Max-Planck-Institut
für Mathematik
in den Naturwissenschaften
Leipzig

On the De Giorgi Conjecture in Half spaces

by

John Andersson

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Abstract

We prove that the De Giorgi conjecture isn't true in half spaces $x_1 > 0$, even if the boundary data is nonzero and the restriction of a one dimensional solution to $x_1 = 0$.

1 Introduction

In 1978 E. de Giorgi made the following conjecture:
Conjecture. *If $u \in C^2$ solves*

$$\Delta u = u^3 - u \tag{1}$$

and

$$|u| \leq 1, \quad \frac{\partial u}{\partial x_n} > 0$$

in the whole $\mathbb{R}^n$. Then all level sets of u are hyper planes, at least if $n \leq 8$.

The conjecture was recently proved by O. Savin under the assumption

$$\lim_{x_n \to \pm\infty} u(x', x_n) = \pm 1,$$

where $x' = (x_1, ..., x_{n-1})$, see [S]. It is easy to see that a solution with flat level sets must be of the form

$$u(x) = \tanh \frac{e \cdot x}{\sqrt{2}}, \tag{2}$$

where e is any unit vector in $\mathbb{R}^n$.

Our objective in these pages is to investigate if the conjecture is true in a half space $\{x_1 > 0\}$ with the restriction of (2) as boundary values on $x_1 = 0$. To be more precise, we will construct a counter example in $\mathbb{R}^2$ with boundary values $\tanh \frac{x_2}{2}$ as boundary values. But let us begin with some observations.

Remark 1: A solution of $\Delta u = u^3 - u$ is a critical point of

$$J(u, \Omega) = \int_\Omega \frac{1}{2}|\nabla u|^2 + \frac{1}{4}(u^2 - 1)^2 dx.$$

Remark 2: If u_j solves $\Delta u_j = r_j^2(u_j^3 - u_j)$ then $\tilde{u}_j(x) \equiv u_j(\frac{x}{r_j})$ solves $\Delta \tilde{u}_j = \tilde{u}_j^3 - \tilde{u}_j$.

Observe that u_j is a critical point of the rescaled energy functional

$$J_{r_j}(u_j, \Omega) = \int_\Omega \frac{1}{2r_j}|\nabla u_j|^2 + \frac{r_j}{4}(u_j^2 - 1)^2 dx.$$

Remark 3: (See [K]) Let $u \in C^{0,1}([0,1] \times [0,1])$. We can define an non-decreasing rearrangement in the x_2 direction in the following way:

$$u^*(x) \equiv \sup(c; x_2 > m(\{u(x_1, \cdot) < c\})),$$

where $m(\cdot)$ is the one dimensional Lebesgue measure. This function is non-decreasing in x_2 for each $x_1 \in [0,1]$, and more interesting (see [K] Theorem 2.13)

$$\int_{[0,1]\times[0,1]} |\nabla u|^2 dx \geq \int_{[0,1]\times[0,1]} |\nabla u^*|^2 dx,$$

also

$$\int_{[0,1]\times[0,1]} (u^2 - 1)^2 dx = \int_{[0,1]\times[0,1]} ((u^*)^2 - 1)^2 dx.$$

2 Main Theorem.

Theorem 1. *There exist a solution to*

$$\begin{array}{ll} \Delta u = u^3 - u & \text{in } \mathbb{R}^2_+ \\ u(0, x_2) = \tanh(\frac{x_2}{2}) & \\ \frac{\partial u}{\partial x_2} > 0 & \text{in } \mathbb{R}^2_+ \\ |u| \leq 1, & \end{array} \tag{3}$$

which isn't reducible to one variable.

Proof: We will construct a solution to (3) and then show that it can't be one dimensional.

Step 1: Let u_j be a minimizer of the following obstacle like problem. Minimize

$$\int_{[0,1]\times[0,1]} \frac{1}{2j}|\nabla u|^2 + \frac{j}{4}(u^2 - 1)^2 dx \tag{4}$$

over the set $K_j = \{u \in H^1;\ u \geq 0,\ u = g_j \text{ on } \partial[0,1] \times [0,1]\}$. Where g is smooth and increasing in x_2, $g_j = \tanh\left(\frac{jx_2}{2}\right)$ for $x_2 < \frac{1}{2}$ and $g(x_1, 1) = 1$. We see that $(u_j - 1)^+$, that is the positive part of $u_j - 1$, must be identically zero. Since it is sub-harmonic with zero boundary values. We also know that the minimizer, u_j, is positive and super-harmonic, and can therefore not attain an interior minimum. This means that $u_j > 0$ in $(0,1) \times (0,1)$.

Now we use remark 3 and non-decreasingly rearrange u_j, to the function u_j^*. By remark 3 u_j^* is also a minimizer, so we can assume that $u_j = u_j^*$. That is

$\frac{\partial u_j}{\partial x_2} \geq 0$. By the moving plane method and Hopf's Lemma we can actually show that $\frac{\partial u_j}{\partial x_2} > 0$.

Step 2: Define $\tilde{u}_j = u_j(\frac{x}{j})$, by remark 2, $\tilde{u}_j$ solves (3) in $[0, j] \times [0, j]$. By step 1, $|\tilde{u}_j| \leq 1$ and $\frac{\partial \tilde{u}_j}{\partial x_2} > 0$. Now standard regularity theory shows that a subsequence of $\tilde{u}_j$ converges locally in $C^{1,\alpha}(\mathbb{R}^2_+)$ to a solution v in the first quadrant.

Step 3: The function v constructed in the previous paragraph has the following boundary values

$$\begin{aligned} v(0, x_2) &= \tanh(\tfrac{x_2}{2}) \\ v(x_1, 0) &= 0. \end{aligned}$$

But this implies that we can define a solution u in the upper half space in the following way

$$u(x_1, x_2) = \begin{cases} v(x_1, x_2) & \text{if } x_2 \geq 0 \\ -v(x_1, -x_2) & \text{if } x_2 < 0. \end{cases}$$

Step 4: It only remains to show that u isn't one dimensional. But if u where one dimensional then it must equal $\tanh(\frac{x_2}{2})$ since $u(x_1, 0) = 0$, and the gradient of a one dimensional solution is perpendicular to it's zero set. However $\tanh(\frac{x_2}{2})$ isn't a solution which means that u is not one dimensional. □

3 Some Generalizations.

The technique in this small note is crude but still admits some generalizations, first and foremost the boundary data on the set $\{x_1 = 0\}$ may be generalized considerably, to any increasing function that is not a one dimensional solution of the partial differential equation. However it is the boundary data that assures our non degeneracy, that is that $\tilde{u}_j$ doesn't converge to zero.

We could also generalize the right hand side of $\Delta u = u^3 - u$ to $\Delta u = f(u)$ where $f(\cdot)$ is anti symmetric, continuous and satisfies

$$\begin{aligned} f(u) &\leq 0 \quad \text{for } 0 < u < 1 \\ f(u) &\geq 0 \quad \text{for } u \geq 1. \end{aligned} \tag{5}$$

The final generalization we would like to point out is that instead of using the lapalcian we could use the p-laplacian, Δ_p, defined in the following way

$$\Delta_p u = \text{div}(|\nabla u|^{p-2} \nabla u).$$

We assume that $p \geq 1$, for $p = 2$ we have the usual laplacian.

A solution of the p-laplace De Giorgi problem in Ω can be found by minimizing

$$\int_\Omega |\nabla u|^p + \frac{1}{4}(u^2 - 1)^2 dx$$

over a suitable functiopn space. We have to convince ourselves that the strong maximum principle and the rearrangement of remark 3 works. For the maximum principle and existence for solutions of the p-laplacian we refer to [HKM], and for the rearrangement we refer to [K]. As noted in step 1 in the proof of Theorem 1 the rearangement of u will only assure that $\frac{\partial u}{\partial x_2} \geq 0$, to get the strict inequality we need another argument. The moving plane method doesn't work here because of the nonlinearity of the Δ_p, so we must use another technique. We will however get the following theorem.

Theorem 2. *There exist a solution to*

$$\begin{aligned} &\Delta_p u = f(u) && \text{in } \mathbb{R}^2_+ \\ &u(0, x_2) = g(x_2) && \\ &\frac{\partial u}{\partial x_2} \geq 0 && \text{in } \mathbb{R}^2_+ \\ &|u| \leq 1, && \end{aligned} \tag{6}$$

which isn't reducible to one variable, for any f satisfying (5) and nonzero increasing g with $|g| \leq 1$. If $p = 2$ the derivative in x_2 may be made strictly increasing.

4 References

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