Max-Planck-Institut
für Mathematik
in den Naturwissenschaften
Leipzig

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by

Xiu-Hong Gao, Sergio Albeverio, Shao-Ming Fei, and Zhi-Xi Wang

Preprint no.: 4 2006

Matrix Tensor Product Approach to the Equivalence of Multipartite States under Local Unitary Transformations

X. H. Gao[1], S. Alberverio[2], S. M. Fei[1,2], Z. X. Wang[1]

[1] Department of Mathematics, Capital Normal University, Beijing 100037
[2] Institut für Angewandte Mathematik, Universität Bonn, D-53115

Abstract

The equivalence of multipartite quantum mixed states under local unitary transformations is studied. A criterion for the equivalence of non-degenerate mixed multipartite quantum states under local unitary transformations is presented.

PACS numbers: 03.67.-a, 02.20.Hj, 03.65.-w

Quantum entangled states are playing fundmental roles in quantum information processing such as quantum computation, quantum teleportation, dense coding, quantum cryptographic schemes quantum error correction, entanglement swapping, and remote state preparation (RSP) etc.. However the theory of quantum entanglement is still far from being satisfied. To quantify the degree of entanglement a number of entanglement measures have been proposed for bipartite states. Most of these proposed measures of entanglement involve extremizations which are difficult to handle analytically. For multipartite case how to give a well defined measure is still under discussion. For general mixed states till now we don't even have an operational criterion to verify whether a state is separable or not. As a matter of fact, the degree of entanglement of a multipartite quantum system remains invariant under local unitary transformations of every subsystems. Therefore the quantum states can be classified according to the local unitary transformations. Nevertheless an explicit picture of the orbits (geometry and topology) under such transformations is not yet ready. We even don't have a general (operational) criterion to verify if two mixed states are equivalent or not under local unitary transformations.

One approach in dealing with the equivalence of quantum states under local unitary transformations is to find the complete set of invariants under local unitary transformations. Two states are equivalent under local unitary transformations if and only if they have the same values of all these invariants. The method developed in [1, 2], in principle, allows one to compute all the invariants of local unitary transformations, though in general it is not operational. The invariants for general two-qubit and three qubits systems have been studied in [3, 4]. In [5] a complete set of invariants is presented for bipartite generic mixed states. In [6] a complete set of invariants under local unitary transformations is presented for rank-2 and multiplicity free mixed (bipartite) states. The invariants for pure tripartite states are also studied in [7, 8].

In [9] another approach is presented in investigating the equivalence problem in terms of fixed point subgroup and tensor decomposability of certain matrices. The problem is reduced to verify whether a certain matrix is rank one or not. A criterion for the equivalence of two non-degenerate mixed bipartite states under local unitary transformations has been presented.

It is rather difficult to deal with the equivalence problem of multipartite states in terms of invariants approach. The number of invariants increases quickly when the subsystems increase. Nevertheless, we find that the approach in [9] can be easily generalized to multipartite case. In this note we investigate the equivalence of multipartite states under local unitary transformations by using the method in [9]. An operational criterion for the equivalence of non-degenerate mixed multipartite states under local unitary transformations is presented.

We first consider tripartite case. Let H_1 (resp. H_2 and H_3) be an M (resp. N and P)-dimensional complex Hilbert space, with $|e_i\rangle$, $i = 1, ..., M$ (resp. $|f_j\rangle$, $j = 1, ..., N$ and $|g_k\rangle$, $k = 1, ..., P$), as an orthonormal basis. A general pure state on $H_1 \otimes H_2 \otimes H_3$ is of the form

$$|\Psi\rangle = \sum_{i=1}^{M}\sum_{j=1}^{N}\sum_{k=1}^{P} a_{ijk}|e_i\rangle \otimes |f_j\rangle \otimes |g_k\rangle, \qquad a_{ijk} \in \mathbb{C} \tag{1}$$

with the normalization $\sum_{i=1}^{M}\sum_{j=1}^{N}\sum_{k=1}^{P} a_{ijk}a_{ijk}^* = 1$ ($*$ denoting complex conjugation). A tripartite quantum mixed state on $H_1 \otimes H_2 \otimes H_3$ is described by a density matrix ρ which can be decomposed according to its eigenvalues and eigenvectors: $\rho = \sum_{i=1}^{MNP} \lambda_i|\nu_i\rangle\langle\nu_i|$, where λ_i are the eigenvalues and $|\nu_i\rangle$, $i = 1, ..., MNP$, the corresponding eigenvectors of the form (1).

Two density matrices ρ and ρ' are said to be equivalent under local unitary transformations if there exist unitary operators U_1 on H_1, U_2 on H_2 and U_3 on H_3 such that

$$\rho' = (U_1 \otimes U_2 \otimes U_3)\rho(U_1 \otimes U_2 \otimes U_3)^\dagger. \tag{2}$$

For a Hermitian matrix A on $H_1 \otimes H_2 \otimes H_3$, the set of commuting matrices B such that $AB = BA$ is called the commutant of A, denoted as $C(A)$. Obviously $C(A)$ is a subalgebra. We call the set of unitary matrices U such that $UA = AU$ the fixed point subgroup of A, denoted as $C_U(A)$, which is a subgroup of the unitary group of all unitary matrices. In the following we say that a matrix V on $H_1 \otimes H_2 \otimes H_3$ is tensor decomposable if it can be written as $V = V_1 \otimes V_2 \otimes V_3$ for $V_1 \in End(H_1)$, $V_2 \in End(H_2)$, $V_3 \in End(H_3)$.

If two density matrices ρ and ρ' are equivalent under local unitary transformations, they must have the same set of eigenvalues λ_i, $i = 1, ..., n$. Let X and Y be the unitary matrices that diagonalize ρ and ρ' respectively,

$$\rho = X\Lambda X^\dagger, \qquad \rho' = Y\Lambda Y^\dagger, \tag{3}$$

where $\Lambda = diag(\lambda_1, \lambda_2, ..., \lambda_{MNP})$.

[Lemma 1]. Let G be the fixed point unitary subgroup associated with ρ. Then ρ' is equivalent to ρ under local unitary transformations if and only if the coset $GXY^\dagger$ contains a unitary tensor decomposable matrix.

[Proof]. Suppose $\rho' = (U_1 \otimes U_2 \otimes U_3)\rho(U_1 \otimes U_2 \otimes U_3)^\dagger$, then $\rho' = Y\Lambda Y^\dagger = YX^\dagger \rho XY^\dagger$. Hence $YX^\dagger\rho XY^\dagger = (U_1 \otimes U_2 \otimes U_3)\rho(U_1 \otimes U_2 \otimes U_3)^\dagger$, or

$$(U_1 \otimes U_2 \otimes U_3)^\dagger YX^\dagger \rho = \rho(U_1 \otimes U_2 \otimes U_3)^\dagger YX^\dagger.$$

That is, $(U_1 \otimes U_2 \otimes U_3)^\dagger YX^\dagger \in C_U(\rho)$, and $C_U(\rho)XY^\dagger = YX^\dagger C_U(\rho)$ contains a unitary tensor decomposable element $U_1 \otimes U_2 \otimes U_3$.

Conversely, assume $GXY^\dagger$ contains a tensor decomposable element $U_1 \otimes U_2 \otimes U_3$. We have then $UXY^\dagger = U_1 \otimes U_2 \otimes U_3$, $U\rho = \rho U$ and U is unitary. Therefore

$$\begin{aligned} \rho' &= Y\Lambda Y^\dagger = (U_1 \otimes U_2 \otimes U_3)^\dagger UX\Lambda X^\dagger U^\dagger (U_1 \otimes U_2 \otimes U_3) \\ &= (U_1 \otimes U_2 \otimes U_3)^\dagger U\rho U^\dagger (U_1 \otimes U_2 \otimes U_3) = (U_1 \otimes U_2 \otimes U_3)^\dagger \rho (U_1 \otimes U_2 \otimes U_3). \end{aligned}$$

□

Let Z be an a matrix on $H_1 \otimes H_2 \otimes H_3$. If we view Z as a matrix on spaces H_1 and $H_2 \otimes H_3$, it is an $M \times M$ block matrix with each block of size $NP \times NP$. Its realigned matrix $\tilde{Z}_{1|23}$ is defined by

$$\tilde{Z}_{1|23} = [vec(Z_{11}), \cdots, vec(Z_{1M}), \cdots, vec(Z_{M1}), \cdots, vec(Z_{MM})]^t,$$

Taking Z as a matrix on spaces $H_1 \otimes H_2$ and H_3, we have that Z is an $MN \times MN$ block matrix with each block of size $P \times P$, and the realigned matrix $\tilde{Z}_{12|3}$ is of the form

$$\tilde{Z}_{12|3} = [vec(Z_{11}), \cdots, vec(Z_{1MN}), \cdots, vec(Z_{MN1}), \cdots, vec(Z_{MNMN})]^t,$$

where for any $M \times N$ matrix A with entries a_{ij}, $vec(A)$ is defined to be

$$vec(A) = [a_{11}, \cdots, a_{1N}, a_{21}, \cdots, a_{2N}, \cdots, a_{M1}, \cdots, a_{MN}]^t.$$

The above operation of realignment could be defined in an alternative way,

$$(\tilde{Z}_{1|23})_{im,jknp} = (Z)_{ijk,mnp}\,, \qquad (\tilde{Z}_{12|3})_{ijmn,kp} = (Z)_{ijk,mnp}.$$

It is shown that a matrix V can be expressed as the tensor product of two matrices V_1 and V_2, $V = V_1 \otimes V_2$, if and only if [10]

$$\tilde{V} = vec(V_1)vec(V_2)^t. \tag{4}$$

Moreover [7], for an $MN \times MN$ unitary matrix U, if U is a unitarily decomposable matrix, then the rank of $\tilde{U}$ is one, $r(\tilde{U}) = 1$. Conversely if $r(\tilde{U}) = 1$, there exists an $M \times M$ matrix U_1 and an $N \times N$ matrix U_2, such that $U = U_1 \otimes U_2$ and

$$U_1U_1^\dagger = U_1^\dagger U_1 = k^{-1}I_M, \qquad U_2U_2^\dagger = U_2^\dagger U_2 = kI_N, \tag{5}$$

where I_N (resp. I_M) denotes the $N \times N$ (resp. $M \times M$) identity matrix, $k > 0$, and U is a unitary tensor decomposable matrix.

[Lemma 2]. For an $MNP \times MNP$ unitary matrix U on $H_1 \otimes H_2 \otimes H_3$, U is a unitary decomposable matrix if and only if the ranks of $\tilde{U}_{1|23}, \tilde{U}_{12|3}$ are one, i.e., $r(\tilde{U}_{1|23}) = r(\tilde{U}_{12|3}) = 1$.

[Proof]. Suppose $U = U_1 \otimes U_2 \otimes U_3$, then $\tilde{U}_{1|23} = vec(U_1)vec(U_2 \otimes U_3)^t$, $\tilde{U}_{12|3} = vec(U_1 \otimes U_2)vec(U_3)^t$, obviously $r(\tilde{U}_{1|23}) = r(\tilde{U}_{12|3}) = 1$.

Conversely, if $r(\tilde{U}_{1|23}) = 1$, there exists an $M \times M$ matrix U_1 and an $NP \times NP$ matrix U_{23}, such that $U = U_1 \otimes U_{23}$. Therefore

$$\tilde{U}_{12|3} = \begin{pmatrix} u_{11}\tilde{U}_{23} \\ u_{12}\tilde{U}_{23} \\ \vdots \\ u_{MM}\tilde{U}_{23} \end{pmatrix}, \tag{6}$$

where $u_{ij}, j = 1, \cdots, M$ are the entries of U_1. $r(\tilde{U}_{12|3}) = 1$ if and only if $r(\tilde{U}_{23}) = 1$, that is, there exists an $N \times N$ matrix U_2 and a $P \times P$ matrix U_3, such that $U_{23} = U_2 \otimes U_3$, and U is a unitary tensor decomposable matrix. $\square$

Let ρ and ρ' be two density matrices with orthonormal unitary matrices X and Y as given in (3). Set

$$V_0 = X \begin{pmatrix} A_{n_1} & 0 & \cdots & 0 \\ 0 & A_{n_2} & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & \cdots & A_{n_r} \end{pmatrix} Y^\dagger, \tag{7}$$

where n_i, $i = 1, 2, ..., r$, stands for the geometric multiplicity of the eigenvalue λ_i of ρ, $\sum_1^r n_r = MNP$, A_{n_i} are some unitary $n_i \times n_i$ complex matrices. The conclusion on bipartite case [9] is still valid, namely, ρ and ρ' are equivalent under local unitary transformations if and only if rank $r(\tilde{V_0}) = 1$ for some unitary matrices A_{n_i}. For the case that ρ has distinct eigenvalues, we have

[Theorem 1]. If ρ has distinct eigenvalues, ρ is equivalent to ρ' under local unitary transformations if and only if

$$V = XDY^\dagger, \tag{8}$$

$D = diag(e^{i\theta_1}, e^{i\theta_2}, ..., e^{i\theta_{MNP}})$, contains a unitary tensor decomposable element for some $\theta_i \in I\!R$.

[Proof]. Let X and Y be the unitary matrices that diagonalize ρ and ρ' respectively, $\rho = X\Lambda X^\dagger$, $\rho' = Y\Lambda Y^\dagger$, where $\Lambda = diag(\lambda_1, \lambda_2, ..., \lambda_{MNP})$.

As ρ (resp. ρ') has distinct eigenvalues, any set X_1 (resp. Y_1) of unitary eigenvectors corresponding to the eigenvalues of ρ (resp. ρ') can then be obtained through the following equation: $X_1 = XU$ (resp. $Y_1 = YV$), where the unitary matrix U (resp. V) has the form: $U = diag(e^{i\alpha_1}, e^{i\alpha_2}, ..., e^{i\alpha_{MNP}})$ (resp. $V = diag(e^{i\beta_1}, e^{i\beta_2}, ..., e^{i\beta_{MNP}})$), then $X_1 D_1 Y_1^\dagger = XUD_1V^\dagger Y^\dagger = XDY^\dagger$, where $D_1 = diag(e^{i\theta'_1}, e^{i\theta'_2}, ..., e^{i\theta'_{MNP}})$, $D = diag(e^{i\theta_1}, e^{i\theta_2}, ..., e^{i\theta_{MNP}})$, $\theta_i = \alpha_i + \theta'_i - \beta_i, i = 1, 2, ..., MNP$, hence ρ is locally unitary equivalent to ρ' if and only if $XDY^\dagger$, $D = diag(e^{i\theta_1}, e^{i\theta_2}, ..., e^{i\theta_{MNP}})$, contains a unitary tensor decomposable element for some $\theta_i \in I\!R$. $\square$

As an example let us consider a density matrix on 2 × 2 × 2,

$$\rho = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 1/a & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1/b & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/c & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & c & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & b & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & a & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \tag{9}$$

$$\rho' = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & a & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & b & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & c & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1/c & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1/b & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1/a & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \tag{10}$$

ρ' is in fact a PPT entangled edge state [11]. ρ' and ρ have the same eigenvalue set. They are not degenerate in the case $a \neq b \neq c \neq 1$ or 2 or 1/2. Calculating the unitary matrices X and Y that diagonalize ρ and ρ', we have $\rho = X\Lambda X^\dagger$, $\rho' = Y\Lambda Y^\dagger$, where $\Lambda = diag(2, 0, 1/a, a, 1/b, b, 1/c, c)$. Denote $D = diag(d_1, d_2, d_3, d_4, d_5, d_6, d_7, d_8)$, we have the matrix V defined by (8),

$$V = \begin{pmatrix} (-d_1+d_8)/2 & (d_1+d_8)/2 & 0 & 0 & 0 & 0 & 0 & 0 \\ (d_1+d_8)/2 & (-d_1+d_8)/2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & d_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & d_7 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & d_3 & 0 & 0 \\ 0 & 0 & 0 & 0 & d_6 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & d_4 \\ 0 & 0 & 0 & 0 & 0 & 0 & d_5 & 0 \end{pmatrix}.$$

Therefore

$$\tilde{V}_{1|23} = \begin{pmatrix} \frac{-d_1+d_8}{2} & \frac{d_1+d_8}{2} & 0 & 0 & \frac{d_1+d_8}{2} & \frac{-d_1+d_8}{2} & 0 & 0 & 0 & 0 & 0 & d_2 & 0 & 0 & d_7 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & d_3 & 0 & 0 & d_6 & 0 & 0 & 0 & 0 & 0 & 0 & d_4 & 0 & 0 & d_5 & 0 \end{pmatrix},$$

$$\tilde{V}_{12|3} = \begin{pmatrix} (-d_1+d_8)/2 & (d_1+d_8)/2 & (d_1+d_8)/2 & (-d_1+d_8)/2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & d_2 & d_7 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & d_3 & d_6 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & d_4 & d_5 & 0 \end{pmatrix},$$

obviously, when $d_1 = -d_2 = d_3 = -d_4 = -d_5 = d_6 = -d_7 = d_8 = 1, r(\tilde{V}_{1|23}) = r(\tilde{V}_{12|3}) = 1$. Hence ρ and ρ' are equivalent under local unitary transformations.

Now we consider the multipartite case. A general pure state on $H_1 \otimes H_2 \otimes \ldots \otimes H_M$ is of the form

$$|\Psi_M\rangle = \sum_{k=1}^{M} \sum_{i_k=1}^{N_k} a_{i_1 i_2 \ldots i_M} |e_{i_1}\rangle \otimes |f_{i_2}\rangle \otimes \ldots \otimes |g_{i_M}\rangle, \qquad a_{i_1 i_2 \ldots i_M} \in \mathbb{C} \tag{11}$$

with $\sum a_{i_1 i_2 \ldots i_M} a^*_{i_1 i_2 \ldots i_M} = 1$, $|e_{i_1}\rangle$, $|f_{i_2}\rangle$, ..., $|g_{i_M}\rangle$, $i_k = 1, 2, ..., N_k$, $k = 1, 2, ..., M$, the corresponding orthonormal basis of complex Hilbert spaces H_1, H_2, ..., H_M. Two density matrices ρ and ρ' are said to be equivalent under local unitary transformations if there exist unitary operators U_1 on H_1, U_2 on H_2, $\cdots$, and U_M on H_M such that $\rho' = (U_1 \otimes U_2 \otimes \cdots \otimes U_M)\rho(U_1 \otimes U_2 \otimes \cdots \otimes U_M)^\dagger$. For any non-degenerate density matrices we have

[Theorem 2]. Let ρ and ρ' be two non-degenerate density matrices on $H_1 \otimes H_2 \otimes \ldots \otimes H_M$ and X and Y the unitary matrices that diagonalize ρ and ρ' respectively, $\rho = X\Lambda X^\dagger$, $\rho' = Y\Lambda Y^\dagger$, where $\Lambda = diag(\lambda_1, \lambda_2, ..., \lambda_{N_1 N_2 \cdots N_M})$. ρ and ρ' are equivalent under local unitary transformations if and only if the $N_1 N_2 \cdots N_M \times N_1 N_2 \cdots N_M$ unitary matrix $V = XDY^\dagger$, $D = diag(e^{i\theta_1}, e^{i\theta_2}, ..., e^{i\theta_{N_1 N_2 \cdots N_M}})$, satisfies

$$r(\tilde{V}_{1|2\cdots M}) = r(\tilde{V}_{12|3\cdots M}) = \cdots = r(\tilde{V}_{12\cdots M-1|M}) = 1,$$

where

$$\begin{aligned} &(\tilde{V}_{1|2\cdots M})_{i_1 i'_1, i_2 \cdots i_M i'_2 \cdots i'_M} = (V)_{i_1 i_2 \cdots i_M, i'_1 i'_2 \cdots i'_M}, \\ &(\tilde{V}_{12|3\cdots M})_{i_1 i_2 i'_1 i'_2, i_3 \cdots i_M i'_3 \cdots i'_M} = (V)_{i_1 i_2 \cdots i_M, i'_1 i'_2 \cdots i'_M}, \\ &\qquad\qquad \cdots\cdots\cdots\cdots \\ &(\tilde{V}_{12\cdots M-1|M})_{i_1 i_2 \cdots i_{M-1} i'_1 i'_2 \cdots i'_{M-1}, i_M i'_M} = (V)_{i_1 i_2 \cdots i_M, i'_1 i'_2 \cdots i'_M}. \end{aligned}$$

We have studied the equivalence of multipartite quantum mixed states under local unitary transformations in terms of analysis of fixed point subgroup and tensor decomposability of certain matrices. A criterion for the equivalence of non-degenerate mixed multipartite quantum mixed states under local unitary transformations has been presented. In fact this approach works for general multipartite mixed states. But then in stead of the rank $r(\tilde{V})$, one has to verify if $r(\tilde{V_0})$ could be one for all possible matrices A_{n_i}, which is again complicated. The problem is dramatically simplified when the degeneracy of the related density matrices is reduced. In particular, for the non-degenerate case, two density matrices are easily verified whether they are equivalent or not under local unitary transformation.

Acknowledgments The first and last named author gratefully acknowledges the support provided by the China-Germany Cooperation Project 446 CHV 113/231, “Quantum information and related mathematical problems”.

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