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Detection of synchronised chaos in coupled map networks using symbolic dynamics.

by

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We present a method based on symbolic dynamics for the detection of synchronization in networks of coupled maps and distinguishing between chaotic and random iterations. The symbolic dynamics are defined using special partitions of the phase space which prevent the occurrence of certain symbol sequences related to the characteristics of the dynamics. Synchrony in a large network can be detected using measurements from only a single node by comparing the transition probabilities with those of the uncoupled function. The method utilizes a relatively short time series of measurements and hence is computationally very fast. Furthermore, it is robust against parameter uncertainties, is independent of the network size, and does not require knowledge of the connection structure.

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Symbolic dynamics is a fundamental tool available to describe complicated time evolution of a chaotic dynamical system, the Smale horseshoe being a most famous prototype [1]. In symbolic dynamics, instead of representing a trajectory by infinite sequences of numbers one watches the alternation of symbols. In doing so one 'loses' some information but certain important invariants and robust properties of the dynamics may be kept [2]. Most studies of symbolic dynamics in the literature are based on the so-called generating partition [3] of the phase space, for which topological entropy of the system achieves its supremum [4]. On the other hand, finding a generating partition for general systems is a challenging open problem and is the issue of the constant research [5]. The misplacement of partition leads to 'diminishing' of the computed entropy of the system [6]. Symbolic dynamics based on generating partitions plays a crucial role in understanding many different properties of dynamical systems. However, as we shall show, dynamics based on certain other partitions also have important practical uses. In the following, we construct specific partitions such that certain symbolic transitions do not appear in the time evolution of a dynamical system. We present two applications, namely distinguishing between chaotic and random iterations and the detection of synchronization in networks. In the first application, the absence of these forbidden transitions in a chaotic iteration distinguishes it from a corresponding random iteration. In the second application, we utilize the fact that in a synchronized network all nodes follow the same evolution as an isolated unit, so that synchrony can be detected by comparing the symbolic dynamics from any single node against the dynamics in isolation.

Synchronization in large ensembles of coupled dynamical units is studied in many different fields [7–12]. The detection of synchronization of an extended system from local measurements has important applications. For instance, certain pathologies in the neural system, such as epileptic seizures, manifest themselves by synchronized brain signals [10] and there is some evidence that they can be predicted by changing levels of synchronization [11]. It is thus of interest to be able to determine different levels of synchronization in a brain area from local measurements, such as EEG recordings. Similarly, fast and easy detection of synchronization in communication or power networks is important for efficient management of such large systems [12]. The methods so far known to detect synchronization are based on the decrease in the topological entropy or on the correlation between the bivariate or multivariate time series observed at different nodes using tools from nonlinear time series analysis [13, 14]. The first method requires the construction of phase space from time series using delay coordinates [13, 15], which itself is not a trivial task and typically requires extensive computational effort [16]. Furthermore, its success depends on the availability of a sufficiently long time series, although it is not clear what length is sufficient [17]. Even with an infinitely long time series, the detection of synchronization relies on the decrease of phase space dimension, and it is not clear whether any decrease is due to the synchronization of nodes or to the unsynchronized low-dimensional chaos. Similarly, distinguishing chaotic and random systems is not always easy, and the issue is of both fundamental and practical importance [18]. In this Letter, we shall address these problems using symbolic dynamics based on specific partitions.

We consider dynamical systems defined by the iteration rule

$$x(t+1) = f(x(t)) \tag{1}$$

Where $t \in \mathbb{Z}$ is the discrete time and $f : S \to S$ is a map on a subset S of $\mathbb{R}^n$. Let $S_i : i = 1, \ldots, m$ be a partition of S, i.e., a collection of mutually disjoint subsets satisfying $\cup_{i=1}^{m} S_i = S$. We assume that the S_i are nonempty and $m \geq 2$ to prevent trivial cases. The symbolic dynamics corresponding to (1) is the sequence of symbols $\{\ldots, s_{t-1}, s_t, s_{t+1}, \ldots\}$ where $s_t = i$ if $x(t) \in S_i$ (shift of the finite type [2]). We say the set S_i *avoids* S_j if

$$f(S_i) \cap S_j = \emptyset. \tag{2}$$

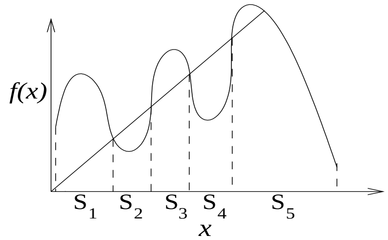

FIG. 1: A partition showing avoiding sets. In the pairs (S_2, S_3),(S_2, S_4), (S_2, S_5), (S_4, S_5), (S_3, S_1), (S_3, S_2), (S_5, S_1), (S_5, S_2), (S_5, S_3), (S_5, S_4), (S_5, S_5), the first set avoids the second.

Clearly, if S_i avoids S_j, so does any of its subsets. We also talk about a *self-avoiding set* if (2) holds with $i = j$. The essence of our method is based on choosing a partition where (2) holds for one or more sets in the partition. The significance is that if S_i avoids S_j, then the symbolic dynamics cannot contain the symbol sequence ij. The notion is extended in a straightforward way to the kth iterate of f. Thus, if $f^k(S_i) \cap S_j = \emptyset$, then the symbol sequence for the dynamics cannot contain the symbols i and j at two positions which are $k - 1$ symbols apart. This constrains the symbol sequences that can be generated by a given map, and provides a robust method to distinguish between different systems by inspecting their symbolic dynamics. As examples of avoiding sets, we mention that for the common unimodal maps of the interval $[0, 1]$, such as the tent or logistic maps, the set $(x^*, 1]$ and its subsets are self-avoiding, where x^* denotes the positive fixed point of f. More generally, if f is a continuous one-dimensional function on the interval $[r_1, r_2]$ having m fixed points $x_1^*, x_2^*, ..x_m^*$, and the derivatives $f^{'}(x^*)$ exist, then one can define a partition with $S_1 = [r_1, x_1^*]$, $S_i = (x_{i-1}^*, x_i^*]$ for $i = 2, ..., m$, and $S_{m+1} = (x_m^*, r_2]$. Then one can have several avoiding pairs, namely (Figure 1),

$$\begin{aligned} &\text{if } f'(x_i^*) \ > \ 0, \ \text{then } f(S_i) \cap S_j = \emptyset, \forall i < j \\ &\text{if } f'(x_i^*) \ < \ 0, \ \text{then } f(S_i) \cap S_j = \emptyset, \forall i > j. \end{aligned}$$

Before turning to their applications, we first show that partitions which contain avoiding sets always exist and give a constructive proof. Suppose one starts with some partition of m sets for which (2) does not hold for any i, j. Now fix some pair $i \neq j$. If $f(S_i) \subset S_j$, then S_i can be arbitrarily partitioned into two parts which avoid each other. If, on the other hand, $f(S_i) \not\subset S_j$, then let $S_{m+1} = f^{-1}(S_j) \cap S_i$, and re-define S_i as $S_i' = S_i \backslash S_{m+1}$ to obtain a new partition with $m + 1$ sets, where S_i' avoids S_j. The same argument can be used to construct self-avoiding sets: Assume $f(S_i) \not\subset S_i$ (otherwise further partition S_i to obtain a set which is not invariant under f)[25] and define $S_{m+1} = f^{-1}(S_i) \cap S_i$ and $S_i' = S_i \backslash S_{m+1}$, so that S_i' is self-avoiding. Hence, it is possible to modify a given partition so that the sequence ij never occurs in the symbolic dynamics.

We first consider the problem of distinguishing chaotic and random iterations. To be specific, suppose the iteration rule is given by the tent map

$$f(x) = (1 - 2|x - \frac{1}{2}|). \qquad (3)$$

Its stationary density on [0,1] is the uniform density, $p(x) \equiv 1$. If we take the generating partition, which is simply a partition of $[0, 1]$ at the middle, then the above tent map and a random sequence from a uniform distribution give equal Kolmogorov-Sinai (KS) entropy [23]. This can be seen by considering the permutation entropy based on transition probabilities of length d,

$$PE = -\frac{1}{d} \sum_{i_1, i_2 \ldots}^{m} p(i_1, i_2 ... i_d) \ln(p(i_1, i_2, ..., i_d)) \qquad (4)$$

where $p(i_1, i_2, \ldots, i_d)$ is the joint probability that $x(1) \in S_{i_1}$, $x(2) \in S_{i_2}$, $\ldots$, $x(d) \in S_{i_d}$. Taking $d = 2$ and $m = 2$, the iteration takes a trajectory to the left or right partition with equal probabilities, and the situation is identical to the tossing of a fair coin. Hence, for this case the symbolic dynamics does not distinguish between the random and chaotic sequences, and the permutation entropy for both systems is equal to $\ln 2$. Now we shift the partition by δ, i.e take the separation point at $l_p = 1/2 + \delta$, and calculate the entropies of the tent map and the random iteration with the same probability distribution, as a function of δ.It turns out that the difference between the two entropies is maximum when l_p is chosen as the fixed point $a/(a+1)$ of the tent map. For the corresponding partition, i.e, $S_1 = [0, l_p]$, $S_2 = (l_p, 1]$, the transition $2 \to 2$ does not occur. So, the random sequence can be identified by the presence of this transition. As the partition point is moved further to the right in the range $a/(a + 1) \leq l_p < 1$, the transition remains forbidden, but the difference between entropies decreases, reaching zero at $l_p = 1$. The implication is that a longer time series would now be needed as the observed occurrence of the symbol 2 becomes less frequent. Hence, the optimal partition would be the one for which the self-avoiding set S_2 is largest, which also yields the largest entropy difference between the chaotic and random maps. Note that the choice of l_p in the range $(1/2, a/(a + 1))$ gives a non-generating partition, but has no two-symbol forbidden sequences. Although some longer sequences may be forbidden, their detection requires longer time series and more computational effort. On the other hand, avoiding sets, which forbid short sequences, is faster and can utilize shorter time-series. We next turn to the detection of synchronization in networks. Consider the well-known coupled map lattice model [22]

$$x_i(t+1) = f(x_i(t)) + \frac{\varepsilon}{k_i} \sum_{j=1}^{N} C_{ij} \left[f(x_j(t)) - f(x_i(t)) \right] \quad (5)$$

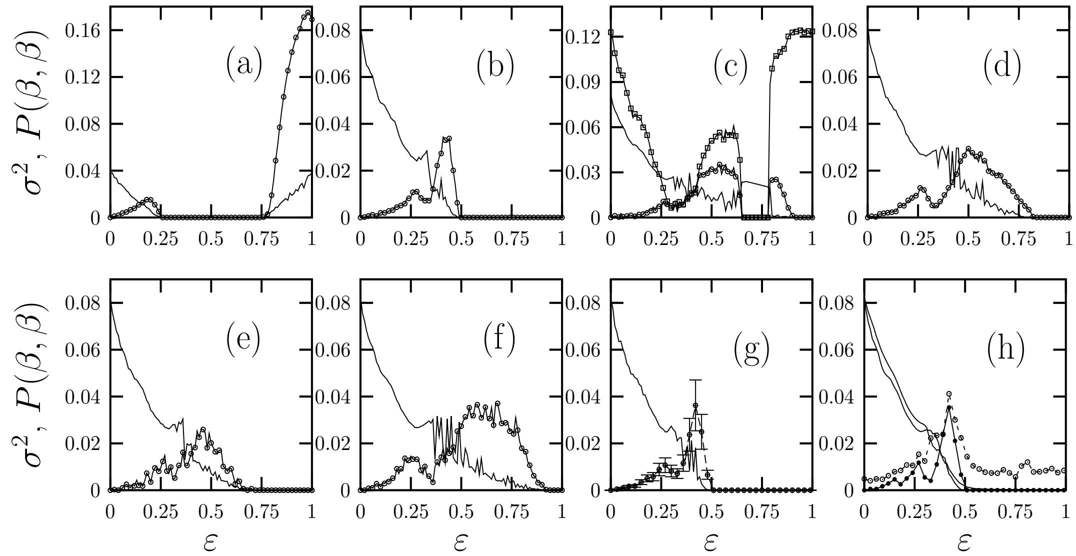

FIG. 2: Detection of synchronization in various networks of coupled maps. Figures (a)-(d), (g)-(h) are plotted for coupled tent maps (3) and (e)-(f) are for coupled logistic maps ($4x(1-x)$). The networks consist of (a) two coupled maps, (b) a globally coupled network of size 50, (c) a small-world network of size 100 and average degree 30, (d) a scale-free network of size 100 and average degree 15, (e) a scale-free network of size 200 and average degree 10, and (f) a random network of size 200 and average degree 8. The horizontal axis is the coupling strength, and the vertical axis gives the synchronization measure σ^2 (–) for the whole network, as well as the transition probability $P(\beta,\beta)$ (∘) calculated using a scalar time series from a randomly selected node. In all cases, the synchronization region ($\sigma^2 = 0$) coincides with the region where $P(\beta,\beta) = 0$ and the other transition probabilities are nonzero, which is the situation for the uncoupled map. Note that in subfigure (c) there is an interval of ϵ roughly between 0.65 and 0.8 such that $P(\beta,\beta) = 0$, but there is no synchronization as $P(\alpha,\alpha)$ (shown by □) is also zero here, unlike the case for the isolated tent map. (g) and (h) are for globally connected networks with $N = 50$, (g) shows the standard error for $P(\beta,\beta)$ for 20 random initial conditions and (h) shows $P(\beta,\beta)$ for Gaussian noise with strength 2% (•) and 5% (∘).

where $x_i(t)$ is the state of the ith node at time t, $i = 1,\ldots,N$, C_{ij} are the elements of the adjacency matrix with value 1 or 0 depending upon whether i and j are connected or not, $\varepsilon \in [0,1]$ is the coupling strength, and k_i is the number of neighbours of the ith unit. Clearly, the system (5) can exhibit a much wider range of behaviour than (1) depending on the coupling structure and the choice of parameters; so the corresponding symbol sequences observed from a node can vary widely. However, at the synchronized state $x_i(t) = x_j(t)$ for all i,j and t, all nodes evolve according to the rule (1). It follows that when the network is synchronized, the symbolic sequence measured from a node will be subject to the same constraints as that generated by (1). And a simple way of detecting synchronization of the network is simply by choosing a random node and calculating the transition probabilities. For logistic and tent maps thing become simpler as one has to look for the presence or absence of symbol subsequences which are forbidden in (1).

For numerical illustration, we take f to be the tent map (3) with the partition $S_1 = [0,x^*]$, $S_2 = (x^*,1]$, at the fixed point $x^* = a/(a+1)$, so that the transition $2 \to 2$ has zero probability for the isolated map. In actual calculations, we use a slightly modified way to define the symbolic dynamics: To two consecutive measurements $x_t x_{t+1}$ we assign the symbol α if $x_{t+1} \geq x_t$ and the symbol β otherwise [19]. This is completely equivalent to the symbol sequences defined by the sets S_1, S_2 because of the specific partition point x^*; thus the transition $\beta \to \beta$ does not occur for the single tent map. The advantage of using α,β is that one only needs to check increases and decreases in the measured signal, which adds more robustness in case the fixed point is not precisely known. We evolve (5) starting from random initial conditions and estimate the transition probabilities using time series of length $\tau = 1000$ from a randomly selected node. Note that the length of the time series is much shorter time than would be required by standard time-series methods which use embedding to reconstruct the phase space for large networks [17]. In our case, however, the length of the series is independent of the network size. We estimate the transition probability $P(i,j)$ by the ratio $\sum_t n(s_t = i, s_{t+1} = j)/\sum_t n(s_t = i)$, where n is a count of the number of times of occurrence. Synchronization is signalled when the variance of variables over the network given by $\sigma^2 = \left\langle \frac{1}{N-1}\sum_i [x_i(t) - \bar{x}(t)]^2 \right\rangle_t$ drops to zero, where $\bar{x}(t) = \frac{1}{N}\sum_i x_i(t)$ denotes an average over the nodes of the network and $\langle \ldots \rangle_t$ denotes an average over time. Fig. 2 summarizes the results. It is seen in all cases that the region for synchronization exactly coincides with the range for which $P(\beta,\beta)$ is zero (and other transition probabilities are nonzero; see subfigure (c)). Hence, regardless of network topology and size, both synchronized and unsynchronized behavior of the network can be accurately detected over the whole range of coupling strengths using only measurements from an arbitrarily selected node. For higher dimensional systems, for example hénon maps, finding optimal partitions corresponding to the maximal permutation entropy difference may be difficult, but the method works for any non-generating partitions and global synchrony is detected by comparing all the transition probabilities measured from a time series of any arbitrary node with those of the isolated function. Results will be presented elsewhere.

We note that the method applies equally well to any diffusive network, say, $[x_i(t+1) = f(x_i(t)) + \sum_{j=1}^{N} w_{ij} g(x_i(t), x_j(t))]$ which may be asymmetrically coupled or weighted, with weights given by w_{ij}, x is n-dimensional, and g is a diffusion function satisfying $g(x,x) = 0\ \forall x$. The basic idea is again the same: So long as the synchronized dynamics is identical to the isolated dynamics, synchrony can be detected by comparing the transition probabilities of symbolic sequences to those for the isolated map. The fact that certain transition probabilities are exactly zero makes this procedure especially robust, and the choice of the partition plays an important role here. For networks including transmission delays, the synchronized dynamics is no longer identical to that of the isolated map [24]. Nevertheless, it can be shown that the properties of the map still induces certain forbidden transitions, whose presence or absence can be used to detect synchrony in the delayed network. Due to space constraints, we defer these results to an extended paper.

In conclusion, we have presented a simple and effective method based on symbolic dynamics for the detection of synchronization in diffusively coupled networks. The method works by taking measurements from as few as one single node, and can utilize rather short sequences of measurements, and hence is computationally very fast. Furthermore, it is robust against noise and parameter uncertainties, is independent of network size, and does not require knowledge of the connection structure. Additionally, the method allows to distinguish between chaotic and random iterations with same underlying probability density.

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