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On the Regularity of a Free Boundary Near Contact Points With a Fixed Boundary

by

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Abstract

We investigate the regularity of a free boundary near contact points with a fixed boundary, with $C^{1,1}$ boundary data, for an obstacle-like free boundary problem. We will show that under certain assumptions on the solution, and the boundary function, the free boundary is uniformly C^1 up to the fixed boundary. We will also construct some examples of irregular free boundaries.

1 Introduction

The Problem: In these pages we will discuss the following problem in $\mathbb{R}^n_+$ ($\equiv \mathbb{R}^n \bigcap \{x_1 > 0\}$), or in a half ball $B_r^+(0) = \{x : |x| < 1,\, x_1 > 0\}$,

$$\left.\begin{array}{ll} \Delta u = \chi_{\Omega_u} & \text{where } \Omega_u = B_r^+ \setminus \{u = |\nabla u| = 0\} \\ u|_\Pi = f(x_2, ..., x_n) & \text{where } \Pi = \{x_1 = 0\} \\ u \in W^{2,p}(B_r^+(0)) & \end{array}\right\} \qquad (1)$$

f in a convenient class of functions ($C^{1,1}$ or $C^{2,Dini}$) and

$$f(0) = |\nabla f(0)| = 0.$$

With the notation $\Lambda_u = \{u = |\nabla u| = 0\}$, we will call the set $\Gamma_u = \Omega_u \cap \Lambda_u$ the free boundary. Our main question is then the regularity of the free boundary. More precisely we will show that under some conditions on the blow up of u, Γ_u is uniformly C^1 near $x^0 \in \Pi \cap \Gamma$ when

$$f \in C^{1,1}(\mathbb{R}^{n-1}) \text{ (or } C^{2,Dini}).$$

As things turn out we must also assume some regularity of $\partial\{f \neq 0\}$, see Theorem 2.

Other questions that will concern us are the regularity of u, and at what angle $\partial\Omega_u$ hits Π at a given point.

Main result: In Theorem 1 and its Corollary (pages 10 and 12) we show that asymptotically we have only two behaviors of solutions to problem (1) near contact points. The main result in this paper is formulated in Theorem 2 (page 14). This Theorem states that if the solution behaves like the larger of the two possibilities, then the free boundary is uniformly differetiable in a nontangential approach region near contact points. We also show that the free boundary approaches the fixed boundary in a uniform manner. Neither of this is true for those solutions which behaves asymptotically as the smaller possibility, a counter example will be given in a forthcoming paper, [AS]. However if the solution behaves assymptotically as the smaller solution then the free boundary is differentialble but not uniformily, and under suitable assumptions on the support of the boundary values we can drop the assumption on a non-tangential approach region.

Background: Problem (1) is a translated form of the dam problem, though somewhat more general. The dam problem is a physical model of filtration of water through a porous medium. The domain is the media, naturally divided in a wet part Ω_u, and a dry part Λ_u separated by a free boundary Γ_u. It is known that the free boundary is a graph $\phi(x_1)$. The pressure in the water u will be zero in the dry part and nonnegative in the entire domain. It also satisfies

$$\nabla \cdot (\nabla u + \chi_{\Omega_u} e_1) = 0, \tag{2}$$

with prescribed boundary values. If the domain is $\mathbb{R} \times \mathbb{R}_+$ and the boundary values are $u(0, x_2) = -2\lambda^2 x_2$. We can define

$$W(x_1, x_2) = -\int_{\phi(x_1)}^{x_2} u(x_1, t)dt.$$

Then W is a solution to (1) in $\mathbb{R}^2_+$ (if we rotate it by $\pi/2$ radians) with boundary values $f(x_2) = \lambda^2 (x_2)^2_+$.

This problem was studied in [AG] in any half space of $\mathbb{R}^2$ with boundary values $-(x_2)_-$ on the boundary. The authors of [AG] show that if the free boundary touches the fixed boundary at the origin then it does so either horizontally or orthogonally. They also consider other boundary values $u = 0$ on the fixed boundary and the normal derivative $\partial u/\partial \eta = 0$ on the free boundary. Although their problem is slightly different from ours (except the case when the fixed boundary is $\{x_2 = 0\}$) we believe that the analysis in these pages may strengthen their results. Firstly in this article we do not assume that u is nonnegative. Secondly, we are considering whether the free boundary approaches the fixed boundary in a uniform manner. We are also explicitly stating our results for a wider class of boundary functions.

The authors of [SU] consider equation (1) with $f \equiv 0$. More exactly they show that the free boundary Γ_u is uniformly C^1 near contact points with the fixed boundary Π_u Theorem 2 obviously strengthens their regularity results.

Questions about interior regularity of the free boundary have been considered earlier in [CKS].

Plan of the paper: In the next section we give some examples. In section 3 we state important tools for our study. In Section 4 we discuss the growth of u near Γ_u, then in Section 5 we make an attempt to classify the global solutions of (1) when f is homogeneous of second degree. By global solutions we mean solutions in $\mathbb{R}^n_+$ with quadratic growth at the infinity (i.e. solutions satisfying $\limsup_{|x|\to\infty} u(x)/|x|^2 \leq C$). We write "an attempt" since we only get some partial results on this issue, however our results are strong enough to continue the analysis. In Section 6 we will discuss the regularity of Γ_u. Section 7 deals with more general domains (C^1 near a contact point) and more general right hand sides than in (1). Then in the last section we state some open questions directly related to this paper.

List of definitions:

$\mathbb{R}^n_+$ is the upper half space: $\mathbb{R}^n \cap \{x_1 > 0\}$.

$B_r(x_0)$ is the open ball of radius r centered at x_0: $\{x; |x - x_0| < r\}$.

$B_r^+(x_0)$ is the ball intersected with $\mathbb{R}^n_+$.

Λ_u is the set where $u = |\nabla u| = 0$.

Ω_u is the set Λ_u^c, the complement of Λ_u.

The free boundary of u, that is $\partial\Lambda_u \cap \Omega_u$, will be denoted Γ_u,

χ_Ω is the characteristic function of Ω, i.e,

$$\chi_\Omega(x) = \begin{cases} 1 & x \in \Omega \\ 0 & x \notin \Omega. \end{cases}$$

By $u_r(x)$, $0 < r < \infty$, we will mean $\frac{u(rx)}{r^2}$, unless otherwise stated.

By the blow up of u we will mean $\lim_{r\to 0} u_r$ through some subsequence in r and denote it u_0 (if there is no ambiguity). If $r_j \to 0$ we will say that $\{u_{r_j}\}$ is a blow up sequence of u.

The shrink down of u we will mean $\lim_{r\to\infty} u_r$ through some subsequence. We denote the shrink down by u_∞ (if there is no ambiguity).

Π is the plane $\{x_1 = 0\}$.

If Ω is a set we will by $r\Omega$ mean $\{x; \frac{1}{r}x \in \Omega\}$.

Ω^c is the complement of Ω.

By x' we will mean the vector $(x_2, x_3, ..., x_n)$, and sometimes write $x = (x_1, x')$.

By spt$u$ we will mean the support of u, that is the closure of $\{u \neq 0\}$.

Acknowledgment: Before we start I would like to thank Henrik Shahgholian for all his help and support, all this papers benefits are due to him, but all its shortcomings are entirely mine.

2 Some examples.

Example 1: Obviously every solution to $\Delta u = 1$ in the upper half ball is a solution of (1).

Example 2: For any unit vector e in $\mathbb{R}^n$, $\frac{1}{2}(e \cdot x)_+^2$ is a solution. Where $(\cdot)_+ \equiv \max(\cdot, 0)$. In this case the free boundary is the plane through the origin with e as normal, and $f(x') = \frac{1}{2}(e' \cdot x')_+^2$.

Example 3: Let us now state a nontrivial example which in some sense shows a substantial difference between our results and the results in [SU]. Let

$$f(x_1, x_2, x_3) = \begin{cases} \frac{1}{2}\frac{x_2^2 x_3^2}{x_2^2+x_3^2} & \text{if } x_2, x_3 > 0 \\ 0 & \text{else.} \end{cases}$$

Let also u be the solution of the obstacle problem in the upper half ball in $\mathbb{R}^3$ with f as boundary values, on ∂B_1^+. By a solution to the obstacle problem we here mean a positive solution to (1). Its existence is known, see [F].

Now we claim that the free boundary touches Π in an angle, more exactly $\overline{\Omega}_u \cap \Pi = \{x_2 \geq 0\} \cup \{x_3 \geq 0\}$. This certainly implies that Γ_u isn't C^1 at the origin.

To show that $\Gamma_u \cap \Pi$ is the angle given in the previous paragraph it is enough to show that $u \leq f$ in $B_1^+(0)$. Since then, by the positivity of f and u, $\text{spt}(u) \subset \text{spt}(f)$. In particular $\text{spt}(u) \cap \Pi = \text{spt}(f) \cap \Pi$.

So lets prove that $u \leq f$. Define $\Omega_+ \equiv \{x; u(x) > f(x)\}$. By the positivity of u and f we know that $\Omega_+ \subset \Omega_u$. Consider the function $v = u - f|_{\Omega_+}$ which is positive with zero boundary values, but on the other hand $\Delta v = \chi_{\Omega_u} - \Delta f \geq 0$. This must imply that Ω_+ is empty, otherwise we would get a positive subharmonic function with zero boundary values.

This example shows that we can have a boundary function f that forces the free boundary to touch the fixed boundary in a set that isn't C^1.

But before we leave this example let us just consider another interesting feature of the solution u. Let us consider a blow up sequence u_{r_j}. By standard elliptic theory, and the quadratic growth of u, we can show that the blow up sequence (or a subsequence) converges (in $C^{1,\alpha}$ sense) to a global homogeneous solution of degree two. But since the boundary values are homogeneous of degree two they are invariant under this blow up. So we can actually find a global homogeneous solution that touches the fixed boundary in a cone. Moreover this also excludes that the free boundary, Γ_u, touches the fixed boundary tangentially.

If the touch where tangentially then

$$u_r \to u_0 = \begin{cases} 0 & x_1 > 0 \\ f & x_1 = 0 \end{cases},$$

in $C^{1,\alpha}$ for some subsequence $r \to 0$. But u_0 is obviously discontinuous contradicting the $C^{1,\alpha}$ convergence.

Example 4: In the previous example we used that $f \in C^{1,1}$ and that $|\Delta f| \leq 1$ to find a solution of (1) whose free boundary hits Π in the set $\partial\text{spt}(f)$. In this example we will use this to construct solutions of (1) with contact set equaling the boundary of any closed set in Π.

We only need to construct an $f \in C^{1,1}$ with spt(f) as the closure of the complement of any given closed set. If we have such an f we can just divide f with the supremum of its laplacian and then follow the construction of u in the previous example.

Let F be a closed set, in Π, then it is possible (see [St]) to find a regularized distance $\delta(x)$ satisfying:

1. $\delta(x) \in C^\infty(\Pi \setminus F)$
2. $c_1 \mathrm{dist}(x, F) \le \delta(x) \le c_2 \mathrm{dist}(x, F)$ c_1 and c_2 are independent of F
3. $|\frac{\partial^\alpha \delta(x)}{\partial x^\alpha}| \le C_\alpha(\mathrm{dist}(x, F))^{1-|\alpha|}$ where α is a multi index and C_α is a constant independent of F.

It is easy to see that $f = \frac{\delta^2}{\sup(|\Delta\delta^2|)}$ is the boundary function that will force the free boundary of u to hit the fixed boundary in ∂F.

It is noteworthy point out that this makes it possible to construct solutions with the Hausdorf dimension of the contact set $\Gamma_u \cap \Pi$ anywhere between $(n-2)$ and $(n-1)$.

The last two examples clearly shows that we have to make an assumption not only on the regularity of f but also the regularity of the boundary of the support of f in order to prove that the free boundary is C^1 near contact points with the fixed boundary.

3 Monotonicity formulas.

In this section we will state two monotonicity lemmas crucial to all results in this paper. The first is due to G.S Weiss [W] and the second due to H.W. Alt, L.A. Caffarelli and A. Friedman [ACF].

To get a reasonable class of functions to work with we make the following definition.

Definition 1. *We say that $u \in P_r(M, f)$ if u satisfies (1) in $B_r(0)$, and* $\sup_{B_r(0)^+} |u| \le M$.

We will also write $u \in P_\infty(M, f)$ if u satisfies (1) in $\mathbb{R}^n_+$ and

$$\limsup_{|x|\to\infty} u(x)/|x|^2 \le M,$$

such solutions of (1) will be called global solutions.

Lemma 1. *(Essentially due to G. S. Weiss) Let $u \in P_R(M, f)$ with f homogeneous of degree two, then the function*

$$\Psi(r, u) \equiv r^{-n-2} \int_{B_r(0)\cap\mathbb{R}^n_+} (|\nabla u|^2 + 2u) - r^{-n-3} \int_{\partial B_r(0)\cap(R)^n_+} 2u^2 \qquad (3)$$

is nondecreasing in $r < R$.

Proof: Set

$$u_r(x) = \frac{u(rx)}{r^2}. \tag{4}$$

Then

$$\Psi(r,u,0) = \int_{B_1^+(0)} (|\nabla u_r|^2 + 2u_r) - \int_{\partial B_1^+(0)\cap\mathbb{R}^n_+} 2u_r^2. \tag{5}$$

It is enough to show $\frac{\partial\Psi}{\partial r} \geq 0$. Calculations of the derivative gives

$$\frac{\partial\Psi}{\partial r} = \int_{B_1^+(0)\cap\mathbb{R}^n_+} (2\nabla u_r \cdot \nabla u_r' + 2u_r') - \int_{\partial B_1^+(0)\cap\mathbb{R}^n_+} 4u_r u_r', \tag{6}$$

where $u_r' \equiv \frac{\partial u_r}{\partial r} = \frac{1}{r}(\nabla u_r \cdot x - 2u_r)$. Using integration by parts on the first term in the above equation we obtain

$$\int_{B_1^+(0)\cap\mathbb{R}^n_+} 2\nabla u_r \cdot \nabla u_r' = \int_{\partial(B_1^+(0)\cap\mathbb{R}^n_+)} (2\nabla u_r \cdot \eta)u_r' - \int_{B_1^+(0)} 2\Delta u_r u_r', \tag{7}$$

where η is the exterior normal of $\partial(B_1^+(0)\cap\mathbb{R}^n_+)$. We can split $\partial B_1^+(0)$ into two parts $\partial B_1^+(0) = \big(\partial B_1^+(0)\backslash\Pi\big) \cup \big(B_1(0)\cap\Pi\big)$. Therefore equation (7) can be written as

$$\int_{B_1^+(0)\cap\mathbb{R}^n_+} (2\nabla u_r \cdot \nabla u_r') = \int_{\partial B_1^+(0)\backslash\Pi} (2\nabla u_r \cdot x)u_r' + \tag{8}$$

$$\int_{B_1(0)\cap\Pi} (2\nabla u_r \cdot (-e_1))u_r' - \int_{B_1^+(0)} 2\Delta u_r u_r'.$$

If we insert equation (8) in (6) we will arrive at

$$\int_{\partial B_1^+(0)\backslash\Pi} (2\nabla u_r \cdot x)u_r' + \int_{B_1(0)\cap\Pi} (2\nabla u_r \cdot (-e_1))u_r' - \tag{10}$$

$$\int_{\partial B_1^+(0)\backslash\Pi} 4u_r u_r'.$$

Here we have used that $\Delta u_r = \chi_{r\Omega}$. If we further realize that $\nabla u_r \cdot x = ru_r' + 2u_r$ in the first integral, the expression may be simplified to

$$\int_{\partial B_1^+(0)} 2r(u_r')^2 + 4\int_{B_1(0)\cap\Pi} \frac{\partial u_r}{\partial x_1} u_r'. \tag{12}$$

But $u|_\Pi$ is a homogeneous polynomial of second degree so $u_r|_\Pi$ is independent of r which implies that $u_r' = 0$. And we arrive at

$$\frac{\partial\Psi}{\partial r} = \int_{\partial B_1^+(0)} 2r(u_r')^2 \geq 0. \tag{13}$$

□

Remark: $\Psi(r,u)$ is constant in r if and only if u is homogeneous of second degree.

To see this we notice that $\frac{\partial \Psi}{\partial r}=0$ if and only if $u_r'=0$ that is if and only if $\nabla u_r \cdot x - 2u_r = 0$ which happens if and only if u_r is homogeneous of second degree.

Lemma 2. *Let h_1 and h_2 be two non-negative continuous sub-solutions of $\Delta u = 0$ in $B_R(0)$. Assume further that $h_1(0)=h_2(0)=h_1(x)h_2(x)=0$. Then the following function is monotone in r*

$$\phi(r,h_1,h_2)=\frac{1}{r^4}\Big(\int_{B_r(0)}\frac{|\nabla h_1|^2}{|x|^{n-2}}\Big)\Big(\int_{B_r(0)}\frac{|\nabla h_2|^2}{|x|^{n-2}}\Big)$$

for $0<r<R$. More exactly, if any of the sets $\mathrm{spt}(h_i)\cap\partial B_r(0)$ *digresses from a half spherical cap by a positive area, then either $\frac{\partial\phi(r)}{\partial r}>0$ or $\phi=0$.*

A proof of the first part can be found in [ACF], [CKS] contains a proof of the last statement.

4 The growth of solutions.

In this section we discuss the growth of solutions, more precisely we will prove that

$$\sup_{B_r(x^0)} u \leq Cr^2, \text{ for } x^0\in\partial\Omega,$$

under certain assumptions on f and the density of Λ_u near x^0. In [CKS] and [SU] it is shown that the solution of similar problems are $C^{1,1}$. This is however not true in our case, even if $f\in C^{1,1}$. Lets construct a counter example. If we take the Poisson integral of $(x_2)_+^2/2$ in B_1^+ we get a harmonic function with $(x_2)_+^2/2$ as boundary values on Π. If we then add $x_1^2/2$ and subtract x_1 we get a solution in $P_1(2,(x_2)_+^2/2)$ with $0\in\Lambda$. Explicitly the solution looks like this:

$$u(x_1,x_2)=\int_{-1}^{1}\frac{c_n x_1}{t^2+x_1^2}\frac{(x_2-t)_+^2}{2}dt+\frac{x_1^2}{2}-x_1.$$

To see that u defined above isn't quadraticly bounded and obviously not $C^{1,1}$, we evaluate the integral and arrive at

$$u(x_1,x_2)=c_n\Big(\frac{-x_1^2+x_2^2}{2}\big(\arctan(\frac{x_2}{x_1})-\arctan(\frac{x_2-1}{x_1})\big)$$
$$+x_1x_2\ln\big(\frac{x_1^2+(x_2-1)^2}{x_1^2+x_2^2}\big)\Big)+\frac{x_1^2}{2}-x_1.$$

It is easy to see that

$$\lim_{s\to 0+}\Big|\frac{u(s,s)}{s^2}\Big|=\infty,$$

that is u isn't quadraticly bounded. Since 0 is a free boundary point this also implies that $u \notin C^{1,1}(\overline{B^+_{1/2}})$.

Before we start to prove a bound for the growth of solutions, we need some notation. We define

$$S(j,u,z) \equiv \sup_{B_{2^{-j}}(z) \bigcap \mathbb{R}^n_+} |u|,$$

and

$$\mathbb{M}(u,z) \equiv \{j \in \mathbb{N};\ 4S(j+1,u,z) \geq S(j,u,z)\}.$$

Lemma 3. *Let $u \in P_1(M,f)$ and that for every $r \leq 1$ the following inequality holds*

$$\frac{\text{cap}(\Lambda_u \cap B^+_r(0))}{\text{cap}(B^+_r(0))} \geq s > 0.$$

Suppose also that $f(x) \leq C|x|^2$, then there exists a constant C such that

$$S(j,u,0) \leq C2^{-2j} \quad j \in \mathbb{M}(u).$$

Proof: We will argue by contradiction. So assume that there exists u_j and $k_j \in \mathbb{M}(u_j)$ such that

$$S(k_j,u_j,0) \geq j2^{-2k_j}. \tag{14}$$

Define

$$\tilde{u}_j(x) \equiv \frac{u_j(2^{-k_j}x)}{S(k_j+1,u_j,0)} \qquad \text{in } B^+_1(0). \tag{15}$$

Then, by (14) and the definition of $\mathbb{M}$, we'll have

$$\|\Delta \tilde{u}_j\|_\infty = \frac{2^{-2j_k}}{S(k_j+1,u_j,0)} \leq \frac{2^{-2j_k}}{\frac{1}{4}S(k_j,u_j,0)} \leq \frac{4\cdot 2^{-2k_j}}{j2^{-2k_j}} = \frac{4}{j} \to 0 \tag{16}$$

By (15) and since $k_j \in \mathbb{M}(u_j)$

$$\sup_{B^+_{1/2}(0)} |\tilde{u}_j| = 1.$$

Hence by standard elliptic theory a subsequence of $\tilde{u}_j$ will converge in the sense of $C^{1,\alpha}$ in the half ball $B^+_1(0)$ to, say, u_0. By our capacity-density condition it will follow (see [KS]) that $\text{cap}(\Lambda_{u_0}) > 0$. But this means that we get a non constant harmonic function with a set of non zero capacity where $u_0 = |\nabla u_0| = 0$. But this is a contradiction, see [RS]. □

Lemma 4. *There exists a constant C such that, for u satisfying the hypothesis of Lemma 3, $S(j,u,z) \leq 4C2^{-2j}$.*

Proof: Let j be the first integer such that the inequality doesn't hold, then

$$S(j-1,u,0) \leq 4C2^{-2(j-1)} \leq 4S(j,u,0) \ \text{ i.e. } j-1 \in \mathbb{M}(u). \tag{17}$$

By Lemma 3 we will have

$$S(j,u,0) \leq S(j-1,u,0) \leq C2^{-(j-1)} = 4C2^{-j} \tag{18}$$

contradicting our assumption that $S(j,u,0) > 4C2^{-2j}$. □

The lemma is important if we want to blow up u. It assures that the blow up sequence is uniformly bounded on compact sets and thus the limit is bounded on compact sets.

If $|u(x)| \leq C|x|^2$ then $\sup|u_r| \leq C$. However, we might have that $\lim_{r\to 0} u_r \equiv 0$. We must assure some growth of u to exclude this, e.g. we need

$$\sup_{B_r(x^0)} |u| \geq cr^2.$$

When we blow up u at a point on the free boundary in the interior of $\mathbb{R}^n_+$ this growth is shown in [CKS]. And if we blow up at a point $x^0 \in \Pi$ we can use the growth of the boundary function to assure that the blow up won't go to zero, if the boundary function satisfies

$$\sup_{B_r(x^0)} f \geq cr^2.$$

However, if the blow up of the boundary function is zero it is possible that $u_r \to 0$. For example

$$\max(\frac{|x-\sqrt{3}e_1|^2}{6} - \frac{1}{|x-\sqrt{3}e_1|} - \frac{2+\sqrt{3}}{2\sqrt{3}}, 0)$$

will be a solution of (1) in $\mathbb{R}^3_+$, with it's restriction to Π as boundary values, whose blow up at the origin vanishes identically. On the other side if there exists a cone in Ω_u with vertex at the origin then $\sup_{B_r} u(x) \geq C|r|^2$ for some C.

With higher regularity on the boundary values we could expect to have quadratic growth even without our capacity assumption. This is indeed the case as the following lemma proves. But first we need a definition.

Definition 2. *We say that a function f is $C^{k,Dini}(\Omega)$ if f is k times continuously differentiable in Ω and the $k:th$ derivatives have a modulus of continuity $\omega = \omega_f$ such that*

$$\int_0^1 \frac{\omega(s)}{s} ds < \infty.$$

Lemma 5. *If $f \in C^{2,Dini}(\Pi \cap B_1(0))$ and $|D^2 f(0)| = |\nabla f(0)| = f(0) = 0$ and $\omega(s)/s^\gamma$ is decreasing for some $\gamma < 1$, then there exist a constant $C(M,f)$ such that*

$$|u(x)| \leq C|x|^2,$$

for all $u \in P_1(M,f)$ with the origin as a free boundary point.

Proof: It is enough to show that the conclusion of Lemma 3 holds under our assumptions. Then the lemma follow by the argument in the proof of Lemma 4.

We will argue by contradiction as in Lemma 3. Assume that we have a sequence u_j of solutions in $P_1(M, f)$, $k_j \in \mathbb{M}(u_j)$ such that equation (14) holds. By continuing as in Lemma 3 we can define a blow up sequence $\tilde{u}_j$ as in (15). Both (16) and (4) will also follow.

But without our capacity assumption we will have to work a little bit harder to get the desired contradiction. We know that (for $e \in \Pi$ and $|e| = 1$) $(D_e f(x'))^{\pm} \leq C|x'|\omega(|x'|)$. But this implies that we can find a sequence of harmonic functions $v_j^{\pm}$ uniformly C^1 up to the boundary in $B_{1/2}^+$ such that $v_j^{\pm} \geq (D_e u_j)^{\pm}$ in $B_{1/2}^+$ (see [Wi]), and $v_j^{\pm} = D_e f^{\pm}$ on Π. Hence

$$\sup_{B_r^+} |D_e u_j| \leq Cr,$$

and we arrive at

$$\sup_{B_r^+} |D_e \tilde{u}_j| \leq \frac{Cr}{j}. \tag{19}$$

A subsequence of $\tilde{u}_j$ will now converge in C^1 to u_0, say. By equation (19) $D_e u_0 = 0$ for all $e \in \Pi$. This implies that $u_0 = ax_1 + b$ but the origin is a free boundary point which implies that $a = b = 0$, contradicting equation (4). This means that Lemma 3 follows with our assumptions on f. □

5 Classification of global solutions.

In this section we will classify all homogeneous global solutions. In a forthcoming paper by the author and H. Shahgholian it will be shown that there exists non-homogeneous global solutions with homogeneous boundary data. In this section we will not discuss that result, but we will describe some of the qualitative behavior of these solutions. We also discuss how the existence of such solutions affect our analysis of the regularity of the free boundary.

We will start with a lemma classifying homogeneous solutions of (1).

Theorem 1. *Let u be a homogeneous global solution to* (1) *with* $f = \lambda^2 (x_2)_+^2$ *where* $\lambda \leq \frac{1}{\sqrt{2}}$. *Then u must be of the following form*

$$u = \Big(\pm \sqrt{\frac{1}{2} - \lambda^2} x_1 + \lambda x_2 \Big)_+^2. \tag{20}$$

Proof: We start by reducing the n-dimensional problem to a two dimensional one. Consider $v \equiv D_e u$ for any unit vector e orthogonal to the $x_1 x_2$-plane. The restriction of v to Π is certainly zero. This makes it possible for us to extend v continuously by zero to the lower half space. Lets for simplicity denote the extended function also by v.

If we set h_1 and h_2 as $v^{\pm}$ they will satisfy the assumptions of Lemma 2. Let ψ be as in Lemma 2. Then

$$\lim_{r\to\infty}\psi(r,v^+,v^-)=\lim_{r\to\infty}\psi(1,v_r^+,v_r^-)=C,$$

and for any $s>0$ and any sequence $r_j\to\infty$ such that $v_{r_j}^{\pm}$ converges, by standard convergence argument to, lets say, $v_\infty^{\pm}$ we will have

$$C=\lim_{r\to\infty}\psi(rs,v^+,v^-)=\lim_{r\to\infty}\psi(s,v_r^+,v_r^-)=\psi(s,v_\infty^+,v_\infty^-).$$

But then by Lemma 2, $C=0$ since $v_\infty^{\pm}=0$ in the lower half space.

It follows from the positivity and monotonicity of ψ that $\psi(r,v^+,v^-)=0$ for any r thus $v^+=0$ or $v^-=0$.

By a similar argument we can show that $D_e u$ never changes sign, for any vector e orthogonal to e_1 but not orthogonal to the x_2-axis. Lets sketch some details. First we assume, for definiteness, that the x_2-component of e is positive. Then we choose

$$h_1=\begin{cases}(D_e u)^- & x_1\geq 0\\ 0 & x_1<0\end{cases}$$

and $h_2(x_1,x_2,...,x_n)\equiv h_1(-x_1,x_2,...,x_n)$. Observe that h_1 and h_2 will be continuous since $(D_e u)^-$ is zero on Π.

Continuity of the first derivatives shows that $\mathrm{spt}(h_1)\cap\partial B_1(0)$ digresses from a spherical cap by positive area and by homogeneity it follows that $\mathrm{spt}(h_1)\cap\partial B_r(0)$ digresses from a spherical cap for any r. This shows that h_1, h_2 have all the characteristics of $v^{\pm}$ used in the proof above. So we can use the same argument as above to conclude that $h_1=0$ or $h_2=0$. But $h_2(x_1,..,x_n)\equiv h_1(-x_1,..,x_n)$ thus $h_1=h_2=0$.

Let us show that u is two dimensional. Pick any $x^0\in\Omega_u$ and any two non-collinear vectors η^1 and ν^1 orthogonal to the x_1-axes. We know that $D_{\eta^1}u(x^0)$ and $D_{-\eta^1}u(x^0)$ have different signs, thus by continuity there must exist a vector e^1 in the space spanned by η^1 and ν^1 such that $D_{e^1}u(x^0)=0$. But $D_{e^1}u$ is harmonic and never changes sign, thus by the maximum principle $D_{e^1}u$ must be identically zero. Then we can pick two new not collinear vectors η^2 and ν^2 orthogonal to both e^1 and the x_1-axes and deduce that there exists a vector e^2 such that $D_{e^2}u\equiv 0$. Continue this process until you can't find two non-collinear vectors η^k and ν^k orthogonal to e^1, e^2,..., e^{k-1} and the x_1-axes. Since u is independent of the directions e^1, e^2,..., e^{k-1} and there is only one vector $e\in\Pi$ orthogonal to $\{e^i\}_{i=1}^{i=k-1}$ u must be two dimensional.

Now we continue showing u has the indicated form (20). Since u is two dimensional and homogeneous of second degree it has the following form in polar coordinates

$$u_\infty(r,\theta)=r^2\Theta(\theta).$$

In particular the free boundary is a line $\{(r,\theta_0); r \in \mathbb{R}_+\}$ for some θ_0. If we rewrite Δ in polar coordinates (1) becomes

$$\begin{aligned}&\Delta u_\infty = 4\Theta(\theta) + \tfrac{\partial^2\Theta(\theta)}{\partial\theta^2} = \chi_{\{0<\theta<\theta_0\}}\\ &\Theta(0) = \lambda^2\\ &\Theta(\theta_0) = 0\\ &\tfrac{\partial\Theta(\theta_0)}{\partial\theta} = 0.\end{aligned}$$

This ODE is easy to solve and gives the desired representation of the solutions. □

This Lemma gives us information about the local behavior of $u \in P_r(M,f)$ near 0 and ∞. But before we state that result (Corollary 1) we need another lemma.

Lemma 6. *Let u be a quadraticly bounded function in $P_R(M,f)$, with the blow up of $f \in C^{1,1}$, $f_0 = \lambda(x_2)_+^2$ for $\lambda \le 1/2$. Then the blow up limit of u is unique.*

Proof: We argue by continuity of the Weiss functional. Since the Weiss functional, Ψ of Lemma 1, is continuous in $r \le R/2$ for all $u \in P_R(M,f)$ we know that for any two converging blow up sequences $u_{s_j} \to u_{s_0}$ and $u_{r_j} \to u_{r_0}$ we have

$$\Psi(1,u_{s_0}) = \Psi(1,u_{r_0}). \tag{21}$$

Also

$$\lim_{j\to\infty}\Psi(r_j t,u) = \lim_{j\to\infty}\Psi(t,u_{r_j}) = \Psi(1,(u_{r_0})_t), \tag{22}$$

but this equals $\Psi(1,u_{r_0})$ by continuity, for all $t>0$. So u_{r_0} is homogeneous of second degree by the remark after Lemma 1. And the same argument shows that u_{s_0} is homogeneous.

By the preceding lemma we know that there are only two homogeneous solutions, lets denote them P_1 (the function in equation (20) with a "-"-sign) and P_2 (dito with the "+"-sign). A simple calculation shows that $\Psi(1,P_2) > \Psi(1,P_1)$. Therefore, by (21), $u_{r_0} = u_{s_0}$. □

Corollary 1. 1. *Let $u \in P_r(M,f)$ be of quadratic growth (the capacity criterion in Lemma 3 holds for instance) and let, any blow up of f, $f_0 = \lambda^2(x_2)_+^2$ then*

$$u_0 \equiv \lim_{r\to 0}\frac{u(rx)}{r^2} = \big(\pm\sqrt{\frac{1}{2}-\lambda^2}x_1 + \lambda x_2\big)_+^2,$$

where the existence of the limit is assured by the preceding Lemma.

2. *If $u \in P_\infty(M,f)$ and the shrink down of f, $f_\infty = \lambda^2(x_2)_+^2$ then*

$$u_\infty \equiv \lim_{r\to\infty}\frac{u(rx)}{r^2} = \big(\pm\sqrt{\frac{1}{2}-\lambda^2}x_1 + \lambda x_2\big)_+^2,$$

where the existence of the limit is assured by the preceding Lemma.

3. If both 1. and 2. above are satisfied then $u_\infty \geq u_0$.

Proof: By the quadratic growth of u, $\Psi(r, u, 0)$ must be bounded (Ψ is as in Lemma 1). Therefore $\lim_{r\to 0}\Psi(r, u, 0)$ converges to a constant C_0. Moreover

$$\lim_{r\to 0}\Psi(rs, u, 0) = \lim_{r\to 0}\Psi(s, u_r, 0) = C_0 \quad \forall s > 0. \tag{23}$$

So u_0 is homogeneous of second degree. This together with Theorem 1 shows the first statement of the Corollary. The same argument works with the shrink down of u (that is the case $r \to \infty$).

The third statement in the corollary is a consequence of the monotonicity of Ψ, $\Psi(1, u_\infty) \geq \Psi(1, u_0)$. A direct calculation of the energies of the two possible values of u_0 and u_∞ will show that $u_\infty \geq u_0$. □

This corollary states that we have only two kinds of behavior of global solutions at the origin and two kinds of behavior at the infinity point. More precisely the behavior is as one of two "half-polynomials", the functions in (20) with "+" and "-" sign respectively. These functions will be used so often hereafter that we give them special notation.

Definition 3. *By* U_1 *and* U_2 *we mean the following functions:*

$$U_1 = \Big(-\sqrt{\frac{1}{2} - \lambda^2}x_1 + \lambda x_2\Big)_+^2,$$

$$U_2 = \Big(+\sqrt{\frac{1}{2} - \lambda^2}x_1 + \lambda x_2\Big)_+^2.$$

It would be more correct to denote the global homogeneous solutions by U_1^λ and U_2^λ, since U_1 and U_2 depend on the parameter λ. But for simplicity of notation we don't explicitly indicate that λ dependence. What particular λ we are using is always clear from context.

A reasonable question to ask is whether there exist solutions which behaves like U_1 near the origin and U_2 near the infinity, i.e. with blow up equal to U_1 and shrink down equal to U_2. As mentioned in the beginning of this section, such a solution was recently constructed and will appear in a forthcoming paper by the author and Henrik Shahgholian. These non-homogeneous solutions will force us to define and work with a new class $\hat{P}_r(M, f)$ in the next section. This new class of functions will be strictly smaller than $P_r(M, f)$ and therefore Theorem 2 will be substantially weaker than it could have been if such inhomogeneous solutions didn't exist, if we at all could conceive a world where our mathematics where different (I leave the last question to the Kantians). Though we won't ponder upon the inhomogeneous solutions let us mention something about their appearance, we will prove that any inhomogeneous solution of quadratic growth lies in between U_1 and U_2.

Lemma 7. *Let* u *be an inhomogeneous global solution of quadratic growth, with homogeneous boundary values, then* $U_1 \leq u \leq U_2$.

Proof: Lets first prove that $u \leq U_2$. Assume that u is a non homogeneous quadraticly bounded solution with blow up U_2 and shrink down U_1. Define $Q_r = (a_r x_1 + \lambda x_2)_+^2$ where a_r is the smallest constant such that $u_r \leq Q_r$ in B_1^+. It is easy to see that Q_r is increasing and its limit as $r \to \infty$ is U_2 which gives the proof.

The proof that $U_1 \leq u$ is done similarly. □

6 The Regularity of the Free Boundary.

In this section we discuss the regularity of the free boundary. To be more exact we will prove that the free boundary is pointwise differentiable up to the fixed boundary. We get uniform C^1-regularity in a non-tangential access region if at touching points where the solution's blow-up is the larger possibility, and non-uniform C^1-regularity at blow-up points where the limit is the smaller possibility.

The idea of the proof comes directly from [SU], where they consider the class $P_r(M, 0)$. In the case of [SU] all global solutions are homogeneous and there exist no problem with non-homogeneous global solutions. Therefore they have no problem to show uniformity of the regularity. Also their assumption that $u = 0$ on Π means that they do not have to consider non-tangential approach regions.

Let us first prove a Theorem stating the non-uniform C^1-regularity of the free boundary, and later provide further assumptions to get stronger results.

Theorem 2. *Suppose $u \in P_1(M, f)$ be quadratically bounded and f satisfying*

$$\lim_{r_j \to 0} \frac{f(r_j x + x^j)}{r_j^2} = \lambda^2 (x_2)_+^2,$$

then Γ_u is a C^1 in $N \cap \{x_1 \geq \epsilon |x'|\}$ for every ϵ and a small neighbourhood N of the origin.

Remark 1: In section 2 we showed that we can find a solution u whose free boundary meets the fixed boundary on the boundary of any closed set. In particular we can find a solution u with a sawtooth function forced between two parabola touching at the origin as contact set. As in section 2 we construct such a function by constructing an f with the sawtooth function as $\partial\{f \neq 0\}$. The free boundary of u is certainly not C^1 up to Π in this case even though the blow up of f has support in a half space, therefore the non-tangentil approach is a neccessary.

Proof: Let u be as in the Theorem, we need to show that near the origin the free boundary is differentiable in the non-tangential approach region and also that the normals converge as we get closer to the origin.

Let $x^j \to 0$ be points in $\Gamma \cap \{x_1 > \delta |x'|\}$. Denote $r_j = |x^j|$ and make the blow-up

$$u_j = \frac{u(r_j x)}{r_j^2} \to U_i \quad \text{for } i = 1 \text{ or } i = 2.$$

In both cases we can deduce that for j large enough the free boundary of u is, near x^j, traped between two planes paralell to Γ_{U_i} that are $o(r_j)$ appart. Using that $x_1 \geq \delta|x'|$, we can consider our solution in $B_{x_1^j}(x^j)$ where the free boundary is trpped betweein the planes at a distance $o(x_1^j)$, interiour regularity results applies [C] and gives the desired result. □

Next we would want to get a result on the uniform regularity of the free boundary. To do that we need to restrict our class of solutions somewhat.

Definition 4. *Let f be a function satisfying*

$$\lim_{r \to 0} \frac{f(rx)}{r^2} = \lambda^2 (x_2)_+^2$$

for $\lambda \in [0, \frac{1}{\sqrt{2}}]$. Then we define $\hat{P}_r(M,f)$ as the subset of $P_r(M,f)$ consisting of functions whose blow ups exist and equals U_2, for the definition of U_2 see Definition 3 on page 13.

Theorem 3. *Let $u \in \hat{P}_1(M,f)$ then there exists an r_0 such that*

1)$\Gamma_u \cap B_{r_0} \subset \{x;\ dist(x, \Gamma_{U_2} \leq |x|\sigma(|x|))\}$ for a uniform modulus of continuity σ

2) the free boundary is uniformly C^1 in the region $B_{r_0} \cap \{x_1 \geq \delta|x'|\}$, the C^1 norm of Γ_u depends on n, δ, λ and M.

Proof: We start by proving 1. We will show that for each ϵ there exists a ρ_ϵ such that if $x^0 \in \Gamma_u$, $x_1^0 \geq \delta|x^{0'}|$ $\|x^0\| < \rho_\epsilon$ then $x^0 \in \{x; \mathrm{dist}(x, \partial\Omega_{P_2}) < \epsilon|x|\} \equiv K$. We can then choose σ such that $\sigma(\rho_\epsilon) = \epsilon$.

Assume the contrary, that is there exists u_j and $x^j \in \Gamma_{u_j}$ such that $|x^j| \to 0$ and $x^j \notin K$. To get the desired contradiction we make the following blow up

$$\tilde{u}_j(x) = \frac{u_j(|x^j|x)}{|x^j|^2}.$$

Obviously

$$\tilde{x}^j := \frac{x^j}{|x^j|} \in \partial\Omega_{\tilde{u}_j} \cap \partial B_1^+(0),$$

with $\mathrm{d}(\tilde{x}^j, \partial\Omega_{P_2}) > \epsilon$. Now a subsequence of $\{\tilde{u}_j\}_j$ will converge in $C^{1,\alpha}$ to a global solution u_0. Also $\tilde{x}^j \to \tilde{x}$ for a subsequence, with $\tilde{x} \notin K$. If we can show that $u_0 = U_2$ we will get a contradiction to $\tilde{x} \notin K$ and the first part of the theorem follows. $\lim_{r\to 0} \frac{u_0(rx)}{r^2}$ is a global homogeneous solution so, by Corollary 1, we have to exclude that the blow up of u_0 is U_1. But this follows by Lemma 6 and that $u \in \hat{P}_r(M,f)$.

So lets prove the second statement of the theorem. We will still follow the main lines of [SU]. Our first goal is to show that Γ_u is $C^{1,\alpha}$ away from the fixed boundary Π. We will need the following lemma (similar to Lemma 5.2 in [SU]).

Lemma 8. *1. Given $\epsilon > 0$, there exists $\rho_\epsilon > 0$ such that if u is as in Theorem 3, with $\lambda > 0$ and $x^0 \in \Gamma_u \cap B^+_{\rho_\epsilon} \cap \{x^0_1 \geq \delta |x^{0'}|\}$, then*

$$\sup_{B_{3r}(0)} |u(x+x^0) - U_2(x+x^0)| \leq \epsilon (r)^2,$$

with $r = x^0_1$ Moreover, $u \equiv 0$ in $S = \{x \in B_{3r/4}(x^0);\ dist(x, \Omega_{U_2}) > \frac{3C\sqrt{\epsilon}r}{4}\}$

2. If $\lambda = 0$ the same is true with $\frac{(x_1 - x^0_1)^2_+}{2}$ instead of U_2.

Proof: Assume the contrary, i.e. there exists $u_j \in \hat{P}_1(M,f)$ and $x^j \in \Gamma_{u_j} \cap B_{1/j}(0)$ such that

$$\sup_{B_{3r_j}(0)} |u_j(x+x^j) - U_2(x+x^j)| > \epsilon (r_j)^2,$$

where $r_j = x^j_1$. Upon scaling u_j

$$\tilde{u}_j(x) = \frac{u_j(r_j x + x^j)}{(r_j)^2}$$

we'll have

$$\sup_{B_3(0)\cap\{x_1 > -1\}} |\tilde{u}_j(x) - U_2(x + \frac{x^j}{r_j})| > \epsilon.$$

We will show that both $\tilde{u}_j$ and $U_2(x + \frac{x^j}{r_j})$ converges to U_2 and thereby get the contradiction.

Since u is quadraticly bounded and $\lambda > 0$, $\tilde{u}_j$ converges to a global solution in $C^{1,\alpha}$. Moreover

$$\tilde{u}_j|_{\{x_1=-1\}} = \frac{f(x^j_1 x' + (x^j)')}{(x^j_1)^2} \to U_2(x)|_{\{x_1=-1\}}.$$

Since $0 \in \Gamma_{\tilde{u}_j}$, for all j, $\tilde{u}_j \to U_2$.

So we are done, with the first part of 1. in the lemma, if we can show that $U_2(x + \frac{x^j}{x^j_1}) \to U_2(x)$. But this is easy, by the first part of Theorem 3

$$\frac{x^j}{x^j_1} \to \hat{x} \in \Gamma_{U_2}.$$

But U_2 is invariant under translations in Γ_{U_2}.

The second part of 1 in the Lemma is an easy consequence of the well known fact that

$$\sup_{B_r(x^0)} u \geq u(x^0) + Cr^2$$

for any ball $B_r(x^0) \subset \Omega_u$, see [CKS] for details.

The proof for $\lambda = 0$ is very similar. All the details may be found in the proof of Lemma 5.2 in [SU]. The statement in [SU] is a little weaker but all the details of the proof works in our case. □

Continuation of the proof of Theorem 3: By Lemma 8 we may use Lemma 6.2 in [CKS] to conclude that $u \geq 0$ in $B_{3x_1^0/4}$. Then by Theorem 7 in [C] it follows that, for small $|x^0|$, Γ_u is a $C^{1,\alpha}$-graph over $\partial\Omega_{P_2}$in $B_{x_1^0/4}(x^0)$ with $C^{1,\alpha}$-norm not larger than C/x_1^0.

So it remains to show that the normal of Γ_u is uniformly continuous up to Π. We will once again use an argument of contradiction. So lets assume that there exists a sequence of functions $u_j \in \hat{P}_1(M,f)$, a sequence of points $x^j \in \Gamma_{u_j}$ such that $|x^j| \to 0$ and angle$(n_{x^j}, e) > \epsilon$ for some $\epsilon > 0$. Where e is the normal of Γ_{P_2} and n_{x^j} is the normal of Γ_{u_j} at x^j.

Consider the blow up

$$\tilde{u}_j(x) = \frac{u_j(r_j x + x^j)}{r_j^2}$$

where $r_j = |x_1^j|$. By the assumption $u_j \in \hat{P}_1(M,f)$, a subsequence of $\tilde{u}_j$ will converge to U_2, in $C^{1,\alpha}$. But (for another subsequence if necessary) $\Gamma_{\tilde{u}_j}$ will converge to Γ_{U_2} in $C^{1,\alpha}(B_1(e))$, that such a subsequence exists follows from Lemma 8. By that Lemma $\tilde{u}_j$ is ϵ-close to U_2 in a ball around e, and by [C] (Theorem 7) it follows that $\Gamma_{\tilde{u}_j}$ is uniformly $C^{1,\alpha}$ when ϵ is small enough. This contradicts angle$(n_{x^j}, e) > \epsilon$. This implies that the normal of Γ_u is uniformly continuous up to the boundary for all $u \in \hat{P}_1(M,f)$. □

Remark: If $\lambda = \frac{1}{2}$ then $\hat{P}_1(M,f) = P_1(M,f)$.

Remark: If we let $\lambda = 0$ then this Theorem is a generalization of the first two statements of Theorem C in [SU].

With more assumptions on the regularity of the boundary values we can deduce more regularity of the free boundary. Let us give one such example as a Corollary to Theorem 2.

Corollary 2. *Let $u \in P_1 M, f \setminus \hat{P}_1(M,f)$. Assume further that $f \in C^2(spt(f))$, $\{f = 0\}$ is a C^1 set and f satesfies the properties in Theorem 2 then there is a small neighbourhood of the origin where Γ_u is C^1.*

Proof: The Weiss energy functional of Lemma 1 is upper semicontinious in the center of the balls where the integrals are taken, therefore all the touching points near the origin will be points where the blow-up converges to U_1. Using the regularity of f and non-tangential regularity of Theorem 2 the result follows. □

7 Generalizations

Theorem 2 may be strengthened further without much effort. Actually without any considerable changes all the details in the proof of Theorem 2 works.

Theorem 4. *Let u solve*

$$\left.\begin{array}{ll} \Delta u = g\chi_{\Omega_u} & \text{where } g \in C^{0,Dini} \\ u = \|\nabla u\| = 0 & \text{on } \Gamma_u \text{ (the free boundary)} \\ u|_{\Upsilon} = f(x_2, ..., x_n) & \text{where } \Upsilon = \{(\upsilon(x'), x'); |x'| \leq 1\},\ \upsilon \in C^1 \\ u \in W^{2,p}(B_r^+(0)), & \end{array}\right\} \quad (24)$$

$\upsilon(0) = 0$, $\nabla\upsilon(0) = 0$ and $g(0) = 1$ (for simplicity), assume furthermore that the assumptions of Theorem 2 holds, with obvious changes in the definition of $\hat{P}_1(M, f)$ (i.e. u solves equation (24) instead of equation (1)). We also need to assume that

$$\liminf_{\epsilon \to 0} \frac{|\Lambda_u \cap B_\epsilon^+(x^0)|}{|B_\epsilon^+(x^0)|} > \frac{1}{4}, \quad (25)$$

for all $x^0 \in \Gamma_u$ in a neighborhood of the origin. Then the conclusion of Theorem 2 also holds, but naturally σ will depend also on g and υ.

Proof: Since the proof is very similar to the proof of Theorem 2 we will just indicate the differences. All the relevant results of section 4 and 5 will work with no substantial changes in the proofs. We only have to realize that the blow up of Υ becomes the hyper plane Π, and in the uniqueness proof for blow ups we have to be little more careful when we show that the Weiss energy functional is continuous, but there is no substantial changes in the proofs. So we may freely use those lemmas.

To prove the second part of the theorem we need an equivalence of Lemma 8. But the lemma follows easy even in this case. Now the theorem easily follows in this more general case in the same way as Theorem 2; we need only to use the results in [B] (Theorem 7.2) to show C^1 regularity of Γ_u away from Π, it is now we need the condition of equation (25). We leave the details to the reader. The interested reader may also consult [BS] for further information on free boundary problems with the g-function appearing on the right side of our equation. □

Remark: Obviously the assumption that $g(0) = 1$ is stronger than we need, a simple normalization allows a more general g, we leave the details to the reader.

8 References

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