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The Free Boundary Near the Fixed Boundary for the Heat Equation

by

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Abstract

We study the regularity of the free boundary, near contact points with the fixed boundary, for a parabolic free boundary problem:

$$\Delta u - \frac{\partial u}{\partial t} = \chi_{\{u \neq 0\}} \quad \text{in } B_r^+.$$

We will show that the free boundary is a C^1 manifold up to the fixed boundary under certain regularity assumptions on the boundary data, the C^1 norm is uniform for a certain, and specified, subclass of solutions.

1 Introduction

In this paper we will investigate the free boundary (see below for definitions) for the heat equation near contact points with the fixed boundary. Mathematically the problem can be formulated in the following way

$$\left.\begin{array}{ll} \Delta u - \partial_t u = \chi_{\Omega_u} & \text{where } \Omega_u = Q_r^+ \setminus \{u = |\nabla u| = 0\} \\ u|_{\Pi} = f(x_2, ..., x_n, t) & \text{where } \Pi = \{x_1 = 0\}. \end{array}\right\} \tag{1}$$

All relevant notation will be defined at the end of the introduction.

Before we state our main results, in the next section, we will shortly describe the mathematical and applicational context of our problem.

Applications: Under the assumption that $u, u_t \geq 0$ this is the well known Stefan problem describing the melting process of ice. For further detail on the Stefan problem see [F].

Mathematical background: The regularity of free boundaries have been extensively studied over the last twenty years and the literature on the Stefan problem is vast. This problem however (without sign restriction) is to the authors knowledge first studied by L.A. Caffarelli, A. Petrosyan and H. Shahgholian in [CPS]. The authors showed that a solution is $C^{1,1}$ in space and $C^{0,1}$in time, and that the free boundary is locally analytic under an assumption on the density of $\{u = 0\}$ backward in time.

The regularity of the free boundary near contact points with the fixed boundary was investigated by D.E. Apushkinskaya, N.N. Uraltseva and H. Shahgholian in [ASU]. The authors of [ASU] works under the assumption of vanishing boundary data. They prove that the free boundary is uniformly C^1 away from the fixed boundary, but only Lipschitz as a graph over the fixed boundary near a contact point. This Lipschitz property of the free boundary is rather surprising and not in line with similar results in the elliptic case where the free boundary approach the fixed boundary tangentially, see [SU].

Our main object in this paper is to extend the results of [ASU] to non-vanishing boundary data. In a sense this is a twin paper to [A] where the same problem is investigated for a corresponding elliptic problem.

Plan of the paper: In the next section, after this short introduction, we will state our main results in theorems one to three. Before we are able to prove these theorems we must however introduce some technical tools, the so-called monotonicity formulas. The monotonicity formulas are well known in this context, but we include them for completeness in section 3.

With these tools at hand we will prove the main theorems in sections four, five and six. Section seven is dedicated to a discussion of how to extend Theorem 3 up to the fixed boundary and also give examples when such an extension fails.

Notation: $\mathbb{R}^{n+1}$ will denote $n+1$ dimensional real space with coordinates $(x_1, x_2, .., x_n, t)$.

$B_r(x^0)$ will denote the n dimensional ball in the x variables.

$Q_r(x^0, t^0)$ will denote $B_r(x^0) \times (-r^2, 0)$, the parabolic cylinder.

$\mathbb{R}^n_+$, $B_r^+(x^0)$ and $Q_r^+(x^0, t^0)$ will denote the corresponding sets intersected with $\{x_1 > 0\}$.

Π will be the plane $\{x_1 = 0\}$.

$\mathcal{H}^n(\Omega)$ is the n-dimensional Hausdorff measure of Ω.

$|\Omega|$ will denote the Lebesgue measure of Ω, $|x|$ will also denote the Euclidian norm of the vector $x \in \mathbb{R}^n$. With another slight abuse of notation we will use $|(x,t)| = \sqrt{|x|^2 + |t|}$, the parabolic distance.

$\mathrm{dist}((x^0, t^0), \Omega)$ will denote the Euclidian distance from x^0 to the set $\Omega \cap \{t = t^0\}$, the t^0-section of Ω.

$\mathrm{pardist}((x^0, t^0), \Omega)$ will denote the parabolic distance between (x^0, t^0) and Ω, that is $\inf_{(x,t)\in\Omega} \sqrt{|x - x^0|^2 + |t - t^0|}$.

χ_Ω will denote the characteristic function of the set Ω.

$u^\pm$ will denote the positive and negative parts of the function u. that is $u^\pm = \max(\pm u, 0)$.

$f|_\Omega$ will denote the restriction of f to the set Ω.

∂_i for $i = 1, .., n$ and ∂_t will denote $\frac{\partial}{\partial x_i}$ and $\frac{\partial}{\partial t}$ respectively.

u_i and u_t will be used for $\partial_i u$ and $\partial_t u$ respectively.

Δ is the spatial Laplacian, $\Delta = \sum_{i=1}^n \partial_i^2$.

∇u is the spatial gradient of u, $\nabla u = (\partial_1 u, ..., \partial_n u)$.

Λ_u is the set where $u = \nabla u = 0$. We will also use the notation $\Lambda_u(-r^2)$ for $\Lambda_u \cap \{t = -r^2\}$.

Ω_u is the complement of Λ_u with respect to the domain of u.

Γ_u, the free boundary of u, is the intersection of the closures of Λ_u and Ω_u.

$C^{1+\alpha,\beta}(\Omega)$ is the parabolic Hölder space of functions $C^{1+\alpha}$ in the spatial variables and C^β in the time variable.

$W_a(r, u)$ is defined in Lemma 1.

$\Psi(t, h_1, h_2)$ is defined in Lemma 2.
$P_r(M, f)$ is defined in Definition 1.
The limit $\lim_{r\to 0} u(rx)/r^2$, maybe normalised differently, through some subsequence will be called *the blow-up of u*.

2 Main Results.

Before we state our main results we need some definitions to simplify our statements.

Definition 1. *$P_r(M, f)$ is the set of all functions solving equation (1) in the sense of distributions and whose L^∞-norm is bounded by M.*

By $P_\infty(f)$ we will mean the set of quadratically bounded functions solving equation (1) in the sense of distributions in the entire space $x_1 > 0$, $t < 0$. By quadratically bounded we mean $\sup_{Q_r^+} |u| < Cr^2$, where C may depend on u but not on r.

Theorem 1. *Let $u \in P_1(M, f)$,*

$$\liminf_{r\to 0} \frac{\mathcal{H}^n(\Gamma_u \cap Q_r)}{r^n} > \epsilon > 0 \tag{2}$$

and

$$\sup_{Q_r\cap\Pi} |f| \le C_1 r^2$$

then

$$\sup_{Q_r^+} |u| \le C_2 M r^2.$$

Theorem 2. *Let $u \in P_\infty^+(f)$, be a homogeneous function then if $f = 1/2(e\cdot x)_+^2|_\Pi$ for a unit vector $e = (e_1, ..., e_n) \in \mathbb{R}^n$, then $u = 1/2(e\cdot x)_+^2$ or $u = 1/2(\hat{e}\cdot x)_+^2$ in Q_∞^+. Here $\hat{e} = (-e_1, e_2, ...e_n)$ is the reflection of e in Π.*

Remark: An interesting question is whether the same result is true without the homogeneity assumption. This is not the case, for a proof see [AS].

Remark: Another interesting question is if the same could be said if f is a polynomial, but this is not true either. Let us sketch some details. We want to construct a solution in Q_1^+ whose blow up is not a polynomial.

Consider the function

$$v = \frac{1}{4}\big((e\cdot x)_+^2 + (-e\cdot x)_+^2\big) + b(x_1 - \sqrt{-t})_+^2,$$

where $e = (\sqrt{1-a}, 2a)$ and $a, b > 0$ are small constants. Straight forward calculations gives

$$\Delta v - \frac{\partial v}{\partial t} \le \frac{3}{4}a + \frac{1}{4} + 3b - \frac{bx_1}{\sqrt{-t}} \le 1,$$

if a and b are small enough. This maximum principle gives that u, the solution of equation (1) with v as boundary data, will satisfy $u \leq v$. But this implies that $\mathrm{spt}(u) \subset \mathrm{spt}(v)$. But the support of v is invariant under blow ups which in particular means that the support of u_0, the blow up of u, will lie in the support of v. But this excludes the possibility for u_0 to be a polynomial.

A total classifications for solutions as the one above is probably a difficult task today and is not even known in the elliptic case (see [AS]). This question will not be addressed in this paper.

Because of the importance of the two half polynomial solutions introduced in the above theorem we will make the following definition.

Definition 2. *Let f be a given function on Π satisfying*

$$\lim_{r\to 0} \frac{f(rx)}{r^2} = (e\cdot x)_+^2|_\Pi,$$

$|e| = 1/\sqrt{2}$. By U_1^f we mean $((-|e_1|, e_2, ..., e_n)\cdot x)_+^2$ and by U_2^f we mean $((|e_1|, e_2, ..., e_n)\cdot x)_+^2$.

We will often write U_1 and U_2 when the f-dependence is given by context.

The final main result in this paper regards the local behaviour of the free boundary near a contact point with the fixed boundary. As in [A] this result depends on the blow-up of u at the contact point. We will need the following definition.

Definition 3. *$\hat{P}_r(M, f)$ will denote the subset of $P_r(M, f)$ of functions whose blow-up at the origin is U_2.*

Here we tacitly assume that the blow up of f is a half polynomial.

Theorem 3. *Let $u \in P_1(M, f)$ be quadratically bounded, and $\epsilon > 0$. Assume further that*

$$\lim_{r\to 0} \frac{f(rx, r^2t)}{r^2} = \lambda(x_2)_+^2,$$

for a $\lambda \leq \frac{1}{2}$, then $\Gamma_u \cap \{x_1 \geq \epsilon(|x'| + \sqrt{|t|})\}$ is a C^1 graph in a neighbourhood of the origin.

If $u \in \hat{P}_1(M, f)$ then we have the following uniformity

1. *there exists a modulus of continuity and a constant r_0 (depending only on f and M) such that*

   $$\Gamma_u \cap \{x_1 \geq \epsilon(|x'|+\sqrt{|t|})\}\cap Q_{r_0} \subset \{x;\ \textit{paradist}((x,t), \Gamma_{U_2}) \leq \sigma_\epsilon(|(x,t)|)|(x,t)|\},$$

2. *the C^1 norm of the Γ_u is uniform in M, f and ϵ.*

Remark: The condition $\{x_1 \geq \epsilon(|x'| + \sqrt{|t|})\}$ is a non-tangential approach condition. It is necessary if the regularity of f or the support of f is irregular. With further assumptions on f and the blow-up of u this will give C^1 regularity up to the fixed boundary. This is expected and similar to Fatou's Lemma for harmonic functions.

We will at the end of this paper prove, as a a corollary to Theorem 3, a C^1 up to the fixed boundary result and also give some examples of when C^1 regularity up to the fixed boundary fails.

3 Technical Tools and Known Results.

An essential role in the theory of free boundaries is played by the so-called monotonicity formulas. We will use two monotonicity formulas in this paper, the first one (essentially) due to G.S. Weiss and the second due to L.A. Caffarelli.

Lemma 1. *Let u be a solution to (1) and let also f be homogeneous of degree two. Then*

$$W_a(r,u) = \frac{1}{r^4}\int_{-4r^2}^{-r^2}\int_{B_a^+}\Big(\frac{1}{2}|\nabla u|^2 + u + \frac{u^2}{2t}\Big)G(x,t)dxdt$$

satisfies

$$\frac{dW_a(r,u)}{dr} = \frac{1}{2r}\int_{-4}^{-1}\int_{B^+_{a/r}}\frac{|u_r'|^2}{-t}G(x,-t)dxdt + J_a(r,u) \tag{3}$$

for $0 < r \le a \le 1$. Here

$$u_r'(x,t) = x\cdot\nabla u_r(x,t) + 2t\partial_t u_r(x,t) - 2u_r(x,t),$$

$$G(x,t) = \frac{exp(-|x|^2/4t)}{(4\pi t)^{n/2}} \textit{ for } t>0 \textit{ and } G(x,t)=0 \textit{ for } t\le 0,$$

$$J_a(r,u) = \int_{-4}^{-1}\int^{+}_{\partial B_{r/a}}\frac{u_r'}{r}(\eta\cdot\nabla u_r)G(x,-t)dxdt \\ -\frac{a}{2r^2}\int_{-4}^{-1}\int_{(\partial B_{a/r})}\Big(|\nabla u_r|^2 + 2u_r + \frac{(u_r)^2}{t}\Big)G(x,-t)dxdt,$$

η is the outward normal of $\partial B^+_{a/r}$.

Remark: It is not difficult to see that

$$|J_a(r,u)| \le \frac{1}{p(r/a)}e^{-\frac{a^2}{16r^2}},$$

where $p(\cdot)$ goes to zero with polynomial speed. This implies in particular that

$$\lim_{r\to 0} J_a(r,u) = 0$$

for any $a > 0$.

It is also easy to see that even if f isn't homogeneous almost the same result holds, we will however get the following extra term on the right hand side in formula (3):

$$\int_{-4}^{-1}\int_{\Pi\cap B_{a/r}}\frac{\partial' f_r}{r}\Big(\frac{\partial u_r}{\partial x_1}\Big)G(x,-t)dxdt. \tag{4}$$

This term will however go to zero uniformly if f_r converges to a homogeneous function uniformly in $C^{1+\alpha,\beta}_{loc}$. A fact that we will use later in Section 6.

Proof: The proof follows the same lines as in [ASU]. By a change of variables we see that

$$W_a(r,u) = W_{a/r}(1,u_r),$$

where $u_r = \frac{u(rx,r^2t)}{r^2}$. From this we will get

$$\begin{aligned}&\tfrac{d}{dr}W_{a/r} = \textstyle\int_{-4}^{-1}\int_{B^+_{a/r}} \Big(\nabla u_r'\cdot\nabla u_r + u_r' + \tfrac{u_r u_r'}{t}\Big)G(x,-t)dxdt- \\ &\tfrac{a}{r^2}\textstyle\int_{-4}^{-1}\int_{\partial B_{a/r}\cap\mathbb{R}^n_+}\Big(\tfrac{1}{2}|\nabla u_r|^2 + u_r + \tfrac{u_r^2}{2t}\Big)G(x,-t)dxdt = I_1 + I_2.\end{aligned}$$

Integrating I_1 by parts will lead to

$$\begin{aligned}&I_1 = \textstyle\int_{-4}^{-1}\int_{B^+_{a/r}} u_r'\Big(-\Delta u_r - \tfrac{x_i}{2t}\partial_i u_r + 1 + \tfrac{u_r}{t}\Big)G(x,-t)dxdt+ \\ &\textstyle\int_{-4}^{-1}\int_{\partial B^+_{a/r}\cap\mathbb{R}^n_+} u_r'\tfrac{\partial u_r}{\partial\eta}G(x,-t)dxdr,\end{aligned} \tag{5}$$

where η is the outward unit normal. Observe that we are here using that the boundary values are homogeneous.

Using the definition of u_r' together with the equality

$$\frac{du}{dr} = \frac{u_r'}{r}$$

will lead us to

$$-\Delta u_r - \frac{x_i}{2t}\partial_i u_r + 1 + \frac{u_r}{t} = -\Delta u_r + \partial_t u_r + 1 - \frac{u_r'}{2t}.$$

Using this in equation (5) will give the desired result. $\square$

Lemma 2. *Let h_1 and h_2 be two sub-caloric functions in $\mathbb{R}^n\times[-1,0]$ with polynomial growth at infinity such that*

$$h_1(0,0) = h_2(0,0) = 0 \;\; \textit{and} \;\; h_1\cdot h_2 = 0,$$

then the functional

$$\Psi(t,h_1,h_2) = \frac{1}{t^2}\int_{-t}^{0}\int_{\mathbb{R}^n}|\nabla h_1|^2 G(x,-s)dxds\cdot\int_{-t}^{0}\int_{\mathbb{R}^n}|\nabla h_2|^2 G(x,-s)dxds$$

is nondecreasing in $t\in(0,1)$.

Proof: The proof of this Lemma is rather technical so we will omit it here. We refer the reader to [C2].

4 Proof of Theorem 1

The proof follows the lines in of the proof of Lemma 3.1 in [ASU], we will sketch some details.

Denote

$$M_k(u) = \sup_{Q_{2^{-k}}} |u|, \quad \text{for } k \in \mathbb{N}.$$

It is sufficient to show that $\exists C_2$ such that

$$4^{k+1}M_{k+1}(u) \leq \max(4M_1(u), ..., 4^k M_k(u), MC_2) \quad \forall k \in \mathbb{N}.$$

Suppose, to get a contradiction, that this fails. That is $\forall j \in \mathbb{N}$ there exists $u_j \in P_1^+(M)$ and $k_j \in \mathbb{N}$ such that

$$4^{k_j+1}M_{k+1}(u_j) \leq \max(4M_1(u_j), ..., 4^{k_j} M_k(u_j), j).$$

We now make the blow up

$$\tilde{u}_j = \frac{u_j(2^{-k_j}x, 2^{-2k_j}t)}{M_{k_j+1}(u_j)}.$$

It is easy to see that $k_j \to \infty$ as $j \to \infty$. It is also easy to verify that each $\tilde{u}_j$ is uniformly quadratically bounded away from the origin and that

$$\Delta\tilde{u}_j - \partial_t\tilde{u}_j \to 0.$$

It will also follow from the definition of $\tilde{u}_j$ that $\sup_{Q^+_{1/2}} |\tilde{u}_j| = 1$ and that $\tilde{u}_j(0, x', t) \to 0$.

Standard compactness theory will imply that a subsequence of $\tilde{u}_j$ will converge to, say, u_0. u_0 will be a caloric, non-zero, quadratically bounded function in Q^+_∞ with zero boundary values on Π.

Now we can use the Liuoville Theorem (Lemma 2.1 in [ASU]) to deduce that u_0 is a quadratic polynomial in x and linear in t. But u_0 will also preserve the Hausdorff criteria which will lead to a contradiction.

5 Proof of Theorem 2.

We will show that $u_t = 0$. Then result follows then from [A].

Form homogeneity it follows that $\|u_t\|_\infty < \infty$. Consider the functions w_R^+ satisfying

$$\begin{array}{ll} \Delta w_R^+ - \frac{\partial w_R^+}{\partial t} = 0 & \text{in } \{t > -R\} \\ w_R^+(x, -R) = \|u_t\|_\infty & \text{if } \{x_1 > 0\} \\ w_R^+(x, -R) = 0 & \text{if } \{x_1 < 0\}. \end{array}$$

w_R^+ is uniformly bounded by $\|u_t^+\|_\infty$ and therefore a subsequence will converge as $R \to \infty$, call the limit w^+. A direct calculation establishes that $w^+ = \|u_t^+\|_\infty/2$.

Now $w_R^+ \geq u_t$ which implies that $u_t \leq \|u_t^+\|_\infty/2$. A similar calculation will yield the opposite inequality. Thus $\|u_t\|_\infty \leq \|u_t\|_\infty/2$.

6 Proof of Theorem 3.

The proof of this Theorem is the most technically difficult in this paper. We will only prove the uniformity in the second part of the Theorem. The proof of the first part is very similar but simpler, the reader can substitute a fixed function u whenever we consider a sequence of functions u_j in the proof below and the first part of the Theorem follows.

We will start to prove a lemma which will help us to use Theorem 2.

Lemma 3. *Let $u \in P_r(m, f)$ and any blow up of f be homogeneous and u have quadratic growth. Then any blow up of u will be homogeneous. Moreover if $\frac{f(rx,r^2t)}{r^2} \to \lambda(x_2)_+^2$ in $C_{loc}^{1+\alpha,\beta}$, then the blow up of u is unique.*

Proof: Let $r_j \to 0$ be a sequence such that

$$\lim_{j\to\infty} \frac{u(r_jx, r_j^2t)}{r_j^2} = u_0(x,t) \quad \text{locally in } Q_\infty^+.$$

Then for arbitrary α and β we will have

$$W_1(\alpha r_j, u) - W_1(\beta r_j, u) \to 0 \text{ as } j \to \infty.$$

But this is equivalent to

$$\begin{aligned} 0 = &\lim_{j\to\infty} W_{1/r_j}(\alpha, u_{r_j}) - W_{1/r_j}(\beta, u_{r_j}) = \\ = &\int_\beta^\alpha \frac{dW_{1/r_j}(\theta, u_{r_j})}{d\theta} d\theta. \end{aligned}$$

The last integral can be estimated by means of Lemma 1 and the remark after that Lemma. We will get

$$\begin{aligned} 0 = &\lim_{j\to\infty} \int_\beta^\alpha \frac{1}{2\theta} \int_{-4}^{-1} \int_{B^+_{1/\theta r_j}} \frac{|u'_{\theta r_j}|^2}{-t} G(x,-t)dxdtd\theta = \\ &\lim_{j\to\infty} \int_\beta^\alpha \frac{1}{2\theta^5} \int_{-4\theta^2}^{\theta^2} \int_{B^+_{1/r_j}} \frac{|u'_{r_j}|^2}{-t} G(x,-t)dxdtd\theta. \end{aligned}$$

This implies that

$$\frac{1}{2\theta^5} \int_{-4\theta^2}^{\theta^2} \int_{\mathbb{R}^n_+} \frac{|u_0'|^2}{-t} G(x,-t)dxdt = 0$$

for almost all $\theta \in (\beta, \alpha)$. This is only true if $u_0' = 0\, a.e.$, but this is equivalent to homogeneity for u_0.

To prove the uniqueness of the blow-up we argue by contradiction. Assume that there exists two subsequences $r_j, s_j \to 0$ such that $\lim_{j\to\infty} u_{r_j} = U_1$ and $\lim_{j\to\infty} u_{s_j} = U_2$. But this is not possible by Lemma 1 and the remark after that lemma. □

We will also need another lemma before we start with the proof of the main Theorem, but before that let us refresh our memory of a definition in [CPS].

Definition 4. *The minimal diameter of a set E in $\mathbb{R}^n$, denoted $md(E)$, is the infimum of distances between two parallel hyper planes such that E is contained in the strip between these planes.*

In [CPS] it is proven that there exists a modulus of continuity μ such that if, for some r

$$\eta(r) = \frac{md(\Lambda_u(-r^2) \cap B_r)}{r} > \mu(r),$$

then Γ_u is C^1 in $Q_{r/2}(x^0, t^0)$. Together with the next Lemma we may use their result to deduce regularity of the free boundary close to a contact point.

Lemma 4. *Let u be as in Theorem 3, then there exists an $\rho = \rho_{\varepsilon,\delta}$ for each $\varepsilon > 0$ and $\delta > 0$ such that if $(x^0, t^0) \in Q_\rho^+ \cap \Gamma_u \cap \{x_1 \geq \epsilon(|x'| + \sqrt{|t|})\}$ then*

$$\frac{md(\Lambda_{\tilde{u}}(-\delta^2 r^2) \cap B_{\delta r}}{\delta r} > \varepsilon,$$

$\tilde{u} = u(x + x^0, t + t^0)$ and $r = x_1^0$.

We will prove the first statement in the uniformity part of Theorem 3 before we prove this Lemma.

Proof of 1) in Theorem 3: Assume that this isn't true. That is we assume that there exists $u_j \in P_1(M, f)$ and $\{x_1 \geq \epsilon(|x'| + \sqrt{|t|}\} \cap \Gamma_{u_j} \ni (x^j, t^j) \to 0$ and $(x^j, t^j) \notin \{(x, t); \text{paradist}((x, t), \partial\Omega_{U_2}) < \delta|(x, t)|\}$. Make a blow up

$$u_j(x, t) = \frac{u(r_j x, r_j^2 t)}{r_j^2} \to u_0 \text{ for a subsequence in } C^{1,\alpha}.$$

By assumption u_0 will be the half space solution U_2. But this contradicts the assumption $(x^j, t^j) \notin \{(x, t); \text{paradist}((x, t), \partial\Omega_{U_2}) < \delta|(x, t)|\}$. □

Proof of Lemma 4: The argument is by contradiction and blow up. Assume that there exists a sequence $u_j \in \hat{P}_1(M, f)$ and $(x^j, t^j) \in \Gamma_{u_j} \cap B_{1/j}(0) \cap \{x_1 \geq \epsilon(|x'| + \sqrt{|t|})\}$ such that

$$\frac{md(\Lambda_{u_j(x+x^j, t+t^j)}(-\delta^2 r^2) \cap B_{r_j})}{\delta r_j} \leq \frac{1}{j}, \tag{6}$$

for $r_j = x_1^j$. Upon scaling u_j

$$\tilde{u}_j = \frac{u_j(r_j x + x^j, r_j^2 t + t^j)}{r_j^2},$$

we will have, for a sub sequence, $\tilde{u}_j \to u_0$. But by part 1 of Theorem 3 u_0 is a half space solution contradicting equation (6). □

Now we are ready to prove the second statement in the uniformity part of Theorem 3.

Proof of 2) in Theorem 3: By [CPS] (Theorem 14.1) and the previous lemma it follows that the free boundary is $C^{1,\alpha}$ near the contact point. However the

$C^{1,\alpha}$-norm may increase to infinity as we approach the contact point. This is however not the case.

Assume that there exists u_j and $0 \leftarrow (x^j, t^j) \in \Gamma_{u_j} \cap \{x_1 \geq \epsilon(|x'| + \sqrt{|t|})\}$ such that the angle of the normal of Γ_{u_j} at (x^j, t^j) is larger than some angle $\delta > 0$. Then we can blow up by

$$\tilde{u}_j = \frac{u_j(|x_1^j|x + x^j, |x_1^j|^2 t + t^j)}{|x_1^j|^2}.$$

By the quadratic bound on u_j, a subsequence of $\tilde{u}_j$ will converge to some u_0 in $C^{1,\alpha}$. By our classification of global solutions $u_0 = U_2$. This is a contradiction since the free boundary will (if we take another subsequence) converge in $C^{1,\alpha}$ to a surface with normal in at least δ radians away from the normal of Γ_{U_2}. □

7 Up to the boundary Regularity.

In the previous section we showed regularity of the free boundary in every non-tangential access cone up to the free boundary. In this section we will show full regularity of the free boundary up to the fixed boundary in a small neighbourhood of the touching point. To do that we need to assume more regularity of the boundary values f. We also need to assume something about the blow-up of u.

Our first result in this direction is a Corollary to Theorem 3

Corollary 1. *Let $u \in P_1(M, f) \setminus \hat{P}_1(M, f)$ satisfy the conditions in Theorem 3. Assume further that spt$(f) \in C^1$ and that $f \in C^2($spt$(f))$ then there is a small parabolic cube Q_r where $\Gamma \in C^1$.*

Proof: Since $f \in C^2(\text{spt}(f))$ the blow-up of f is continuously changing in the centre of the blow-up. By Theorem 3 we need to show that the blow-ups of u also changes continuously. That is, we need to show that the blow-up of u is U_1 at every point in a neighbourhood of the origin.

We notice that $W_a(r, U_2) \geq W_a(r, U_1)$, with equality only if $\lambda = 1/2$. The Corollary follows from the upper semi-continuity of the Weiss energy functional $W_a(0, u)$ in the centre of the ball where the integrals are taken in. □

Remark: In this Corollary we have to assume that $u \in P_1(M, f) \setminus \hat{P}_1(M, f)$. The same result would be uniformly true (that is with uniform C^1 regularity in a uniform cube depending only on f and M) if we assume that all the blow-ups of u equals U_2. Whether we have such a continuity in the blow-ups is not known and does not seem to follow from the techniques in this paper.

Finally we would like to draw attention to two examples related to the appearance of irregular free boundaries. The first is just a reminder of an example from [A] and the second one shows another phenomenon where the free boundary becomes irregular.

Example: Given a closed set one can construct a positive $C^{1,1}$ function that is C^∞ in its support. This is easy using powers regularised distance functions.

By the boundedness of the second derivatives we can multiply by a constant so the function is a super-solution to

$$\Delta v = \chi_{\{v \geq 0\}}.$$

Using this function as boundary values we can construct a solution u such that $0 \leq u \leq v$. Therefore the boundary of any closed set can be the touching set of the free boundary. In particular we need the assumption on the regularity of the support of f in Corollary 1.

Example: In this example we want to point out another phenomena where the regularity up to the fixed boundary fails. Let u be a positive solution to

$$\begin{array}{l} \Delta u - \frac{\partial u}{\partial t} = \chi_{\{u \geq 0\}} \quad \text{in } D = B_1^+ \times (0, \infty) \\ u = f \quad \text{on } \partial D, \end{array}$$

where f is the following function

$$\begin{array}{l} f = U_2 \quad \text{on } B_1^+ \times \{0 \leq t \leq 1\} \\ f = 0 \quad \text{on } \{x_1 > 0\} \times \{t > 1\} \\ f = U_2 \quad \text{on } \{x_1 = 0\} \times \{t > 1\}. \end{array}$$

Then $u = U_2$ when $t < 1$. By comparison, for t_ϵ large enough $u \leq U_1(x_1, x_2 + \epsilon, x_3, ...)$ for any $\epsilon > 0$. By the uniformity result in Theorem 3 this implies that the blow-up at free boundary points for large t must equal U_1. Therefore there is smallest t_0 such that if $t \leq t_0$ then the blow-up of u is U_2 and if $t \geq t_0$ the blow-up of u at free boundary points are U_1. In particular the normal of the free boundary jumps at the point $(0, t_0)$.

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