (2)— implies

$$U^B = U^A + \upsilon \otimes N^*, \qquad \text{for some} \quad \upsilon \in \mathbb{R}^d; \tag{31}$$

we say that U^A and U^B are *rank-one connected* [3, 26]. By virtue of the second one of the Rankine-Hugoniot relations (5), the function $\psi(\rho) := W(U(\rho))$ with

$$U(\rho) = U^A + \rho\, \upsilon \otimes N^* = U^B - (1 - \rho)\, \upsilon \otimes N^*$$

satisfies $\psi'(0) = \psi'(1) = 0$. Therefore if the Legendre-Hadamard condition (30) is satisfied at any $U(\rho)$ with $\rho \in [0, 1]$, for example at U^A and U^B, then also

$$0 \ > \ \psi''(\tilde{\rho}) = \sum \frac{\partial^2 W}{\partial U_{ij} \partial U_{hk}} (U(\tilde{\rho})) \upsilon_i \upsilon_h N_j^* N_k^*,$$

for an open set of $\tilde{\rho} \in (0, 1)$, i. e., the Legendre-Hadamard condition is violated along the way. This region where hyperbolicity is lost is sometimes called the *spinodal region* [34].

3.4. **Subsonicity.**

Definition 3.5. (i) A speed $s \in \mathbb{R}$ is called subsonic with respect to a direction $N \in S^{d-1}$ and a state $U \in \mathbb{R}_+^{d \times d}$ $N \in S^{d-1}$ if

$$s^2 < \min\{\kappa_j(N, U) : j = 1, \dots, m\}.$$

(ii) A phase boundary (3) is called subsonic if its speed s is subsonic with respect to both (N, U^-) and (N, U^+).

Lemma 3.6. *With o_-, o_+, l denoting the number of outgoing characteristics on the left, the number of outgoing characteristics on the right, and the degree of undercompressivity, respectively, (cf. Sec. 2), a subsonic phase boundary of speed $s > 0$ [$s < 0$] has*

$$o_- = d, o_+ = d^2, l = 1 \quad [o_- = d^2, o_+ = d, l = 1].$$

Proof. This is a direct consequence of Corollary 3.4. ☐

3.5. **Choice of the kinetic rule.** The fact that $l = 1$ is the reason why one takes the function g in the kinetic rule (7) with values in $\mathbb{R}^1$ (as opposed to $\mathbb{R}^l$ with some other l). Clearly, the existence and the stability behavior of a phase boundary solution (3) depend crucially on the actual shape of g. The material-sciences literature provides significant proposals regarding this choice; cf., e. g., [2, 33, 11]. The present paper does not enter this question at all. For the application of its results to a well-motivated general class of kinetic rules, the reader is referred to [15].

4. Normal-modes analysis

We study modes of the matrix field

$$\mathcal{A}(U, s, \lambda, \tilde{\xi}) = C(s)^{-1}(\lambda I + i\sum_{j\neq 1} \xi_j A_j(U))(A_1(U) - sI)^{-1}C(s) \tag{32}$$

with

$$C(s) := \begin{pmatrix} I_d & 0 & 0 \\ 0 & s\, I_{d^2-d} & 0 \\ 0 & 0 & I_d. \end{pmatrix}, \tag{33}$$

assuming that U satisfies hypothesis (H1) of local hyperbolicity and s is subsonic with respect to $((1, 0, \ldots, 0), U)$. The spatio-temporal frequency vector $(\lambda, \tilde{\xi}) = (\lambda, \xi_2, \ldots, \xi_d)$ ranges in

$$\Gamma = \{(\lambda, \tilde{\xi}) \in \mathbb{C} \times \mathbb{R}^{d-1} : \operatorname{Re}\lambda \geq 0, |\lambda|^2 + |\tilde{\xi}|^2 = 1\}$$

For convenience we extend the definition of the acoustic tensor to allow complex directions. Let $(\omega, \tilde{\omega}) \in \mathbb{C} \times \mathbb{C}^{d-1}$, $\omega_1 = \omega, \tilde{\omega} = (\omega_2, \ldots, \omega_d)$ and define

$$\begin{aligned}\tilde{\mathcal{N}}(\omega, \tilde{\omega}, U) &:= \sum_{i,j=1}^{d} \omega_i \omega_j B_i^j(U) \\ &= \omega^2 B_1^1(U) + \omega \sum_{j\neq 1} \omega_j (B_j^1(U) + B_1^j(U)) + \sum_{i,j\neq 1} \omega_i \omega_j B_i^j(U).\end{aligned}$$

We use the short-cut $\tilde{\mathcal{N}}(\omega, \tilde{\omega}) = \tilde{\mathcal{N}}(\omega, \tilde{\omega}, U)$.

Lemma 4.1. *For every $(\lambda, \tilde{\xi}) \in \Gamma$, the $2d$-dimensional linear space*

$$\mathbb{G}(\lambda, \tilde{\xi}) := \{(\lambda Y, i\xi_2 Y, \cdots, i\xi_d Y, Z)^\top \,:\, Y, Z \in \mathbb{C}^d\} \subseteq \mathbb{C}^{d^2+d}, \tag{34}$$

is invariant for $\mathcal{A}(U, s, \lambda, \tilde{\xi})$. The matrix $\mathbb{M} : \mathbb{C}^{2d} \to \mathbb{C}^{2d}$ that expresses the action

$$\mathcal{A}(U, s, \lambda, \tilde{\xi})(\lambda Y, i\xi_2 Y, \cdots, i\xi_d Y, Z)^\top = (\lambda \tilde{Y}, i\xi_2 \tilde{Y}, \cdots, i\xi_d \tilde{Y}, \tilde{Z})^\top$$

of $\mathcal{A}$ on $\mathbb{G}$ as

$$\mathbb{M}(U, s, \lambda, \tilde{\xi}) \begin{pmatrix} Y \\ Z \end{pmatrix} := \begin{pmatrix} M_1^1 & M_1^2 \\ M_2^1 & M_2^2 \end{pmatrix} \begin{pmatrix} Y \\ Z \end{pmatrix} = \begin{pmatrix} \tilde{Y} \\ \tilde{Z} \end{pmatrix}, \tag{35}$$

has the $d \times d$-block components

$$M_1^1 := -\hat{B}(\lambda s I + i\sum_{j\neq 1} \xi_j B_j^1), \tag{36}$$

$$M_1^2 := \hat{B}, \tag{37}$$

$$M_2^1 := (\lambda s I + i\sum_{j\neq 1}\xi_j B_1^j)\hat{B}(\lambda s I + i\sum_{j\neq 1}\xi_j B_j^1) - \lambda^2 I - \sum_{i,j\neq 1}\xi_i\xi_j B_j^i, \qquad (38)$$

$$M_2^2 := -(\lambda s I + i\sum_{j\neq 1}\xi_j B_1^j)\hat{B} \qquad (39)$$

where

$$\hat{B}(s) := (s^2 - B_1^1)^{-1} \qquad (40)$$

and is well defined for all subsonic s including 0.

Remark 4.2. (i) This shows that, while $\mathcal{A}$ is defined only for $s \neq 0$, its restriction

$$\mathcal{A}(U,s)|\mathbb{G} : \mathbb{G} \to \mathbb{G}$$

has a unique continuous/analytic extension to all values (U,s) such that s is subsonic with respect to U, including $s = 0$. (ii) Regarding (40), note that the invertibity of $s^2 - B_1^1$ follows from subsonicity.

For the proof and later we will use

Lemma 4.3.

$$C(s)^{-1}(A_1 - sI) = \begin{pmatrix} -sI & 0 & \cdots & 0 & -I \\ 0 & -I & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & -I & 0 \\ -B_1^1 & -B_2^1 & \cdots & -B_d^1 & -sI \end{pmatrix} \qquad (41)$$

and

$$(A_1 - sI)^{-1}C(s) = \begin{pmatrix} -s\hat{B} & -\hat{B}B_2^1 & \cdots & -\hat{B}B_d^1 & \hat{B} \\ 0 & -I & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & -I & 0 \\ \hat{B}B_1^1 & s\hat{B}B_2^1 & \cdots & s\hat{B}B_d^1 & -s\hat{B} \end{pmatrix}, \qquad (42)$$

continuous/analytic functions including $s = 0$.

Proof. By direct block computation. ☐

Proof of Lemma 4.1. Let $r = (\lambda Y, i\xi_2 Y, \cdots, i\xi_d Y, Z)^\top \in \mathbb{G}$, for some $Y, Z \in \mathbb{C}^d$. With the aid of (42) one can compute

$$(A_1 - sI)^{-1}C(s)r = \begin{pmatrix} \hat{B}(Z - (\lambda s I + i\sum_{j\neq 1}\xi_j B_j^1)Y) \\ -i\xi_2 Y \\ \vdots \\ -i\xi_d Y \\ \hat{B}((\lambda B_1^1 + is\sum_{j\neq 1}\xi_j B_j^1)Y - sZ) \end{pmatrix},$$

where $\hat{B}$ is defined by (40). Multiplying on the left by $C(s)^{-1}(\lambda I + i\sum_{j\neq 1}\xi_j A_j)$ we obtain

$$\mathcal{A}(\lambda, \tilde{\xi}, s)r = \begin{pmatrix} \lambda\tilde{Y} \\ i\xi_2\tilde{Y} \\ \vdots \\ i\xi_d\tilde{Y} \\ \tilde{Z} \end{pmatrix},$$

with

$$\tilde{Y} = \hat{B}(Z - (\lambda s I + i \sum_{j\neq 1} \xi_j B_j^1) Y), \tag{43}$$

and

$$\tilde{Z} = (\lambda s I + i \sum_{j\neq 1} \xi_j B_1^j) \hat{B}((\lambda s I + i \sum_{j\neq 1} \xi_j B_j^1) Y - Z) - \lambda^2 Y - \sum_{i,j\neq 1} \xi_i \xi_j B_j^i Y, \tag{44}$$

showing $\mathcal{A}\,\mathbb{G} \subseteq \mathbb{G}$, as claimed. Clearly, $\dim \mathbb{G} = 2d$. Let us take a look at the mapping $(Y, Z) \mapsto (\tilde{Y}, \tilde{Z})$ defined by (43), (44), which can be written in matrix form as (35) - (39). We are interested in the eigenvalues $\beta = -i\mu \in \mathbb{C}$ of $\mathbb{M}$. Assuming $(Y, Z)^\top \in \mathbb{C}^{2d}$ is an eigenvector, then

$$\begin{aligned} M_1^1 Y + M_1^2 Z &= -i\mu Y, \\ M_2^1 Y + M_2^2 Z &= -i\mu Z. \end{aligned}$$

Hence, $Z = -(M_1^2)^{-1}(i\mu I + M_1^1)Y$ and $Y \neq 0$. Upon substitution,

$$(M_2^1 - M_2^2 (M_1^2)^{-1} M_1^1 - i\mu (M_2^2 (M_1^2)^{-1} + (M_1^2)^{-1} M_1^1) + \mu^2 (M_1^2)^{-1}) Y \;=\; 0.$$

Plugging the expressions for M_j^i into the matrix acting on Y in the last equation and simplifying we obtain

$$\begin{aligned} (\lambda s I + i \sum_{j\neq 1} \xi_j B_1^j) \hat{B} (\lambda s I + i \sum_{j\neq 1} \xi_j B_j^1) - \lambda^2 I - \sum_{i,j\neq 1} \xi_i \xi_j B_j^i + \\ - (\lambda s I + i \sum_{j\neq 1} \xi_j B_1^j) \hat{B} (\lambda s I + i \sum_{j\neq 1} \xi_j B_j^1) + \\ + i\mu (\lambda s + i \sum_{j\neq 1} \xi_j B_j^1) + i\mu (\lambda s + i \sum_{j\neq 1} \xi_j B_1^j) + \mu^2 (s^2 - B_1^1) = \\ = -(\mu^2 B_1^1 + \mu \sum_{j\neq 1} \xi_j (B_j^1 + B_1^j) + \sum_{i,j\neq 1} \xi_j \xi_i B_j^i) - (i\mu s - \lambda)^2 I = \\ = -(\tilde{\mathcal{N}}(\mu, \tilde{\xi}) + (i\mu s - \lambda)^2 I), \end{aligned}$$

yielding

$$(\tilde{\mathcal{N}}(\mu, \tilde{\xi}) + (i\mu s - \lambda)^2 I) Y = 0.$$

□

We will investigate only those modes of $\mathcal{A}(U, s, \cdot, \cdot)$ the amplitudes of which lie in $\mathbb{G}$.

Lemma 4.4. *For $(\lambda, \tilde{\xi}) \in \Gamma$ and s subsonic, the eigenvalues $-i\mu$ of $\mathbb{M}(U, s, \lambda, \tilde{\xi})$ satisfy*

$$\det(\tilde{\mathcal{N}}(\mu, \tilde{\xi}, U) + (i\mu s - \lambda)^2 I) = 0, \tag{45}$$

and $(Y, Z)^\top \in \mathbb{C}^{2d}$ is an eigenvector of $\mathbb{M}$ if and only if

$$\begin{aligned} &Y \in \ker(\tilde{\mathcal{N}}(\mu, \tilde{\xi}) + (i\mu s - \lambda)^2 I), \;\; Y \neq 0, \;\; \text{and} \\ &Z = \Big(s(\lambda - i\mu s) I + i\mu B_1^1 + i \sum_{j\neq 1} \xi_j B_j^1 \Big) Y. \end{aligned} \tag{46}$$

Moreover, for $\operatorname{Re}\lambda > 0$, d of these eigenvalues (counting multiplicities) have $\operatorname{Im}\mu > 0$, while the remaining d of them have $\operatorname{Im}\mu < 0$.

Proof. Clearly, (45) and (46) follow from the proof of Lemma 4.1. The last assertion comes essentially from Hersh' lemma [16]. For completeness, we recall the original argument of Hersh. Suppose $\mu \in \mathbb{R}$ is a solution to (45). Since $\tilde{\mathcal{N}}(\mu, \tilde{\xi}) = \mathcal{N}(\mu, \tilde{\xi})$ is the real acoustic tensor, by hyperbolicity (H1), $-(i\mu s - \lambda)^2$ must be real and positive, implying $\operatorname{Re}\lambda = 0$. Therefore, the roots of (45) in $\operatorname{Re}\lambda > 0$ must all have $\operatorname{Im}\mu \neq 0$. By continuity of the roots and connectedness of Γ, it suffices to count them for $\lambda = \eta \in \mathbb{R}^+$, $\tilde{\xi} = 0$. This yields $\tilde{\mathcal{N}}(\mu, 0) = \mu^2 B_1^1$, and consequently $\mu = i\eta/(\pm\sqrt{\kappa} - s)$, where $\kappa > 0$ is an eigenvalue of B_1^1. By hypothesis, s is subsonic, thus,

$$\operatorname{Im}\mu = \frac{\eta}{+\sqrt{\kappa} - s} > 0, \qquad \text{and,} \quad \operatorname{Im}\mu = \frac{\eta}{-\sqrt{\kappa} - s} < 0,$$

lead us to count d unstable and d stable frequencies. □

Lemma 4.5. *There exist continuous mappings (analytic for* $\operatorname{Re}\lambda > 0$*)*

$$\begin{aligned} &\hat{R}_s^u(U) : \Gamma \to \mathbb{C}^{2d\times d}, \quad \hat{L}_s^u(U) : \Gamma \to \mathbb{C}^{d\times 2d}, \\ &\hat{R}_s^s(U) : \Gamma \to \mathbb{C}^{2d\times d}, \quad \hat{L}_s^s(U) : \Gamma \to \mathbb{C}^{d\times 2d}, \end{aligned} \tag{47}$$

with $\hat{L}_s^u(U)\hat{R}_s^u(U) = I_d$, $\hat{L}_s^s(U)\hat{R}_s^s(U) = I_d$*, spanning right and left invariant spaces of* $\mathbb{M}(U, s, \lambda, \tilde{\xi})$*, spaces that are unstable, respectively stable (at least) for* $\operatorname{Re}\lambda > 0$*. The matrix fields*

$$\hat{R}_s^u(U), \hat{L}_s^u(U), \hat{R}_s^s(U), \hat{L}_s^s(U)$$

depend continuously on U *and* $s \in (-\sqrt{\kappa_{min}(e_1, U)}, \sqrt{\kappa_{min}(e_1, U)})$.

Proof. By Lemma 4.4, it is clear that for $\operatorname{Re}\lambda > 0$ and subsonic s (including $s = 0$) the matrix $\mathbb{M}$ is hyperbolic in the sense that its eigenvalues $-i\mu$ have non-zero real part, and additionally, they split into d stable (with $\operatorname{Im}\mu < 0$), and d unstable (with $\operatorname{Im}\mu > 0$) ones. By standard matrix perturbation theory [18], the stable and unstable spaces are analytic in $(\lambda, \tilde{\xi})$ and we can choose bases arranged in analytic matrix fields (47), for $\operatorname{Re}\lambda > 0$. The next lemma will show that $\mathbb{M}$ satisfies Majda's block structure assumption. This allows us to extend the matrix fields continuously to the imaginary axis $\operatorname{Re}\lambda = 0$, as claimed. □

For a precise statement of the block structure condition see [22, 24] and the references therein.

Lemma 4.6. *The matrix* $\mathbb{M}$ *defined in* (35) *satisfies the block structure condition of Majda on a neighborhood of any point* $(\underline{\lambda}, \underline{\tilde{\xi}}, \underline{U}, \underline{s}) \in \mathbb{C} \times \mathbb{R}^{d-1} \times \mathbb{R}_+^{d\times d} \times \mathbb{R}$*, with* $\underline{U}$ *near* U^A *or* U^B*,* $-\sqrt{\kappa_{min}(e_1, \underline{U})} < \underline{s} < \sqrt{\kappa_{min}(e_1, \underline{U})}$*, and* $\underline{\lambda} = i\underline{\tau}$*,* $\underline{\tau} \in \mathbb{R}$*,* $|\underline{\tau}|^2 + |\underline{\tilde{\xi}}|^2 = 1$.

Proof. We follow Métivier's arguments in [24] closely. Let us denote $\lambda = \eta + i\tau$, with $\eta, \tau \in \mathbb{R}$ and by $z = (U, s, \eta, \tau, \tilde{\xi})$ the parameters in $\mathbb{R}_+^{d\times d} \times \mathbb{R}^{d+2}$. Define the sets

$$\begin{aligned} \Sigma &:= \{(\eta, \tau, \tilde{\xi}) : \ \eta^2 + \tau^2 + |\tilde{\xi}|^2 = 1, \ \eta \geq 0\}, \\ \Sigma_0 &:= \Sigma \cap \{\eta = 0\}, \quad \text{(imaginary axis)}. \end{aligned}$$

$\mathbb{M}(z)$ is a $2d \times 2d$ matrix, defined on a neighborhood $\mathcal{O}$ of $\underline{z} \in \mathbb{R}_+^{d\times d} \times \mathbb{R} \times \Sigma$, and C^∞ in z, where $\underline{U}$ is near U^A or U^B. It suffices to show that $\mathbb{M}$ satisfies the following conditions:

(i) When $\underline{\eta} > 0$, then $\det(i\mu I + \mathbb{M}(\underline{z})) \neq 0$, for all $\mu \in \mathbb{R}$.

(ii) When $\underline{z} \in \mathbb{R}_+^{d\times d} \times \mathbb{R} \times \Sigma_0$, then for all $\underline{\mu} \in \mathbb{R}$ such that $\det(i\underline{\mu} I + \mathbb{M}(\underline{z})) = 0$, there are a positive integer $\alpha \in \mathbb{Z}^+$ and C^∞ functions $\nu(\mu, \tilde{\xi}, U, s)$ and $\theta(z, \mu)$ defined on neighborhoods of $(\underline{\mu}, \underline{\tilde{\xi}}, \underline{U}, \underline{s})$ in $\mathbb{C} \times \mathbb{R}^{d-1} \times \mathbb{R}_+^{d\times d} \times \mathbb{R}$, and $(\underline{z}, \underline{\mu}) \in \mathcal{O} \times \mathbb{C}$, respectively, holomorphic in μ and such that

$$\det(i\mu I + \mathbb{M}(z)) = \theta(z, \mu)(\eta + i\tau + i\nu(\mu, \tilde{\xi}, U, s))^\alpha. \tag{48}$$

Moreover, ν is real when μ is real, and $\theta(\underline{z}, \underline{\mu}) \neq 0$. In addition, there is a C^∞ matrix-valued function $\mathbb{P}(\mu, \tilde{\xi}, U, s)$ on a neighborhhod of $(\underline{\mu}, \underline{\tilde{\xi}}, \underline{U}, \underline{s})$, holomorphic in μ, such that $\mathbb{P}$ is a projection of rank α and

$$\ker(i\mu I + \bar{\mathbb{M}}(z)) = \mathbb{P}(\mu, \tilde{\xi}, U, s)\mathbb{C}^{2d}, \tag{49}$$

when $\eta + i\tau + i\nu(\mu, \tilde{\xi}, U, s) = 0$.

By hyperbolicity, (i) holds. Indeed, suppose $\underline{\eta} > 0$. If $-i\mu$ is an eigenvalue of $\mathbb{M}(\underline{z})$ with $\mu \in \mathbb{R}$, then by Lemma 4.4,

$$\det(\mathcal{N}(\mu, \underline{\tilde{\xi}}, \underline{U}) + (\underline{\eta} + i\underline{\tau} - i\mu\underline{s})^2 I) = 0,$$

where $\mathcal{N}$ is the real acoustic tensor. By assumption (H1), $(\underline{\eta} + i\underline{\tau} - i\mu\underline{s})^2$ must be real and negative, yielding a contradiction with $\underline{\eta} > 0$.

To verify (ii), suppose $\underline{\eta} = 0$. If $\underline{\mu} \in \mathbb{R}$ is such that $\det(i\underline{\mu} I + \mathbb{M}(\underline{z})) = 0$, then by Lemma 4.4,

$$\det(\mathcal{N}(\underline{\mu}, \underline{\tilde{\xi}}, \underline{U}) - (\underline{\tau} - \underline{\mu}\underline{s})^2 I) = 0.$$

Since $(\underline{\tau}, \underline{\tilde{\xi}}) \neq (0,0)$, then $(\underline{\mu}, \underline{\tilde{\xi}}) \neq (0,0)$. Indeed, if $\underline{\tilde{\xi}} = 0$ then $\mathcal{N}(\underline{\mu}, 0) = \underline{\mu}^2 B_1^1$ and $\det(\underline{\mu} B_1^1 - (\underline{\tau} - \underline{\mu}\underline{s})^2 I) = 0$ implies $\underline{\mu} \neq 0$. (In particular, $\underline{\tau} - \underline{\mu}\underline{s} \neq 0$ holds.) Therefore, by (H1) and (H2), there exists a unique $\kappa_j(\underline{\mu}, \underline{\tilde{\xi}}, \underline{U}) > 0$ such that $(\underline{\tau} - \underline{\mu}\underline{s})^2 = \kappa_j(\underline{\mu}, \underline{\tilde{\xi}}, \underline{U})$, or equivalently, there exists a unique root (depending on the sign of $\underline{\tau} - \underline{\mu}\underline{s}$), $a_j = \underline{\mu}\underline{s} + \sqrt{\kappa_j}$ or $a_j = \underline{\mu}\underline{s} - \sqrt{\kappa_j}$, such that

$$\underline{\tau} + a_j(\underline{\mu}, \underline{\tilde{\xi}}, \underline{U}, \underline{s}) = 0.$$

The characteristic speeds a_j are real analytic functions of μ, which can be extended to the complex domain. In addition, the factorization

$$\begin{aligned}\det(i\underline{\mu} I + \mathbb{M}(\underline{z})) &= \theta(\underline{z}, \underline{\mu}) \prod_{l=1}^m (\underline{\tau} - \underline{\mu}\underline{s} + \sqrt{\kappa_l})^{\alpha_l} (\underline{\tau} - \underline{\mu}\underline{s} - \sqrt{\kappa_l})^{\alpha_l} \\ &= \tilde{\theta}(\underline{z}, \underline{\mu})(\underline{\tau} + a_j(\underline{\mu}, \underline{\tilde{\xi}}, \underline{U}, \underline{s}))^{\alpha_j},\end{aligned}$$

with $\theta(\underline{z}, \underline{\mu}) \neq 0$, also extends to a complex neighborhood of $\underline{\mu}$ and to $\lambda = i\tau + \eta \in \mathbb{C}$ (see [19]). Indeed, there exists $\delta > 0$ such that a_j are extended to analytic functions $\nu_j(\mu, \tilde{\xi}, U, s)$ defined for complex μ such that $|\mathrm{Im}\, \mu| \leq \delta(|\mathrm{Re}\, \mu| + |\tilde{\xi}|)$, with $\nu_j = a_j$ whenever μ is real (see [25].) The factorization can also be complexified in a possibly smaller neighborhood of $\underline{\mu}$ and to $\lambda = i\tau + \eta$ where

$$\det(i\mu I + \mathbb{M}(z)) = \tilde{\theta}(z, \mu)(\eta + i\tau + i\nu_j(\mu, \tilde{\xi}, U, s))^{\alpha_j},$$

when $\eta + i\tau + i\nu_j = 0$, i.e. where (48) holds.

Since for each $(\mu, \tilde{\xi}) \neq (0,0)$, $\mu \in \mathbb{R}$, κ_j is a real, positive and semi-simple eigenvalue of $\mathcal{N}(\mu, \tilde{\xi}, U)$ with local constant multiplicity α_j, then the matrix $\Pi_j : \mathbb{C}^d \to \mathbb{C}^d$, defined as

$$\Pi_j(\mu, \tilde{\xi}, U) := -\frac{1}{2\pi i} \int_{|\zeta - \kappa_j(\mu,\tilde{\xi},U)| \leq \varepsilon} (\mathcal{N}(\mu, \tilde{\xi}, U) - \zeta)^{-1} \, d\zeta,$$

with $\varepsilon > 0$ sufficiently small, is a projector of constant rank α_j, C^∞ function of $(\mu, \tilde{\xi}, U)$, for $(\mu, \tilde{\xi}) \neq (0,0)$. Thus,

$$\ker(\mathcal{N}(\underline{\mu}, \underline{\tilde{\xi}}, \underline{U}) - (\underline{\tau} - \underline{\mu}\underline{s})^2 I) = \Pi_j(\underline{\mu}, \underline{\tilde{\xi}}, \underline{U})\mathbb{C}^d.$$

By analytic continuation, the projectors Π_j extend analytically to μ in a small neighborhood of $\underline{\mu}$. Thus, if we define $\mathbb{P}_j(\underline{\mu}, \underline{\tilde{\xi}}, \underline{U}, \underline{s}) : \mathbb{C}^{2d} \to \mathbb{C}^{2d}$ as

$$\mathbb{P}_j(\underline{\mu}, \underline{\tilde{\xi}}, \underline{U}, \underline{s}) := \begin{pmatrix} \Pi_j(\underline{\mu}, \underline{\tilde{\xi}}, \underline{U}) & 0 \\ i(\underline{s}(\underline{\tau} - \underline{\mu}\underline{s})I + \underline{\mu}B_1^1 + \sum_{k\neq 1} \underline{\xi}_k B_k^1)\Pi_j(\underline{\mu}, \underline{\tilde{\xi}}, \underline{U}) & 0 \end{pmatrix},$$

then it is clearly a projector of constant rank α_j, which can be extended analytically to some small complex neighborhood of $\underline{\mu}$ as well. By Lemma 4.4, $\mathbb{M}$ has an eigenvector $(Y, Z)^\top \in \mathbb{C}^{2d}$ with eigenvalue $-i\underline{\mu}$ if and only if

$$Z = (\underline{s}(\underline{\tau} - \underline{\mu}\underline{s})I + i\underline{\mu}B_1^1 + \sum_{k\neq 1} \underline{\xi}_k B_k^1)Y, \quad \text{and,} \quad (\mathcal{N}(\underline{\mu}, \underline{\tilde{\xi}}, \underline{U}) - (\underline{\tau} - \underline{\mu}\underline{s})^2 I)Y = 0.$$

Hence $Y \in \Pi_j(\underline{\mu}, \underline{\tilde{\xi}}, \underline{U})\mathbb{C}^d$, and by construction, it is then clear that

$$\ker(i\underline{\mu}I + \mathbb{M}(\underline{z})) = \mathbb{P}_j(\underline{\mu}, \underline{\tilde{\xi}}, \underline{U}, \underline{s})\mathbb{C}^{2d}.$$

Analogously, this relation and the projector $\mathbb{P}_j$ extend to a small complex neighborhood of $\underline{\mu}$ such that (49) holds.

In this fashion, we have shown that $\mathbb{M}$ satisfies the generic Assumption 1.4 in [24]. By Theorem 1.5 in the same reference, and taking the parameter a in [24] as $a := (U, s)$, we can conclude that $\mathbb{M}$ satisfies the block structure condition on a neighborhood of $\underline{z}$, as claimed. □

Remark 4.7. Continued inspection shows that the characteristic polynomial of $\mathcal{A}$ is

$$\pi(\mu) = (i\mu s - \lambda)^{d^2-d} \det(\tilde{\mathcal{N}}(\mu, \tilde{\xi}) + (i\mu s - \lambda)^2 I).$$

Eqs. (1) thus possess also a Lopatinski frequency

$$\beta_* = -i\mu_* = -\frac{\lambda}{s}.$$

This frequency creates a bad singularity around $s = 0$.

5. Proofs of Theorems 1 and 2

In principle, we could compose the original $(d^2 + d + 1) \times (d^2 + d + 1)$ Lopatinski determinant as in (22). However, Theorems 1,2 establish determinants of distinctly smaller order and, more importantly, in them (i) the singular mode mentioned at the end of Sec. 4 does not appear, while (ii) the characteristic case $s = 0$ is not singular.

The key point for proving Theorems 1,2 is the observation that due to the constraints (2), the whole Fourier analysis can be restricted to a $2d$-dimensional bundle (over Γ) of amplitudes and: this bundle is $\mathbb{G}$!

We assume without loss of generality that N is the positive direction of the x_1-axis.

Lemma 5.1. *Consider any solution to (1),(2) of the form*

$$(U,V)(x,t) = (\hat{U}(x_1 - st), \hat{V}(x_1 - st)) \exp(i\tilde{\xi}\cdot\tilde{x} + \lambda t),$$

where $x = (x_1, \tilde{x})$, $\tilde{x} = (x_2, \dots, x_d) \in \mathbb{R}^{d-1}$ *and* $(\lambda, \tilde{\xi}) \in \Gamma$. *Then, necessarily,*

$$C(s)^{-1}(A_1 - sI)(\hat{U}(\cdot), \hat{V}(\cdot))^\top \in \mathbb{G}(\lambda, \tilde{\xi}).$$

Proof. Constraints (2) account for $\partial_j U_k = \partial_k U_j$ for all $j,k = 1,\dots,d$. Hence, for $j,k \neq 1$, we have $i\xi_j \hat{U}_k = i\xi_k \hat{U}_j$, which, in turn, implies that

$$\hat{U}_j = -i\xi_j Y, \quad \text{for some } \ Y \in \mathbb{C}^d, \quad \text{all } \ j \neq 1. \tag{50}$$

Equations (2) also imply $\partial_1 U_j = \partial_j U_1$, for $j \neq 1$, which leads to

$$\hat{U}_j' = i\xi_j \hat{U}_1. \tag{51}$$

From the first equations in (1) we have $\partial_t U_{ij} = \partial_j V_i$ for all i,j, implying

$$\lambda \hat{U}_1 - s\hat{U}_1' = \hat{V}', \tag{52}$$

$$\lambda \hat{U}_j - s\hat{U}_j' = i\xi_j \hat{V}, \quad \text{for all } j \neq 1. \tag{53}$$

From (50), (51), and (53), we obtain for $j \neq 1$,

$$\begin{aligned} i\xi_j \hat{V} &= \lambda \hat{U}_j - s\hat{U}_j' \\ &= -i\xi_j \lambda Y - is\xi_j \hat{U}_1, \end{aligned}$$

or simply,

$$\hat{V} = -(\lambda Y + s\hat{U}_1).$$

Hence $(\hat{U}, \hat{V})^\top(\cdot)$ has the form

$$\begin{pmatrix} \hat{U}_1 \\ \hat{U}_2 \\ \vdots \\ \hat{U}_d \\ \hat{V} \end{pmatrix} = \begin{pmatrix} \hat{U}_1 \\ -i\xi_2 Y \\ \vdots \\ -i\xi_d Y \\ -(\lambda Y + s\hat{U}_1) \end{pmatrix}.$$

Multiplying on the left by $C(s)^{-1}(A_1 - sI)$ we get

$$\begin{aligned} C(s)^{-1}(A_1 - sI)(\hat{U}, \hat{V})^\top(\cdot) &= \begin{pmatrix} -sI & 0 & \cdots & 0 & -I \\ 0 & -I & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & -I & 0 \\ -B_1^1 & -B_2^1 & \cdots & -B_d^1 & -sI \end{pmatrix} \begin{pmatrix} \hat{U}_1 \\ -i\xi_2 Y \\ \vdots \\ -i\xi_d Y \\ -(\lambda Y + s\hat{U}_1) \end{pmatrix} \\ &= \begin{pmatrix} \lambda Y \\ i\xi_2 Y \\ \vdots \\ i\xi_d Y \\ Z \end{pmatrix} \in \mathbb{G}(\lambda, \tilde{\xi}), \end{aligned}$$

where $Z := (s^2 - B_1^1)\hat{U}_1 + (i\sum_{j\neq 1}\xi_j B_j^1 - \lambda s I)Y$. This proves the result. □

Proof of the Theorems. At every point $(\lambda,\tilde{\xi}) \in \Gamma$, the isomorphism $\mathcal{J}(\lambda,\tilde{\xi}) : \mathbb{C}^{2d} \to \mathbb{G}$, with matrix representation

$$\mathcal{J} = \begin{pmatrix} \lambda I & 0 \\ i\xi_2 I & 0 \\ \vdots & \vdots \\ i\xi_d I & 0 \\ 0 & I \end{pmatrix}, \tag{54}$$

translates between $\mathbb{G}$ and its natural coordinates representation we have introduced in Sec. 4. For example, the stable and unstable right bundles of $\mathbb{M}$ readily lift to stable and unstable bundles of $\mathcal{A}$ as

$$\begin{aligned} \check{R}^s(\lambda,\tilde{\xi}) &:= \mathcal{J}(\lambda,\tilde{\xi})\hat{R}^s(\lambda,\tilde{\xi}), \\ \check{R}^u(\lambda,\tilde{\xi}) &:= \mathcal{J}(\lambda,\tilde{\xi})\hat{R}^u(\lambda,\tilde{\xi}). \end{aligned} \tag{55}$$

Note that $\check{R}^s$ and $\check{R}^u$ are not both the full stable and unstable bundle of $\mathcal{A}$, since amplitudes associated with the singular frequency μ_* are not captured. However, these latter are exactly the ones which are not compatible with the constraint (2), while $\check{R}^s$ and $\check{R}^u$ comprise all stable and unstable amplitudes which are compatible with the constraint. Consequently, we simply work directly with $\hat{R}^s, \hat{R}^u$.

Using the jump conditions (5), (6), here

$$\begin{aligned} -s[U_1] - [V] &= 0, \\ -s[V] - [\sigma(U)_1] &= 0, \\ [U_j] &= 0, \quad \text{for all } j \neq 1, \end{aligned}$$

we find the jump vector

$$Q = \begin{pmatrix} \lambda[U_1] \\ is\xi_2[U_1] \\ \vdots \\ is\xi_d[U_1] \\ -(\lambda s[U_1] + i\sum_{j\neq 1}\xi_j[\sigma(U)_j]) \end{pmatrix}.$$

Thus,

$$Q = C(s)\mathcal{J}\hat{Q} \quad \text{with } \hat{Q} = \begin{pmatrix} [U_1] \\ -(\lambda s[U_1] + i\sum_{j\neq 1}\xi_j[\sigma(U)_j]) \end{pmatrix},$$

and we work directly with $\hat{Q}$.

These considerations together with Lemma 5.1 and the findings of Sec. 4 on the matrix field $\mathbb{M}$ show that $\hat{\Delta}$ indeed controls the linear stability in the affirmative ($(\mathrm{i})_1$) as well as in the negative ($(\mathrm{i})_2$) — in coordinates, on state space, which differ from the original ones, i. e., the conserved quantities, by the linear transformation

$$C(s)^{-1}(A_1(U) - sI).$$

By Lemma 4.3, this transformation is regular, also if $s = 0$. Together with (H4), this uniformity makes the whole involved nonlinear analysis of [22, 23, 9] applicable, also if $s = 0$, and allows to define, in turn,

$$\mathcal{K}(U^\pm) := (A_1(U^\pm) - sI)^{-1}C(s)\mathcal{J}.$$

Theorem 1 is proved.

Theorem 1 given, Theorem 2 is proved in exactly the same way as Lemma 2.1. Finally, Corollary 1 is a special case of Theorem 1, of course exactly with $s = 0$, which we have accentuated because of its importance.

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