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**On the Non-Tangential Touch Between the Free and Fixed Boundaries for the Two-Phase Obstacle-Like Problem**

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# ON THE NON-TANGENTIAL TOUCH BETWEEN THE FREE AND THE FIXED BOUNDARIES FOR THE TWO-PHASE OBSTACLE-LIKE PROBLEM

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ABSTRACT. In this paper we consider the following two-phase obstacle-problem-like equation in the unit half-ball

$$\Delta u = \lambda_+ \chi_{\{u>0\}} - \lambda_- \chi_{\{u<0\}}, \ \lambda_\pm > 0.$$

We classify the angles of touch between the free boundary and the fixed one if the boundary data $f$ satisfies certain conditions. This results are connected to the results from [AMM].

## 1. INTRODUCTION

1.1. **The Problem.** The following two-phase analogue of classical obstacle problem was suggested by G. S. Weiss in [W2] and then considered by N.N. Uraltseva in [U] and H. Shahgholian, N.N. Uraltseva and G.S. Weiss in [SUW]. Study properties of a weak solution $u \in W^{1,2}(D)$ of

$$\Delta u = \lambda_+ \chi_{\{u>0\}} - \lambda_- \chi_{\{u<0\}}, \tag{1}$$

in the domain $D$, such that $u - f \in W_0^{1,2}(D)$ for a given $f \in W^{1,2}(D)$. In our paper we always assume $\lambda_\pm > 0$, and we consider the cases $D = B_1$ and $D = B_1^+$, as well as the case of the so-called global solutions $D = \mathbb{R}^n_+$.

Obviously (1) is the Euler-Lagrange equation of the energy functional

$$J(u) = \int_D |\nabla u|^2 + 2\lambda_+ \max(u,0) + 2\lambda_- \max(-u,0)dx.$$

Note that if the boundary data $f$ is non-negative (non-positive) then the solution $u$ is so, too, and we arrive at classical obstacle problem (see [C]). In the two-phase case we do not have the property that the gradient vanishes on the free boundary $\Gamma_u$ (see Section 1.2 for definition), as it was in the classical case; this causes difficulties.

We consider the following problem: Let $u$ be a weak solution of (1) in $B_1^+$, $0 \in \overline{\Gamma}_u$, $f := u|_\Pi \in C^{1,1}(B_1 \cap \Pi)$, and

$$\frac{f(rx')}{r^2} \to_{r\to 0} a_+(x_2^+)^2 - a_-(x_2^-)^2, \ a_\pm > 0 \tag{2}$$

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and

$$f(x)^{\pm} \in C^{2,Dini}(\text{spt} f^{\pm}).$$

We classify the angles of touch between the fixed and free boundaries at 0. It should be mentioned also that in [AMM] authors and N. Matevosyan proved that if $a_{\pm} = 0$ then the free boundary of $u$ approaches the fixed one at 0 tangentially. Under some growth assumptions they proved that this approach is uniform.

Several papers have been investigating similar problems using the same techniques (see for instance [A], [AMM], [SU]). However, in this paper we run into some new difficulties, and have to invent some new techniques, that renders this publication some value. The main new contribution is in the proof of Lemma 15. There we give a boundary regularity result, in the sense of a growth estimate, by using compactness and energy considerations together with geometric properties of the free boundary. We consider this to be the heart of the paper and believe that it will be of interest to experts in this field.

Before we state our main result let us briefly fix notation and give a some definitions. First we would want to recall the definition of $C^{2,Dini}(U)$; these are functions from $C^2(U)$ such that

$$|D^2 f(x) - D^2 f(y)| \leq \omega(|x-y|),$$

where $D^2 f$ is the Hessian of $f$ and $\omega$ is a Dini modulus of continuity, i.e.,

$$\int_0^1 \frac{\omega(s)}{s} ds < \infty.$$

1.2. **Notations.** In the sequel we use following notations:

| | |
|---|---|
| $\mathbf{R}^n_+$ | $\{x \in \mathbf{R}^n : x_1 > 0\}$ |
| $\mathbf{R}^n_-$ | $\{x \in \mathbf{R}^n : x_1 < 0\}$, |
| $B(z,r)$ | $\{x \in \mathbf{R}^n : \lvert x-z\rvert < r\}$, |
| $B_r$ | $B(0,r)$, |
| $B_r^+$ | $\mathbf{R}^n_+ \cap B_r$, |
| $\Pi$ | $\{x \in \mathbf{R}^n : x_1 = 0\}$, |
| $x'$ | $(x_2, \ldots, x_n)$, |
| $K_\epsilon$ | $\{x \in \mathbf{R}^n_+ : x_1 > \epsilon\lvert x'\rvert\}$, |
| $\lVert \cdot \rVert_\infty$ | canonical norm, |
| $e_1, \ldots, e_n$ | standard basis in $\mathbf{R}^n$, |
| $\nu,\ e$ | arbitrary unit vectors, |
| $D_\nu,\ D_{\nu e}$ | first and second directional derivatives, |
| $v^+,\ v^-$ | $\max(v,0),\ \max(-v,0)$, |
| $\chi_D$ | characteristic function of the set $D$, |
| $\partial D$ | boundary of the set $D$, |
| $\Omega_u^+$ | $\{x \in D : u(x) > 0\}$, |
| $\Omega_u^-$ | $\{x \in D : u(x) < 0\}$, |

$\Lambda_u$ $\qquad \big\{x \in B_1^+ : u(x) = |\nabla u(x)| = 0\big\}$,
$\Gamma_u$ $\qquad (\partial\Omega_u^+ \cup \partial\Omega_u^-) \cap D$, the free boundary.

**Definition 1.** *We will say that 'the free boundary hits the fixed boundary with an angle $\theta$ at $x_0$', 'the angle of touch between the free and fixed boundaries is $\theta$' and equivalent formulations if*

*(a) $x_0 \in \partial D$, that is $x_0$ is a point on the boundary of our domain*

*(b) $\Gamma_u \cap B_r(x_0) \neq \emptyset$ for $r > 0$, that is the free boundary touches the fixed boundary at $x_0$*

*(c) $x_0$ is in the reduced boundary of $\Gamma_u$, and the angle between its measure theoretic normal and the normal of $\partial D$ equals $\vartheta$.*

*We will also use 'angles of touch' etc. if the free boundary locally has two or more components with well defined angles of touch.*

## 2. Main result

**Theorem A.** *Let $u$ be a solution of (1) in $B_1^+$ with boundary data $f$ on $\Pi$, condition (2) is satisfied, $\nabla u(0) = 0$ and*

$$a_+ \leq \frac{\lambda_+}{2} \text{ and } a_- \leq \frac{\lambda_-}{2}. \tag{3}$$

*Then the solution $u$ is quadratically bounded near 0, that is $|u| \leq C|x|^2$, and the free boundary can 'hit' the fixed one only with angles $\theta_0$, $\theta_j^\pm$, $j = 1, 2$ (to be specified later, see Figure 1). If the condition (3) fails then the solution is not quadratically bounded and free boundary approaches the fixed one orthogonally in every non-tangential access cone. If $\nabla u(0) \neq 0$ then the free boundary approaches the fixed one tangentially.*

**Remark 2.** *By orthogonal approach in a non-tangential access cone we mean that in each cone $x_1 > \epsilon|x'|$ with $\epsilon > 0$ the free boundary $\Gamma_u$ is trapped in a cusp $\Gamma_u \in \{|x_2| \leq \sigma_\epsilon(x_1)x_1\}$ where $\sigma_\epsilon$ is a modulus of continuity depending on $n$, $\epsilon$ and $\|u\|_{L^\infty}$.*

*This formulation of the Theorem is actually a little weaker than what we can prove. In fact, if say $a_+ > \lambda_+/2$ then this is true in a small neighbourhood of the origin so we can apply the Theorem on each touching point in that set, under the assumption that $|\nabla u| = 0$ along the entire touching set, and deduce a stronger statement without referring to a non-tangential approach region.*

**Remark 3.** *There is a slight difference between the formulations of the result in the orthogonal and non-orthogonal part of the Theorem. This is for technical reasons. Even if the result seems weaker in the orthogonal case, it is only slightly so since we can always make the non-tangential access cone as close as the half space as we would like.*

**Remark 4.** *The final statement of the Theorem, that $|\nabla u| \neq 0$ implies tangential touch, follows easily from classical $C^{1,\alpha}$ estimates of the solution.*

**Remark 5.** *In the formulation of the theorem we have chosen to consider the case when $|\nabla f(0)| = 0$. This assumption is probably not necessary to analyse the local behaviour of the free boundary near a 'touching point.' We have not chosen to consider the more general case though but will briefly describe some features that will play a role in a description of that problem.*

*If we have a point where the free boundary touches the fixed boundary and $u$ is in $C^{1,1}$ at that point; then $u(x) - \nabla u(0) \cdot x$ is quadratically bounded and we can make a blow-up whose limit is a homogeneous solution to a partial differential equation similar to our original one. A classification of global solutions will give possible angles of touch, and therefore also determine a discrete set of possible values of $\partial u(0)/\partial x_n$. The only problem is to characterise the touching points where $u$ is $C^{1,1}$.*

**Remark 6.** *Since the main tool we use proving Theorem A is the blow-up argument, these results could be generalised for domains with smooth enough boundary.*

## 3. Technicalities

3.1. **Monotonicity formulae.** Here we introduce two monotonicity formulae in the following two lemmas, which play crucial role in our proofs. The first one was presented by H. W. Alt, L. A. Caffarelli and A. Friedman in [ACF] and was developed then in [CKS]. The second one is due to G. Weiss [W1], [SUW]. In [A] the first author adapted it to the half-space case and our representation is analogous.

**Lemma 7. (ACF monotonicity formula)**

*Let $h_1$, $h_2$ be two non-negative continuous sub-solutions of $\Delta u = 0$ in $B_R$. Assume further that $h_1 h_2 = 0$ and that $h_1(0) = h_2(0) = 0$. Then the following function is non-decreasing in $r \in (0, R)$*

(4)
$$\varphi(r) \equiv \varphi(r, h_1(x), h_2(x)) = \frac{1}{r^4}\left(\int_{B_r} \frac{|\nabla h_1|^2 dx}{|x|^{n-2}}\right)\left(\int_{B_r} \frac{|\nabla h_2|^2 dx}{|x|^{n-2}}\right).$$

*More exactly, if any of the sets $spt(h_j) \cap \partial B_r$ digresses from a spherical half-cap by a positive area, then either $\varphi'(r) > 0$ or $\varphi(r) = 0$.*

Let us point out an important scaling invariance of the monotonicity functional $\varphi$, namely

$$\varphi(r, h_1(x), h_2(x)) = \varphi(1, h_1(rx)/r, h_2(rx)/r).$$

In the next Lemma we will make the assumption that our solution is asymptotically homogeneous on the boundary. Observe that a function $g$ is homogeneous of order 2 if $x \cdot \nabla g - 2g = 0$.

**Lemma 8. (Weiss' monotonicity formula)**
*Let $u$ solve (1) in $B_R^+$ and $u|_{\Pi\cap B_R} = g$, where $|x\cdot\nabla g - 2g| = o(|x|)$. Then the function*
(5)
$$\Phi(r) = r^{-n-2}\int_{B_r\cap\mathbb{R}^n_+}(|\nabla u|^2+2\lambda_+u^++2\lambda_-u^-)-r^{-n-3}\int_{\partial B_r\cap\mathbb{R}^n_+}2u^2d\mathcal{H}^{n-1}$$
*satisfies*
$$\Phi(\sigma)-\Phi(\rho) > -o(\sigma), \qquad \text{for } 0<\rho<\sigma<R.$$
.

*Moreover, if $\Phi(\rho)=\Phi(\sigma)$ for any $0<\rho<\sigma<R$ then $\Phi$ is homogeneous of degree two in $(B_\sigma\backslash B_\rho)\cap\mathbb{R}^n_+$.*

This Lemma says that $\Phi$ is almost monotone in its argument. If $g$ is homogeneous then $\Phi$ will be fully homogeneous in its argument. The proof is analogous to the proof of the Lemma 1 in [A].

## 4. Global solutions

**Lemma 9.** *Let $u$ solve (1) in $\mathbb{R}^n_+$ with boundary data $u|_\Pi = a_+(x_2^+)^2 - a_-(x_2^-)^2$. Then if $|u|\le C(1+|x|^2)$ for any finite $C$ then $u$ is two-dimensional, i.e., in some system of coordinates*
$$u(x)=u(x_1,x_2),$$
*where the $e_1$ direction is normal to $\Pi$.*

**Proof.** Let us take any $e\in\Pi$ and consider functions $(D_eu)^\pm$. In [U] Uraltseva proved that these functions are sub-harmonic. If $\langle e,e_2\rangle>0$ then the boundary values of $u$ are monotone increasing in $e$ direction we can extend $(D_eu)^-$ by zero to $\mathbb{R}^n_-$; let us denote the new function by $h$. If we now take $g(x):=h(-x_1,x')$, we can apply ACF monotonicity formula on pair $h,g$. For $r<s$ we have
$$\varphi(r,h,g)\le\varphi(s,h,g)\le\lim_{s\to\infty}\varphi(s,h,g)=:C_e.$$
By standard regularity theory for elliptic equations and the bound $|u|\le C(1+|x|^2)$, thus we can find a sequence $u_{r_j}=\frac{u(r_jx)}{r_j^2}\to u_\infty$, uniformly on compact subsets and in $(W^{2,p}_{loc}\cap C^{1,\alpha}_{loc})(\mathbb{R}^n_+\cup\Pi)$, for any $1<p<\infty$ and $0<\alpha<1$. Denoting by $h_{r_j}=\frac{h(r_jx)}{r_j}$, $g_{r_j}=\frac{g(r_jx)}{r_j}$ and their limits as $j\to\infty$ by $h_\infty,g_\infty$. It is easy to see that $h$ and $g$ satisfies the conditions in the ACF monotonicity formula. Using this and the scaling invariance of the monotonicity functional $\varphi$ we may deduce
$$C_e=\lim_{r_j\to\infty}\varphi(sr_j,h,g)=\lim_{r_j\to\infty}\varphi(s,h_{r_j},g_{r_j})=\varphi(s,h_\infty,g_\infty),\ \ \forall s>0.$$
From $\{x_1<0\}\subset\{h=0\}$ and continuity it follows that $\varphi(r,h_\infty,g_\infty)\equiv 0$ or $\varphi'(r,h_\infty,g_\infty)>0$ for all $r>0$, thus $C_e=0$ and we get $D_eu\ge 0$.

Now let us take any $e\in\Pi$ orthogonal to $e_2$ and consider unit vector $e(\phi)=\cos\phi\, e_2+\sin\phi\, e\in\Pi$, $\phi\in[0,\pi]$. From the $C^1$-continuity we have that the sets $\{\phi : \Omega^{\pm}_{D_{e(\phi)}u} \neq \emptyset\}$ are relatively open in $[0,\pi]$. On the other hand they are both non-empty and have empty intersection in $[0,\pi]\backslash\{\frac{\pi}{2}\}$; this means that $D_{e(\frac{\pi}{2})}u = D_e u \equiv 0$ and we are done. $\square$

**Proposition 10.** *Let $u$ be homogeneous of degree two and solve (1) in $\mathbb{R}^n_+$ with boundary data $u|_\Pi = a_+(x_2^+)^2 - a_-(x_2^-)^2$. Then $a_\pm \le \frac{\lambda_\pm}{2}$ and the function $u(x)$ coincides with one of the following described below. Take $\vartheta^\pm = \frac{1}{2}\arccos(1-\frac{4a_\pm}{\lambda_\pm})$ and assume $\vartheta^- \le \vartheta^+$ then*

$$u_1(x)\frac{\lambda_+}{2}((x\cdot e_1^+)^+)^2 - \frac{\lambda_-}{2}((x\cdot e_1^-)^+)^2,$$

$$u_2(x) = \frac{\lambda_+}{2}((x\cdot e_2^+)^+)^2 - \frac{\lambda_-}{2}((x\cdot e_2^-)^+)^2,$$

*where* $e_1^+ = (-\sin\vartheta^+, \cos\vartheta^+, 0, \dots, 0)$, $e_1^- = e_2^- = (-\sin\vartheta^-, -\cos\vartheta^-, 0, \dots, 0)$, $e_2^+ = (\sin\vartheta^+, \cos\vartheta^+, 0, \dots, 0)$. *The third possible solution will be in case $\vartheta^- \le \vartheta^+$ as follows. Take $\vartheta_0 = \frac{1}{2}\arccos(1-\frac{4(a_++a_-)}{(\lambda_++\lambda_-)})$ and*

$$u_3(r,\phi) = r^2(\frac{\lambda_+}{4} + (a_+ - \frac{\lambda_+}{4})\cos 2\phi + \beta_+\sin 2\phi), \quad \textit{if } 0 \le \phi \le \vartheta_0$$

*and*

$$u_3(r,\phi) = r^2(-\frac{\lambda_-}{4} - (a_- - \frac{\lambda_-}{4})\cos 2\phi + \beta_-\sin 2\phi), \quad \textit{if } \vartheta_0 \le \phi \le \pi,$$

*where*

$$\beta_\pm = \frac{\mp\frac{\lambda_\pm}{4} \mp (a_\pm - \frac{\lambda_\pm}{4})\cos 2\vartheta_0}{\sin 2\vartheta_0}$$

*The case $\vartheta^+ < \vartheta^-$ is analogous.*

*It follows that if $a_- > \frac{\lambda_-}{2}$ or $a_+ > \frac{\lambda_+}{2}$ then no homogeneous solution exists.*

**Proof.** By Lemma 9 it is enough to consider two-dimensional functions $u$. So let us rewrite $u$ in radial coordinates as

$$u(x) = u(r,\theta) = r^2\phi(\theta), \;\; r \in [0,\infty), \;\; \theta \in [0,\pi].$$

Then we get the ODE

$$\phi'' + 4\phi = \lambda_+\chi_{\{\phi>0\}} - \lambda_-\chi_{\{\phi<0\}}$$

in the interval $[0,\pi]$ with boundary data $\phi(0) = a_+$ and $\phi(\pi) = -a_-$. The analysis of the ODE shows that only if $a_- > \frac{\lambda_-}{2}$ or $a_+ > \frac{\lambda_+}{2}$ then the ODE has no solution.

In the case $a_\pm \le \frac{\lambda_\pm}{2}$ the simple analysis of the ODE gives that the only solutions are those which are described in the formulation of the Proposition. These are illustrated in the Figure 1 in terms of positivity and negativity sets for the case $\vartheta_- < \vartheta_+$.

Let us just mention that in the case $\vartheta^+ < \vartheta^-$ we will take $e_1^\pm$ as before, $e_2^+ = e_1^+$, $e_2^- = (\sin\vartheta^-, -\cos\vartheta^-, 0, \dots, 0)$ and $\vartheta_0 = \pi - \frac{1}{2}\arccos(1-\frac{4(a_++a_-)}{(\lambda_++\lambda_-)})$. $\square$

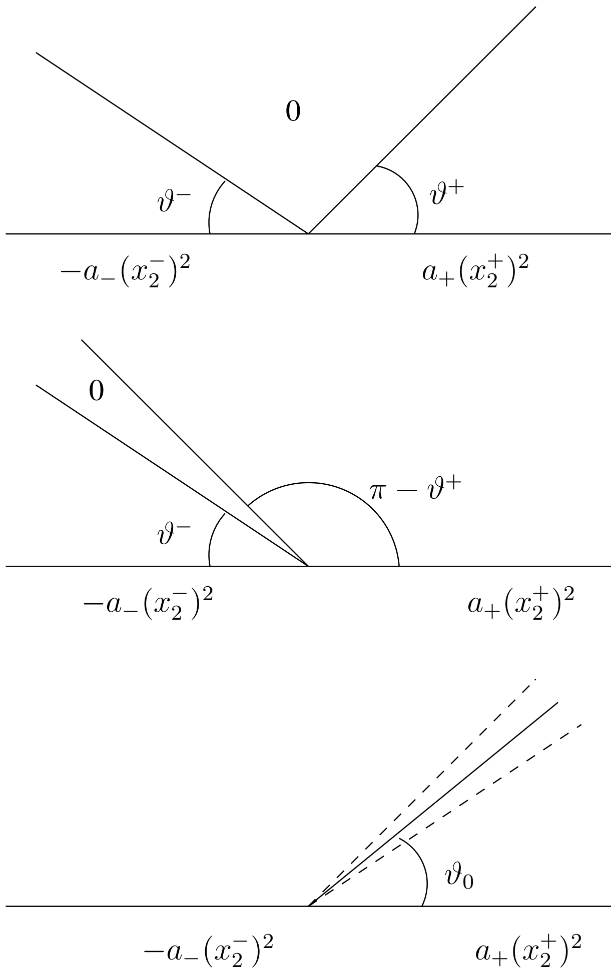


Figure 1. Global solutions

**Remark 11.** *There exist global solutions which are not homogeneous. These were constructed in* [AS] *for the one phase case* $u \geq 0$.

## 5. Proof of Theorem A

In this section $u$ is a solution of (1) satisfying $|\nabla u(0)| = 0$. First we will prove Theorem A under the assumption that $u$ is quadratically bounded.

**Lemma 12.** *If* $a_\pm \leq \frac{\lambda_\pm}{2}$ *and* $u$ *is a solution satisfying* $|u(x)| \leq C|x|^2$ *for some finite* $C$, *then the angles of touch between the free and the fixed boundaries are as in the case of homogeneous global solutions.*

**Proof.** Let us take $u_r(x) = r^{-2}u(rx)$ and observe that for a (sub)sequence $r_n \to 0$ this converges to global solution $u_0$, which is quadratically bounded. What remains to show is that $u_0$ is homogeneous of order two. This follows immediately from Lemma 8,

$$\Phi_{u_0}(s) = \lim_{j\to\infty} \Phi_{u_{r_j}}(s) = \lim_{j\to\infty} \Phi_u(r_j s) = \Phi_u(0+).$$

□

In the next Lemma we will prove the second statement of the main Theorem.

**Lemma 13.** *If $a_+ > \frac{\lambda_+}{2}$ or $a_- > \frac{\lambda_-}{2}$, then $u$ is not quadratically bounded near $0$ and the free boundary approaches the fixed boundary orthogonally in a nontangential approach region.*

**Proof.** We will argue by contradiction. Therefore we assume that there is a sequence of free boundary points $x^j \in \Gamma_u$ approaching the origin in a nonorthogonal way. That is $x_i^j/|x^j| \geq \epsilon$ for $i = 1, 2$. We will set $r_j = |x^j|$, $r_j \to 0$.

If $u(r_j x)/r_j^2$ is uniformly bounded on compact sets then we can repeat the proof of the previous Lemma and obtain a function $u_0$, which is a homogeneous global solution with boundary data $a_+(x_2^+)^2 - a_-(x_2^-)^2$; a contradiction, since $a_+ > \frac{\lambda_+}{2}$ or $a_- > \frac{\lambda_-}{2}$. Therefore $u(r_j x)/r_j^2$ diverges, since the argument in this paragraph works for all sequences $r_j \to 0$ we can conclude the first part of the Lemma.

Now assume that $u$ is not quadratically bounded. For a sequence $r_j \to 0$ let us consider functions $u_j(x) = u(r_j x)/S_j$, where we denote $S_j = \sup_{B^+_{r_j}} |u|$.

For a sub-sequence $u_j$ (that we may assume to be the full sequence) converges to a function $u_0$. Since $\frac{\sup_{B^+_{r_j}} |u|}{r^2} \to \infty$ we get that $u_0$ is harmonic and has zero boundary data on $\Pi$.

We want to show that $u_0$ is a quadratic polynomial. This follows from the Weiss energy functional. First observe that $u_j$ satisfies

$$\Delta u_j = \frac{\lambda_+}{S_j}\chi_{\{u_j>0\}} - \frac{\lambda_-}{S_j}\chi_{\{u_j<0\}},$$

and thus satisfies a monotonicity formula of Weiss type that we will denote $\Phi_j$ to indicate the dependence of $S_j$. Moreover $\Phi_j(1/(2r_j)) \leq Cr_j^2/S_j \to 0$ and also $|\Phi_j(1)| \leq C$, the first inequality follows from the rescaling and the second from elliptic estimates of $u_j$. By the monotonicity it follows that $|\Phi_j(r)| \leq C$ for $r \geq 1$.

In the limit it follows that

$$\left| r^{-n-2}\int_{B_r\cap\mathbb{R}^n_+} |\nabla u_0|^2 dx - r^{-n-3}\int_{\partial B_r\cap\mathbb{R}^n_+} 2u_0^2 d\mathcal{H}^{n-1}\right| \leq C \quad \text{for } r \geq 1.$$

By the conditions $u(0) = |\nabla u(0)| = 0$ we can exclude that $u_0$ is a constant or a plane therefore it must be a second order polynomial, this since higher growth of $u_0$ would contradict the above inequality.

Using that $u_0 = 0$ on $\Pi$ we may conclude that $u_0 = x_1 \sum_{i=2}^n c_i x_i$, for some constants $c_i$.

We need to show that $c_i = 0$ for $i = 3...n$. This is done with a barrier argument for the $x_2$ derivatives. Let us sketch some details.

We know that $u_2 \equiv \partial u/\partial x_2$ is subharmonic so we can use a harmonic function $v$ with the same boundary values as a barrier. This barrier is easy to estimate close to the origin where $v = 2a_+(x_2^+)^2 - 2a_-(x_2^-)^2 + o(|x|)$ by means of the Poisson integral. Using that $u_2 \leq v$ we can deduce that $c_i = 0$ for $i = 3...n$.

Now we have shown that the blow up $u_j \to cx_1x_2$, this contradicts that $x_j \in \Gamma_u$ approaches the origin non-orthogonally, since that would imply that $u_j(x^j/r_j) = 0$ and $u_0$ should have a point $z$ on $(\partial B_1)^+$ where it is zero with $z \cdot e_2 \neq 0$ and $z \cdot e_1 > 0$. □

**Remark 14.** *In the final part of this proof we see why we have to, in the main Theorem, formulate the orthogonal result differently than the non-orthogonal. If the sequence approached the contact point tangentially to* $\Pi$ *we would not get a contradiction since the blow-up is identically zero on* $\Pi$.

What remains to prove Theorem A is the following.

**Lemma 15.** *If* $a_\pm \leq \frac{\lambda_\pm}{2}$ *and* $\nabla u = 0$ *then* $u$ *is quadratically bounded near* $0$.

**Proof.** The proof is inspired by the method of continuity. Actually we prove something more. Let us denote by $Q = Q(N, \eta) \subset \overline{\mathbb{R}}_+^2$ the subset of all pairs $(a_+, a_-)$ such that all solutions of (1) are quadratically bounded provided that the following additional conditions are satisfied

(1) $|\nabla u(0)| = 0$

(2) boundary data $f$ on $\Pi$ satisfying,

$$\lim_{r\to 0} \frac{f(rx)}{r^2} = a_+(x_2^+)^2 - a_-(x_2^-)^2,$$

(3) $\|f\|_{C^2} \leq N$ and $f^\pm \in C^{2,Dini}(\mathrm{spt} f^\pm)$ with a uniform Dini modulus of continuity $\eta$.

We prove that

$$Q = [0, \frac{\lambda_-}{2}] \times [0, \frac{\lambda_+}{2}].$$

We divide the proof into following four claims:

*Claim 1:* $Q$ is non-empty

*Claim 2:* $Q$ is closed

*Claim 3:* $Q$ is relatively open in $[0, \frac{\lambda_-}{2}) \times [0, \frac{\lambda_+}{2})$

*Claim 4:* $Q \subset [0, \frac{\lambda_-}{2}] \times [0, \frac{\lambda_+}{2}]$

**Proof of Claim 1:** This can be done in a similar way as was done in [AMM], where it was proven that if $a_+ = a_- = 0$ along the touching set then $u$ is quadratically bounded.

We will show that $(0, 0) \in Q$.

We use a barrier argument on the derivatives in the $\{x_1 = 0\}$-plane. By the $C^{2,Dini}$ assumption on $f$ in the supports of $f^{\pm}$ we know that $|e\cdot\nabla f(x')| \le C|x'|\eta(|x'|) \equiv R(x')$ for any unit vector in $\Pi$. We can therefore estimate the derivatives $e\cdot\nabla u$ from above by a harmonic function $v$ with boundary data $R$ on $\Pi$.

Since $\eta$ is Dini we know that $v$ has linear growth away from the origin (see [Wi]). This directly implies that $(e\cdot\nabla u)^+$ has linear growth. By methods previously introduced (see the proof of Lemma 13) it is easy to see that if $u$ is not quadratically bounded then (after a rotation of the coordinate system)

$$\frac{u(r_j x)}{\sup_{B^+_{r_j}}|u|} \to c x_1 x_2,$$

for some sequence $r_j \to 0$ and $\sup_{B^+_{r_j}}|u|/r_j^2 \to \infty$. But by the linear estimate on the derivatives we have

$$\frac{\partial u(r_j x)/S_j}{\partial x_2} \le C\frac{r_j^2|x|}{S_j} \to 0.$$

Which is a contradiction, therefore $(0,0)\in Q$.

**Proof Claim 2:** Assume, aiming for a contradiction, that $Q \ni a^k_{\pm} \to a_{\pm} \notin Q$. Since $(a_+, a_-) \notin Q$ there exists boundary values $f_0$ and quadratically non-bounded solution $u_0$ such that $|\nabla u(0)| = 0$. Let us take a sequence $f_k \to f_0$ on $\Pi$ with

$$\lim_{r\to 0}\frac{f_k(rx)}{r^2} = a^k_+(x_2^+)^2 - a^k_-(x_2^-)^2$$

and denote the solutions with boundary values $f_k$ by $u_k$. By stability we may choose boundary values on $(\partial B_1)^+$ in such a way that $|\nabla u_k(0)| = 0$. Moreover we can choose the boundary values $u_k$ so that $u_k \to u_0$ in $C^{1,\alpha}$.

We now use Weiss' monotonicity formula. Since $u_k$ are quadratically bounded on one hand (this follows by the assumption that $(a_+, a_-) \in Q$) $\Phi(u_k, r) \ge 0$, for all $0 < r < 1$ but on the other hand, by assumption $u_0$ is quadratically unbounded, for small enough $r_0$ we can get $\Phi(u_0, r_0) < -1$, a contradiction.

**Proof of Claim 3:** Once again we argue indirectly and assume that the claim is false. Then there exists a sequence $(a^k_+, a^k_-)$ from the complement of $Q$ converging to an $(a_+, a_-) \in Q \cap [0, \frac{\lambda_-}{2}) \times [0, \frac{\lambda_+}{2})$. For each pair $a^k_{\pm}$ we may choose boundary values $f_k$ and such that the corresponding solution $u_k$ has zero gradient at the origin but it is not quadratically bounded.

Let us denote by $K$ a conical neighbourhood of $x_1$-axis in the $x_1 \times x_2$-half-plane which we will denote by $\Pi_{12}$. We will consider the intersections of the free boundary with $\Pi_{12}$. Since the functions $u_k$ are not quadratically bounded we know that $\Gamma_{u_k} \cap \Pi_{12}$ lies in $K$ near the origin, say for $|x| \leq r_k$, and let $r_k$ be the smallest such $r$.

That such $r_k$ exists, and that $r_k \to 0$, follows from $u_k \to u$ in $C^{1,\alpha}$, the assumption that $(a_+, a_-) \in Q$ and the classification of angels of touch. If no such sequence $r_k \to 0$ exists then $\Gamma_{u_k} \cap \Pi_{12} \cap B_\epsilon \subset K \cap B_\epsilon$ that is not possible by the classification of touching angles.

We consider now the blow-up

$$\tilde{u}_k(x) = \frac{u_k(r_k x)}{\sup_{B^+_{r_k}} |u_k|}.$$

Since $\sup_{B^+_{r_k}} |u_k| \geq \frac{\max(a_\pm)}{2} r_k^2$ two cases are possible

$$\limsup r_k^{-2} \sup_{B^+_{r_k}} |u_k| = \infty$$

or is finite and positive. In the first case the limit of $\tilde{u}_k$ (for some subsequence) will be a quadratically bounded harmonic function with zero boundary data, thus it will be $cx_1x_2$, which by construction, should vanish at some point on the boundary of the cone $K \subset \Pi_{12}$, a contradiction.

In the other case we consider the blow-up

$$\tilde{u}_k(x) = \frac{u_k(r_k x)}{r_k^2}.$$

The boundary values then converge to $a_+(x_2^+)^2 - a_-(x_2^-)^2$ and the limit function $u_0$ (again for some subsequence) will be quadratically bounded. By the classification of the hit angle of quadratically bounded solutions we know that for fixed $a_\pm$ we can choose the cone $K$ small enough, in order to avoid any points of the free boundary of $u_0$ in $K \subset \Pi_{12}$ near the origin. On the other hand from the construction we have that $\Gamma_{u_0} \cap \Pi_{12} \cap B_1^+ \subset K$, a contradiction.

**Proof of Claim 4:** This is shown as in the proof of Lemma 13. $\square$

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## References

[ACF] H.W.Alt, L.A. Caffarelli, A. Friedman *Variational problems with two phases and their free boundaries* Trans. of AMS, Vol. 282, Nr. 2, 431-461

[A] J. Andersson *On the regularity of a free boundary near contact points with a fixed boundary* Preprint

[AMM] J. Andersson, N. Matevosyan, H. Mikayelyan *On the tangential touch between the free and the fixed boundaries for the two-phase obstacle-like problem* to appear in Arkiv för Matematik

[AS] J. Andersson, H. Shahgholian *Global solutions of the obstacle problem in half-spaces, and their impact on local stability* to appear in Cal.Var. Partial Differential Equations

[C] L.A. Caffarelli *The obstacle problem revisited* J. Fourier Analysis and Appl. 4 (1998), no. 4-5, 383-402

[CKS] L.A. Caffarelli, L. Karp, H. Shahgholian *Regularity of a free boundary with application to the Pompeiu problem* Ann. of Math. (2) 151 (2000), no. 1, 269-292

[KKS] A.L. Karakhanyan, C.E. Kenig, H. Shahgholian *The behavior of the free boundary near the fixed boundary for a minimization problem* to appear in Cal.Var. Partial Differential Equations

[SU] H. Shahgholian, N.N. Uraltseva, *Regularity properties of a free boundary near contact points with the fixed boundary*, Duke math. J. 116 (2003), no. 1, 1-34

[SUW] H. Shahgholian, N.N. Uraltseva, G.S. Weiss *Global solutions of an obstacle-problem-like equation with two phases* (to appear in Monatshefte für Mathematik)

[U] N.N. Uraltseva *two-phase obstacle problem*, Journal of Mathematical Sciences, Vol. 106, No. 3, 2001

[W1] G.S. Weiss *Partial regularity for a weak solutions of an elliptic free boundary problem* Commun. in PDE, 23(3&4), 439-455 (1998)

[W2] G.S. Weiss *An obstacle-problem-like equation with two phases: pointwise regularity of the solution and an estimate of the Hausdorff dimension of the free boundary* Interfaces and Free Boundaries, Vol. 3(2001), 121-128

[Wi] K.-O. Widman *Inequalities for the Green function and boundary continuity of the gradient of solutions of elliptic differential equations* Math. Scand. 21, 1967, 17-37 (1968)

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