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Elastic energy stored in a crystal induced by screw dislocations: from discrete to continuum

by

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ELASTIC ENERGY STORED IN A CRYSTAL INDUCED BY SCREW DISLOCATIONS: FROM DISCRETE TO CONTINUUM

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Abstract. This paper deals with the passage from discrete to continuum in modeling the static elastic properties, in the setting of anti-planar linear elasticity, of vertical screw dislocations in a cylindrical crystal.

We study, in the framework of Γ-convergence, the asymptotic behavior of the elastic stored energy induced by dislocations as the atomic scale ε tends to zero, in the regime of dilute dislocations, i.e., rescaling the energy functionals by $1/\varepsilon^2|\log\varepsilon|$.

First we consider a continuum model, where the atomic scale is introduced as an internal scale, usually called core radius. Then we focus on a purely discrete model. In both cases, we prove that the asymptotic elastic energy as $\varepsilon \to 0$ is essentially given by the number of dislocations present in the crystal. More precisely the energy per unit volume is proportional to the length of the dislocation lines, so that our result recovers in the limit as $\varepsilon \to 0$ a line tension model.

Keywords: Crystals, Analysis of microstructure, Stress concentration, Calculus of variations.

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Contents

1. Introduction

This paper deals with energy minimization methods to model static elastic properties of *dislocations* in crystals. We are interested in the asymptotic behavior of the elastic energy stored in a crystal, induced by a configuration of dislocations, as the atomic scale tends to zero. Our approach is completely variational, and is based on Γ-*convergence*. First we consider a continuum model, where the atomic scale is introduced as an internal scale, usually called *core radius.* Then we focus on a purely discrete model.

We consider the setting of *anti-planar* linear elasticity, so that all the physical quantities involved in our model will be defined on a domain $\Omega \subset \mathbb{R}^2$, which represents a horizontal section of an infinite cylindrical crystal. The elastic energy associated with a vertical *displacement* $u : \Omega \to \mathbb{R}$, in absence of dislocations, is given by [1]

$$E(\nabla u) := \int_\Omega |\nabla u(x)|^2 \, dx.$$

Now we assume that vertical *screw dislocations* are present in the crystal. To model the presence of dislocations we follow the general theory of eigenstrains; [2] namely to any dislocation corresponds a pre-existing strain in the reference configuration. In this framework a configuration of screw dislocations in the crystal can be represented by a measure on Ω which is a finite sum of *Dirac masses* of the type $\mu := \sum_i z_i |\mathbf{b}| \, \delta_{x_i}$. Here x_i's represent the intersection of the dislocation lines with Ω, $\mathbf{b}$ is the so-called *Burgers vector*, which in this anti-planar setting is a vertical fixed vector whose modulus depends on the specific crystal lattice, and $z_i \in \mathbb{Z}$ represent the multiplicity of the dislocations. The class of admissible strains associated with a dislocation μ is given by the fields whose circuitation around the dislocations x_i are equal to $z_i|\mathbf{b}|$. These fields by definition have a singularity at each x_i and are not in $L^2(\Omega; \mathbb{R}^2)$. To set up a variational formulation it is then convenient to introduce an internal scale ε called core radius, which is comparable with the atomic scale, and to remove balls of radius ε around each point of singularity x_i. More precisely to any admissible strain ψ we associate the elastic energy

$$E_\varepsilon(\psi) := \int_{\Omega_\varepsilon(\mu)} |\psi(x)|^2 \, dx,$$

where $\Omega_\varepsilon(\mu) := \Omega \setminus \cup_i B_\varepsilon(x_i)$. Given a dislocation μ, the elastic energy induced by μ, in the absence of external forces, is given by minimizing $E_\varepsilon(\psi)$ among all admissible strains.

This variational formulation has been recently considered in [4] to study the limit of the elastic energy induced by a fixed configuration of dislocations as the atomic scale ε tends to zero. The authors prove in particular that the energy [3] is of the order $|\log \varepsilon|$.

In this paper we study the asymptotic behavior of the elastic energy induced by the dislocations in terms of Γ-convergence, in this regime of energies, i.e., rescaling the energy functionals by $|\log \varepsilon|$, without assuming the dislocations to be fixed, uniformly bounded in mass nor *well separated.* Let us describe our continuum model in more details.

Given a dislocation μ, the class of admissible strains $\mathcal{AS}_\varepsilon(\mu)$ associated with μ is given (we consider for simplicity $|\mathbf{b}| = 1$) by

$$\mathcal{AS}_\varepsilon(\mu) := \{\psi \in L^2(\Omega_\varepsilon(\mu); \mathbb{R}^2) : \text{curl } \psi = 0 \text{ in } \Omega_\varepsilon(\mu) \text{ in the sense of distributions,}$$
$$\int_{\partial A} \psi(s) \cdot \tau(s) \, ds = \mu(A) \text{ for every open set } A \subset \Omega \text{ with } \partial A \text{ smooth and with } \partial A \subset \Omega_\varepsilon(\mu)\}.$$

[1] For simplicity we will assume the *shear modulus* of the crystal equal to 1.

[2] We refer the reader to [12], [13] for an exhaustive treatment of the subject.

[3] Though the Burgers vector should be rescaled by ε, in this and in the following results the Burgers vector is kept fixed. The relevant physical case can be recovered simply introducing a supplementary rescaling term of the order $1/\varepsilon^2$ in the energy functionals.

Here $\tau(s)$ is the oriented tangent vector to ∂A at the point s, and the integrand $\psi(s)\cdot\tau(s)$ is intended in the sense of traces (see Theorem 2, page 204 in [6]). The (rescaled) elastic energy associated with μ is given by

$$\mathcal{F}_\varepsilon(\mu) := \frac{1}{|\log\varepsilon|}\Big(\min_{\psi\in\mathcal{AS}_\varepsilon(\mu)} E_\varepsilon(\psi) + |\mu|(\Omega)\Big). \tag{1.1}$$

The first term in the energy represents the elastic energy far from the dislocations, where the crystal is assumed to have a linear hyper-elastic behavior (see Remark 2.6 for a partial justification of the use of linear elasticity in this region far from dislocations). The second addendum, $|\mu|(\Omega)$, is the *total variation* of μ on Ω, and represents the elastic energy stored in the region surrounding the dislocations (the introduction of this energy in the continuum model will be fully justified by our discrete model; see Remark 2.6 and Remark 3.2 for more details).

In Theorem 2.3 we prove that the Γ-limit of the functionals $\mathcal{F}_\varepsilon$, with respect to the *flat convergence* of the dislocations (see (2.2)), is given by the functional $\mathcal{F}$ defined by

$$\mathcal{F}(\mu) := \frac{1}{2\pi}|\mu|(\Omega). \tag{1.2}$$

The asymptotic elastic energy per unit volume is essentially proportional to the number (and hence to the length) of the screw dislocations. Then we recover in the limit as $\varepsilon\to 0$ a *line tension model.*

A similar result was obtained in [8], [9], where the authors considered a phase field model for dislocations proposed by [10]. They study the asymptotic behavior, in different rescaling regimes, of the elastic energy given by the interaction of a non-local $H^{1/2}$ elastic energy, a non-linear *Peierls potential* and a *pinning condition*, under the assumption that only one *slip system* is active. In particular, in the energy regime corresponding to a rescaling of the order $1/|\log\varepsilon|$, their Γ-limit is given by the sum of a bulk term, taking in account the pinning condition, and a surface term concentrated on the dislocation lines.

More in general, energy concentration phenomena as a result of the logarithmic rescaling are nowadays classical in the theory of *Ginzburg-Landau* type functionals, to model vortices in superfluidity and superconductivity. We refer the reader to [3], [1] and to the references therein.

Even if we do not assume the dislocations to be fixed, our analysis shows that, as $\varepsilon_n\to 0$, the most convenient way to approximate a dislocation μ with multiplicity $z_i\equiv 1$, is the constant sequence $\mu_n\equiv\mu$. In this respect the main point is that there is no homogenization process able to approach an energy less then $1/2\pi(\lim_{\varepsilon_n}|\mu_n^+(\Omega)-\mu_n^-(\Omega)|)$, where μ_n^+ (respectively μ_n^-) represents the positive (respectively the negative) part of μ_n. The latter term can be interpreted (see Remark 2.4) as the quantity usually referred to as *geometrically needed dislocations*. We conclude that in this energy regime there is no energetic advantage for the crystal to create micro-patterns of dislocations.

These considerations become trivial if one assume a priori an uniform bound for the number of dislocations. However sequences $\{\mu_n\}$ with uniformly bounded energy (i.e., such that $\mathcal{F}_{\varepsilon_n}(\mu_n)\le C$)) are not in general bounded in mass. The main reason is that one can easily construct a *short dipole* $\mu_n := \delta_{x_n}-\delta_{y_n}$, with $|\mu_n|(\Omega)=2$, $|x_n-y_n|\to 0$, and whose energetic contribution is vanishing. On the other hand, it is clear that the flat norm of these dipoles is also vanishing. This is the reason why we study the Γ-convergence with respect to the flat convergence, instead of weak convergence of measures. We prove that the *equi-coercivity* property holds with respect to the flat convergence: sequences μ_n with uniformly bounded energy, up to a subsequence, converge with respect to the flat norm. The proof of this result represents the main difficulty in our analysis.

Our strategy is to divide the dislocations in clusters such that in each cluster the distance between the dislocations is of order ε_n^δ, for some $0 < \delta < 1$. The family of clusters with *zero effective multiplicity*, namely such that the sum of the multiplicities in the cluster is equal to zero, will play the role of short dipoles. Using the estimate $|\mu_n|(\Omega) \leq E|\log \varepsilon_n|$, which follows directly from $\mathcal{F}_{\varepsilon_n}(\mu_n) \leq E$ and from the second addendum in (1.1), we deduce that these clusters give a vanishing contribution to the flat norm, of order $|\log \varepsilon_n|^2 \varepsilon_n^\delta \to 0$. We identify the remaining clusters (with non zero effective multiplicity) with Dirac masses, obtaining a sequence of measures $\tilde{\mu}_n := \sum_i z_i \delta_{x_i}$. Assume for a while that $\tilde{\mu}_n$ is uniformly bounded in mass, so that (up to a subsequence) $\tilde{\mu}_n$ weakly converge to a measure μ. We prove that $\mu_n - \tilde{\mu}_n$ has vanishing flat norm and we deduce the convergence of μ_n to μ with respect to the flat norm.

The main point in the previous argument is that $\tilde{\mu}_n$ is uniformly bounded in mass. This will be a consequence of the key Lemma 2.5, where we prove that each cluster with non zero effective multiplicity gives a positive energetic contribution. It is in this step that we have to prevent the possibility of a homogenization process, able to approach a vanishing energy through a sequence μ_n with non zero geometrically needed dislocations. This analysis will be performed through an iterative process, which will require the introduction of several meso-scales. The choice of the number of meso-scales involved in this analysis as $\varepsilon_n \to 0$ will play a fundamental role in our proof.

The last part of the paper is devoted to a purely discrete model. We consider the illustrative case of a *square lattice* of size ε with *nearest neighborhood interactions*, following along the lines of the more general theory introduced in [2].

In this framework a displacement u is a function defined on the set Ω^0_ε of points of the lattice; the strains β are defined on the bonds of the lattice, i.e., on the class Ω^1_ε of the oriented segments of the square lattice; finally a dislocation is represented by an integer function α defined on the class Ω^2_ε of the oriented squares of the lattice.

The class of admissible strains associated with a dislocation α is given by the strains $\xi : \Omega^1_\varepsilon \to \mathbb{R}$ satisfying

$$\mathbf{d}\,\xi = \alpha, \tag{1.3}$$

where the operator $\mathbf{d}$ is defined in (3.3). The condition expressed in (1.3) means that for every $Q \in \Omega^2_\varepsilon$, the discrete circuitation of ξ on ∂Q is equal to $\alpha(Q)$. The rescaled elastic energy induced by α is given by

$$\mathcal{F}^d_\varepsilon(\alpha) := \frac{1}{|\log \varepsilon|} \min_{\xi:\Omega^1_\varepsilon \to \mathbb{R}:\, \mathbf{d}\,\xi = \alpha} E^d_\varepsilon(\xi), \tag{1.4}$$

where the discrete elastic energy $E^d_\varepsilon(\xi)$ is defined in (3.2).

Every dislocation $\alpha : \Omega^2_\varepsilon \to \mathbb{Z}$ is induced by a function $\beta : \Omega^1_\varepsilon \to \mathbb{Z}$ defined on the bonds of the lattice, such that $\mathbf{d}\,\beta = \alpha$. The class of admissible strains can be then written in the equivalent form

$$\{\beta + \mathbf{d}\,u,\ u : \Omega^0_\varepsilon \to \mathbb{R}\},$$

where $\mathbf{d}\,u : \Omega^1_\varepsilon \to \mathbb{R}$ is now the discrete gradient of u defined in (3.1). In this respect β can be interpreted as a discrete eigenstrain inducing the dislocation α. If $\alpha = \mathbf{d}\,\beta = 0$, then β is a compatible strain, i.e, $\beta = \mathbf{d}\,v$ for some displacement v, and the associated stored energy is equal to zero. Therefore α measures the degree of incompatibility of the eigenstrain β.

In Theorem 3.4 we restate our Γ-convergence result given in Theorem 2.3 in this discrete setting. The proof can be obtained as an immediate consequence of the results achieved in the continuum model, introducing an interpolation procedure with suitable commutative properties with respect to the chains $u \xrightarrow{\mathbf{d}} \xi \xrightarrow{\mathbf{d}} \alpha$ and $u \xrightarrow{\nabla} \psi \xrightarrow{\text{curl}} \mu$ (see Proposition 3.3).

In the discrete model the behavior of the elastic stored energy is controlled by the lattice size ε, and it is not necessary (see Remark 3.2) to introduce a supplementary internal scale, as the core radius in the continuum case, to divide the stored elastic energy into two contributions, one concentrated in a region surrounding the dislocations, and the other one far away. In this respect the discrete model seems very natural, and provides a theoretical justification of the continuum model.

2. The continuum model

Here we introduce our continuum model for vertical screw dislocations in an infinite cylindrical crystal, in the setting of anti-planar elasticty. We will study, in terms of Γ-convergence, the asymptotic behavior of the elastic stored energy in the crystal induced by the screw dislocations as the atomic internal scale ε tends to zero. For the definition and the basic properties of Γ-convergence, we refer the reader to [5].

2.1. **Description of the continuum model.** In this section we introduce the space of *screw dislocations* X, and the elastic energy functionals $\mathcal{E}_\varepsilon : X \to \mathbb{R}$. We are in the setting of anti-planar elasticty, so that the physical quantities involved in the model will be defined on a horizontal section $\Omega \subset \mathbb{R}^2$ of the infinite cylindrical crystal.

2.1.1. *The space X of screw dislocations.* Let Ω be an open bounded subset of $\mathbb{R}^2$. For any $x \in \Omega$ we denote by δ_x the Dirac mass centered at x. Let us denote by $\mathcal{M}(\Omega)$ the class of Radon measures on Ω. The space of screw dislocations X is given by

$$X := \{\mu \in \mathcal{M}(\Omega) : \mu = \sum_{i=1}^{M} z_i \delta_{x_i},\, M \in \mathbb{N},\, x_i \in \Omega,\, z_i \in \mathbb{Z}\}. \tag{2.1}$$

The support set of μ defined by $\mathrm{supp}(\mu) := \{x_1, \dots, x_M\}$ represents the set where the dislocations are present, while the leading coefficients z_i in (2.1) are the *multiplicities* of the dislocations at the points x_i's. We endow X with the *flat norm* $\|\mu\|_f$ [4] defined by

$$\|\mu\|_f = \inf\{|S|,\, S \in \mathcal{S} : \partial S \llcorner \Omega = \mu\} \qquad \text{for every } \mu \in X. \tag{2.2}$$

Here $\mathcal{S}$ denotes the family of finite formal sum of oriented segments L_i in $\overline{\Omega}$, with extreme points p_i and q_i, and with integer multiplicity m_i; the mass of $S = \sum_{i=1}^{M} m_i L_i$ is given by

$$|S| := \sum_{i=1}^{M} |m_i||L_i| = \sum_{i=1}^{M} |m_i||q_i - p_i|,$$

and ∂S is defined by

$$\partial S := \sum m_i(\delta_{q_i} - \delta_{p_i}). \tag{2.3}$$

We will denote by $\mu_n \xrightarrow{f} \mu$ the convergence of μ_n to μ with respect to to the flat norm.

[4]Here we are adapting the classical definition of the flat norm to our context of Dirac masses confined in an open bounded set. For the canonical definition of the flat norm and its main properties we refer the reader to [7], [11].

2.1.2. *Admissible strains.* Let us fix $\varepsilon > 0$. Given $\mu \in X$, we denote by

$$\Omega_\varepsilon(\mu) := \Omega \setminus \bigcup_{x_i \in \operatorname{supp}(\mu)} B_\varepsilon(x_i),$$

where $B_\varepsilon(x_i)$ denotes the ball of center x_i and radius ε.

The class $\mathcal{AS}_\varepsilon(\mu)$ of *admissible strains* associated with μ is given by

(2.4) $\mathcal{AS}_\varepsilon(\mu) := \{\psi \in L^2(\Omega_\varepsilon(\mu); \mathbb{R}^2) : \operatorname{curl}\, \psi = 0$ in $\Omega_\varepsilon(\mu)$ in the sense of distributions,

$\int_{\partial A} \psi(s) \cdot \tau(s)\, ds = \mu(A)$ for every open set $A \subset \Omega$ with ∂A smooth and with $\partial A \subset \Omega_\varepsilon(\mu)\}$.

Here $\tau(s)$ is the oriented tangent vector to ∂A at the point s, and the integrand $\psi(s) \cdot \tau(s)$ is intended in the sense of traces (see Theorem 2 page 204 in [6]).

Remark 2.1. Let $\psi \in \mathcal{AS}_\varepsilon(\mu)$. By the definition (2.4), we have in particular that the circuitation of ψ along ∂A is equal to zero for every $A \subset \Omega_\varepsilon(\mu)$, which is consistent with curl $\psi = 0$ in $\Omega_\varepsilon(\mu)$ in the sense of distributions.

2.1.3. *The elastic energy.* The elastic energy associated with a strain $\psi \in \mathcal{AS}_\varepsilon(\mu)$ is given by

$$E_\varepsilon(\psi) := \|\psi(x)\|^2_{L^2(\Omega_\varepsilon(\mu);\mathbb{R}^2)}.$$

The elastic energy functional $\mathcal{E}_\varepsilon : X \to \mathbb{R}$ is defined by

$$\mathcal{E}_\varepsilon(\mu) := \min_{\psi \in \mathcal{AS}_\varepsilon(\mu)} E_\varepsilon(\psi) + |\mu|(\Omega) \qquad \text{for every } \mu \in X. \tag{2.5}$$

The first addendum represents the elastic energy stored in a region far from the dislocations. The second addendum is the *total variation* of μ on Ω, and represents the so-called *core energy*, namely the energy stored in the balls $B_\varepsilon(x_i)$ (see Remark 2.6 and Remark 3.2 for some comment on the core energy in this model).

Remark 2.2. Note that the minimum problem in (2.5) is well posed. In fact, following the *direct method of calculus of variations* let ψ_n be a minimizing sequence. We have that $\|\psi_n\|_{L^2(\Omega_\varepsilon(\mu);\mathbb{R}^2)} \le C$ for some positive constant C. Therefore (up to a subsequence) $\psi_n \rightharpoonup \psi$ for some $\psi \in L^2(\Omega_\varepsilon(\mu); \mathbb{R}^2)$. Moreover (see Theorem 2 page 204 in [6]) we have $\psi \in \mathcal{AS}_\varepsilon(\mu)$. By the fact that the L^2 norm is lower semicontinuous with respect to weak convergence we deduce that ψ is a minimum point.

2.2. **The Γ-convergence result.** In this section we study the asymptotic behavior, as $\varepsilon \to 0$, of the elastic energy functionals $\mathcal{E}_\varepsilon$ defined in (2.5) in terms of Γ-convergence. To this aim let us rescale the functionals $\mathcal{E}_\varepsilon$ setting

$$\mathcal{F}_\varepsilon := \frac{1}{|\log \varepsilon|} \mathcal{E}_\varepsilon, \tag{2.6}$$

and let us introduce the candidate Γ-limit $\mathcal{F} : X \to \mathbb{R}$ defined by

$$\mathcal{F}(\mu) := \frac{1}{2\pi} |\mu|(\Omega) \qquad \text{for every } \mu \in X. \tag{2.7}$$

Theorem 2.3. *The following Γ-convergence result holds.*

i) *Equi-coercivity. Let $\varepsilon_n \to 0$, and let $\{\mu_n\}$ be a sequence in X such that $\mathcal{F}_{\varepsilon_n}(\mu_n) \le E$ for some positive constant E independent of n. Then (up to a subsequence) $\mu_n \xrightarrow{f} \mu$ for some $\mu \in X$.*

ii) Γ-*convergence. The functionals* $\mathcal{F}_{\varepsilon_n}$ Γ*-converge to* $\mathcal{F}$ *as* $\varepsilon_n \to 0$ *with respect to the flat norm, i.e., the following inequalities hold.*

Γ*-liminf inequality:* $\mathcal{F}(\mu) \leq \liminf \mathcal{F}_{\varepsilon_n}(\mu_n)$ *for every* $\mu \in X$, $\mu_n \xrightarrow{f} \mu$ *in* X.

Γ*-limsup inequality: given* $\mu \in X$, *there exists* $\{\mu_n\} \subset X$ *with* $\mu_n \xrightarrow{f} \mu$ *such that* $\limsup \mathcal{F}_{\varepsilon_n}(\mu_n) \leq \mathcal{F}(\mu)$.

Remark 2.4. Let $\mu_n \xrightarrow{f} \mu$ in X. Let us denote by μ_n^+ (and respectively by μ_n^-) the positive (and respectively the negative) part of the measure μ_n. Assume that $\mathrm{supp}(\mu_n) \subset U \subset \overline{U} \subset \Omega$, for some open set U. Then we have

$$\liminf \mathcal{F}_{\varepsilon_n}(\mu_n) \geq \frac{1}{2\pi}|\mu|(\Omega) \geq \frac{1}{2\pi} \lim_n \left(|\mu_n^+(\Omega) - \mu_n^-(\Omega)|\right).$$

Therefore the limit of the quantity $|\mu_n^+(\Omega) - \mu_n^-(\Omega)|$, usually referred as *geometrically needed dislocations*, is (up to the constant $1/2\pi$) a lower bound for the Γ-limit $\mathcal{F}$.

2.2.1. *Equi-coercivity.* The prove of the equi-coercivity property is quite technical and requires some preliminary result. Before giving the rigorous proof, let us recall the main steps of our strategy.

The first step is to divide the dislocations in clusters of the type

$$A_r(\mu_n) := \bigcup_{x \in \mathrm{supp}(\mu_n)} B_r(x), \tag{2.8}$$

such that in each cluster the distance between the dislocations is of order $r = \varepsilon_n^\delta$, for some $0 < \delta < 1$. The main point is that the family of clusters of dislocations with zero *effective multiplicity*, namely such that the sum of the multiplicities of the dislocations in each cluster is equal to zero, gives a vanishing contribution to the flat norm, while the number of the remaining clusters with non zero effective multiplicity is uniformly bounded. This latter fact is more delicate, and will be done in the following key lemma, which states that each cluster with non zero effective multiplicity gives a positive energetic contribution.

Lemma 2.5. *Let* $0 < \delta < 1$ *be fixed. Let* $\varepsilon_n \to 0$ *and let* $\{\mu_n\}$ *be a sequence such that* $\mathcal{F}_{\varepsilon_n}(\mu_n) \leq E$ *for some positive* E *independent of* n. *Moreover assume that for every* n *there exists a connected component* C_n *of* $A_{\varepsilon_n^\delta}(\mu_n)$ *(defined according to* (2.8)*) with* $C_n \subset \Omega$, *and* $\mu_n(C_n) \neq 0$. *Then*

$$\liminf_n \frac{1}{|\log \varepsilon_n|} \int_{C_n} |\psi_n(x)|^2 \, dx \geq \frac{1}{2\pi}(1-\delta) \quad \textit{for every sequence } \{\psi_n\} \subset \mathcal{AS}_{\varepsilon_n}(\mu_n).$$

Before giving the formal proof of the lemma, let us explain its main ideas. Let C_n be a cluster of dislocations with effective multiplicity equal to $\lambda \neq 0$, and let γ be a closed curve surrounding C_n, and which does not intersect any other cluster of dislocations. Then the circuitation of every admissible strain $\psi_n \in \mathcal{AS}_{\varepsilon_n}(\mu_n)$ on γ is equal to λ. We get an estimate of the tangential component of ψ_n on γ, and hence of the L^2 norm of ψ_n on γ. Extending this estimate on an annular neighborhood F_n of the cluster C_n, by means of polar coordinates, we want to obtain an estimate of the elastic energy stored around C_n independently of ε_n. However the rigorous proof will require some additional effort. The main obstruction to the previous argument is that in general F_n may intersect other clusters of dislocations. Our strategy is then to iterate the previous construction in sub-clusters of C_n. We consider a certain number of exponents $0 < \delta = \delta_0 < \delta_1 < \dots \delta_{N_\varepsilon} = 1$, where $N_\varepsilon \to \infty$ as $\varepsilon_n \to 0$. For *almost every scale* ε^{δ_i} we find a sub-cluster of C_n of size ε^{δ_i} with non zero effective multiplicity, surrounded by some annulus F^i_n, such that the sets F^i_n are pairwise disjoint, and the elastic energy stored in each F^i_n is of the order $\delta_i - \delta_{i-1}$. We deduce that the elastic energy stored in C_n is at least of the order $1-\delta$.

The starting point of our analysis is the following estimate, which easily follows by the second addendum in (2.5) and by (2.6).

$$\sharp\, \mathrm{supp}(\mu_n) \le E|\log \varepsilon_n| \qquad \text{for every } n \in \mathbb{N}, \tag{2.9}$$

where $\sharp\, \mathrm{supp}(\mu_n)$ denotes the number of elements in $\mathrm{supp}(\mu_n)$.

Proof of Lemma 2.5. For every n, let us set

$$s^i_n := \delta + \frac{i}{H|\log(\log \varepsilon_n)|} \qquad \text{for every } 0 \le i \le M_n,$$

where $H>0$ is a fixed positive constant, and M_n is the integer part of $H(1-\delta)|\log(\log\varepsilon_n)|$. For every n, and for every $0 \le i \le M_n$, let us set

$$A^i_n := A_{\varepsilon_n^{s^i_n}}(\mu_n) \cap C_n = \bigcup_{x \in \mathrm{supp}(\mu_n)\cap C_n} B_{\varepsilon_n^{s^i_n}}(x).$$

Let $\mathcal{C}^i_n$ be the family of the connected components $C^{i,j}_n$ of A^i_n with $\mu_n(C^{i,j}_n) \ne 0$. Let us now split the indices $i \in [0, M_n)$ into two families J_n and I_n by setting $i \in J_n$ if every element in $\mathcal{C}^i_n$ contains at least two elements of $\mathcal{C}^{i+1}_n$; $i \in I_n$ otherwise. Let us prove that

$$\liminf_n \frac{\sharp\, I_n}{M_n} = 1 - o(1/H), \tag{2.10}$$

where $\sharp E$ denotes the number of elements of a set E, and $o(1/H) \to 0$ as $H \to \infty$. To this aim, note that if $i \in J_n$, then $\sharp\,\mathcal{C}^{i+1}_n \ge 2\sharp\,\mathcal{C}^i_n$, and hence, using that $\sharp\,\mathcal{C}^i_n$ is non decreasing with respect to i, and recalling that by assumption $\sharp\,\mathcal{C}^0_n = \sharp\,\{C_n\} = 1$, we have

$$\sharp\,\mathrm{supp}(\mu_n) \ge \sharp\,\mathcal{C}^{M_n}_n \ge 2^{\sharp\, J_n}\sharp\,\mathcal{C}^0_n = 2^{\sharp\, J_n}.$$

By (2.9) we obtain

$$E|\log\varepsilon_n| \ge \sharp\,\mathrm{supp}(\mu_n) \ge 2^{\sharp\, J_n}.$$

Therefore $\sharp\, J_n \le C|\log(\log\varepsilon_n)|$ for some positive constant C independent of H. We deduce that

$$\limsup_n \frac{\sharp\, J_n}{M_n} \le \limsup_n \frac{C|\log(\log\varepsilon_n)|}{H(1-\delta)|\log(\log\varepsilon_n)| - 1} = \frac{C}{H(1-\delta)},$$

which together with $\sharp\, J_n + \sharp\, I_n \equiv M_n$ gives (2.10).

Let $i \in I_n$. By definition there exists C^{i,j_i}_n in $\mathcal{C}^i_n$ which contains exactly one element $C^{i+1,k_{i+1}}_n$ in $\mathcal{C}^{i+1}_n$. Let $p^i_n \in C^{i+1,k_{i+1}}_n$ be chosen arbitrarily. Let us define

$$R^i_1 := E|\log\varepsilon_n|\varepsilon_n^{s^{i+1}_n}, \qquad R^i_2 := \frac{1}{2}\varepsilon_n^{s^i_n}.$$

Moreover, for every connected component $C^{i+1,l}_n$ of A^{i+1}_n different from $C^{i+1,k_{i+1}}_n$ let us set

$$r^l_1 := \min_{x \in C^{i+1,l}_n} |x - p^i_n| - \varepsilon_n^{s^i_n+1}, \qquad r^l_2 := \max_{x \in C^{i+1,l}_n} |x - p^i_n| + \varepsilon_n^{s^i_n+1}. \tag{2.11}$$

Let us set

$$L^i_n := (R^i_1, R^i_2) \setminus \cup_l \overline{(r^l_1, r^l_2)}, \qquad F^i_n := \{x \in \mathbb{R}^2 : |x - p^i_n| \in L^i_n\}. \tag{2.12}$$

The set L_n^i is a finite union of open intervals, and hence it can be written in the form

$$L_n^i = \bigcup_{l=1}^{N_n^i} (\varepsilon_n^{\alpha_{i,n}^l}, \varepsilon_n^{\beta_{i,n}^l}). \tag{2.13}$$

Let us claim the following properties concerning L_n^i and F_n^i.

a) For every i, n and for every $r \in L_n^i$, we have $\mu_n(B_r(p_n^i)) \neq 0$;
b) For every n the sets F_n^i are pairwise disjoint;
c) For every i we have $\sum_{l=1}^{N_n^i} |\alpha_{i,n}^l - \beta_{i,n}^l| = (1+o(1/n))/\big(H|\log(\log\varepsilon_n)|\big)$, where $o(1/n)$ is a function independent of i tending to zero as $n \to \infty$.

Let $\psi_n \in \mathcal{AS}_{\varepsilon_n}(\mu_n)$. Using the claim, we are in position to estimate the L^2 norm of ψ_n on each F_n^i using polar coordinates. By property $a)$, and by the fact that ψ_n is an admissible strain (see (2.4)) we have

$$\int_{(0,2\pi)} |\psi_n(r,\theta)| r d\theta \geq |\mu_n(B_r(p_n^i))| \geq 1 \qquad \text{for every } r \in L_n^i.$$

Using Jensen's inequality and property $c)$ above, we deduce the following estimate.

$$\int_{F_n^i} |\psi_n(x)|^2 \, dx = 2\pi \sum_{l=1}^{N_n^i} \int_{(\varepsilon_n^{\alpha_{i,n}^l}, \varepsilon_n^{\beta_{i,n}^l})} r \left(\frac{1}{2\pi} \int_{(0,2\pi)} |\psi_n(r,\theta)|^2 \, d\theta \right) dr \geq$$
$$2\pi \sum_{l=1}^{N_n^i} \int_{(\varepsilon_n^{\alpha_{i,n}^l}, \varepsilon_n^{\beta_{i,n}^l})} r \left(\frac{1}{2\pi} \int_{(0,2\pi)} |\psi_n(r,\theta)| \, d\theta \right)^2 dr \geq \frac{1}{2\pi} \sum_{l=1}^{N_n^i} \int_{(\varepsilon_n^{\alpha_{i,n}^l}, \varepsilon_n^{\beta_{i,n}^l})} \frac{1}{r} dr =$$
$$\frac{1}{2\pi} \sum_{l=1}^{N_n^i} |\alpha_{i,n}^l - \beta_{i,n}^l| |\log\varepsilon_n| = \frac{1}{2\pi} \frac{1+o(1/n)}{H|\log(\log\varepsilon_n)|} |\log\varepsilon_n|.$$

Summing the previous inequality over all $i \in I_n$ and dividing by $|\log\varepsilon_n|$, in view of (2.10) and property $b)$ above, we deduce

$$\liminf_n \frac{1}{|\log\varepsilon_n|} \int_{C_n} |\psi_n(x)|^2 \, dx \geq \liminf_n \frac{1}{|\log\varepsilon_n|} \sum_{i\in I_n} \int_{F_n^i} |\psi_n(x)|^2 \, dx \geq$$
$$\liminf_n \frac{1}{2\pi} \frac{\sharp I_n (1+o(1/n))}{H|\log(\log\varepsilon_n)|} = \liminf_n \frac{1}{2\pi} \frac{M_n(1-o(1/H))}{H|\log(\log\varepsilon_n)|}.$$

Letting $H \to \infty$, and recalling that M_n is the integer part of $H(1-\delta)|\log(\log\varepsilon_n)|$, we obtain

$$\liminf_n \frac{1}{|\log\varepsilon_n|} \int_{C_n} |\psi_n(x)|^2 \, dx \geq \liminf_n \frac{1}{2\pi} \frac{H(1-\delta)|\log(\log\varepsilon_n)| - 1}{H|\log(\log\varepsilon_n)|} = \frac{1}{2\pi}(1-\delta). \tag{2.14}$$

In order to conclude the proof, we have to prove the claim.

Property $a)$ follows directly by the construction of the intervals L_n^i. More precisely one can easily check that by construction every cluster of dislocations $C_n^{i+1,l} \in A_n^{i+1}$ intersecting $B_r(p_n^i)$ is actually contained in $B_r(p_n^i)$, and since $i \in I$, then all these clusters except one have zero effective multiplicity.

Let us pass to the proof of property (b). Let $i_1, i_2 \in I_n$, with $i_1 < i_2$. We will use the notation used in the previous constructions, with i replaced respectively by i_1, i_2. In particular let $p_n^{i_1} \in C_n^{i_1+1,k_{i_1+1}}$, $p_n^{i_2} \in C_n^{i_2+1,k_{i_2+1}}$. We divide the proof into two cases.

In the first case we assume that $p_n^{i_2} \in C_n^{i_1+1,k_{i_1+1}}$. In this case, since $|p_n^{i_1} - p_n^{i_2}| \leq \varepsilon_n^{s_n^{i_1+1}}$, we have

$$R_2^{i_2}(p_n^{i_2}) + |p_n^{i_1} - p_n^{i_2}| = \frac{1}{2}\varepsilon_n^{s_n^{i_2}} + |p_n^{i_1} - p_n^{i_2}| \leq 2\varepsilon_n^{s_n^{i_1+1}} < E|\log\varepsilon_n|\varepsilon_n^{s_n^{i_1+1}} = R_1^{i_1}(p_n^{i_1}),$$

and hence $B_{R_2^{i_2}}(p_n^{i_2}) \subset B_{R_1^{i_1}}(p_n^{i_1})$, so that $F_n^{i_1}$ and $F_n^{i_2}$ are disjoint.

Let us consider now the case $p_n^{i_2} \notin C_n^{i_1+1,k_{i_1+1}}$. In this case we have $p_n^{i_2} \in C_n^{i_1+1,l}$ for some connected component $C_n^{i_1+1,l}$ of $A_n^{i_1+1}$ different from $C_n^{i_1+1,k_{i_1+1}}$. Therefore by (2.11), (2.12) we deduce that

$$B_{\varepsilon_n^{s_n^{i_1+1}}}(p_n^{i_2}) \cap F^{i_1} = \emptyset.$$

On the other hand

$$R_2^{i_2} = \frac{1}{2}\varepsilon_n^{s_n^{i_2}} \leq \frac{1}{2}\varepsilon_n^{s_n^{i_1+1}}.$$

We deduce that

$$B_{R_2^{i_2}}(p_n^{i_2}) \cap F^{i_1} = \emptyset.$$

This concludes the proof of b).

Let us pass to the proof of c). For every i we have

$$|L_n^i| \geq (R_2^i - R_1^i) - \tilde{C}|\log\varepsilon_n|\varepsilon_n^{s_n^{i+1}} \geq \frac{1}{2}\varepsilon_n^{s_n^i} - C|\log\varepsilon_n|\varepsilon_n^{s_n^{i+1}} = \varepsilon_n^{s_n^i}(\frac{1}{2} - C|\log\varepsilon_n|\varepsilon_n^{(s_n^{i+1}-s_n^i)}), \tag{2.15}$$

where C is a constant depending only on E. On the other hand, fixing the quantity

$$\sum_{l=1}^{N_n^i} |\varepsilon_n^{\alpha_{i,n}^l} - \varepsilon_n^{\beta_{i,n}^l}|$$

in (2.13), and maximizing $|L_n^i|$ with respect to the position of the indices $\alpha_{i,n}^l$, $\beta_{i,n}^l$, we obtain

$$|L_n^i| \leq \varepsilon_n^{s_n^i}(\frac{1}{2} - \varepsilon_n^{\sum_{l=1}^{N_n^i}|\alpha_{i,n}^l - \beta_{i,n}^l|}). \tag{2.16}$$

From (2.15) and (2.16) we deduce that

$$\varepsilon_n^{s_n^i}(\frac{1}{2} - \varepsilon_n^{\sum_{l=1}^{N_n^i}|\alpha_{i,n}^l - \beta_{i,n}^l|}) \geq \varepsilon_n^{s_n^i}(\frac{1}{2} - C|\log\varepsilon_n|\varepsilon_n^{(s_n^{i+1}-s_n^i)}).$$

Therefore

$$\varepsilon_n^{\sum_{l=1}^{N_n^i}|\alpha_{i,n}^l - \beta_{i,n}^l|} \leq C|\log\varepsilon_n|\varepsilon_n^{(s_n^{i+1}-s_n^i)} = C|\log\varepsilon_n|\varepsilon_n^{\frac{1}{H|\log(\log\varepsilon_n)|}},$$

from which it easily follows c). □

We are now in a position to prove the equi-coercivity. The idea is to identify the clusters of dislocations with non zero effective multiplicity with Dirac masses, obtaining a sequence of measures $\tilde{\mu}_n := \sum_i z_i \delta_{x_i}$. By Lemma 2.5 we will deduce that $\tilde{\mu}_n$ is uniformly bounded in mass, so that (up to a subsequence) $\tilde{\mu}_n$ weakly converge to a measure μ. We prove that $\mu_n - \tilde{\mu}_n$ has vanishing flat norm and we deduce the convergence of μ_n to μ with respect to the flat norm.

Proof of the equi-coercivity property. Let $0 < S < T < 1$. For every $S < \delta < T$, let us consider the set $A_{\varepsilon_n^\delta}(\mu_n)$ defined as in (2.8). Let us denote by $\mathcal{C}_{\delta,n}$ the family of connected components $C^1_{\delta,n}, \dots, C^{M_{\delta,n}}_{\delta,n}$ of $A_{\varepsilon_n^\delta}(\mu_n)$ which are contained in Ω, and satisfying $\mu(C^l_{\delta,n}) \neq 0$. By Lemma 2.5 we deduce that $\sharp\mathcal{C}_{\delta,n} = M_{\delta,n}$ is bounded by a constant M independent of n and δ. For every n, let us consider the finite family of indices

$$I_n := \{t^1_n, \dots, t^{M_n}_n\},\ S \leq t^1_n < t^2_n < \ldots < t^{M_n}_n \leq T, \quad M_n \leq M,$$

given by the discontinuity points of the function $\delta \to \sharp\mathcal{C}_{\delta,n}$. Up to a subsequence, we have that the set of accumulation points of I_n is of the type

$$I_\infty := \{\delta^1, \dots, \delta^H\},\ S \leq \delta^1 < \delta^2 < \ldots < \delta^H \leq T, \quad H \leq M.$$

Let

$$[\delta_1, \delta_2] \subset (S, T) \setminus I_\infty.$$

For n big enough we have that the function $\delta \to \sharp\mathcal{C}_{\delta,n}$ is constant on $[\delta_1, \delta_2]$. Since each element of $\mathcal{C}_{\delta_1,n}$ contains at least one element of $\mathcal{C}_{\delta_2,n}$, we deduce that actually each element $C^l_{\delta_1,n} \in \mathcal{C}_{\delta_1,n}$ contains exactly one element $C^l_{\delta_2,n} \in \mathcal{C}_{\delta_2,n}$.

We want to prove that for every sequence $\{H_n\} \subset \mathcal{C}_{\delta_1,n}$ we have

$$\limsup_n |\mu_n(H_n)| \leq K \tag{2.17}$$

for some positive constant K independent of n.

Let G_n be the only element of $\mathcal{C}_{\delta_2,n}$ contained in H_n. The idea, as in the proof of Lemma 2.5, is to evaluate the elastic energy of every admissible strain $\psi_n \in \mathcal{AS}_{\varepsilon_n}(\mu_n)$, stored in the region between G_n and H_n, using polar coordinates. To this aim, let $p_n \in G_n$, and let us define

$$R_1 := E|\log \varepsilon_n|\varepsilon_n^{\delta_2}, \qquad R_2 := \varepsilon_n^{\delta_1}.$$

Moreover, for every connected component C^l_n of $A_{\varepsilon_n^{\delta_2}}(\mu_n)$ different from G_n let us set

$$r^l_1 := \min_{x \in C^l_n} |x - p_n| - \varepsilon_n^{\delta_2}, \qquad r^l_2 := \max_{x \in C^l_n} |x - p_n| + \varepsilon_n^{\delta_2}.$$

Let us set

$$L_n := (R_1, R_2) \setminus \cup_l \overline{(r^l_1, r^l_2)}, \qquad F_n := \{x \in \mathbb{R}^2 : |x - p_n| \in L_n\}.$$

The set L_n is a finite union of open intervals, and hence it can be written in the form

$$L_n = \bigcup_{l=1}^{N_n} (\varepsilon_n^{\alpha^l_n}, \varepsilon_n^{\beta^l_n}). \tag{2.18}$$

Arguing as in the proof of properties *a*) and *c*), in the proof of Lemma 2.5, we deduce that the following properties hold.

i) For every n and for every $r \in L_n$, we have $\mu_n(B_r(p_n)) = \mu_n(G_n)$.
ii) $\sum_{l=1}^{N_n} |\alpha^l_n - \beta^l_n| = (1 + o(1/n))(\delta_2 - \delta_1)$, where $o(1/n) \to 0$ as $n \to \infty$.

Let $\psi_n \in \mathcal{AS}_{\varepsilon_n}(\mu_n)$. By the fact that ψ_n is an admissible strain and by property *i*) we deduce that

$$\int_{(0,2\pi)} |\psi_n(r, \theta)| r d\theta \geq |\mu(G_n)| \qquad \text{for every } r \in L_n.$$

Using Jensen's inequality and property $ii)$ above we obtain

$$\int_{F_n} |\psi_n(x)|^2\,dx = 2\pi \sum_{l=1}^{N_n} \int_{(\varepsilon_n^{\alpha_n^l},\varepsilon_n^{\beta_n^l})} r \left(\frac{1}{2\pi} \int_{(0,2\pi)} |\psi_n(r,\theta)|^2\,d\theta \right) dr \geq$$
$$2\pi \sum_{l=1}^{N_n} \int_{(\varepsilon_n^{\alpha_n^l},\varepsilon_n^{\beta_n^l})} r \left(\frac{1}{2\pi} \int_{(0,2\pi)} |\psi_n(r,\theta)|\,d\theta \right)^2 dr \geq \frac{1}{2\pi}|\mu(G_n)|^2 \sum_{l=1}^{N_n} \int_{(\varepsilon_n^{\alpha_n^l},\varepsilon_n^{\beta_n^l})} \frac{1}{r}dr =$$
$$\frac{1}{2\pi}|\mu(G_n)|^2 \sum_{l=1}^{N_n} |\alpha_n^l - \beta_n^l||\log\varepsilon_n| = \frac{1}{2\pi}|\mu(G_n)|^2(1+o(1/n))(\delta_2-\delta_1)|\log\varepsilon_n|.$$

Dividing by $|\log\varepsilon_n|$ in the previous inequality, and noticing that $\mu_n(G_n) = \mu_n(H_n)$, we deduce

$$E \geq \limsup_n \mathcal{F}_n(\mu_n) \geq \frac{1}{2\pi}(\delta_2-\delta_1)\limsup_n(\mu_n(G_n))^2 = \frac{1}{2\pi}(\delta_2-\delta_1)\limsup_n(\mu_n(H_n))^2, \tag{2.19}$$

and this concludes the proof of (2.17).

Now we construct the sequence $\{S_n\}$ of oriented segments in $\mathcal{S}$ (see (2.2)) of the form $S_n = F_n + N_n$, such that

$$\partial S_n \llcorner \Omega = \mu_n, \quad |\partial F_n| \leq 2MK, \quad |N_n| \to 0, \tag{2.20}$$

which is clearly enough to guarantee the compactness of the sequence μ_n.

To this aim, in every element $H_n^l \in \mathcal{C}_{\delta_1,n}$ fix a point p_n^l and consider the measure

$$\nu_n := \sum_{l=1}^{\sharp\mathcal{C}_{\delta_1,n}} \mu(H_n^l)\delta_{p_n^l}.$$

We have that $|\nu_n| \leq MK$, and hence we can find $F_n \in \mathcal{S}$ satisfying

$$|\partial F_n| \leq 2MK, \quad \partial F_n \llcorner \Omega = \nu_n. \tag{2.21}$$

Now let us denote by $\mathcal{I}_n$ the union of the connected components of $A_{\varepsilon_n^{\delta_1}}(\mu_n)$ strictly contained in Ω, and by $\mathcal{K}_n$ the union of the connected components of $A_{\varepsilon_n^{\delta_1}}(\mu_n)$ intersecting $\partial\Omega$. We clearly have

$$\mathrm{supp}(\mu_n) \subset \mathcal{I}_n \cup \mathcal{K}_n. \tag{2.22}$$

Moreover by construction $(\mu_n - \nu_n)(I_n^l) = 0$ for every $I_n^l \in \mathcal{I}_n$. Therefore, using that $\sharp\,\mathrm{supp}(\mu_n) \leq E|\log\varepsilon_n|$, and that for every $I_n^l \in \mathcal{I}_n$ we have $\mathrm{diam}(I_n^l) \leq E|\log\varepsilon_n|\varepsilon_n^{\delta_i}$, we can easily find $V_n \in \mathcal{S}$ such that

$$\partial V_n = (\mu_n - \nu_n)\llcorner \mathcal{I}_n, \qquad |V_n| \leq \mathrm{diam}(I_n^l)\,\sharp\,\mathrm{supp}(\mu_n) \leq \varepsilon_n^{\delta_1} E^2 |\log\varepsilon_n|^2. \tag{2.23}$$

On the other hand, since for every $x \in \mathrm{supp}(\mu_n) \cap \mathcal{K}_n$ we have $d(x,\partial\Omega) \leq E|\log\varepsilon_n|\varepsilon_n^{\delta_1}$, we can also find $W_n \in \mathcal{S}$ (joining each $x \in \mathrm{supp}(\mu_n) \cap \mathcal{K}_n$ with a point of $\partial\Omega$) such that

$$\partial W_n = \mu_n \llcorner \mathcal{K}_n \qquad |W_n| \leq \varepsilon_n^{\delta_1} E^2 |\log\varepsilon_n|^2. \tag{2.24}$$

Setting $N_n := V_n + W_n$, by (2.21), (2.22), (2.23) and (2.24) we deduce that (2.20) holds true. $\square$

2.2.2. Γ-*convergence.* Here we prove the Γ-convergence result.

Proof of the Γ-*limsup inequality.* It is enough to prove the Γ-limsup inequality assuming that $|\mu(x)| = 1$ for every $x \in \operatorname{supp}(\mu)$. In fact the class of measures satisfying this assumption is dense in energy and with respect to the flat convergence in X.

The recovering sequence is given by the constant sequence $\mu_n \equiv \mu$. We have to construct a sequence of admissible strains $\psi_n \in \mathcal{AS}_\varepsilon(\mu)$ satisfying

$$\mathcal{F}(\mu) \geq \limsup \frac{1}{|\log \varepsilon_n|} \int_{\Omega_{\varepsilon_n}(\mu)} |\psi_n|^2. \tag{2.25}$$

To this aim, for every $x_i \in \operatorname{supp}(\mu) \cap \Omega$, we consider the field ψ_{x_i}, which in polar coordinates is defined by

$$\psi_{x_i}(r,\theta) := \frac{1}{2\pi r}\tau(r,\theta)$$

where $\tau(r,\theta)$ is the tangent vector to $\partial B_r(x_0)$ at the point with coordinates (r,θ).

The recovering sequence ψ_n is defined by

$$\psi_n := \sum_{x_i \in \operatorname{supp}(\mu)} \psi_{x_i} \llcorner \Omega_{\varepsilon_n}. \tag{2.26}$$

It can be easily proved that $\psi_n \in \mathcal{AS}_\varepsilon(\mu)$, and

$$\limsup \frac{1}{|\log \varepsilon_n|} \int_{\Omega_{\varepsilon_n}(\mu)} |\psi_n|^2 = \limsup \sum_{x_i \in \operatorname{supp}(\mu)} \frac{1}{|\log \varepsilon_n|} \int_{\Omega_{\varepsilon_n}(\mu)} |\psi_{x_i}|^2 = \mathcal{F}(\mu),$$

and this concludes the proof of (2.25). $\square$

Proof of the Γ-*liminf inequality.* Let $\mu \in X$ and let $\mu_n \xrightarrow{f} \mu$ in X. We can assume that $\liminf \mathcal{F}_n(\mu_n) < \infty$. Let us fix $0 < S < T < 1$, and let us consider the set $A_{\varepsilon_n^T}(\mu_n)$ defined as in (2.8). By Lemma 2.5 we deduce that there exists a finite number of connected component $C_n^1, \dots, C_n^{L_n}$ of $A_{\varepsilon_n^T}(\mu_n)$, with L_n uniformly bounded by a constant M independent of n, such that $C_n^l \subset \Omega$ and $\mu_n(C_n^l) \neq 0$. Let us denote by $\sharp_n : (S,T) \to \{1, \dots M\}$ the function which counts the number of connected components of $A_{\varepsilon_n^s}(\mu_n)$ containing at least one C_n^l. For every n, let us consider the finite family of indices

$$I_n := \{t_n^1, \dots, t_n^{M_n}\},\ S \leq t_n^1 < t_n^2 < \dots < t_n^{M_n} \leq T, \quad M_n \leq M,$$

given by the discontinuity points of $\sharp_n$. Up to a subsequence, we have that the set of accumulation points of I_n contained in (S,T) is of the type

$$I_\infty := \{\delta^1, \dots, \delta^H\},\ \ S < \delta^1 < \delta^2 < \dots < \delta^H < T, \quad H \leq M.$$

Let us set $\delta^0 = S$, $\delta^{H+1} = T$, and for every $0 \leq i \leq H$ consider intervals (a^i, b^i) with $\delta^i < a^i < b^i < \delta^{i+1}$. Fixed any $\tau > 0$, we can always assume that

$$\sum_i (b^i - a^i) \geq T - S - \tau.$$

For every $s \in (a^i, b^i)$, we have exactly $\sharp_n(b^i)$ connected components $K_n^{s,1}, \dots, K_n^{s,\sharp_n(b^i)}$ of $A_{\varepsilon_n^s}(\mu_n)$ containing at least one C_n^l. For every $1 \leq j \leq \sharp_n(b^i)$ we arbitrarily fix a point $p_n^{i,j} \in K_n^{b^i,j}$. Let us define

$$R_1^{i,n} := E|\log \varepsilon_n| \varepsilon_n^{b^i}, \qquad R_2^{i,n} := \varepsilon_n^{a^i}.$$

Moreover, for every connected component $C_n^{i,l}$ of $A_{\varepsilon_n^{b^i}}(\mu_n)$ different from $K_n^{b^i,j}$ let us set

$$r_{1,n}^{i,j,l} := \min_{x\in C_n^{i,l}} |x - p_n^{i,j}| - \varepsilon_n^{b_i}, \qquad r_{2,n}^{i,j,l} := \max_{x\in C_n^{i,l}} |x - p_n^{i,j}| + \varepsilon_n^{b_i}.$$

Let us set

$$L_n^{i,j} := (R_1^{i,n}, R_2^{i,n}) \setminus \cup_l \overline{(r_{1,n}^{i,j,l}, r_{2,n}^{i,j,l})}, \qquad F_n^{i,j} := \{x \in \mathbb{R}^2 : |x - p_n^{i,j}| \in L_n^{i,j}\}.$$

The sets $L_n^{i,j}$ are finite union of open intervals, and hence they can be written in the form

$$L_n^{i,j} = \bigcup_{l=1}^{N_n^{i,j}} (\varepsilon_n^{\alpha_n^{i,j,l}}, \varepsilon_n^{\beta_n^{i,j,l}}).$$

Let us denote by $\mathcal{H}_n^i$ the family of sets $F_n^{i,j}$ which are strictly contained in Ω. The following properties concerning $L_n^{i,j}$ and $F_n^{i,j}$ and $\mathcal{H}_n^i$ can be readily verified by the reader.

a) For every i, j, for n big enough, and for every $r \in L_n^{i,j}$, we have $\mu_n(B_r(p_n^{i,j})) \equiv \mu(F_n^{i,j})$;
b) For n big enough, the sets $F_n^{i,j}$ are pairwise disjoint;
c) For every i, $\mu_n \llcorner \left(\cup_{F_n^{i,j}\in\mathcal{H}_n^i} F_n^{i,j}\right) \xrightarrow{f} \mu$
d) For every i, j, $\sum_{l=1}^{N_n^{i,j}} |\alpha_n^{i,j,l} - \beta_n^{i,j,l}| = (1+o(1/n))(b^i - a^i)$, where $o(1/n) \to 0$ as $n \to \infty$.

Arguing as in the proof of (2.19) we obtain that for every $\psi_n \in \mathcal{AS}_{\varepsilon_n}(\mu_n)$ and for every $F_n^{i,j} \in \mathcal{H}_n^i$

$$\frac{1}{|\log \varepsilon_n|} \int_{F_n^{i,j}} |\psi_n|^2\, dx \geq \frac{1}{2\pi} \big(\mu_n(F_n^{i,j})\big)^2 (1 + o(1/n))(b^i - a^i).$$

Summing the previous inequality over all $F_n^{i,j} \in \mathcal{H}_n^i$, we obtain

$$\frac{1}{|\log \varepsilon_n|} \sum_{F_n^{i,j}\in\mathcal{H}_n^i} \int_{F_n^{i,j}} |\psi_n|^2\, dx \geq \frac{1}{2\pi}(1 + o(1/n)) \sum_{F_n^{i,j}\in\mathcal{H}_n^i} |\mu_n(F_n^{i,j})|(b^i - a^i). \tag{2.27}$$

Recalling that the diameter of $F_n^{i,j}$ tends to 0 as $n \to \infty$, by property *c)* we easily deduce that for every fixed i

$$\liminf_n \sum_{F_n^{i,j}\in\mathcal{H}_n^i} |\mu_n(F_n^{i,j})| \geq |\mu|(\Omega). \tag{2.28}$$

Letting $n \to \infty$ in (2.27), and using (2.28) we obtain

$$\liminf_n \mathcal{F}(\mu_n) \geq \liminf_n \frac{1}{|\log \varepsilon_n|} \int_{\Omega_{\varepsilon_n}(\mu)} |\psi_n|^2\, dx \geq \liminf_n \frac{1}{|\log \varepsilon_n|} \sum_i \sum_{F_n^{i,j}\in\mathcal{H}_n^i} \int_{F_n^{i,j}} |\psi_n|^2\, dx \geq$$
$$\liminf_n \frac{1}{2\pi} \sum_i \sum_{F_n^{i,j}\in\mathcal{H}_n^i} |\mu_n(F_n^{i,j})|(b^i - a^i) \geq (T - S - \tau)\frac{1}{2\pi}|\mu|(\Omega).$$

Letting $S \to 0$, $T \to 1$ and $\tau \to 0$ we deduce the Γ-liminf inequality. □

Remark 2.6. Let $C, C' > 0$ be fixed positive constants. Here we observe that nothing changes in our Γ-convergence result if in the definition of $\Omega_\varepsilon(\mu)$ we remove balls of radius $C\varepsilon$

instead of ε, and if we multiply the second addendum $|\mu|(\Omega)$ in (2.5) by C'. More precisely given $\mu \in X$, define

$$\Omega_\varepsilon^C(\mu) := \Omega \setminus \bigcup_{x_i \in \operatorname{supp}(\mu)} B_{C\varepsilon}(x_i). \tag{2.29}$$

Define consequently the space of admissible strains $\mathcal{AS}_\varepsilon^C(\mu)$ associated with μ as follows

$$\text{(2.30)}\quad \mathcal{AS}_\varepsilon^C(\mu) := \{\psi \in L^2(\Omega_\varepsilon^C(\mu);\mathbb{R}^2) : \ \operatorname{curl}\psi = 0 \text{ in } \Omega_\varepsilon^C(\mu) \text{ in the sense of distributions,}$$
$$\int_{\partial A} \psi(s)\cdot\tau(s)\,ds = \mu(A) \text{ for every open set } A \subset \Omega \text{ with } \partial A \text{ smooth and with } \partial A \subset \Omega_\varepsilon^C(\mu)\}.$$

Finally let $\mathcal{E}_\varepsilon^{C,C'}(\mu)$ be defined by

$$\mathcal{E}_\varepsilon^{C,C'}(\mu) := \min_{\psi \in \mathcal{AS}_\varepsilon^C(\mu)} \int_{\Omega_\varepsilon^C(\mu)} |\psi(x)|^2\,dx + C'|\mu|(\Omega), \tag{2.31}$$

and let $\mathcal{F}_\varepsilon^{C,C'} := 1/|\log\varepsilon|\mathcal{E}_\varepsilon^{C,C'}$ be the corresponding rescaled functionals. Then Theorem 2.3 still holds true with $\mathcal{F}_\varepsilon$ replaced by $\mathcal{F}_\varepsilon^{C,C'}$.

In this respect the choice of the core radius and of the core energy does not play an essential role in the asymptotic behavior of the functionals $\mathcal{F}_\varepsilon$ as $\varepsilon \to 0$. This fact gives also a partial justification of the use of linearized elasticity in $\Omega_\varepsilon^C(\mu)$. In fact, the recovering sequence ψ_n given in (2.26) satisfies

$$\|\psi_n\|_{L^\infty(\Omega_{\varepsilon_n}^C(\mu_n);\mathbb{R}^2)} \le \frac{1}{2\pi C\varepsilon_n} + O(n),$$

where $O(n)$ is uniformly bounded with respect to n. Recalling that the admissible strains should be rescaled by ε_n (because the Burgers vector has to be rescaled by ε_n), we deduce that the modulus of the gradient of the rescaled recovering sequence can be chosen arbitrarily small, choosing C big enough. This is our partial justification of the use of linear elasticity.

3. The discrete model

Here we give a Γ-convergence result in a discrete model for the stored energy associated with a configuration of screw dislocations, as the atomic distance ε tends to zero. The model follows the general theory of eigenstrains (we refer the reader to [12]): a dislocation in the crystal is associated with a pre-existing plastic strain in the reference lattice. In the next section we will describe our discrete model, which follows the lines of the more general theory introduced in [2].

3.1. **Description of the discrete model.** We will consider the illustrative case of a square lattice, with nearest neighborhood interactions. Let $\Omega \subset \mathbb{R}^2$ be a horizontal section of the region occupied by the cylindrical crystal. We will assume for simplicity Ω to be polygonal. In the reference configuration, the lattice of atoms is given by the set

$$\Omega_\varepsilon^0 := \{x \in \varepsilon\mathbb{Z}^2 \cap \Omega\}.$$

We denote by Ω_ε^1 the class of *bonds* in Ω, i.e., the class of oriented ε-segments $[x, x+\varepsilon e_i]$, where e_1, e_2 is the canonical basis of $\mathbb{R}^2$, and x, $x+\varepsilon e_i \in \Omega_\varepsilon^0$.

Given a function $u : \Omega_\varepsilon^0 \to \mathbb{R}$, let us introduce the (rescaled) *discrete gradient* of u, $\mathbf{d}\,u : \Omega_\varepsilon^1 \to \mathbb{R}$, defined by

$$\mathbf{d}\,u([x,x+\varepsilon e_i]) := u(x+\varepsilon e_i) - u(x) \qquad \text{for every } [x,x+\varepsilon e_i] \in \Omega_\varepsilon^1. \tag{3.1}$$

Given a strain $\xi : \Omega^1_\varepsilon \to \mathbb{R}$, the elastic energy associated with ξ, is given by

$$E^d(\xi) := \sum_{v \in \Omega^1_\varepsilon} a(v)(\xi(v))^2, \tag{3.2}$$

where the function $a(v) \in \{1/2,\, 1\}$, introduced only to simplify some interpolation procedure (see property b) of Proposition 3.3), is defined by

$$a(v) := \begin{cases} \frac{1}{2} & \text{if } \partial v \subset \partial \Omega; \\ 1 & \text{otherwise.} \end{cases}$$

The elastic energy associated with a displacement $u : \Omega^0_\varepsilon \to \mathbb{R}$, in absence of dislocations, is given by $E^d(\mathbf{d}\, u)$.

To model the presence of dislocations, following [2] we introduce the class Ω^2_ε of oriented ε-squares $[x, x+\varepsilon e_2, x+\varepsilon e_1+\varepsilon e_2]$ with x, $x+\varepsilon e_2$, $x+\varepsilon e_1+\varepsilon e_2, x+\varepsilon e_1 \in \Omega^0_\varepsilon$. For simplicity we will always assume that $\Omega = \cup_{Q \in \Omega^2_\varepsilon} Q$.

In this discrete setting, a dislocation is represented by a function $\alpha : \Omega^2_\varepsilon \to \mathbb{Z}$. The squares in the support of α represent the zone where a dislocation is present, while the value of α on these squares represents the multiplicity of the dislocation.

Given $\xi : \Omega^1_\varepsilon \to \mathbb{R}$, the function $\mathbf{d}\, \xi : \Omega^2_\varepsilon \to \mathbb{R}$ is defined by

$$\begin{aligned} &\mathbf{d}\, \xi([x, x+\varepsilon e_2, x+\varepsilon e_1+\varepsilon e_2]) := \xi([x, x+\varepsilon e_2]) + \xi([x+\varepsilon e_2, x+\varepsilon e_1+\varepsilon e_2]) - \\ &\xi([x+\varepsilon e_1, x+\varepsilon e_1+\varepsilon e_2]) - \xi([x, x+\varepsilon e_1]) \qquad \text{for every } [x, x+\varepsilon e_2, x+\varepsilon e_1+\varepsilon e_2] \in \Omega^2_\varepsilon. \end{aligned} \tag{3.3}$$

The elastic energy associated with a dislocation $\alpha : \Omega^0_\varepsilon \to \mathbb{Z}$ is given by

$$\mathcal{E}^d_\varepsilon(\alpha) := \min_{\xi : \Omega^1_\varepsilon \to \mathbb{R} :\, \mathbf{d}\, \xi = \alpha} E^d_\varepsilon(\xi). \tag{3.4}$$

Remark 3.1. Note that if α is a dipole of the type

$$\alpha(Q) := \begin{cases} -1 & \text{if } Q = [x, x+\varepsilon e_2, x+\varepsilon(e_1+e_2)]; \\ +1 & \text{if } Q = [x+\varepsilon z e_1, x+\varepsilon(e_2+z e_1), x+\varepsilon(e_1+e_2+z e_1)]; \\ 0 & \text{otherwise,} \end{cases}$$

for some $x \in \Omega^0_\varepsilon$, $z \in \mathbb{Z}$, then $\alpha = \mathbf{d}\, \beta$, with β defined by

$$\beta(v) := \begin{cases} 1 & \text{if } v = [x+\varepsilon s e_1, x+\varepsilon(s e_1+e_2)] \text{ with } s \in \{1, \dots, z\}; \\ 0 & \text{otherwise.} \end{cases}$$

Actually for every $\alpha : \Omega^2_\varepsilon \to \mathbb{R}$ we can find $\beta :\, \mathbf{d}\, \beta = \alpha$. By linearity, it is sufficient to check it in the case

$$\alpha(Q) := \begin{cases} 1 & \text{if } Q = [x, x+\varepsilon e_1, x+\varepsilon(e_1+e_2)]; \\ 0 & \text{otherwise,} \end{cases}$$

We have $\alpha = d\beta$, where

$$\beta(v) := \begin{cases} 1 & \text{if } v = [x-\varepsilon s\, e_1, x-\varepsilon s\, e_1+\varepsilon\, e_2] \text{ with } s \in \{0 \cup \mathbb{N}\}; \\ 0 & \text{otherwise.} \end{cases}$$

Note that there are many β inducing the same α (such that $\mathbf{d}\, \beta = \alpha$). More precisely if $\mathbf{d}\, \beta = \alpha$, then α is induced exactly by

$$\{\beta + \mathbf{d}\, u,\ u : \Omega^0_\varepsilon \to \mathbb{R}\}.$$

This follows by the fact that if $\xi : \Omega^1_\varepsilon \to \mathbb{R}$ is such that $\mathbf{d}\,\xi = 0$, then there exists $u : \Omega^0_\varepsilon \to \mathbb{R}$ such that $\xi = \mathbf{d}\,u$, and $\mathbf{d}\,\mathbf{d}\,u(Q) = 0$ for every $u : \Omega^0_\varepsilon \to \mathbb{R},$ for every $Q \in \Omega^2_\varepsilon$.

We deduce that if $d\beta = \alpha$, then

$$\mathcal{E}^d_\varepsilon(\alpha) := \min_{u:\Omega^0_\varepsilon \to \mathbb{R}} E^d_\varepsilon(\mathbf{d}\,u - \beta).$$

Therefore β can be interpreted as an eigenstrain associated with the dislocation α. However we stress that the energy depends on α and not on the particular choice of the eigenstrain inducing α.

3.2. **The Γ-convergence result.** To study the asymptotic behavior of the elastic energy functionals $\mathcal{E}^d_\varepsilon$ as $\varepsilon \to 0$ in terms of Γ-convergence, it is convenient to define a common space of configurations of dislocations independent of ε. To this aim, to every dislocation $\alpha : \Omega^2_\varepsilon \to \mathbb{Z}$ we associate the measure

$$\mu(\alpha) := \sum_{Q \in \Omega^2_\varepsilon} \alpha(Q)\delta_{x(Q)},$$

where $x(Q)$ denotes the center of Q. Therefore, as in the continuum case, the space of dislocations is the space X defined in (2.1). Moreover we denote by X_ε the subspace of X given by the measures μ such that $\mu = \mu(\alpha)$ for some $\alpha \in \Omega^2_\varepsilon$. Finally, given $\mu \in X_\varepsilon$, we will denote by $\alpha(\mu) : \Omega^2_\varepsilon \to \mathbb{Z}$ the (unique) dislocation satisfying $\mu(\alpha(\mu)) = \mu$.

The class of discrete admissible strains associated with ε and $\mu \in X_\varepsilon$ is defined by

$$\mathcal{AS}^d_\varepsilon(\mu) := \{\xi : \Omega^1_\varepsilon \to \mathbb{Z} : \, \mathbf{d}\,\xi = \alpha(\mu)\}.$$

The rescaled energy functionals take the form

$$\mathcal{F}^d_\varepsilon(\mu) := \begin{cases} \frac{1}{|\log\varepsilon|}\,\mathcal{E}_\varepsilon(\alpha(\mu)) & \text{if } \mu \in X_\varepsilon; \\ +\infty & \text{in } X \setminus X_\varepsilon. \end{cases} \tag{3.5}$$

Remark 3.2. Here we notice that in the discrete model, we do not need to introduce the core energy $|\mu|(\Omega)$ as in the continuum case to obtain an estimate similar to (2.9). The term $|\mu|(\Omega)$, in the continuum model, represents the energy stored in a region surrounding the dislocations, whose diameter is comparable to the atomic distance. This interpretation is fully justified by the following easy computation. Let $\mu \in X_\varepsilon$, let $x \in \mathrm{supp}(\mu)$, and let $Q_\varepsilon(x)$ be the ε-square centered at x. For every admissible strain $\xi \in \mathcal{AS}^d_\varepsilon(\mu)$, we have by definition

$$\sum_{v \in \partial Q_\varepsilon(x)} \xi(v) = \mu(x).$$

We deduce that

$$\sum_{v \in \partial Q_\varepsilon(x)} |\xi(v)|^2 \geq C, \tag{3.6}$$

where C is a constant independent of ε. Therefore, in the discrete model, the energy stored in the bonds near to the dislocations turns out to be controlled from below by $|\mu|(\Omega)$. As observed in Remark 2.6, a sharp computation of this energy becomes unnecessary in the continuum model, in the study of the asymptotic behavior of the elastic energy as $\varepsilon \to 0$.

By (3.6) we deduce (as in (2.9)) that if $\varepsilon_n \to 0$, and $\{\mu_n\}$ is a sequence in X such that, for every $n \in \mathbb{N}$, $\mathcal{F}^d{}_{\varepsilon_n}(\mu_n) \leq E$ for some positive constant E, then

$$\sharp\,\mathrm{supp}(\mu_n) \leq CE|\log(\varepsilon_n)| \qquad \text{for every } n \in \mathbb{N}, \tag{3.7}$$

where C is a fixed positive constant independent of ε.

The candidate Γ-limit of the functionals $\mathcal{F}^d_\varepsilon$, as in the continuum case (see (2.7)), is the functional $\mathcal{F} : X \to \mathbb{R}$ defined by

$$\mathcal{F}(\mu) := \frac{1}{2\pi}|\mu|(\Omega) \qquad \text{for every } \mu \in X. \tag{3.8}$$

Now we provide some interpolation procedures which will be used in the proof of the Γ-convergence result. Let $u : \Omega^0_\varepsilon \to \mathbb{R}$. Let us introduce its extension $\tilde{u} : \Omega \to \mathbb{R}$, defined in the following way. We divide every $Q \in \Omega^2_\varepsilon$, in two triangle. In each triangle T, $\tilde{u}$ is the only affine function coinciding with u on the vertices of T. In a similar way, given a function $\xi : \Omega^1_\varepsilon \to \mathbb{R}$ we define $\tilde{\xi} : \Omega \to \mathbb{R}^2$ imposing on each triangle

$$\tilde{\xi} \equiv (\xi(v_1(T)), \xi(v_2(T))), \tag{3.9}$$

where $v_1(T)$ and $v_2(T)$ are the horizontal (parallel to e_1) and the vertical (parallel to e_2) edges of T. We collect in the following proposition some properties satisfied by the interpolated functions introduced above.

Proposition 3.3. *The following facts hold.*

a) *For every $u : \Omega^0_\varepsilon \to \mathbb{Z}$, we have $\nabla\tilde{u} = \frac{1}{\varepsilon}\tilde{\mathbf{d}u}$;*

b) *For every $\xi : \Omega^1_\varepsilon \to \mathbb{Z}$, we have $E^d(\xi) = \|\tilde{\xi}\|^2_{L^2(\Omega;\mathbb{R}^2)}$;*

c) *The function $\tilde{\xi}$ belongs to the class $\mathcal{AS}^C_\varepsilon(\mu(\mathbf{d}\,\xi))$ defined in* (2.30) *for every $C \geq 2^{1/2}$.*

Now we are in a position to give our Γ-convergence result in this discrete model, for the elastic energy functionals ${\mathcal{F}^d}_\varepsilon$ as $\varepsilon \to 0$.

Theorem 3.4. *The following Γ-convergence result holds.*

i) *Equi-coercivity. Let $\varepsilon_n \to 0$, and let $\{\mu_n\}$ be a sequence in X such that ${\mathcal{F}^d}_{\varepsilon_n}(\mu_n) \leq E$ for some positive constant E independent of n. Then $\mu_n \xrightarrow{f} \mu$ for some $\mu \in X$.*

ii) *Γ-convergence. The functionals ${\mathcal{F}^d}_{\varepsilon_n}$ Γ-converge to $\mathcal{F}$ as $\varepsilon_n \to 0$ with respect to the flat norm, i.e., the following inequalities hold.*

Γ-liminf inequality: $\mathcal{F}^d(\mu) \leq \liminf {\mathcal{F}^d}_{\varepsilon_n}(\mu_n)$ for every $\mu \in X$, $\mu_n \xrightarrow{f} \mu$ in X.

Γ-limsup inequality: given $\mu \in X$, there exists $\{\mu_n\} \subset X$ with $\mu_n \xrightarrow{f} \mu$ such that $\limsup \mathcal{F}_{\varepsilon_n}(\mu_n) \leq \mathcal{F}(\mu)$.

Proof. We begin by proving the equi-coercivity property.

Equi-coercivity. Let $\xi_n \in \mathcal{AS}^d_{\varepsilon_n}(\mu_n)$ be such that

$$\frac{1}{|\log\varepsilon_n|}E^d_{\varepsilon_n}(\xi_n) \leq E + 1.$$

Let $C \geq 2^{1/2}$. By Proposition 3.3 we have that the functions $\tilde{\xi}_n$ introduced in (3.9) are in the class $\mathcal{AS}^C_\varepsilon(\mu_n)$ defined in (2.30). By Proposition 3.3 and by (3.7) we deduce that $\mathcal{F}^{C,1}_{\varepsilon_n}(\mu_n) \leq K$ for some positive constant $K > 0$. Therefore by Theorem 2.3 and by Remark 2.6 we deduce that the equi-coercivity property holds.

Γ-liminf inequality. Let $\mu_n \xrightarrow{f} \mu$ in X, and let $\xi_n \in \mathcal{AS}^d_{\varepsilon_n}(\mu_n)$ be such that

$$\liminf \frac{1}{|\log\varepsilon_n|}E^d_{\varepsilon_n}(\xi_n) = \liminf \mathcal{F}^d(\mu_n).$$

By Proposition 3.3 we have that $\tilde\xi_n \in \mathcal{AS}^C_\varepsilon(\mu_n)$, with $C \geq 1/2$. By the Γ-liminf inequality of Theorem 2.3, and by Remark 2.6 we deduce that for every positive constant $C' > 0$ we have

$$\mathcal{F}^d(\mu) \leq \liminf \frac{1}{|\log\varepsilon_n|}\left(\int_{\Omega^C_{\varepsilon_n}(\mu_n)} |\tilde\xi_n|^2 + C'|\mu_n|(\Omega)\right).$$

By the arbitrariness of C', by Remark 3.2 and by Proposition 3.3 we deduce

$$\mathcal{F}^d(\mu) \leq \liminf_n \frac{1}{|\log\varepsilon_n|}\int_{\Omega^C_{\varepsilon_n}(\mu_n)} |\tilde\xi_n|^2 \leq \liminf_n \frac{1}{|\log\varepsilon_n|} E^d_{\varepsilon_n}(\xi_n) = \liminf_n \mathcal{F}^d(\mu_n),$$

that is the Γ-liminf inequality holds true.

Γ-*limsup inequality.* It is enough to prove the Γ-limsup inequality assuming that $|\mu(x)| = 1$ for every $x \in \operatorname{supp}(\mu)$. In fact the class of measures satisfying this assumption is dense in energy and with respect to the flat convergence in X.

The recovering sequence is given by the constant sequence $\mu_n \equiv \mu$. We have to construct a sequence of admissible strains $\xi_n \in \mathcal{AS}^d_{\varepsilon_n}(\mu_n)$ satisfying

$$\mathcal{F}^d(\mu) \geq \limsup \frac{1}{|\log\varepsilon_n|} E^d_{\varepsilon_n}(\xi_n). \tag{3.10}$$

Let us fix $x_i \in \operatorname{supp}(\mu)\cap\Omega$. For every $v := [v_1, v_2] \in \Omega^1_\varepsilon$, let as denote by $T(v)$ the triangle whose vertices are x_i, v_1 and v_2, and by $\theta_{x_i}(v) \in [0, 2\pi)$, its angle at the point x_i. We consider the field $\xi^n_{x_i} : \Omega^1_\varepsilon \to \mathbb{R}$ defined by

$$\xi^n_{x_i}(v) := \theta_{x_i}(v) o(T) \qquad \text{for every } v \in \Omega^1_{\varepsilon_n},$$

where $o(T) \in \{-1, 1\}$ is equal to 1 if the oriented segments $[x, v_1]$, $[v_1, v_2]$, $[v_2, x]$ induce a clock-wise orientation to ∂T; $o(T) = -1$ otherwise.

Let us fix $0 < \delta < 1$. We set

$$A^n_i := \{x \in \Omega : \, |x - x_i| < \varepsilon_n^{1-\delta}\},$$
$$B^n_i := \Omega \setminus \overline{A}^n_i.$$

Let us consider the function $\tilde\xi^n_{x_i} : \Omega \to \mathbb{R}^2$ of $\mathbb{R}^n$ defined in (3.9). By construction, for n big enough $\tilde\xi^n_{x_i}$ satisfies as follows.

i) For every $x \in A^n_i$

$$|\tilde\xi^n_{x_i}(x)| \leq \frac{C}{\max\{|x - x_i|, \varepsilon_n\}},$$

where C is a positive constant independent of ε_n;

ii) For every $x \in B^n_i$

$$|\tilde\xi^n_{x_i}(x)| = \frac{1 + o(\varepsilon_n)}{|x - x_i|},$$

where $o(\varepsilon_n) \to 0$ as $\varepsilon_n \to 0$.

The recovering sequence ξ_n is defined by

$$\xi_n := \sum_i \xi^n_{x_i}.$$

It can be easily proved that $\xi_n \in \mathcal{AS}^d_{\varepsilon_n}(\mu_n)$. By Proposition 3.3 and by properties *i*), *ii*) above it follows that

$$\lim_{\varepsilon_n\to 0}\frac{1}{|\log\varepsilon_n|}E^d_{\varepsilon_n}(\xi_n)=\lim_{\varepsilon_n\to 0}\frac{1}{|\log\varepsilon_n|}\|\tilde{\xi}_n\|^2_{L^2(\Omega;\mathbb{R}^2)}=\frac{1}{2\pi}\sharp\,\mathrm{supp}(\mu)(1+o(\delta))=\mathcal{F}^d(\mu)(1+o(\delta)),$$

where $o(\delta)\to 0$ as $\delta\to 0$, and this concludes the proof of (3.10) and of the Γ-convergence result.

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