Max-Planck-Institut
für Mathematik
in den Naturwissenschaften
Leipzig

Domain walls and vortices in thin ferromagnetic films

by

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Preprint no.: 45 2006

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May 2, 2006

1 Introduction

1.1 Multiscale problems in micromagnetics

Ferromagnetic materials show a fascinating variety of magnetization patterns on scales ranging from a few nanometers to hundreds of microns. The formation and evolution of these patterns is at the heart of numerous magnetic devices, including the ubiquitous magnetic storage media. Somewhat surprisingly, this large variety of patterns can be understood as (local) minimizers of a simple, yet subtle, functional, the micromagnetic energy. Their dynamics is described by an associated evolution equation, the Landau-Lifshitz (or Landau-Lifschitz-Gilbert) equation, which combines Hamilitonian and dissipative aspects.

Until recently the micromagnetic energy was mostly analyzed in one of two ways. The first approach is to consider special ansatz functions (inspired by physical intuition) with a few free parameters and then to optimize over these parameters. While this approach has lead to valuable insights, it is also limited in its scope. In particular one cannot detect something which has not been put in the ansatz. The second approach is large scale computation. This has been successful for answering specific questions for submicron devices. Due to the wide separation of the relevant scales, direct numerical simulation is, however, restricted to the smallest scales and cannot cover the full picture. Perhaps even more importantly, it answers specific questions, but provides little insight in general principles and understanding.

In the last decade a new approach to micromagnetics has emerged, and the SPP 1095 has had an important impact in shaping it. This approach is based on two ideas. First, considerable insight can be gained by the identification of optimal scaling laws involving the natural parameters, such as material constants or geometric quantities, and the corresponding magnetization patterns. This amounts to establishing upper and lower bounds on the micromagnetic energy.

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While for the former one can often rely on intuition and test functions developed in the physics community, the latter requires in general new mathematical ideas. The second approach is to derive simplified theories in certain limiting parameter regimes, e.g., for thin films. These theories reduce the complexity of the magnetic energy landscape (and the dynamics in that landscape) and allow one to get a better insight into the essential structures.

In this paper we focus on the mathematical analysis of the statics and dynamics of magnetization structures in thin films. For a broader review of recent developments, written for a more general science audience, we refer to [12].

1.2 The micromagnetic energy and associated variational problems

We first discuss the functional from which all our results are ultimately derived, the micromagnetic energy. This energy is a sum of terms of various types. Depending on certain material parameters and on the shape and size of the ferromagnetic sample, any of these terms can play a dominant role, or an interplay between several of them can take place. This explains the multitude of different patterns derived from this theory.

We consider an open domain $\Omega \subset \mathbb{R}^3$ which represents the ferromagnetic body that we study. The magnetization of this body is given by a vector field $\mathbf{m} : \Omega \to \mathbb{R}^3$. Below the Curie temperature, the magnetization is saturated, which means that $\mathbf{m}$ is of constant length. We use a normalization such that $\mathbf{m}$ has values in the unit sphere $\mathbb{S}^2$. Now we consider several energies associated to $\mathbf{m}$.

The so-called exchange energy models the tendency towards parallel alignment of neighboring magnetization vectors in the underlying atomic lattice. It is given by the functional

$$\frac{d^2}{2} \int_\Omega |\nabla \mathbf{m}|^2 \, d\mathbf{x}.$$

Here d is a material constant, called the exchange length.

The ferromagnetic material may have crystalline anisotropies which prefer certain directions of $\mathbf{m}$. An integral of the form

$$Q \int_\Omega \phi(\mathbf{m}) \, d\mathbf{x}$$

represents such anisotropies. Here $\phi : \mathbb{S}^2 \to [0, \infty)$ is a fixed function and Q is another material constant. Usually ϕ is assumed to be smooth, often even a polynomial.

The magnetization induces a magnetic field, often called the stray field or demagnetizing field, which obeys the static Maxwell equations. It can be represented by a potential $u : \mathbb{R}^3 \to \mathbb{R}$ which solves the equation

$$\Delta u = \operatorname{div}(\chi_\Omega \mathbf{m}) \quad \text{in } \mathbb{R}^3.$$

Here χ_Ω is the characteristic function of Ω (in other words, we extend $\mathbf{m}$ by 0 outside of Ω). If Ω is bounded, the saturation condition guarantees that $\mathbf{m} \in L^p(\Omega, \mathbb{R}^3)$ for every $p \in [1, \infty]$. Hence there exists a unique solution of this equation in the Sobolev space $H^1(\mathbb{R}^3)$. The induced field is then represented by $-\nabla u$. Its energy is called the magnetostatic energy and given by

$$\frac{1}{2}\int_{\mathbb{R}^3} |\nabla u|^2 \, d\mathbf{x}.$$

The interaction with an external field $h : \mathbb{R}^3 \to \mathbb{R}^3$ induces an energy term

$$-\int_\Omega \mathbf{h} \cdot \mathbf{m} \, d\mathbf{x}$$

that prefers alignment of the magnetization with the external field. In this paper, applied fields appear as driving forces in the context of moving domain walls.

The micromagnetic energy is the sum of all four energies, that is,

$$E(\mathbf{m}) = \frac{d^2}{2}\int_\Omega |\nabla \mathbf{m}|^2 \, d\mathbf{x} + Q\int_\Omega \phi(\mathbf{m}) \, d\mathbf{x} + \frac{1}{2}\int_{\mathbb{R}^3} |\nabla u|^2 \, d\mathbf{x} - \int_\Omega \mathbf{h} \cdot \mathbf{m} \, d\mathbf{x}. \quad (1)$$

The exchange term is of leading order in this functional, but the constant in front of it is typically small. The other three terms are of order 0, but one of them (the magnetostatic energy) involves the non-local pseudo-differential operator $\nabla \Delta^{-1} \operatorname{div}$. In some situations, its behavior is quite different from the behavior of the other terms. Under certain conditions, these energies may be in competition with one another. For instance, the exchange energy favors constant magnetizations, whereas the magnetostatic energy prefers vector fields which are divergence free (in $\mathbb{R}^3$). Because of the jump at the boundary, the two conditions cannot be satisfied simultaneously. The anisotropy term, on the other hand, may not penalize a varying vector field in principle, at least not if the function ϕ has several minima (which is usually the case), but it favors rapid transitions between different states—unlike the exchange energy. An analysis of such interplays can explain some of the observed patterns in ferromagnets.

We study two types of variational problems associated to the micromagnetic energy. Minimizers of E, or more generally, local minimizers and (stable) critical points represent the stable magnetization patterns of our ferromagnet. If we write

$$\nabla_{L^2} E(\mathbf{m}) = -d^2 \Delta \mathbf{m} + Q \nabla \phi(\mathbf{m}) + \nabla u - \mathbf{h} \quad (2)$$

for the L^2-gradient of E (without the saturation constraint), then these variational problems give rise to the Euler-Lagrange equation

$$(\mathbf{1} - \mathbf{m} \otimes \mathbf{m}) \nabla_{L^2} E(\mathbf{m}) = \nabla_{L^2} E(\mathbf{m}) - (\mathbf{m} \cdot \nabla_{L^2} E(\mathbf{m}))\mathbf{m} = 0 \quad \text{in } \Omega.$$

(Here $\mathbf{1}$ denotes the identity (3×3)-matrix.) That is, the projection of $\nabla_{L^2} E(\mathbf{m})$ onto the tangent space $T_\mathbf{m}\mathbb{S}^2$ vanishes. This equation can also be expressed in the form

$$d^2(\Delta \mathbf{m} + |\nabla \mathbf{m}|^2 \mathbf{m}) - Q \nabla \phi(\mathbf{m}) + (\mathbf{1} - \mathbf{m} \otimes \mathbf{m})(\mathbf{h} - \nabla u) = 0 \quad \text{in } \Omega.$$

Moreover, we have homogeneous Neumann boundary conditions

$$\frac{\partial \mathbf{m}}{\partial \nu} = 0 \quad \text{on } \partial\Omega.$$

A model for the dynamical behavior of the magnetization is given by the Landau-Lifshitz equation

$$\frac{\partial \mathbf{m}}{\partial t} + \gamma \mathbf{m} \wedge \nabla_{L^2} E(\mathbf{m}) + \alpha \mathbf{m} \wedge \frac{\partial \mathbf{m}}{\partial t} = 0, \tag{3}$$

also called the Landau-Lifshitz-Gilbert equation. Here $\wedge$ denotes the vector product in $\mathbb{R}^3$. Both γ and α are fixed constants, and we require that $\alpha\gamma > 0$. Another common way to write the equation is

$$\frac{\partial \mathbf{m}}{\partial t} = \hat{\gamma} \mathbf{m} \wedge \nabla_{L^2} E(\mathbf{m}) + \hat{\alpha} \mathbf{m} \wedge (\mathbf{m} \wedge \nabla_{L^2} E(\mathbf{m})). \tag{4}$$

The two versions are equivalent for

$$\hat{\alpha} = \frac{\alpha\gamma}{\alpha^2 + 1} \quad \text{and} \quad \hat{\gamma} = -\frac{\gamma}{\alpha^2 + 1}.$$

The terms in (3) and (4) with coefficients γ and $\hat{\gamma}$, respectively, describe a magnetic precession. We call them the gyromagnetic terms. The terms with coefficients α and $\hat{\alpha}$, respectively, are damping terms (hence the sign condition on $\alpha\gamma$, giving rise to the condition $\hat{\alpha} > 0$). From the mathematical point of view, the damping terms are the more important ones, because they make the problem parabolic. Without them, the equations would be of the type of a nonlinear Schrödinger equation.

Taking the vector product with $\mathbf{m}$ in all terms of (3), we obtain a third equivalent version of the equation,

$$\tilde{\alpha} \frac{\partial \mathbf{m}}{\partial t} + \tilde{\gamma} \mathbf{m} \wedge \frac{\partial \mathbf{m}}{\partial t} = (\mathbf{m} \otimes \mathbf{m} - \mathbf{1}) \nabla_{L^2} E(\mathbf{m}), \tag{5}$$

where

$$\tilde{\alpha} = \frac{\alpha}{\gamma} \quad \text{and} \quad \tilde{\gamma} = -\frac{1}{\gamma}.$$

Representing the equation in this form is convenient because it underlines the similarity to the negative L^2-gradient flow for the functional E subject to the constraint $|\mathbf{m}| = 1$. (In fact this gradient flow is (5) for $\tilde{\alpha} = 1$ and $\tilde{\gamma} = 0$.) We will normally use the Landau-Lifshitz equation in the form (5).

It is natural to impose a homogeneous Neumann boundary condition also for the Landau-Lifshitz equation.

Apart from the obvious quantities d, Q, ϕ, and $\mathbf{h}$, the qualitative behavior of E and of solutions of the above variational problems also depends on the shape and the size of the sample Ω. Some idea of the dependence on the size can be gained by studying the scaling properties of the four terms which contribute to the micromagnetic energy. Suppose for a number $\lambda > 0$, we replace Ω by $\lambda\Omega$ and the vector field $\mathbf{m}$ by $\mathbf{m}(x/\lambda)$ (and similarly $\mathbf{h}$ by $\mathbf{h}(x/\lambda)$). Then the exchange energy is multiplied by the factor λ, whereas the other energy terms are multiplied by λ^3. Thus it is to be expected that for a very small sample, the exchange energy determines the behavior of $\mathbf{m}$ to a large extent; that is, a minimizer of E is nearly constant. For a very large sample, on the other hand, the exchange energy is insignificant, and the behavior of $\mathbf{m}$ is ruled by the other terms.

On the other hand, we can use rescalings to eliminate one of the parameters in our problem. Replacing Ω by $\lambda\Omega$, and replacing simultaneously d by λd, we obtain a functional whose energy landscape differs only by a constant. This way we can normalize the problem such that either d or Q become 1, or such that Ω has unit size.

1.3 Thin films

The results in this paper are concerned with ferromagnetic bodies in the shape of thin films. That is, we consider domains of the form

$$\Omega_\delta = \Omega \times (0, \delta),$$

where Ω is now a two-dimensional domain and $\delta > 0$ is small compared with the size of Ω. We either study the limit behavior of the micromagnetic energy and its variational problems as $\delta \searrow 0$, or we use the thinness of Ω_δ as a justification for working directly in two dimensions and with an energy that approximates the micromagnetic energy for thin films. In both cases, the projection of $\mathbf{m}$ onto the plane $\mathbb{R}^2 \times \{0\}$ and the third component of $\mathbf{m}$ play different roles. It is therefore convenient to use the notation $\mathbf{m} = (m, m_3)$ for the magnetization vector field, where $m = (m_1, m_2)$. Similarly we often write $\mathbf{x} = (x, x_3) = (x_1, x_2, x_3)$ for a generic point in $\mathbb{R}^3$. Sometimes, however, it is more convenient to use coordinates (x, y, z) in $\mathbb{R}^3$.

The reduction to two dimensions—whether by a rigorous asymptotic analysis or formally—decreases the complexity of the problems that we study. Nevertheless, a rich variety of patterns can still be observed, and there exist different asymptotic regimes for the thin-film limit which give rise to different reduced theories. These regimes are determined by certain relations between the parameters involved in the problem, of which we have now four (under the assumption that the shape of Ω is fixed, but the size can still be varied by scaling): In addition to the material constants d and Q, we have the thickness δ and a length scale L of the cross-section Ω. When we are not interested in the behavior of $\mathbf{m}$ near $\partial\Omega$, we may assume $\Omega = \mathbb{R}^2$, and then L need not be considered. If we choose to neglect certain terms of the micromagnetic energy, this may of course reduce the number of parameters further. For instance, if we consider only the exchange energy and the magnetostatic energy (which we do in a substantial part of this paper), then the asymptotic regime depends on the behavior of the ratio d/L as we let the aspect ratio δ/L converge to 0. Some asymptotic regimes for this thin-film limit have been studied by Gioia and James [17]; Carbou [8]; DeSimone, Kohn, Müller, and Otto [11]; and Kohn and Slastikov [22, 23]. Another regime is discussed in this paper, first through a simplified model in two dimensions, then by an asymptotic analysis for the micromagnetic energy on Ω_δ for $\delta \searrow 0$. This theory also establishes a link to the theory of Ginzburg-Landau vortices, which were first studied by Bethuel, Brezis, and Hélein [4, 5].

A further dimensional reduction is made in the context of parametrized domain wall models, that we investigate in detail. Such models represent the basic building blocks within larger domain patterns or more complex domain wall structures. Of particular interest is the regime of weakly anisotropic (soft) thin films, where such transition layers significantly differ from those more common in phase transitions.

2 Domain Walls: Internal Structure and Dynamics

The primary phenomenon that one associates with magnetic pattern formation is the decomposition of a magnetic body into almost uniformly magnetized regions. The so-called magnetic domains are separated by thin transition layers, called domain walls, that interact in a complex network. The structure of such domain walls is among the central concerns of micromagnetic theory. While the analysis of domain walls is mathematically an interesting matter of its own, the physical relevance relies in the resulting mutual interaction having large impact on the global magnetic microstructure, especially when nonlocal effects dominate. In reduced thin-film theories, domain walls often emerge as line singularities while fine structural properties no longer have any effect. Breaking the resulting degeneracy by means of transparent selection principles rising from higher order contributions remains a major challenge. On the other hand, magnetic domain walls can exhibit internal substructures themselves or can be made up as a complex composite, such as the *cross-tie wall*, cf. [19] pp. 240-241.
The simplest domain wall patterns are one-dimensional and appear as extreme cases in a hierarchy of domain wall models that emerge in diverse parameter regimes: Within a *Bloch wall*, the magnetization vector performs a rotation perpendicular to the transition axis. The main feature is the avoidance of magnetic volume charges, so that this wall type is energetically favorable in bulk situations and essentially equivalent to the transition problem arising from Cahn-Hilliard models. Such models exhibit sharply localized and rapidly decaying transition profiles. Our analysis shows that this behavior can largely change when nonlocal interactions dominate and internal length scales fail to be determined by dimensional analysis. The *Néel wall*, where the transition proceeds in-plane, is preferred in suitable thin-film regimes and characterized by the avoidance of magnetic surface charge. The presence of three energy components with different scaling behavior gives rise to multiple length scales. The typical feature of a Néel wall is the very long logarithmic tail of transition profiles. Such behavior has been predicted by heuristic arguments and numerical simulation (cf. [39, 16]) in order to explain long-range interaction of Néel walls, when neighboring tails overlap. Here we demonstrate rigorously how the main analytical feature of the variational principle, a critical regularity property, gives rise to the typical logarithmic decay behavior [30, 31, 32]. This global approach served in addition to resolve the spatial scaling laws in terms of all involved parameters and to derive a somewhat universal limiting profile that reflects the decay.
The evolution of magnetic patterns in the presence of applied fields is closely related to the motion of domain walls. Gyrotropic domain wall motion is based on the Landau-Lifshitz-Gilbert (LLG) equations, that describe a damped precession of the magnetization vector about the effective field, i.e. mathematically a hybrid heat and Schrödinger flow for the free energy. An appropriate local description relies on the concept of moving fronts that propagate with constant speed. Traveling wave solutions for the associate LLG dynamics represent a natural dynamic counterpart to static domain walls. As it turns out they provide valuable insight into the mechanisms and properties of domain wall dynamics, where besides energetics and spatial structures, kinematic quantities as *wall mobility* and *wall mass* come into play.

Whereas in the equilibrium case the magnetization path is dictated by energetics and particularly stray-field interactions, a second mechanism effects the shape of a moving domain wall: the precession dynamics as prescribed by LLG pushes the magnetization vector away from its optimal path, taking into account a gain in stray-field energy. Many interesting effects originate from this balance of energetic and dynamic forces, especially when enhanced by strong shape anisotropy in the regime of thin films. The bulk situation, however, can surprisingly be solved explicitly by means of a famous construction by Walker, i.e. a tilted version of the Landau and Lifshitz solution for the standard Bloch wall, cf. [44]. Again all spatial and temporal scales can be read off by dimensional reasoning. The natural question on whether such a construction can be perturbed to the regime of finite layers has been answered in the affirmative [33]. Indeed, a suitable choice of canonical coordinates transforms the associate LLG system into a weakly coupled Schrödinger/reaction-diffusion system and makes it accessible for spectral methods. The analysis also demonstrates that the finite layer perturbation is indeed a singular one and how this relates to slow decay.
In the regime of thin films the competition between stray field and precession is singular, that is to say, the asymptotically hard constraint of in-plane magnetization is geometrically incompatible with LLG. In order to derive an effective evolution equation, the change of spatial scaling has to be accompanied with a change of times scale. An effective thin-film limit for LLG with finite Gilbert damping has been carried out in [15, 22] and in Theorem 12, where the gyromagnetic precession term effectively turns into a large damping term as well. While the overall relaxation dynamics is captured correctly, oscillatory phenomena, such as spin waves or domain wall resonances, are suppressed in such a limit. In order to account for these effects we consider the complementary regime where Gilbert damping is comparable to the relative thickness. We show that in this regime LLG keeps its oscillatory features and turns into a damped nonlocal wave map equation [7]. In the context of domain wall motion it provides a mechanical analogy and sheds some new light on the notion of wall mass. For small applied fields, the traveling wave problem, modeled on this wave-type dynamics, reduces to the question of linear stability for stationary Néel walls. Then the implicit function theorem provides existence and determines the mobility of traveling Néel walls.

2.1 Mathematical framework for planar domain walls

Let us consider an infinitely extended uniaxial magnetic film that is represented by $\Omega_\delta = \mathbb{R}^2 \times (0, \delta)$ and oriented by the anisotropy (easy) axis $\mathbb{R}\,\hat{\mathbf{e}}_\mathbf{2}$. We consider parameterized transitions along $\mathbb{R}\,\hat{\mathbf{e}}_\mathbf{1}$ (that we call transition axis) that connect antipodal states on the easy axis, i.e.

$$\mathbf{m} : \mathbb{R} \to \mathbb{S}^2 \quad \text{with} \quad \mathbf{m}(\pm\infty) = (0, \pm 1, 0).$$

In the following we denote the transition parameter by x and the vertical coordinate by z, i.e. we set $x_1 = x$ and $x_3 = z$. Under the hypothesis that, within the film, the magnetization varies only along the transition axis, we identify the associated global magnetization field $\mathbf{m}(\mathbf{x}) = \mathbf{m}(x)\chi_{(0,\delta)}(z)$ that is defined for $\mathbf{x} \in \mathbb{R}^3$. Then $\mathbf{m} = \mathbf{m}(\mathbf{x})$ induces the stray field ∇u determined by the potential equation $\Delta u = \nabla \cdot \mathbf{m}$ in $\mathcal{D}'(\mathbb{R}^3)$. We observe that $u = u(x,z)$ where the z dependence only stems from shape anisotropy. Thus the micromagnetic energy induces the following averaged domain wall energy per unit length:

$$E(\mathbf{m}) = \frac{1}{2} \fint_0^\delta \left\{ d^2 \int |\mathbf{m}'|^2 \, dx + \int \nabla u \cdot \mathbf{m} \, dx + Q \int (1 - m_2^2) \, dx \right\} \, dz. \quad (6)$$

The vertical average in (6) is redundant for the *exchange* and *anisotropy* portion. In order to perform a dimensional reduction for the *stray field*, we introduce a reduced stray-field operator

$$\mathcal{S}_\delta : \mathbf{m} \mapsto \fint_0^\delta \nabla u \, dz \quad \text{where} \quad \Delta u = \nabla \cdot \mathbf{m} \quad \text{in} \quad \mathcal{D}'(\mathbb{R}^3).$$

Changing the order of integration, the averaged stray-field energy can be expressed as

$$E_{\text{stray}}(\mathbf{m}) = \frac{1}{2} \fint_0^\delta \int \nabla u \cdot \mathbf{m} \, dx \, dz = \frac{1}{2} \int_{\mathbb{R}} \mathcal{S}_\delta(\mathbf{m}) \cdot \mathbf{m} \, dx.$$

A straightforward calculation shows that the operator $\mathcal{S}_\delta$ has an interpretation in terms of Fourier multiplication operators. Indeed, we have

$$\mathcal{S}_\delta(\mathbf{m}) = \Big[\sigma(\delta D) \, m_1 , 0 \, , \big(1 - \sigma(\delta D)\big) \, m_3 \Big] : \mathbb{R} \to \mathbb{R}^3,$$

where $\sigma(D) f = \mathcal{F}^* \Big(\xi \mapsto \sigma(\xi) \hat{f}(\xi) \Big)$. The basic Fourier multiplier $\sigma(\xi)$ is given by

$$\sigma(\xi) = \left(1 - \frac{1 - \exp(-|\xi|)}{|\xi|} \right) \sim \begin{cases} \frac{1}{2}|\xi| & \text{for low frequencies } \xi \\ 1 & \text{for high frequencies } \xi. \end{cases} \quad (7)$$

The reduced stray-field operator can equivalently be described by means of convolution kernel, cf. [16], [31] for a derivation and a detailed discussion. Accordingly, the reduced stray-field energy can be written as

$$E_{\text{stray}}(\mathbf{m}) = \frac{1}{2} \int \sigma(\delta\xi) \, |\hat{m}_1(\xi)|^2 \, d\xi + \frac{1}{2} \int \big(1 - \sigma(\delta\xi)\big) \, |\hat{m}_3(\xi)|^2 \, d\xi. \quad (8)$$

The advantage of the Fourier representation is that one can easily read off the asymptotic form of interaction from the asymptotic behavior of Fourier multipliers. From (8) one can separate a local contribution and a nonlocal one that vanishes in the bulk regime

$$E_{\text{stray}}(\mathbf{m}) = \frac{1}{2} \, \|m_1\|_{L^2}^2 + \frac{1}{2} \int \big(1 - \sigma(\delta\xi)\big) \Big\{ |\hat{m}_3(\xi)|^2 - |\hat{m}_1(\xi)|^2 \Big\} \, d\xi. \quad (9)$$

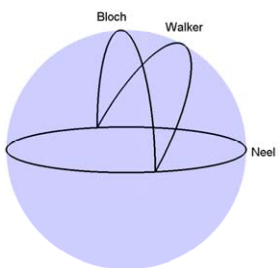
Indeed, from (7) we deduce that $\sigma(\delta \, \xi) \to 1$ in the regime when $\delta|\xi| \to \infty$. Thus, for a corresponding family of transitions $\mathbf{m}$ so that m_1 and m_3 are uniformly bounded in $L^2(\mathbb{R})$, we infer that

$$E_{\text{stray}}(\mathbf{m}) = \frac{1}{2} \, \|m_1\|_{L^2}^2 + o(1). \quad (10)$$

We observe that in the bulk regime, the stray-field interaction reduces to a local contribution having the form of an additional anisotropy term that penalizes magnetizations that point along the transition axis. In the complementary thin-film regime when $\delta|\xi| \to 0$ we have $\sigma(\delta\,\xi) \to \delta|\xi|$ and, for m_1 and m_3 uniformly bounded in $H^1(\mathbb{R})$, an asymptotic expansion

$$E_{\text{stray}}(\mathbf{m}) = \frac{1}{2}\,\|m_3\|_{L^2}^2 + \frac{\delta}{4}\,\|m_1\|_{\dot{H}^{1/2}}^2 + o(\delta), \tag{11}$$

where $\|f\|_{\dot{H}^{1/2}}^2 = \int |\xi||\hat{f}(\xi)|^2\,d\xi$ denotes the homogeneous $H^{1/2}$-norm. The zero order contribution can be interpreted as the residual surface charge interaction having the form of an additional anisotropy that penalizes vertical magnetizations. The first order term corresponds to residual volume charge interaction. From a variational point of view the leading order stray-field contribution in (10) and (11), respectively, determines asymptotically a geodesic magnetization path. Whereas in the bulk situation the stray field interaction can be eliminated completely by choosing a path perpendicular to the transition axis, i.e. $m_1 = 0$ (Bloch walls), the penalty on the vertical component as $\delta \to 0$ enforces in-plane rotations, i.e. $m_3 = 0$ (Néel walls), taking into account internal stray fields that typically appear to the leading order.



The figure shows the **Bloch** *and the* **Néel** *wall path as perpendicular geodesic connections of antipodal states. It also shows an intermediate geodesic that corresponds to a moving domain wall in the bulk regime where dynamic forces lead to an inclination towards the Néel wall path, the so-called* **Walker** *path. We will refer to the polar angle φ between the Bloch wall and the Walker path as the Walker angle.*

Complete elimination of stray-field interaction cannot be achieved by means of one-dimensional transition modes. In somewhat thicker films, however, the symmetric Néel wall would lead to comparatively large stray-field contribution. But for an attempt to construct a stray-field free transition layer one has to abandon the symmetry assumption and to permit variations in the vertical direction. Such an object, referred to as **asymmetric Bloch wall**, has been discovered by Hubert, cf. [19] pp. 245-249, where volume charges are avoided by a vortex construction in the wall center. At the same time numerical simulations have confirmed a dramatic decrease of energy by breaking the wall symmetry. Recently, a rigorous verification based on an ansatz-free interpolation argument has been provided by Otto in [38].

2.1.1 Bloch walls versus Néel walls

The infinite Bloch wall in bulk samples has been the first micromagnetic object proposed and calculated in the seminal work by Landau and Lifshitz, cf. [27]. Once the stray-field energy is fully eliminated by choosing an appropriate path, the corresponding optimal profile and minimal energy can be found by nowadays standard variational methods. Indeed, from (6) we get for $\mathbf{m} = (0, m_2, m_3)$:

$\mathbb{R} \to \mathbb{S}^2$ with $m_2(\pm\infty) = \pm 1$,

$$E(\mathbf{m}) = \frac{d^2}{2} \int |\mathbf{m}'|^2 \, dx + \frac{Q}{2} \int (1 - m_2^2) \, dx.$$

The length scale $w = \sqrt{d^2/Q}$ defines the typical Bloch wall domain width. Rescaling by w and renormalizing the energy by the factor $4\sqrt{d^2 Q}$ yields

$$E(\mathbf{m}) = \frac{1}{4} \int |\mathbf{m}'|^2 \, dx + \frac{1}{4} \int (1 - m_2^2) \, dx.$$

Using the identity $|\mathbf{m}'|^2 = (m_2')^2/(1 - m_2^2)$ we deduce the optimality relation

$$m_2' = 1 - m_2^2 \quad \text{with} \quad m_2(\pm\infty) = \pm 1 \tag{12}$$

that is uniquely solved by $m_2(x) = \tanh(x)$. Thus in the original scaling

$$\mathbf{m}(x) = \big(0, \tanh(x/w), \operatorname{sech}(x/w)\big). \tag{13}$$

Moreover we deduce from (12) and Young's inequality that

$$E(\mathbf{m}) \geq \frac{1}{2} \int |m_2'| \, dx \geq 1,$$

that is attained under equipartition for $m_2(x) = \tanh(x)$. Thus we recover from scaling that the Bloch wall energy per unit length is given by $e_0 = 4\sqrt{d^2 Q}$. One may wonder whether the Bloch wall path is indeed optimal; but this is a simply consequence of $|\mathbf{m}'|^2 \geq (m_2')^2/(1-m_2^2)$ that holds true for any $\mathbf{m} : \mathbb{R} \to \mathbb{S}^2 \in H^1$ and Young's inequality that imply the same lower energy bound.
Twenty years later, Louis Néel realized that in a regime where the film thickness becomes comparable to the Bloch wall width, a transition mode within the film plane can lower the total energy decisively, cf. [37]. New ideas, however, had to be developed in order to provide a satisfactory analysis of this multiscale object. Indeed, for an in-plane rotation $m : \mathbb{R} \to \mathbb{S}^1$ the thin-film approximation of (6) yields

$$E(\mathbf{m}) = \frac{d^2}{2} \int |m'|^2 \, dx + \frac{\delta}{4} \int |\xi| |\hat{m}_1(\xi)|^2 \, d\xi + \frac{Q}{2} \int |m_1|^2 \, dx. \tag{14}$$

Unlike the Bloch wall problem where only two energy components remain that can be balanced by a single length scale, the Néel wall problem incorporates two characteristic length scales. Those are connected with the competition of two energy components, respectively: In order to highlight the competition between *stray field* and *anisotropy* we rescale by the tail width $w = \delta/(2Q)$. With the small aspect ratio $\mathcal{Q} = 4\kappa^2 Q$ where $\kappa = d/\delta \simeq 1$, we obtain the following singular perturbation problem

$$E_{\mathcal{Q}}(m) = \frac{\mathcal{Q}}{2} \|m\|_{\dot{H}^1}^2 + \frac{1}{2} \|u\|_{\dot{H}^{1/2}}^2 + \frac{1}{2} \|u\|_{L^2}^2 \to \min \tag{15}$$
$$m = (u, v) : \mathbb{R} \to \mathbb{S}^1 \quad \text{with} \quad u(0) = 1$$

that captures the logarithmic decay behavior as we will see below. There is a second characteristic length scale that is smaller than the tail width and related to the width of the core in the very center of the Néel wall. In the above regime it coincides merely with the exchange length d, and rescaling yields an expression

$$\frac{1}{2}\|m\|^2_{\dot H^1}+\frac{1}{4\kappa}\|u\|^2_{\dot H^{1/2}}+\frac{Q}{2}\|u\|^2_{L^2}$$

that, as Q tends to zero, highlights the competition between *exchange* and reduced *stray-field* energy.

2.2 The logarithmic tail of Néel walls

The main analytical feature of the variational problem (15) is that the energy gives only uniform control of the $H^{1/2}$-norm as $\mathcal{Q}$ tends to zero. Since the $H^{1/2}(\mathbb{R})$ norm just fails to control the modulus of continuity, the pointwise constraint $u(0) = 1$ is delicate and one might expect a logarithmic singularity in a renormalized setting. Logarithmic tails of Néel walls have indeed been predicted by heuristic arguments and the resulting very long range interaction between different Néel walls has important consequences, cf. [19] pp. 242-245, [39] with some extensions in [16]. Logarithmic scaling for the energy has recently been established in [16] and the following refined version is announced in [10]:

Theorem 1. *As $\mathcal{Q}$ tends to zero the minimal energies behave like*

$$\inf E_{\mathcal{Q}}=\frac{\pi}{2}\big(1+o(1)\big)\ln(1/\mathcal{Q})^{-1}$$

where the infimum is taken over transitions that are admissible according to (15).

Similar scaling laws have been derived in the case of periodic Néel wall arrays, where tails are confined by those of neighboring walls, cf. [13]. They were used to heuristically quantify their mutual repulsive force, that is particularly interesting in the context of optimal spacing for the cross-tie wall.
It is remarkable that, in case of finite Néel walls, the above energy asymptotics in Theorem 1 holds true when the infimum is taken over y-periodic transitions $m = m(x, y)$, cf. [9]. Here, the quality factor Q is replaced by the aspect ratio δ/w, i.e. film thickness by tail width, so that $\mathcal{Q} = 4\,\kappa^2\,\frac{\delta}{w}$. The proof is based on a dynamic system argument and a sharp interpolation inequality between L^∞ and BV. The result proves in particular (nonlinear) stability of the one-dimensional Néel wall with respect to two-dimensional variations in the plane, a result yet unknown for infinite Néel walls.

The proof of Theorem 1 is based on a perturbation argument; its shows that minimal energies exhibit the same asymptotics as the minimal energies for the relaxed problem (17) to be introduced below. It turns out to provide an pointwise logarithmic lower bound as well and motivates the main result [31, 32]:

Theorem 2. *Let $u_{\mathcal{Q}}$ be a minimizing profile for the variational principle (15). Then $u_{\mathcal{Q}}$ is symmetric-decreasing and exhibits a logarithmic tail in the sense that*

$$u_{\mathcal{Q}}(x)\simeq\frac{\ln(1/x)}{\ln(1/\mathcal{Q})}\quad \textit{for all}\ \ \mathcal{Q}\lesssim x\lesssim 1\ \ \textit{and}\ \ 0<\mathcal{Q}<1/4.$$

The notations $a \lesssim b$ and $a \simeq b$ mean that, for some universal constant $0 < c < \infty$ we have $a \le c\, b$ and $\frac{1}{c} b \le a \le c\, b$, respectively. It can also be shown that beyond the logarithmic tail a Néel wall profile decays only quadratically $\ln(1/\mathcal{Q})\, u_{\mathcal{Q}}(x) \simeq x^{-2}$ as $|x| \to \infty$, cf. [30], a fact that is related to the limited regularity of associated Fourier multipliers. Renormalization yields in addition a universal limiting profile that captures the essential decay behavior:

Theorem 3. *For any sequence $\mathcal{Q} \to 0$ so that the corresponding sequence of renormalized profiles $U_{\mathcal{Q}} = \ln(1/\mathcal{Q})\, u_{\mathcal{Q}}$ converges in the sense of distributions, the weak limit U_0 is a multiple of the fundamental solution of the operator $(-\Delta)^{1/2} + 1$, and the convergence is strong in L^2_{loc}.*

For fractional derivatives of order $s > 0$ we use the notation $(-\Delta)^{s/2} f = \mathcal{F}^*(\xi \mapsto |\xi|^s \hat{f}(\xi))$, where, for $s = 2$, we have $(-\Delta) f = -\frac{d^2 f}{dx^2}$. The fundamental solution $G \in L^2(\mathbb{R})$ with $(-\Delta)^{1/2} G + G = \delta_0$ is well known as a Fourier integral. It is smooth away from the origin, symmetrically decreasing and has the following expansion:

$$G(x) = \frac{1}{\pi} \begin{cases} \ln(1/|x|) - \gamma + \mathcal{O}(x) & \text{as } |x| \to 0 \\ x^{-2} + \mathcal{O}(x^{-4}) & \text{as } |x| \to \infty, \end{cases} \tag{16}$$

where γ denotes Euler's constant.

2.2.1 Logarithmic lower bounds

We introduce a linear comparison problem arising from relaxation that can be solved explicitly. We have shown in [32] that relaxation of (15) leads to

$$E^*_{\mathcal{Q}}(u) = \frac{\mathcal{Q}}{2} \|u\|^2_{\dot{H}^1} + \frac{1}{2} \|u\|^2_{\dot{H}^{1/2}} + \frac{1}{2} \|u\|^2_{L^2} \to \min \quad \text{in} \quad \{u(0) = 1\}. \tag{17}$$

A standard convexity argument implies the existence of a unique minimizer that satisfier the Euler-Lagrange equation

$$\mathcal{Q}(-\Delta) u^* + (-\Delta)^{1/2} u^* + u^* = \Lambda(\mathcal{Q})\, \delta_0 \quad \text{in} \quad \mathcal{D}'(\mathbb{R}). \tag{18}$$

Expanding the associate Fourier multiplier into partial fractions, (18) can be solved in terms of the fundamental solution (16), so that the following properties can be read off:

Proposition 1. *The unique solutions $u^*_{\mathcal{Q}}$ of the relaxed variational principles (17) exhibit logarithmic tails in the sense that for $0 < \mathcal{Q} < 1/4$:*

$$u^*_{\mathcal{Q}}(x) = \frac{\Lambda(\mathcal{Q})}{\pi} \big(1 + o(1)\big) \big[\ln(1/x) + r_{\mathcal{Q}}(x)\big] \quad \textit{for} \quad \mathcal{Q} < x < 1/e,$$

where the functions $r_{\mathcal{Q}}(x)$ are uniformly bounded in the above regime. The Lagrange multiplier $\Lambda(\mathcal{Q})$ agrees with twice the minimal energy and has the asymptotic behavior (cf. Theorem 1)

$$\Lambda(\mathcal{Q}) = \pi\big(1 + o(1)\big) \ln(1/\mathcal{Q})^{-1}.$$

Surprisingly, the relaxed variational principle not only provides an energetic but also a pointwise lower bound, thus, in view of Proposition 1, a logarithmic lower bound.

Proposition 2. *Let $u_\mathcal{Q}$ be a minimizing profile for variational principle (15). Then the solution $u^*_\mathcal{Q}$ of the relaxed variational principle (17) is a pointwise lower bound.*

Proof. The idea is to derive suitable pseudo-differential inequalities for the profiles to be compared. We observe that $|m'|^2 = |u'|^2/(1-u^2)$ and deduce the following Euler-Lagrange equation

$$\mathcal{Q}\left\{-\frac{d}{dx}\left(\frac{u'}{1-u^2}\right) + \left(\frac{u'}{1-u^2}\right)^2 u\right\} + (-\Delta)^{1/2}u + u = 0 \tag{19}$$

that holds true for a Néel wall profile u in $\{u \neq 1\}$. By Proposition 4 this equation holds true in $\mathbb{R}\setminus\{0\}$. Now the essential ingredients are symmetric convexity of comparison profiles stated in Proposition 1 and the following global maximum principle for the nonlocal field operator $(-\Delta)^{1/2}$:

Lemma 1. *Suppose that the function $u \in H^1(\mathbb{R})$ is smooth in $\mathbb{R}\setminus\{0\}$ and that u attains a global maximum at $x_0 \neq 0$. Then $(-\Delta)^{1/2}u$ is smooth in a neighborhood of x_0 and $(-\Delta)^{1/2}u(x_0) \geq 0$.*

We consider $w = u^* - u \in H^1(\mathbb{R})$ with $w(0) = 0$. From (18) and (19) we deduce, with the positive coefficient $a(x) = \mathcal{Q}/(1-u^2(x))$, that

$$a(x)(-\Delta)w + (-\Delta)^{1/2}w + w \leq 0 \quad \text{in} \quad \mathbb{R}\setminus\{0\}.$$

Since w is smooth away from the origin, the Lemma 1 applies and excludes a global maximum in $\mathbb{R}\setminus\{0\}$. But from Propositions 1 and 4 we infer that $w(x)$ decays as $|x|\to\infty$, and the proof is complete. □

2.2.2 Logarithmic upper bounds

The key observation is that logarithmic upper bounds are captured by sharp elliptic regularity bounds that are uniform in $\mathcal{Q}$. For magnetizations $m_\mathcal{Q}$ of bounded Néel wall energy (15) we have $\|u_\mathcal{Q}\|^2_{H^{1/2}} \leq E_\mathcal{Q}(m_\mathcal{Q})$. Then Sobolev embedding implies, for any $p \in (2,\infty)$, a bound $\|u_\mathcal{Q}\|^2_{L^p} \leq c(p)E_\mathcal{Q}(m_\mathcal{Q})$. A PDE argument, however, shows that for any such p the purely energetic argument misses the optimal scaling by a full factor $E_\mathcal{Q}(m_\mathcal{Q})$ and provides in addition an estimate on the growth of optimal constants. Qualitatively, the same is true for fractional Sobolev norms $H^{1/2}_q$ that are strictly weaker than $H^{1/2}$.

Proposition 3. *For a critical point $m_\mathcal{Q} = (u_\mathcal{Q}, v_\mathcal{Q})$ of (15) we have*

$$\big\|u_\mathcal{Q}\big\|_{L^p} \leq c\, p\, E_\mathcal{Q}(m_\mathcal{Q}) \quad \textit{for each} \quad p \in (1,\infty) \tag{20}$$

for some universal constant $c > 0$ and

$$\big\|u_\mathcal{Q}\big\|_{H^{1/2}_q} \leq c(q)\; E_\mathcal{Q}(m_\mathcal{Q}) \quad \textit{for each} \quad q \in (1,2), \tag{21}$$

where the constant $c(q) > 0$ only depends on q.

Proof. We outline the main steps: Projection of the Euler-Lagrange system

$$\nabla E_\mathcal{Q}(m) = \nabla E_\mathcal{Q}(m)\; m \otimes m \tag{22}$$

onto its first component equation yields, after a suitable decomposition of the non-linearity, an equation of the form

$$\mathcal{Q}\,(-\Delta)u + (-\Delta)^{1/2}u + u = e_{\mathcal{Q}}[m]\,u + r[u], \tag{23}$$

where $e_{\mathcal{Q}}[m] = \mathcal{Q}\,|m'|^2 + |(-\Delta)^{1/4}u|^2 + |u|^2$ is twice the energy density and $r[u]$ is a defect distribution arising from the in compatibility of nonlocal interaction and the geometric constraint $|m| = 1$. In fact, for any test function φ, we have

$$\big\langle r[u], \varphi \big\rangle = \Big\langle (-\Delta)^{1/4}u, \big[(-\Delta)^{1/4}, (u\,\varphi)\big]\,u \Big\rangle.$$

The operator on the left hand side of (23) is uniformly first-order elliptic, while the right hand side is essentially L^1-bounded by the energy, and in that case the claim would follow from a simple Fourier argument. Commutator estimates, however, show that uniform bounds for $r[u]$ are slightly weaker than L^1 and rather distributional, i.e in $H_q^{-1/2}$ for $q \in (1, 2)$. By means of elliptic regularity theory we get a uniform bounds in $H_q^{1/2}$, and the claim follows from asymptotic inequalities for fractional integration. □

From strict rearrangement inequalities that are valid for fractional Sobolev norms and a simple bootstrap argument based on the Euler-Lagrange system (22), we deduce the following symmetry and smoothness result for optimal profiles $u = u_{\mathcal{Q}}$.

Proposition 4. *A Néel wall profile is smooth and symmetrically decreasing.*

Remark 1. *The above proposition is strict in the sense that a Néel wall profile $m = e^{i\theta}$ cannot have a plateau at 0, and the associate phase function θ is strictly increasing, cf. Lemma 2.*

Proof of Theorem 2

In view of Propositions 1 and 2 it remains to prove the logarithmic upper bound. Let $u = u_{\mathcal{Q}}$ be a Néel wall profile. Proposition 4 implies that the pointwise values are below the local averages. Thus Hölder's inequality and Proposition 3 yield

$$0 \le u(x) \le \fint_0^x u\;dy \le \left(\fint_0^x |u|^p\;dy\right)^{1/p} \le c\,p\,\left(\frac{1}{x}\right)^{1/p}\,\inf_{\mathcal{M}} E_{\mathcal{Q}}$$

that is a family of upper bounds parameterized by p. The pointwise optimal choice of p, given by $p(x) = \ln(1/x)$, and Theorem 1 yield the logarithmic upper bound.

Proof of Theorem 3

From Theorem 1 and Proposition 3 we deduce that $U_{\mathcal{Q}}$ is uniformly bounded in, say, $H^{1/2}_{3/2}(\mathbb{R})$, so we can assume that $U_{\mathcal{Q}} \rightharpoonup U_0$ weakly in $L^2(\mathbb{R})$ and strongly on bounded intervals. From the profile equation (23) we get

$$\mathcal{Q}\,(-\Delta)U_{\mathcal{Q}} + (-\Delta)^{1/2}U_{\mathcal{Q}} + U_{\mathcal{Q}} = \ln(1/\mathcal{Q})\Big(e_{\mathcal{Q}}[m_{\mathcal{Q}}]\,u_{\mathcal{Q}} + r[u_{\mathcal{Q}}]\Big).$$

Obviously, the left hand side converges to $(-\Delta)^{1/2}U_0 + U_0$ in the sense of distributions. Thus it remains to show that

(i) the distribution $(-\Delta)^{1/2}U_0 + U_0$ is supported at the origin

(ii) the distribution $\ln(1/\mathcal{Q})r[u_\mathcal{Q}]$ converges to a finite measure.

Claim (i) can be deduced from (19) using the uniform pointwise convergence of Néel wall profiles to zero away from the origin. Claim (ii) follows from an iteration of the arguments in of Proposition 3. Indeed, for $u = u_\mathcal{Q}$ we have a decomposition of the remainder distribution $\langle r[u], \varphi\rangle = \langle f[u], \varphi\rangle + \langle g[u], \varphi\rangle$, where

$$\begin{aligned}\big\langle f[u], \varphi\big\rangle &= \big\langle (-\Delta)^{1/4}u, [(-\Delta)^{1/4}, u^2]\varphi\big\rangle, \\ \big\langle g[u], \varphi\big\rangle &= \big\langle (-\Delta)^{1/4}u, u^2(-\Delta)^{1/4}\varphi\big\rangle.\end{aligned}$$

Product and commutator estimates for fractional derivatives (cf. [18, 21]) and the energy estimate in Theorem 1 show that

$$\big|\big\langle f[u], \varphi\big\rangle\big| \le c\, \big\|u\big\|_{L^\infty} \big\|u\big\|^2_{H^{1/2}} \big\|\varphi\big\|_{C^0} \le c\, \ln(1/\mathcal{Q})^{-1}\, \big\|\varphi\big\|_{C^0}.$$

Thus we find that the contribution coming from $\ln(\mathcal{Q})\, f[u]$ is asymptotically a finite measure. On the other hand, according to the $H^{1/2}_q$ estimate in Proposition 3,

$$\big|\big\langle g[u], \varphi\big\rangle\big| \le c\, \big\|u\big\|_{L^\infty} \big\|u\big\|_{L^6} \big\|u\big\|_{H^{1/2}_{3/2}} \big\|\varphi\big\|_{H^{1/2}_6} \le c\, \ln(1/\mathcal{Q})^{-2}\, \big\|\varphi\big\|_{H^{1/2}_6},$$

so that the contribution from $\ln(\mathcal{Q})g[u]$ vanishes as a distribution as $\mathcal{Q}$ tends to 0.

2.3 Domain wall motion in finite layers

When an external magnetic field $\mathbf{h} = H\,\hat{\mathbf{e}}_2$ is applied that points towards the easy axis, the end-states are no longer equally preferred. Consequently, one expects the domain wall to become unstable and start to move. In gyrotropic domain wall models the evolution of magnetization distributions is characterized by the Landau-Lifshitz-Gilbert equation

$$\begin{aligned}\mathbf{m} \wedge \partial_t \mathbf{m} + \alpha\, \partial_t \mathbf{m} + \gamma\, (1 - \mathbf{m} \otimes \mathbf{m})\nabla E(\mathbf{m}) = (1 - \mathbf{m} \otimes \mathbf{m})\, \mathbf{h} \qquad (24)\\ \mathbf{m} : \mathbb{R} \times (0, \infty) \to \mathbb{S}^2 \quad \text{with} \quad \mathbf{m}(\pm\infty, t) = (0, \pm 1, 0) \quad \text{for} \quad t \in (0, \infty).\end{aligned}$$

where $E(\mathbf{m})$ is the internal domain wall energy. We introduce the aspect ratio $\kappa = d/\delta$ and assume for simplicity that $Q = 1$. Renormalizing space and energy by the exchange length d, we get from (6) and (9) an (internal) domain wall energy of the form

$$E_\kappa(\mathbf{m}) = \frac{1}{2}\int |\mathbf{m}'|^2 + \frac{1}{2}\int (1 - m_2^2)\, dx + \frac{1}{2}\int m_1^2\, dx + G_\kappa(\mathbf{m}), \qquad (25)$$

where the nonlocal portion of the stray-field energy is given by

$$G_\kappa(\mathbf{m}) = \frac{1}{2}\int \big(1 - \sigma(\kappa\xi)\big)\Big(|\hat{m}_3(\xi)|^2 - |\hat{m}_1(\xi)|^2\Big)\, d\xi. \qquad (26)$$

Regarding the dynamic problem, a special class of solutions to constant coefficient systems are traveling wave solutions, i.e. solutions of the form $\mathbf{m} =$

$\mathbf{m}(x + c\,t)$, that describe a motion of constant speed c. In the case $\kappa = 0$ the traveling wave ansatz turns (24) into a constrained nonlinear system of ordinary differential equations. Surprisingly, these equations can be solved explicitly. The solutions are referred to as Walker's exact solutions, see [44, 19]. Our goal is to show that this situation is indeed generic and can be perturbed to layers of large but finite diameter. As this corresponds to the case of small nonlocal interaction, the nonlocal character of the equations will play a minor role. We will show that, in suitable coordinates, domain wall motion according to Landau-Lifshitz dynamics fits into the context of nonlocal, weakly coupled reaction-diffusion systems. But first we review Walker's construction.

2.3.1 Walker's exact solutions

For $\kappa = 0$ we have a transition energy

$$E_0(\mathbf{m}) = \frac{1}{2}\int |\mathbf{m}'|^2\,dx + \frac{1}{2}\int (1 - m_2^2)\,dx + \frac{1}{2}\int m_1^2\,dx.$$

The traveling wave ansatz $\mathbf{m} = \mathbf{m}(x + c\,\gamma\,t)$ yields the system

$$c\,\alpha\,\mathbf{m}' + c\,\mathbf{m}\wedge\mathbf{m}' + \big(1 - \mathbf{m}\otimes\mathbf{m}\big)\nabla E_0(\mathbf{m}) = \big(1 - \mathbf{m}\otimes\mathbf{m}\big)\cdot\mathbf{h}. \tag{27}$$

The above system can be solved explicitly, a calculation that has been first carried out by Walker. The original calculations become more transparent, when the equation is considered in the canonical orthogonal frame $\{\mathbf{m}', \mathbf{m}\wedge\mathbf{m}'\}$ on the tangent space of $\mathbb{S}^2$ along $\mathbf{m}$. It turns out that the following assumptions can be met:

(I) Under the assumption that dissipation compensates the driving force, the system decomposes into three equations,

$$c\,\alpha\,|\mathbf{m}'|^2 = H\,m_2', \quad \nabla E_0(\mathbf{m})\cdot\mathbf{m}' = 0, \quad \text{and} \quad \nabla E_0(\mathbf{m})\cdot\mathbf{m}\wedge\mathbf{m}' = c\,|\mathbf{m}'|^2.$$

(II) Under the assumption that the wall moves with constant polar inclination angle φ, i.e. $\mathbf{m}\wedge\mathbf{m}' = |\mathbf{m}'|\nu$ for some constant unit vector ν, the energy is the Bloch wall energy with increased anisotropy $Q(\varphi) = 1 + \sin^2\varphi$,

$$E_0(\mathbf{m}) = \frac{1}{2}\int |\mathbf{m}'|^2\,dx + \frac{Q(\varphi)}{2}\int (1 - m_2^2)\,dx.$$

Moreover $|\mathbf{m}'|^2 = (m_2')^2/(1 - m_2^2)$ and $\frac{\partial\mathbf{m}}{\partial\varphi} = |\mathbf{m}'|\nu$ holds for such $\mathbf{m}$.

We deduce that, up to scaling, all equations in (I) have the form $|\mathbf{m}'|^2 = 1 - m_2^2$, i.e. $m_2' = 1 - m_2^2$, and can be solved jointly. Matching parameters gives the transition profile, inclination angle, and propagation speed

$$m_2(x) = \tanh\Big[\sqrt{1 + \sin^2\varphi}\,x\Big], \quad \sin(2\varphi) = \frac{H}{\alpha}, \quad \text{and} \quad c = \frac{\sin(2\varphi)}{2\sqrt{1 + \sin^2\varphi}}. \tag{28}$$

Obviously a peak velocity ~ 0.4 (i.e. about $\sim 100\frac{m}{sec}$ for a typical garnet material) is reached at for finite field-strength H beyond which the construction breaks down.

Walker's construction shows that the dynamics of domain walls in bulks samples is accompanied by a decrease of domain wall width by a factor $(1+\sin^2\varphi)^{-\frac{1}{2}}$ and an increase of domain wall energy by the inverse factor. From Walker's construction we deduce that the first order correction for domain wall energies vanishes at small velocities:

$$e_H = e_0\,(1+\sin^2\varphi)^{\frac{1}{2}} = e_0 + \frac{1}{2}\,M\,c^2 + o\left(c^2\right).$$

The second order correction can be viewed as a kinetic energy contribution. Thus, the factor $M = \partial_c^2 e_H$ at $c = 0$ is referred to as the wall mass, a notion that has been introduced by Döring, cf. [14]. In the original scaling the wall mass is given by $M = e_0/(2d^2)$. We will encounter the wall mass again later in the context of traveling Néel walls rising naturally from a wave-type interpretation of Landau-Lifshitz-Gilbert dynamics in thin films.

2.3.2 Stability and perturbation of Walker's construction

Theorem 4. *For sufficiently small field strength H there is a threshold $\kappa(H) > 0$ such that, whenever $\kappa < \kappa(H)$, there is a traveling wave for the Landau-Lifshitz-Gilbert dynamics that connects antipodal states.*

Proof. We perform a stability analysis based on a suitable choice of canonical coordinates that transforms (24) into a weakly coupled 2×2 system of reaction-diffusion type. For this purpose we combine standard stereographic coordinates with a polar rotation by the Walker angle φ that maps the Walker path into the Bloch wall path. In these coordinates $\mathbb{C} \ni z \mapsto \mathbf{m}[z] \in \mathbb{S}^2$, the Walker path is given by the line segment $z_0 : \mathbb{R} \to \{0\} \times [-1, 1]$. For $\mathbf{m} = \mathbf{m}[z]$, functional gradients transform according to $\lambda^{-2}(z)\nabla_z E_\kappa(\mathbf{m}) = \nabla E_\kappa(\mathbf{m})$, where $\lambda(z)$ is the conformal factor. Moreover, (24) becomes a damped Schrödinger equation

$$\alpha\,\partial_t z + i\,\partial_t z - \gamma\,D_x\partial_x z + \gamma\,\Big(f(H,z) + b(\kappa,z)\Big) = 0, \tag{29}$$

where $z(\cdot,t) : \mathbb{R} \to \mathbb{C}$ with $z(\pm\infty,t) = (0,\pm 1)$. The mapping

$$f(H,z) = \lambda^{-2}(z)\,\nabla_z \int \big[\frac{1}{2}(m_1^2 - m_2^2) - H\,m_2\big]\,dx \quad \text{for} \quad \mathbf{m} = \mathbf{m}[z]$$

involves anisotropy, applied field, and the limiting (local) portion of stray-field interaction. For small enough H, it is bi-stable in its second component. With the notation in (26), the map $b(\kappa,z) = \lambda^{-2}(z)\,\nabla_z G_\kappa(\mathbf{m})$ is a nonlocal perturbation from stray-field interaction, that is continuous in κ with $b(0,\cdot) = 0$.

Remark 2. *An important observation is that the mapping $\kappa \mapsto b(\kappa,\cdot)$, considered as a family of nonlinear operators on suitable function spaces, is not differentiable at $\kappa = 0$. Indeed,*

$$\frac{d}{d\kappa} E_{\text{stray}}(\mathbf{m})\big|_{\kappa=0} = \frac{1}{2}\|m_1\|^2_{\dot{H}^{-1/2}} - \frac{1}{2}\|m_3\|^2_{\dot{H}^{-1/2}}$$

with singular behavior at low frequencies that conflicts with slow decay properties in the presence of internal stray fields. The perturbation at $\kappa \sim 0$ is therefore singular, and only continuous versions of the implicit function theorem are at our disposal.

Finally,

$$D_x \partial_x z = \partial_x^2 z + \Gamma(z)\langle \partial_x z, \partial_x z \rangle$$

denotes the second covariant derivative. Surprisingly, the form of Walker solutions remains almost unchanged:

$$z_0(x) = \left(0, \tanh\left[\frac{1}{2}\left(1+\sin^2\varphi\right)^{1/2} x\right]\right).$$

Recall that the associate propagation speed c_0 inherits the Walker angle φ, so for each H we identify the Walker solution with the pair (z_0, c_0). Introducing a moving frame $x \mapsto x + c\,\gamma\, t$, (29) reads like

$$G\big((z,c),\kappa\big) = -D_x \partial_x z + c(\alpha + i)\partial_x z + f(H,z) + b(\kappa, z) = 0.$$

It turns out that our choice of canonical stereographic coordinates provides an almost triangulation for the linearized problem. Its spectral properties can be summarized as follows:

Proposition 5. *For sufficiently small field strength H the linearization at the Walker solution (z_0, c_0), has the form*

$$\frac{\partial G}{\partial z}\big((z_0,c_0),0\big) = \begin{bmatrix} L_1 & M_2 \\ M_1 & L_2 \end{bmatrix} : H^2(\mathbb{R};\mathbb{C}) \to L^2(\mathbb{R};\mathbb{C}),$$

where $L_1 : H^2(\mathbb{R}) \to L^2(\mathbb{R})$ has a bounded inverse while L_2 and its L^2-adjoint have zero as a simple eigenvalue with eigenfunctions $v_0' = Im\, z_0'$ and ψ_0, respectively, so that the integral $\int v_0\, \psi_0 \, dx > 0$ exists. Moreover, $\|M_1\|$ can be made arbitrarily small by choosing H small.

Since $z_0'(x) = \frac{d}{d\lambda}|_{\lambda=0} z_0(x+\lambda)$ can be seen as the infinitesimal generator of translation symmetry, the proposition suggests that degeneracy only stems from translation invariance. Thus we introduce the extended functional equation

$$\mathcal{G}\big((z,c),\kappa\big) = \Big[G\big((z,c),\kappa\big), \operatorname{Im} z(0)\Big] = (0,0). \tag{30}$$

Its linearization with respect to (z,c) at the Walker solution (z_0, c_0) has the form

$$\mathcal{L}_0 = \begin{bmatrix} L_1 & M_2 & -v_0' \\ M_1 & L_2 & \alpha\, v_0' \\ 0 & \delta_0 & 0 \end{bmatrix}.$$

In view of Proposition 5, the invertibility of $\mathcal{L}_0 : H^2(\mathbb{R};\mathbb{C}) \times \mathbb{R} \to L^2(\mathbb{R};\mathbb{C}) \times \mathbb{R}$ for sufficiently small H would follow from a Schur-type argument once we have shown invertibility of the 2×2 matrix on the lower right. But this follows from a standard Fredholm argument (cf. the proof of Theorem 6), taking into account Proposition 5 and the positivity of v_0'. Now the continuous version of the implicit function theorem implies the solvability of (30) for sufficiently small $\kappa > 0$. □

2.4 Domain wall motion in thin films

2.4.1 A wave-type limit for Landau-Lifshitz-Gilbert

Gyromagnetic precession is geometrically incompatible with the asymptotic constraint of in-plane magnetization that is imposed by stray-filed interaction, in other words, the competition between energetic and dynamic forces becomes singular in a thin-film limit. Thus domain wall motion in thin films should be governed by a suitable effective limit for LLG

$$\partial_t \mathbf{m} + \gamma\, \mathbf{m} \wedge \nabla E(\mathbf{m}) + \alpha\, \mathbf{m} \wedge \partial_t \mathbf{m} = 0 \tag{31}$$

as the relative thickness δ/d tends to zero. We recall that Gilbert damping α is a small parameter as well, that is to say, precession proceeds much faster than relaxation. Prior work on thin-film reductions for LLG, leading to enhanced dissipation, cf. [15, 22] and Theorem 12, consider the regime when $\delta/d \ll \alpha$. In order to preserve the oscillatory features of LLG dynamics we take into account small Gilbert damping as well. As it turns out, the effective dynamics depends on asymptotic regime as α and the relative thickness δ/d tend to zero. Rescaling space by the tail width $w = \delta/(2Q)$ and energy by the quality factor, we get, for 3-dimensional transitions $\mathbf{m} = (m, m_3) : \mathbb{R} \to \mathbb{S}^2$, a domain wall energy of the form

$$E_\varepsilon(\mathbf{m}) = E_0(m) + \frac{\mathcal{Q}}{2} \int |m_3'|^2\, dx + \frac{1}{2\varepsilon^2} \|m_3\|_{L^2}^2, \tag{32}$$

where $\mathcal{Q} = 4\,\kappa^2 Q$ and $\varepsilon^2 = Q$. The in-plane portion of the energy $E_0(m)$ is given by

$$E_0(m) = \frac{\mathcal{Q}}{2} \int |m'|^2 + \frac{1}{2} \left\| m_1 \right\|_{\dot{H}^{1/2}}^2 + \frac{1}{2} \int (1 - m_2^2)\, dx. \tag{33}$$

For in-plane magnetizations, it agrees with the standard Néel wall energy that we have considered before. We investigate the regime when $\varepsilon \to 0$ while $\mathcal{Q}$ is uniformly bounded from above and below; in other words $\varepsilon \sim \delta/d$ can be considered as a relative thickness. Let us consider the associated LLG equation in the asymptotic regime when $\alpha(\varepsilon)/\varepsilon \to \nu$. Rescaling time by ε/γ, the system (31) becomes

$$\mathbf{m}_t + \varepsilon\, \mathbf{m} \wedge \nabla E_\varepsilon(\mathbf{m}) + \alpha\, \mathbf{m} \wedge \mathbf{m}_t = 0. \tag{34}$$

Theorem 5. *Let $\mathbf{m}_\varepsilon : \mathbb{R} \times (0, \infty) \to \mathbb{S}^2$ be a family of global solutions of (34) with uniformly bounded initial energy $E_\varepsilon(\mathbf{m}_\varepsilon(0)) \le c$. Suppose that $\alpha(\varepsilon)/\varepsilon \to \nu$ and the in-plane components $m_\varepsilon \rightharpoonup m$ converge locally in L^2. Then*

$$\left[\partial_t^2 m + \nu\, \partial_t m + \nabla E_0(m)\right] \wedge m = 0. \tag{35}$$

Proof. We let $\mathbf{m} = (m, \varepsilon v)$, i.e. we blow-up the vertical component. Then the energy can be written as $E_\varepsilon(\mathbf{m}) = E_0(m) + G_\varepsilon(v)$, where $G_\varepsilon(v) = \frac{1}{2} \int \varepsilon^2 |v'|^2 + |v|^2\, dx$. For $\nu_\varepsilon = \alpha(\varepsilon)/\varepsilon$, the Landau-Lifshitz-Gilbert system (34) can be written as

$$\partial_t \begin{pmatrix} m \\ v \end{pmatrix} + \begin{bmatrix} 0 & -\varepsilon^2 v & m_2 \\ \varepsilon^2 v & 0 & -m_1 \\ -m_2 & m_1 & 0 \end{bmatrix} \begin{pmatrix} \nabla E(m) + \nu_\varepsilon\, \partial_t m \\ \nabla G_\varepsilon(v) + \varepsilon^2 \nu_\varepsilon\, \partial_t v \end{pmatrix} = 0.$$

The energy inequality implies the requisite a priori estimates

$$\nu_\varepsilon \int_0^T \|\partial_t m_\varepsilon\|_{L^2}^2 + E_0(m_\varepsilon(T)) \le E_\varepsilon(\mathbf{m}_\varepsilon(0)),$$
$$\varepsilon^2\, \nu_\varepsilon \int_0^T \|\partial_t v_\varepsilon\|_{L^2}^2 + G_\varepsilon(v_\varepsilon(T)) \le E_\varepsilon(\mathbf{m}_\varepsilon(0)),$$

and passing to the limit yields the following set of equations

$$\partial_t m - v\, m^\perp = 0,$$
$$\partial_t v + m^\perp \cdot \big[\nabla E(m) + \nu\, \partial_t m\big] = 0.$$

From that system the vertical blow-up function v can be eliminated. Indeed, the first equation is equivalent to $\partial_t v = \partial_t^2 m \cdot m^\perp$. Substitution into the second equation yields the result. □

Remark 3. *Under further regularity assumptions, especially the validity of the energy inequality, the asymptotic limit holds true in higher dimensional situations as well.*

2.4.2 Traveling waves and kinematic properties of Néel walls

The latter asymptotic limit suggests the following dynamic model for the evolution of Néel walls in thin films under the influence of a constant applied field $h = H\hat{\mathbf{e}}_2$ that points towards one of the end states determined by anisotropy:

$$\Big(\partial_t^2 m + \nu\, \partial_t m + \nabla E_0(m)\Big) \wedge m = h \wedge m, \tag{36}$$
$$m : \mathbb{R} \times [0,T) \to \mathbb{S}^1 \;\text{ with }\; m(\pm\infty, t) = (0, \pm 1).$$

Representing the transition vector in polar coordinates $m = e^{i\theta}$, the transition energy becomes

$$\mathcal{E}(\theta) = \frac{\mathcal{Q}}{2} \int |\theta'|^2\, dx + \frac{1}{2}\|\cos\theta\|_{H^{1/2}}^2 \quad \text{for} \quad \theta(\pm\infty) = \pm\frac{\pi}{2},$$

where $\|f\|_{H^{1/2}}^2 = \int \big(1 + |\xi|\big)|\hat{f}(\xi)|^2\, d\xi$ denotes the full $H^{1/2}$ norm incorporating anisotropy and stray-field interaction. Then the reduced dynamic equation (36) reads

$$\partial_t^2 \theta + \nu\, \partial_t \theta + \nabla\mathcal{E}(\theta) = h \cdot i e^{i\theta}, \tag{37}$$
$$\theta : \mathbb{R} \times [0,T) \to \mathbb{R} \;\text{ with }\; \theta(\pm\infty, t) = (0, \pm\pi/2).$$

The latter damped wave dynamics invites for a kinematic interpretation for the wall center as a point mass with constant force and dynamic friction. The argument will be rather informal; asymptotically, however, the kinematic findings will be justified rigorously in the context of the traveling wave result below. Indeed, if H is assumed to be suitably small, the moving phase $\theta = \theta(x,t)$ is presumably close to the stationary phase profile θ_0 shifted by $q(t)$, the center of the wall at time t. Hence we make an ansatz $\theta(x,t) = \theta_0(x,t) + \theta_1(x,t)$ where, with a slight abuse of notation, $\theta_0(x,t) = \theta_0(x + q(t))$ and $\theta_1(x,t)$ is assumed to be a small perturbation. Then we approximate

$$\nabla\mathcal{E}(\theta) = \nabla\mathcal{E}(\theta_0) + L_0\theta_1, \quad \text{where} \quad L_0 = D\nabla\mathcal{E}(\theta_0),$$

and $\cos(\theta) = \cos(\theta_0) - \sin(\theta_0)\theta_1$. Now if θ_1 is a solution of the linearized problem $\partial_t^2\theta_1 + \nu\partial_t\theta_1 + L_0\theta_1 + H\sin(\theta_q)\,\theta_1 = 0$, then

$$\theta_0''\,|\dot q|^2 + \theta_0'\,\ddot q + \nu\,\theta_0'\,\dot q = H\,\cos\theta_0.$$

Thus, the associate momentum $p(t) = M\,\dot q(t)$, where $M = \frac{1}{2}\int |\theta_0'|^2\,dx$ can be interpreted as the wall mass, satisfies the equation $\dot p + \nu\,p = H$ with terminal momentum $p^* = H/\nu$. We observe that M is consistent with the wall mass we encountered in section 2.3.1 rising from the infinitesimal increase of energy. Accordingly, the mobility, i.e. the rate of change of propagation speed with respect to H, is given by $\beta = 1/(M\nu)$, consistent with our rigorous perturbation result:

Theorem 6. *For sufficiently small field strength H there is a traveling wave for the reduced Landau-Lifshitz-Gilbert dynamics*

$$c^2\,\theta'' + c\,\nu\,\theta' + \nabla\mathcal{E}(\theta) = H\cos\theta$$

that connects antipodal states at infinity $\theta(\pm\infty) = \pm\pi/2$. Moreover, the propagation speed has an expansion $c = \beta H + o(H)$ where the wall mobility $\beta = 1/(M\nu)$ is related to the wall mass $M = \frac{1}{2}\int |\theta_0'|^2\,dx$ taken from a stationary Néel wall θ_0.

Proposition 6. *Suppose that θ_0 is a critical point of $\mathcal{E}(\theta)$ subjected to center and boundary conditions $\theta(0) = 0$ and $\theta(\pm\infty) = \pm\pi/2$. Then the Hessian*

$$D^2\mathcal{E}(\theta_0)\langle\varphi,\varphi\rangle \geq \int |\theta'|^2|\varphi|^2\,dx + \left\|\varphi\sin\theta\right\|_{H^{1/2}}^2 \geq 0$$

is non-negative for any admissible variation φ.

Proof. Let $\langle f,g\rangle_{H^{1/2}} = \mathrm{Re}\int\big(1+|\xi|\big)\hat f(\xi)\,\bar{\hat g}(\xi)\,d\xi$ be the $H^{1/2}$ inner product. Then

$$D^2\mathcal{E}(\theta_0)\langle\varphi,\varphi\rangle = \mathcal{Q}\int|\varphi'|^2 - \big\langle\cos\theta_0,\varphi^2\cos\theta_0\big\rangle_{H^{1/2}} + \left\|\varphi\sin\theta\right\|_{H^{1/2}}^2.$$

In order to estimate the middle term, we deduce from the Euler-Lagrange equation

$$\mathcal{Q}\int\theta_0'\left(\varphi^2\cot\theta_0\right)'\,dx = \big\langle\cos\theta_0,\varphi^2\cos\theta_0\big\rangle_{H^{1/2}}.$$

Recalling that $-d(\cot\theta)/d\theta = 1+\cot^2\theta$, then the claim follows immediately . ∎

The Proposition states in particular that any critical point of $\mathcal{E}$ is in fact the phase of a minimizing Néel wall.

Corollary 1. *For centered Néel walls θ_0, the linearization $L_0 = D\nabla\mathcal{E}(\theta_0)$ extends to a self-adjoint operator on $L^2(\mathbb{R})$ having zero as a simple eigenvalue with eigenspace spanned by θ_0'.*

We need the following refinement of Proposition 4 that in particular rules out a plateau of Néel wall profiles:

Lemma 2. *The phase θ_0 of a stationary Néel wall is strictly increasing.*

Now the proof of Theorem 6 follows closely the one carried out in [3]. We let $G\big((\theta,c),H\big) = c^2\,\theta'' + c\,\nu\,\theta' + \nabla\mathcal{E}(\theta) - H\cos\theta$ and consider the extended functional equation

$$\mathcal{G}\big((\theta,c),H\big) = \big[G\big((\theta,c),H\big),\theta(0)\big] = (0,0).$$

Observe that for a stationary Néel wall θ_0 and $\theta = \theta_0 + \delta\theta$, the mapping $\big((\theta,c),H\big) \mapsto \mathcal{G}\big((\theta,c),H\big)$ is smooth. The linearization with respect to the first two components (θ,c) at the stationary Néel wall $(\theta_0,0)$ reads like

$$\mathcal{L}_0 = \begin{bmatrix} L_0 & \nu\,\theta_0' \\ 0 & \delta_0 \end{bmatrix}.$$

As a mapping $\mathcal{L}_0 : H^2(\mathbb{R}) \times \mathbb{R} \to L^2(\mathbb{R}) \times \mathbb{R}$ it has a bounded inverse, i.e. for every $(f,b) \in L^2(\mathbb{R}) \times \mathbb{R}$ there is $(\phi,c) \in H^2(\mathbb{R}) \times \mathbb{R}$ so that $L_0\psi + \nu\,\theta_0'\,c = f$ and $\phi(0) = b$. According to the Fredholm alternative and Corollary 1 the first equation is solvable provided $c\,\nu\int|\theta_0'|^2\;dx = \int f\,\theta_0'\;dx$ that fixes c and determines ϕ up to a multiple of θ_0'. But in view of Lemma 2, the second equation provides uniqueness. Now the implicit function theorem implies the existence of a differentiable branch $H \mapsto (\theta(H),c(H))$, so that $c'(0)\,\nu\int|\theta_0'|^2\,dx = 2H$, and the claim follows.

3 Boundary vortices in a 2D model

In this section, we present results on a specific thin-film limit of the magnetic energy for a special regime of rather small films. We will analyze a limit of a two-dimensional functional derived by Kohn and Slastikov [23]. They considered *soft* magnetic films without an external field, which corresponds to a functional that uses only the exchange and magnetostatic terms. They studied the asymptotic behavior of the corresponding version of (1) in a thin film $\Omega_\delta = \Omega \times (0,\delta)$ with $\operatorname{diam}\Omega = 1$ for $\delta \to 0$ and $\frac{d^2}{\delta\log\frac{1}{\delta}} \to \frac{\varepsilon}{2\pi} \in (0,\infty)$. The energy divided by $\frac{2\pi\delta^2}{\varepsilon d^2}$ then Γ-converges to the limit functional

$$E^\varepsilon_{KS}(m) = \frac{1}{2}\int_\Omega |\nabla m|^2 + \frac{1}{2\varepsilon}\int_{\partial\Omega}(m\cdot\nu)^2 \tag{38}$$

defined on maps $m \in H^1(\Omega,S^1)$ (so $m_3 = 0$). Here ν is a unit normal to $\partial\Omega$. The Kohn-Slastikov theorem shows that for this special scaling, the nonlocal contribution arising as the energy of the induced field reduces to a *local* term charging the boundary. The reason for this simplification of the functional lies in a separation of scales between the energy contribution of volume and surface charges to the field energy.

In the following, we present results of Kurzke [26, 24, 25] on the limit of (38) as $\varepsilon \to 0$, for a simply connected domain Ω. The results share some features with those of Moser [34, 35] that are presented in Section 4. In particular, sequences of minimizers develop vortices on the boundary.

Due the two-scale process of first letting $\delta \to 0$ to obtain (38), a two-dimensional problem, and then letting $\varepsilon \to 0$, our approach can be seen as a simplified model for the boundary vortices in Section 4. Since our functional here is local, the results are more detailed, especially for the asymptotic dynamics.

The sequence of functionals E^ε_{KS} is rather similar to the Ginzburg-Landau functional for superconductivity of [5]. With $m_0 = \tau$ being a continuous unit tangent field to $\partial\Omega$, we are (after rescaling and renaming variables) considering the variational problem for $m : \Omega \to \mathbb{R}^2$: Minimize

$$\frac{1}{2}\int_\Omega |\nabla m|^2 + \frac{1}{2\varepsilon}\int_{\partial\Omega}(1-(m\cdot m_0)^2)d\mathcal{H}^1$$

subject to $|m| = 1$ in Ω as $\varepsilon \to 0$. This problem has an interior constraint and a boundary penalty.

Bethuel, Brezis and Hélein [5] studied the behavior as $\varepsilon \to 0$ of

$$\frac{1}{2}\int_\Omega |\nabla m|^2 + \frac{1}{4\varepsilon^2}\int_\Omega (1-|m|^2)^2$$

subject to $m = m_0$ on $\partial\Omega$, so this problem has a boundary constraint and an interior penalty.

Common to both problems is that, as long as m_0 has nonzero topological degree, there is no map in $H^1(\Omega;\mathbb{R}^2)$ that satisfies the constraint and makes the penalty term zero. This is due to the fact that a continuous map $w : \partial\Omega \to S^1$ can be extended to a continuous map $\overline{w} : \overline{\Omega} \to S^1$ if and only if $\deg(w) = 0$. Although H^1 maps need not be continuous, the argument still carries through to show that there is not even an extension of finite H^1 energy. Both problems are thus forced to develop singularities as $\varepsilon \to 0$, and the minimum energy will become unbounded.

We call the singularities of both problems *vortices*, since minimizers converge as $\varepsilon \to 0$ to maps that have the form $\frac{z-a_i}{|z-a_i|}$ near the singularities a_i. In the Ginzburg-Landau case, these vortices are interior and each carries a topological degree of 1; in the Kohn-Slastikov case, the singularities lie on the boundary, and we only see one half of the vortex. Each "boundary vortex" corresponds to a transition from m_0 to $-m_0$ or vice versa, and can be viewed as carrying $\frac{1}{2}$ topological charge.

It was shown in [26, 24] that a single boundary vortex carries an energy of $\frac{\pi}{2}\log\frac{1}{\varepsilon}$ (see Theorem 7), and that the interaction of these vortices is governed by the next order term in the energy expansion, a renormalized energy that can be calculated by the solution of a linear boundary problem (see 8). In [25] it was shown that this renormalized energy actually governs the motion of the vortices in the natural time scaling, when time is accelerated by a factor of $\log\frac{1}{\varepsilon}$ (see Theorem 9).

In the following, we will explain these results in more detail, and in the proper frameworks of two orders of Γ-convergence and Γ-convergence of gradient flows.

A major advantage of the simplified energy (38) is that the problem can be made *scalar* since $m \in S^1$ can be written as $m = e^{iv}$. The energy functional can then be rewritten as

$$\mathcal{E}_\varepsilon(v) = \frac{1}{2}\int_\Omega |\nabla v|^2 + \frac{1}{2\varepsilon}\int_{\partial\Omega}\sin^2(v-g), \tag{39}$$

where g is a function with $ie^{ig} = \nu$. Since Ω is simply connected, the degree of ν as a map from $\partial\Omega$ (which is homeomorphic to S^1) to S^1 is 1, and so g needs to have a jump of height -2π, but can otherwise be chosen as smooth as $\partial\Omega$. As $\varepsilon \to 0$, minimizers v_ε of $\mathcal{E}_\varepsilon$ will now satisfy $\sin^2(v_\varepsilon - g) \approx 0$ on most of $\partial\Omega$, but due to the jump of g, this will not be possible everywhere, and so singularities will develop that correspond precisely to the fast transition from $m \approx \tau$ to $m \approx -\tau$.

We can obviously generalize this to g having a jump of height $-2\pi D$, $D \in \mathbb{Z}$, corresponding to $\deg(e^{ig}) = D$, which will we do in the following. Without loss of generality, we will assume $D \geq 0$. The magnetic case of the Kohn-Slastikov functional is given by $D = 1$.

3.1 Highest order asymptotics

The following calculation gives an upper bound for the energy of a single boundary vortex. We will use a localized energy $\mathcal{E}_\varepsilon(v; B)$ that is defined by

$$\mathcal{E}_\varepsilon(v; B) = \frac{1}{2} \int_{\Omega \cap B} |\nabla v|^2 + \frac{1}{2\varepsilon} \int_{\partial\Omega \cap \overline{B}} \sin^2(v - g). \tag{40}$$

To simplify matters we will assume that $\Omega \cap B_R(0) = B_R^+(0)$ is a half-ball and that $g = 0$, corresponding to the constant tangent. We set $\Gamma_R = \partial\Omega \cap \overline{B_R(0)}$ for the flat boundary which we assume to be part of the x-axis in $z = x + iy$-plane.

Proposition 7. *There is a sequence $v_\varepsilon \in H^1(B_R(0))$ with $v_\varepsilon|_{\Gamma_R} \to v_* = \pi\chi_{x<0}$ in all $L^p(\Gamma_R)$ for $1 \leq p < \infty$ and*

$$\mathcal{E}_\varepsilon(v_\varepsilon; B_R) \leq \frac{\pi}{2} \log \frac{R}{\varepsilon} + C. \tag{41}$$

Proof. Set $v_\varepsilon = \arg(z)$ in $B_R^+(0) \setminus B_\varepsilon^+(0)$. Choose any H^1 continuation w with $0 \leq w \leq \pi$ of $\arg|_{\partial B_1^+(0)}$ to $B_1^+(0)$ and set $v_\varepsilon(z) = w(\varepsilon z)$ inside $B_\varepsilon^+(0)$. This sequence obviously satisfies the claims. □

This shows that for $R = O(1)$, a typical vortex has an energy of approximately $\frac{\pi}{2} \log \frac{1}{\varepsilon} + O(1)$. A combination of $2D$ such vortices to counter the jump of g leads to the following upper bound:

Proposition 8. *Minimizers v_ε of $\mathcal{E}_\varepsilon$ satisfy*

$$\mathcal{E}_\varepsilon(v_\varepsilon) \leq \pi D \log \frac{1}{\varepsilon} + C(\Omega). \tag{42}$$

A different interpretation of "every vortex carries an energy of $\frac{\pi}{2} \log \frac{1}{\varepsilon}$" is given by the following Γ-convergence theorem:

Theorem 7. *Assume (v_ε) is a sequence of functions with $\mathcal{E}_\varepsilon(v_\varepsilon) \leq M \log \frac{1}{\varepsilon}$. Then there exists a sequence of $a_\varepsilon \in 2\pi\mathbb{Z}$ such that the boundary traces $w_\varepsilon = (v_\varepsilon - a_\varepsilon)|_{\partial\Omega}$ are bounded in an Orlicz space of type e^L, and in particular, $\|w_\varepsilon\|_{L^p(\partial\Omega)} \leq C(M)$.*

The sequence w_ε is then precompact in the strong topology of $L^1(\partial\Omega)$, and every cluster point w satisfies $w - g \in \mathrm{BV}(\partial\Omega; \pi\mathbb{Z})$. In addition, we have the lower bound inequality

$$\liminf_{\varepsilon \to 0} \frac{1}{2 \log \frac{1}{\varepsilon}} \int_\Omega |\nabla v_\varepsilon|^2 \geq \frac{1}{2} \int_{\partial\Omega} |D(w - g)|. \tag{43}$$

Conversely, for every u with $u - g \in \mathrm{BV}(\partial\Omega; \pi\mathbb{Z})$ there exists a sequence $u_\varepsilon \in H^1(\Omega)$ such that the trace satisfies $u_\varepsilon \to u$ in $L^1(\partial\Omega)$ and with

$$\lim_{\varepsilon\to 0} \frac{1}{2\log\frac{1}{\varepsilon}} \int_\Omega |\nabla u_\varepsilon|^2 = \frac{1}{2}\int_{\partial\Omega} |D(u-g)|. \tag{44}$$

We will not prove 7 here. A proof based on a nonlocal representation using the $H^{1/2}$ seminorm and rearrangement inequalities is given in [26], and an extension to higher dimensions via slicing that utilizes the Orlicz bound is shown in [25]. Both proofs are based on ideas of Alberti-Bouchitté-Seppecher [1, 2] for similar functionals with a coercive instead of periodic potential.

Remark 4. *Theorem 7 shows that, since the* BV *type limit functional has a lower bound of πD, the energy of minimizers of $\mathcal{E}_\varepsilon$ is $\pi D \log\frac{1}{\varepsilon} + o(\log\frac{1}{\varepsilon})$. For other converging sequences, we obtain just the number of jump points with multiplicities, but the limit functional is independent of the position of these jump points.*

3.2 Separation of vortices and renormalized energy

We will relate the dependence of the energy on the position of the singularities to a renormalized energy given as follows:

Definition 1 (Possible limit functions). *Let $d_i \in \mathbb{Z}$ with $\sum_i d_i = 2D$ and $a_i \in \partial\Omega$ be distinct points. We define the canonical limit function $v_* = v_*(a_i, d_i)$ to be a harmonic function with $\sin^2(v_* - g) = 0$ such that its trace on $\partial\Omega$ jumps by $-\pi d_i$ at the point a_i.*

The renormalized energy *is defined to be*

$$W(a_i, d_i) = \frac{1}{2}\lim_{\rho\to 0}\left(\int_{\Omega\setminus\bigcup_i B_\rho(a_i)} |\nabla v_*|^2 - \pi\sum_i d_i^2 \log\frac{1}{\rho}\right). \tag{45}$$

The renormalized energy can be expressed via the solution of a linear boundary value problem for the Laplacian, see [24].

With the renormalized energy, we can formulate the following second-order Γ-convergence type theorem. For minimizers, we will have $N = 2D$ as above.

Theorem 8. *If (v_ε) is a sequence of functions with $\mathcal{E}_\varepsilon(v_\varepsilon) \le M\log\frac{1}{\varepsilon}$ and $v_\varepsilon \to v_*(a_i, d_i)$ in $L^2(\partial\Omega)$, with $d_i \in \{\pm 1\}$ and a_i distinct, $i = 1, \ldots, N$, then*

$$\liminf_{\varepsilon\to 0}\left(\mathcal{E}_\varepsilon(v_\varepsilon) - \frac{\pi N}{2}\left(\log\frac{1}{\varepsilon} + 1 - \log 2\right)\right) \ge W(a_i, d_i). \tag{46}$$

If additionally (v_ε) are stationary points of $\mathcal{E}_\varepsilon$, then $v_\varepsilon \to v_*$ in $W^{1,p}(\Omega)$ for $p < 2$ and in H^1_{loc} away from the a_i, and (46) holds with equality.

Furthermore, for any $d_i \in \{\pm 1\}$ and a_i distinct, $i = 1, \ldots, N$, there exists a sequence of functions w_ε such that $w_\varepsilon \to v_*(a_i, d_i)$ and

$$\lim_{\varepsilon\to 0}\left(\mathcal{E}_\varepsilon(v_\varepsilon) - \frac{\pi N}{2}\left(\log\frac{1}{\varepsilon} + 1 - \log 2\right)\right) = W(a_i, d_i). \tag{47}$$

We will prove this theorem only partially, for the special case of minimizers, where the bounds and the convergence follow from a comparison argument. The general case can be shown by means of some extra PDE estimates and a regularization technique of Yosida type, replacing the sequence by an improved sequence that minimizes a modified functional, see [25].

Proposition 9 (Euler-Lagrange equations). *Stationary points v_ε of $\mathcal{E}_\varepsilon$ satisfy the equations*

$$\Delta v_\varepsilon = 0 \quad \text{in } \Omega, \tag{48}$$

$$\frac{\partial v_\varepsilon}{\partial \nu} = -\frac{1}{2\varepsilon} \sin 2(v_\varepsilon - g) \quad \text{on } \partial\Omega. \tag{49}$$

Lemma 3 (Rellich-Pohožaev identity). *For a Lipschitz domain G and a harmonic function $v \in H^2(G)$, there holds*

$$\int_{\partial G} \frac{\partial v}{\partial \nu}(z \cdot \nabla v) = \frac{1}{2} \int_{\partial G} z \cdot \nu |\nabla v|^2. \tag{50}$$

Proof. This is easily seen using by testing $\Delta v = 0$ with $z \cdot \nabla v$. □

An easy consequence is

Lemma 4. *For a starshaped Lipschitz domain G, there exists constants such that every harmonic function $v \in H^2(G)$ satisfies*

$$c \int_{\partial G} \left|\frac{\partial v}{\partial \tau}\right|^2 \leq \int_{\partial G} \left|\frac{\partial v}{\partial \nu}\right|^2 \leq C \int_{\partial G} \left|\frac{\partial v}{\partial \tau}\right|^2. \tag{51}$$

Following ideas of [5] and [42], we relate the penalty term $\frac{1}{2\varepsilon} \int_{\partial\Omega} \sin^2(v - g)$ to a radial derivative of the energy:

Definition 2. *For $z_0 \in \partial\Omega$ and $v \in H^2(\Omega)$ we set*

$$A(\rho) = A_{v,\varepsilon,z_0}(\rho) = \rho \int_{\partial B_\rho(z_0) \cap \Omega} |\nabla v|^2 d\mathcal{H}^1 + \frac{\rho}{\varepsilon} \int_{\partial B_\rho(z_0) \cap \partial\Omega} \sin^2(v - g) d\mathcal{H}^0. \tag{52}$$

For stationary points of the energy, A can be used to bound the penalty term:

Proposition 10. *There exists $\varepsilon_0 > 0$ and $C > 0$ such that for all $\varepsilon < \varepsilon_0$, $\rho < \varepsilon^{3/4}$, any stationary point v of $\mathcal{E}_\varepsilon$, and any $z_0 \in \partial\Omega$ there holds*

$$\frac{1}{2\varepsilon} \int_{\Gamma_\rho(z_0)} \sin^2(v - g) \leq A(\rho) + C\sqrt{\varepsilon}. \tag{53}$$

Proof. For simplicity, we show this only for $g = 0$ and a flat boundary. We use $z_0 = 0$ and apply (50) on the domain $\omega_\rho = \Omega \cap B_\rho(0)$, which shows

$$\frac{1}{2} \int_{\partial\omega_\rho} z \cdot \nu |\nabla v|^2 = \rho \int_{\partial B_\rho \cap \Omega} \left|\frac{\partial v}{\partial \nu}\right|^2 + \int_{\Gamma_\rho} \frac{\partial v}{\partial \nu} z \cdot \nabla v. \tag{54}$$

Using the Euler-Lagrange equations, we obtain

$$\frac{\rho}{2}\int_{\partial B_\rho\cap\Omega}|\nabla v|^2 = \rho\int_{\partial B_\rho\cap\Omega}\left|\frac{\partial v}{\partial \nu}\right|^2 - \frac{1}{2\varepsilon}\int_{\Gamma_\rho}\sin 2v(z\cdot\nabla v). \tag{55}$$

Integrating by parts, we see that

$$\frac{1}{2\varepsilon}\int_{\Gamma_\rho}\sin^2(v) = \frac{\rho}{2\varepsilon}\int_{\partial B_\rho\cap\partial\Omega}\sin^2(v)d\mathcal{H}^0 - \frac{1}{2\varepsilon}\int_{\Gamma_\rho}\sin 2v(z\cdot\nabla v) \tag{56}$$

$$= \frac{\rho}{2\varepsilon}\int_{\partial B_\rho\cap\partial\Omega}\sin^2(v)d\mathcal{H}^0 + \frac{\rho}{2}\int_{\partial B_\rho\cap\Omega}|\nabla v|^2 - \rho\int_{\partial B_\rho\cap\Omega}\left|\frac{\partial v}{\partial\nu}\right|^2 \tag{57}$$

$$\leq \frac{1}{2}A(\rho). \tag{58}$$

□

We obtain the following criterion for vortex-free parts of the boundary:

Proposition 11. *There exist constants $\gamma > 0$ and $C > 0$ such that for every $z_0 \in \partial\Omega$, $\varepsilon < \varepsilon_0$, $\rho < \varepsilon^{3/4}$ and every stationary point v of $\mathcal{E}_\varepsilon$ with $A(\rho) < \gamma$ there holds*

$$\sup_{\Gamma_{\rho/2}}\sin^2(v-g) < \frac{1}{4} \tag{59}$$

and

$$\frac{1}{2\varepsilon}\int_{\Gamma_{\rho/2}}\sin^2(v-g) \leq C. \tag{60}$$

Proof. By Lemma 4, we can estimate

$$\int_{\Gamma_\rho}\left|\frac{\partial v}{\partial\tau}\right|^2 \leq C\int_{\partial\omega_\rho}\left|\frac{\partial v}{\partial\nu}\right|^2 \leq C\int_{\partial B_\rho\cap\Omega}|\nabla v|^2 + C\int_{\Gamma_\rho}\left|\frac{\partial v}{\partial\nu}\right|^2. \tag{61}$$

We thus can estimate, using Sobolev embedding in one dimension

$$\begin{aligned}[v]^2_{C^{0,1/2}(\Gamma_\rho)} &\leq C\int_{\Gamma_\rho}\left|\frac{\partial v}{\partial\tau}\right|^2 \\ &\leq C\left(\frac{1}{\rho}A(\rho) + \frac{1}{\varepsilon^2}\int_{\Gamma_\rho}\sin^2(v-g)\right) \leq \frac{C}{\varepsilon}(2\gamma + C\sqrt{\varepsilon_0}).\end{aligned}$$

Assuming now that $\sin^2(u(z)-g(z)) \geq \frac{1}{4}$ for some $z \in \Gamma_{\rho/2}$ and choosing γ and ε_0 sufficiently small then leads to a contradiction. □

Disintegrating the energy radially, we see

Lemma 5. *Let (v_ε) be a sequence of stationary points of $\mathcal{E}_\varepsilon$ satisfying the logarithmic energy bound $\mathcal{E}_\varepsilon(v_\varepsilon) \leq M\log\frac{1}{\varepsilon}$. Then for any $z_0 \in \partial\Omega$, the function $A(\rho) = A_{v_\varepsilon,\varepsilon,z_0}(\rho)$ defined above satisfies*

$$\inf_{\varepsilon^{6/7}\leq\rho\leq\varepsilon^{5/6}}A(\rho) \leq \frac{84}{\log\frac{1}{\varepsilon}}\mathcal{E}_\varepsilon(v_\varepsilon;\Omega\cap B_{\varepsilon^{5/6}}(z_0)) \leq 84M \tag{62}$$

and

$$\inf_{5\varepsilon^{5/6}\leq\rho\leq5\varepsilon^{4/5}}A(\rho) \leq 60M. \tag{63}$$

Using Vitali's covering lemma, we can use this and (53) to show a local upper bound on the penalty term near an almost singularity and in a second step a covering of the set of almost singularities. This leads to

Proposition 12. *There is a constant $N = N(g, \Omega, M)$ such that for any sequence of stationary points v_ε satisfying the energy bound $E_\varepsilon(v_\varepsilon) \le M \log \frac{1}{\varepsilon}$, the approximate vortex set $S_\varepsilon = \{z \in \partial\Omega : \sin^2(v_\varepsilon(z) - g(z)) \ge \frac{1}{4}\}$ can be covered by at most N balls of radius ε, such that the $\varepsilon/5$ balls around the same centers are disjoint.*

For comparison arguments we shall need the following lower bound for the energy on half-annuli whenever $v - g$ is in different wells of $\sin^2$ on both parts of the boundary:

Proposition 13. *Let $0 < \rho < R \le R_0$, R_0 sufficiently small, $z_0 \in \partial\Omega$, w.l.o.g. $z_0 = 0$. We examine the "half-annulus" $D_{R,\rho} = (B_R \setminus \overline{B_\rho}) \cap \Omega$, which can be described by choosing functions $\vartheta_1(r), \vartheta_2(r)$ as $\{re^{i\vartheta} : \vartheta_1(r) < \vartheta < \vartheta_2(r), \rho < r < R\}$ with $|\vartheta_2(r) - \vartheta_1(r) - \pi| \le Cr$. Assume also that for $j = 1, 2$ there holds $(v - g)(re^{i\vartheta_j(r)}) \in (k_j\pi - \delta, k_j\pi + \delta)$ for some $k_j \in \mathbb{Z}$ and some small δ. Then*

$$\mathcal{E}_\varepsilon(v; D_{R,\rho}) \ge \frac{\pi}{2}(k_2 - k_1)^2 \log \frac{R}{\rho} - C(k_2 - k_1)^2(R + \frac{\varepsilon}{\rho}). \tag{64}$$

Proof. For simplicity, we will assume $g = 0$, $\vartheta_1 = 0$ and $\vartheta_2 = \pi$, corresponding to a flat boundary. We will set $v_j(r) = v(re^{i\vartheta_j})$ for the function on the two boundary components. We also assume w.l.o.g. $k_1 = k$ and $k_2 = 0$. Using polar coordinates, disregarding the radial derivative and by use of Hölder's inequality, we calculate

$$\int_{D_{R,\rho}} |\nabla v|^2 \ge \int_\rho^R \frac{1}{r} \int_0^\pi \left|\frac{\partial u}{\partial \vartheta}\right|^2 d\vartheta dr$$
$$\ge \frac{1}{\pi} \int_\rho^R \left(\int_0^\pi \left|\frac{\partial v}{\partial \vartheta}\right|\right)^2 \ge \frac{1}{\pi} \int_\rho^R \frac{(v_1 - v_2)^2}{r} dr.$$

We rewrite $v_1 - v_2 = k\pi - (v_1 - k\pi) - v_2$. Using the lower bound $\sin^2(t - k_i\pi) \ge \sigma t^2$ valid for $|t| < \delta$ with some $\sigma = \sigma(\delta)$, we can estimate

$$\mathcal{E}_\varepsilon(v; D_{R,\rho}) \ge \frac{1}{2} \int_\rho^R \frac{1}{\pi r}\big(k\pi - ((v_1 - k\pi) - v_2)\big)^2 + \frac{\sigma}{\varepsilon}\big(v_1^2 + v_2^2\big)^2 dr.$$

On the last term, we use the inequality $v_1^2 + v_2^2 \ge \frac{1}{2}(v_1 - k\pi + v_2)^2$. Then we use the inequality $\alpha(A - B)^2 + \beta B^2 \ge \frac{1}{\frac{1}{\alpha} + \frac{1}{\beta}} A^2$ on $A = k\pi$ and $B = (v_1 - k\pi - v_2)$. The claim then follows by integration. □

We now recall that v_ε are minimizers of $\mathcal{E}_\varepsilon$ satisfying the upper bound

$$\mathcal{E}_\varepsilon(v_\varepsilon) \le \pi D \log \frac{1}{\varepsilon} + C_0 \tag{65}$$

for some constant C_0, where D is the degree of e^{ig}. We will use an appropriate lower bound for the energy away from the vortex set to show convergence by a comparison argument. The same arguments also hold for stationary points satisfying (65).

By Proposition 12, there exist $a_j^\varepsilon \in \partial\Omega$, $1 \leq j \leq N_\varepsilon \leq N$ such that the approximate vortex set S_ε satisfies $S_\varepsilon \subset \bigcup_{1\leq j\leq N_\varepsilon} B_\varepsilon(a_j^\varepsilon)$. Passing to a subsequence of $\varepsilon \to 0$, we can assume that $N_\varepsilon = N_0$ is constant and $a_j^\varepsilon \to a_j^0$ as $\varepsilon \to 0$. Note that the a_j^0 need not be distinct. We define for $0 < \sigma < \frac{1}{2}\min_{a_j^0 \neq a_{j'}^0} \operatorname{dist}(a_j^0, a_{j'}^0)$ the sets $\Omega_\sigma^\varepsilon = \Omega \setminus \bigcup_j B_\sigma(a_j^\varepsilon)$ and $\Omega_\sigma^0 = \Omega \setminus \bigcup_j B_\sigma(a_j^0)$. With this setup (and this subsequence), the lower bound of Proposition 13 can be combined with the arguments of Struwe [42] to show

Proposition 14. *There is a constant $C = C(g, \Omega, C_0)$ such that $\mathcal{E}_\varepsilon(v_\varepsilon; \Omega_\sigma^\varepsilon) \leq \pi D \log \frac{1}{\sigma} + C$.*

We obtain convergence to the canonical harmonic function:

Proposition 15. *Let (v_ε) be a sequence of critical points satisfying the energy bound*

$$\mathcal{E}_\varepsilon(v_\varepsilon) \leq \pi D \log \frac{1}{\varepsilon} + C_0.$$

Then there is a subsequence and $N = 2D$ points $a_1, \ldots, a_N \in \partial\Omega$ such that

$$\int_{\Omega'} |\nabla v_\varepsilon|^2 \leq M(\Omega') < \infty \tag{66}$$

for all open Ω' with $\overline{\Omega'} \subset \overline{\Omega} \setminus \{a_1, \ldots, a_N\}$. Additionally, there hold the bounds

$$\int_{\Omega} |\nabla v_\varepsilon|^p \leq C(p) \tag{67}$$

uniformly in ε for all $1 \leq p < 2$. In particular, after adding a suitable $z_\varepsilon \in 2\pi\mathbb{Z}$, a subsequence of (v_ε) converges weakly in H^1_{loc} and $W^{1,p}$, $p < 2$, to a harmonic function v_*. The limit has the properties that $(v_* - g)$ is piecewise constant on $\partial\Omega \setminus \{a_1, \ldots, a_N\}$, with values in $\pi\mathbb{Z}$, and jumps by $-\pi$ at the points a_j.

Proof. We use the setup described above, in particular, we use the points a_j^0 as defined there. Note that for $\varepsilon < \varepsilon_0(\sigma)$, there holds $\Omega_\sigma^0 \subset \Omega_{\sigma/2}^\varepsilon$ and so by Proposition 14,

$$\int_{\Omega_\sigma^0} |\nabla v_\varepsilon|^2 \leq 2\mathcal{E}_\varepsilon(v_\varepsilon; \Omega_{\sigma/2}^\varepsilon) \leq 2\pi D \log \frac{2}{\sigma} + C, \tag{68}$$

which proves (66). To obtain the L^p bounds (67), fix a $\sigma > 0$ and $1 \leq p < 2$. Then by Hölder's inequality and Proposition 14,

$$\begin{aligned}
\int_\Omega |\nabla v_\varepsilon|^p &\leq \int_{\Omega_\sigma^\varepsilon} |\nabla v_\varepsilon|^p + \sum_{\ell=1}^\infty \int_{\Omega_{2^{-\ell}\sigma}^\varepsilon \setminus \Omega_{2^{-\ell+1}\sigma}^\varepsilon} |\nabla v_\varepsilon|^p \\
&\leq C + \sum_{\ell=1}^\infty |\Omega_{2^{-\ell}\sigma}^\varepsilon \setminus \Omega_{2^{-\ell+1}\sigma}^\varepsilon|^{1-p/2} \left(\int_{\Omega_{2^{-\ell}\sigma}^\varepsilon} |\nabla v_\varepsilon|^2 \right)^{p/2} \\
&\leq C + c \sum_{\ell=1}^\infty 2^{-(1-p/2)\ell} \left(2\pi D \log \frac{1}{2^\ell \sigma} + C \right)^{p/2} \\
&\leq C.
\end{aligned}$$

From this L^p gradient bound, we obtain the weak compactness up to translation. The weak limit v_* is harmonic since $\int_\Omega \nabla v_* \cdot \nabla\varphi = \lim_{\varepsilon\to 0}\int_\Omega \nabla v_\varepsilon \cdot \nabla\varphi = 0$ for all $\varphi \in C_c^\infty(\Omega)$. That the boundary values satisfy $v_* - g \in \pi\mathbb{Z}$ with possible jumps at the a_i follows from $\int_{\partial\Omega} \sin^2(v_\varepsilon - g) \to 0$ and $(v_\varepsilon - g)$ being close to $\pi\mathbb{Z}$ outside the approximate vortex set S_ε.

That the vortices are indeed single and $N = 2D$ can then be shown by some refined arguments which prove that higher-order vortices must have far higher energy. □

The energy of these limit functions v_* away from a_i is the renormalized energy of (45), and the energy of v_ε on Ω_ρ converges to that of v_*. To prove equality in (46), we thus need to calculate the energy of v_ε close to a_i. This is done by an ε-scale blowup, which leads to a half-space problem. The solutions of the half-space problem are explicitly known and essentially unique (Toland [43], see also Cabré and Sola-Morales [6] for a more general uniqueness theorem). Comparing v_ε with the rescaled half-space solution and some estimates (see [24]) then show the rest of Theorem 8. One can even show

Proposition 16. *For a sequence v_ε of stationary points of $\mathcal{E}_\varepsilon$, the configuration of vortex points (a_i) is stationary for the renormalized energy $W(a_i, d_i)$. For minimizers v_ε, it is minimizing.*

3.3 Motion of vortices

Theorem 8 and the previous proposition show that the renormalized energy W governs the interaction of the vortices on an energetic level. It can be shown that also the motion of the vortices by the gradient flow (which corresponds to the LLG flow since we restrict possible magnetizations to a plane) is given by the renormalized energy:

Theorem 9. *Let $0 < T \le \infty$ and let (v_ε) be a sequence of solutions of*

$$\lambda_\varepsilon \partial_t v_\varepsilon = \Delta v_\varepsilon \quad \text{in } \Omega \times (0,T) \tag{69}$$

$$\frac{\partial v_\varepsilon}{\partial \nu} = -\frac{1}{2\varepsilon}\sin 2(v_\varepsilon - g) \quad \text{on } \partial\Omega \times (0,\infty). \tag{70}$$

For the initial conditions we assume that $v_\varepsilon(0) \to v_*(a_i, d_i)$ for $d_i \in \{\pm 1\}$ and distinct a_i. Furthermore, v_ε is supposed to be initially well-prepared, *meaning that*

$$\mathcal{E}_\varepsilon(v_\varepsilon(0)) - \frac{\pi N}{2}\log\frac{1}{\varepsilon} - \frac{\pi N}{2}(1 - \log 2) \le W(a_i, d_i) + o(1) \tag{71}$$

as $\varepsilon \to 0$.

Depending on the asymptotic behavior of λ_ε, we then have:

(i) *If $\lambda_\varepsilon = \frac{1}{\log\frac{1}{\varepsilon}}$, then there exists a time $T^* > 0$ such that for all $t \in [0, T^*)$, there holds $v_\varepsilon(t) \to v_*(a_i(t), d_i(0)))$. Furthermore, the $a_i(t)$ satisfy the motion law*

$$\frac{da_i}{dt} = -\frac{2}{\pi}\frac{\partial}{\partial a_i} W(a_i(t), d_i(0)) \tag{72}$$

in the tangent space at a_i to $\partial\Omega$. If $T^* < T$ is the maximal time with these properties, then as $t \to T^*$, there exist $i \ne j$ such that $a_i(t)$ and $a_j(t)$ converge to the same point.

The energy of $v_\varepsilon(t)$ satisfies the expansion

$$\mathcal{E}_\varepsilon(v_\varepsilon(t)) = \frac{\pi N}{2}\log\frac{1}{\varepsilon} + \frac{\pi N}{2}(1-\log 2) + W(\vec{a}(t),\vec{d}) + o(1) \tag{73}$$

as $\varepsilon \to 0$.

(ii) *If $\lambda_\varepsilon \log\frac{1}{\varepsilon} \to 0$ as $\varepsilon \to 0$, then for almost every $t \in [0,T)$ we have $v_\varepsilon(t) \to v_*(a_i(0), d_i(0))$, so there is no motion.*

(iii) *If $\lambda_\varepsilon \log\frac{1}{\varepsilon} \to \infty$ as $\varepsilon \to 0$, then for almost every $t \in [0,\infty)$ we have $v_\varepsilon(t) \to v_*(b_i, d_i)$ with $\nabla W(b_i, d_i) = 0$, so the system instantaneously jumps into a critical point.*

Again, there are strong similarities between this result for the motion of boundary vortices and those in the theory of gradient flow motion of interior Ginzburg-Landau type vortices as studied by Jerrard and Soner [20] and Lin [28, 29].

The proof in [25] is based on the technique of Γ-convergence of gradient flows of Sandier and Serfaty [40], applied to the functionals

$$\mathcal{F}^\varepsilon(u) = \mathcal{E}^\varepsilon(u) - \frac{\pi N}{2}(\log\frac{1}{\varepsilon} + 1 - \log 2) \tag{74}$$

and the limit functional

$$\mathcal{F}(a_i) = W(a_i, d_i). \tag{75}$$

We need some additional definitions:

Definition 3. *We say that functionals $\mathcal{F}^\varepsilon$ Γ-converge to $\mathcal{F}$ along the trajectory $u_\varepsilon(t)$ with respect to the convergence "$\overset{S}{\rightharpoonup}$" if there exist $u(t)$ and a subsequence such that for all t, $u_\varepsilon(t)\overset{S}{\rightharpoonup}u(t)$ and*

$$\liminf_{\varepsilon\to 0} \mathcal{F}^\varepsilon(u_\varepsilon(t)) \geq \mathcal{F}(u(t)). \tag{76}$$

The energy excess *$D_\varepsilon(t)$ and the* limiting energy excess *$D(t)$ for a sequence $u_\varepsilon(t)$ are defined via*

$$D_\varepsilon(t) = \mathcal{E}^\varepsilon(u_\varepsilon(t)) - \mathcal{E}(u(t)), \qquad D(t) = \limsup_{\varepsilon\to 0} D_\varepsilon(t). \tag{77}$$

If $u_\varepsilon(t)$ are solutions to the gradient flow for $\mathcal{E}^\varepsilon$ that satisfy $D(0) = 0$, they are said to be initially well-prepared.

The proof of Theorem 9 relies on the following version of Sandier and Serfaty's theorem on the Γ-convergence of gradient flows:

Theorem 10 (Sandier-Serfaty [40]). *Assume $\mathcal{F}^\varepsilon \in C^1(\mathcal{M})$ and $\mathcal{F} \in C^1(\mathcal{N})$. Let u_ε be a sequence of solutions of the gradient flow for $\mathcal{F}^\varepsilon$ on $[0,T)$ with respect to the metric structure X_ε that satisfy*

$$\mathcal{F}^\varepsilon(u_\varepsilon(0)) - \mathcal{F}^\varepsilon(u_\varepsilon(t)) = \int_0^t \|\partial_t u_\varepsilon(s)\|^2_{X_\varepsilon}\, ds. \tag{78}$$

Assume $u_\varepsilon(0)\overset{S}{\rightharpoonup}u_0$, that $\mathcal{F}^\varepsilon$ Γ-converges to $\mathcal{F}$ along the trajectory $u_\varepsilon(t)$, and that (u_ε) is initially well-prepared. Furthermore, assume that (LB) and (CON) hold:

(LB) *For a subsequence such that* $u_\varepsilon(t) \overset{S}{\rightharpoonup} u(t)$*, we have* $u \in H^1((0,T);\mathcal{N})$ *and there exists* $f \in L^1(0,T)$ *such that for every* $s \in [0,T)$ *there holds*

$$\liminf_{\varepsilon\to 0} \int_0^s \|\partial_t u_\varepsilon(t)\|^2_{X_\varepsilon}\,dt \geq \int_0^s \Big(\|\partial_t u\|^2_{T_{u(t)}\mathcal{N}} - f(t)D(t)\Big)\,dt. \tag{79}$$

(CON) *If* $u_\varepsilon(t) \overset{S}{\rightharpoonup} u(t)$*, there exists a locally bounded function* g *on* $[0,T)$ *such that for any* $t_0 \in [0,T)$ *and any* v *defined in a neighborhood of* t_0 *that satisfies* $v(t_0) = u(t_0)$ *and* $\partial_t v(t_0) = -\nabla_{T_{u(t_0)}\mathcal{N}}\mathcal{E}(u(t_0))$*, there exists a sequence* $v_\varepsilon(t)$ *such that* $v_\varepsilon(t_0) = u_\varepsilon(t_0)$ *and the following inequalities hold:*

$$\limsup_{\varepsilon\to 0} \|\partial_t v_\varepsilon(t_0)\|^2_{X_\varepsilon} \leq \|\partial_t v(t_0)\|^2_{T_{v(t_0)}\mathcal{N}} + g(t_0)D(t_0) \tag{80}$$

$$\liminf_{\varepsilon\to 0} \Big(-\tfrac{d}{dt}\Big|_{t=0}\mathcal{F}^\varepsilon(v_\varepsilon(t))\Big) \geq -\tfrac{d}{dt}\Big|_{t=0}\mathcal{F}(v(t)) - g(t_0)D(t_0). \tag{81}$$

Then $u_\varepsilon(t) \overset{S}{\rightharpoonup} u(t)$ *which is the solution of the gradient flow for* $\mathcal{E}$ *with respect to the structure of* $T\mathcal{N}$*.*

This is applicable to our case since (69) with the nonlinear boundary condition (70) is the gradient flow of $\mathcal{F}^\varepsilon$ with respect to the norm $\sqrt{\lambda_\varepsilon}\,\|\cdot\|_{L^2}$, which we will use as the spaces X_ε in the terminology of the theorem above. The functionals $\mathcal{F}^\varepsilon$ are defined on $\mathcal{M} = H^1(\Omega)$. As the sense of convergence, we use $v_\varepsilon \overset{S}{\rightharpoonup} (a_i)$ if $v_\varepsilon \to v_*(a_i, d_i)$ in $L^2(\partial\Omega)$. The necessary Γ-convergence for $\overset{S}{\rightharpoonup}$ follows from (46).

The limit functional is defined on $\mathcal{N} = \{(a_i)_{i=1,\dots,N} : a_i \neq a_j \text{ for } i \neq j\}$, which is an open subset of the (flat) Riemannian manifold $(\partial\Omega)^N$. The approach of [40] for Euclidean limit spaces carries over to this situation without changes. As the limiting norm on the tangent spaces $T_a\mathcal{N}$ which are identified with $\mathbb{R}^N$ we use the constant Riemannian metric $\sqrt{\frac{\pi}{2}}\,\|\cdot\|_{\mathbb{R}^N}$.

Theorem 10 allows us to break up the proof of Theorem 9 into two separate parts, a lower bound and a construction. The proof of the lower bound relies on an anisotropic version of (43) in higher dimensions. This leads to a product estimate like the one of Sandier-Serfaty [41], which can then be used to separate space- and time-derivatives to show (79).

The construction used to show the upper bound inequalities is done by taking a well-prepared sequence and "pushing" the vortices along the boundary with the flow of a vector field that is conformal close to the vortex. The conformality ensures that the highest order of the energy does not change by the flow. With some more detailed local estimates related to (46), the estimates (80) and (81) then follow, as is detailed in [25].

4 Boundary vortices in a refined model

In this section we discuss a thin-film regime that is related to the theory of the the previous section; indeed the theory of Section 3 can be regarded as a simplified version of what is to follow (but, as mentioned earlier, it can also be seen as an asymptotic analysis of a model arising in the thin-film theory of Kohn and Slastikov [23]). We now examine the development of boundary vortices in thin films for a model that is closer to the actual micromagnetic model than the one discussed previously, although we still use some simplifications. In particular we consider now domains that are three-dimensional (but thin in one dimension) and magnetization vector fields with values in $\mathbb{S}^2$ (not $\mathbb{S}^1$). We continue to neglect the anisotropy term in the micromagnetic energy functional and the external magnetic field, but we consider the exchange energy and the magnetostatic energy in the form that they have in the functional E.

Naturally, the problem becomes more difficult when we drop some of the simplifications. It is not surprising, therefore, that we need more assumptions to obtain less information about the asymptotic behavior of the magnetization. But the results we find for this model are consistent with those for the more simplified model, which shows that the latter does indeed describe the significant features of the thin-film limit in the asymptotic regime we study.

We consider the family of domains

$$\Omega_\delta = \Omega \times (0, \delta)$$

for $\delta > 0$, where $\Omega \subset \mathbb{R}^2$ is open and bounded. We also assume that Ω is simply connected and that its boundary is smooth. The outer normal vector on $\partial\Omega$ is denoted by ν. The energy functional (without the anisotropy term and the external field) is then

$$E(\mathbf{m}) = \frac{d^2}{2} \int_{\Omega_\delta} |\nabla \mathbf{m}|^2 \, d\mathbf{x} + \frac{1}{2} \int_{\mathbb{R}^3} |\nabla u|^2 \, d\mathbf{x},$$

where, as usual, the function $u \in H^1(\mathbb{R}^3)$ is determined by the condition

$$\Delta u = \operatorname{div}(\chi_{\Omega_\delta} \mathbf{m}) \quad \text{in } \mathbb{R}^3.$$

We assume for the moment that the shape of Ω is fixed, whereas its size can still be varied by scaling. The problem then involves three length scales: the exchange length d, the thickness δ, and the length scale of the cross-section, measured, e.g., by $L = \operatorname{diam} \Omega$. The asymptotic regime we consider is characterized by the condition that d^2 is of the same magnitude as $L\delta$. For simplicity, we assume $d^2 = L\delta$. Rescaling Ω allows us to normalize $L = 1$, which gives rise to the relation $d^2 = \delta$ between the exchange length and the thickness (and which means that Ω is fixed henceforth).

Since we study the asymptotic behavior of variational problems associated to the micromagnetic energy as $\delta \searrow 0$, we now denote a generic magnetization vector field in Ω_δ by $\mathbf{m}^\delta = (m^\delta, m_3^\delta) \in H^1(\Omega_\delta, \mathbb{S}^2)$. The corresponding potential for the induced magnetic field is then the unique solution $u^\delta \in H^1(\mathbb{R}^3)$ of

$$\Delta u^\delta = \operatorname{div}(\chi_{\Omega_\delta} \mathbf{m}^\delta) \quad \text{in } \mathbb{R}^3. \tag{82}$$

We also consider the maps

$$\overline{\mathbf{m}}^\delta = \frac{1}{\delta} \int_0^\delta \mathbf{m}^\delta(x, s) \, ds, \quad x \in \Omega,$$

so that we can pass to a limit in certain spaces of functions on Ω (usually Sobolev spaces). The limit will then be a map $\mathbf{m} = (m, m_3) : \Omega \to \mathbb{S}^2$. If we consider the functions $\delta^{-1}u^\delta$ and apply equation (82) to a test function $\phi \in C_0^\infty(\mathbb{R}^3)$, we can formally pass to the limit. We obtain (formally) a limit function $u : \mathbb{R}^3 \to \mathbb{R}$ with

$$\int_{\mathbb{R}^3} \nabla u \cdot \nabla \phi \, d\mathbf{x} = \int_\Omega \mathbf{m}(x) \cdot \nabla \phi(x, 0) \, dx \tag{83}$$

for all $\phi \in C_0^\infty(\mathbb{R}^3)$. For the problems we consider here, we have typically $m_3 = 0$ in Ω and $m \cdot \nu = 0$ on $\partial\Omega$. If we have furthermore $m \in W^{1,4/3}(\Omega, \mathbb{S}^1)$, then equation (83) does in fact determine a function $u \in H^1(\mathbb{R}^3)$ uniquely. We will see that this function describes in part the limit of the magnetostatic energy. Also important is its trace on $\Omega \times \{0\}$. We denote this trace by u_0.

It is convenient to divide the micromagnetic energy by δ^2. That is, we consider the family of functionals

$$E_\delta(\mathbf{m}^\delta) = \frac{1}{2\delta} \int_{\Omega_\delta} |\nabla \mathbf{m}^\delta|^2 \, d\mathbf{x} + \frac{1}{2\delta^2} \int_{\mathbb{R}^3} |\nabla u^\delta|^2 \, d\mathbf{x}.$$

Critical points of E_δ satisfy the Euler-Lagrange equation

$$\delta(\Delta \mathbf{m}^\delta + |\nabla \mathbf{m}^\delta|^2 \mathbf{m}^\delta) - (\mathbf{1} - \mathbf{m}^\delta \otimes \mathbf{m}^\delta) \nabla u^\delta = 0 \quad \text{in } \Omega_\delta. \tag{84}$$

It is natural to impose homogeneous Neumann boundary conditions, i.e.,

$$\frac{\partial \mathbf{m}^\delta}{\partial x_3} = 0 \quad \text{in } \Omega \times \{0, \delta\}, \tag{85}$$

$$\frac{\partial \mathbf{m}^\delta}{\partial \nu} = 0 \quad \text{on } \partial\Omega \times (0, \delta). \tag{86}$$

Stable critical points also satisfy

$$\left.\frac{d^2}{ds^2}\right|_{s=0} E_\delta \left(\frac{m + s\psi}{|m + s\psi|} \right) \geq 0$$

for all $\psi \in C^\infty(\overline{\Omega_\delta}, \mathbb{R}^3)$. Standard calculations transform this inequality into

$$0 \leq \int_{\Omega_\delta} (|\nabla \dot{\mathbf{m}}^\delta|^2 - |\dot{\mathbf{m}}^\delta|^2 (|\nabla \mathbf{m}^\delta|^2 + \delta^{-1} \mathbf{m}^\delta \cdot \nabla u^\delta) \, d\mathbf{x} + \frac{1}{\delta} \int_{\mathbb{R}^3} |\nabla u^\delta|^2 \, d\mathbf{x}, \tag{87}$$

where $\dot{\mathbf{m}}^\delta = (\mathbf{1} - \mathbf{m}^\delta \otimes \mathbf{m}^\delta)\psi$.

For the Landau-Lifshitz equation, there exist two interesting time scales, similarly as in the previous section. The first one gives rise to the equation

$$\frac{\partial \mathbf{m}^\delta}{\partial t} = -\hat{\gamma} \mathbf{m}^\delta \wedge (\Delta \mathbf{m}^\delta - \delta^{-1} \nabla u^\delta) - \hat{\alpha} \mathbf{m}^\delta \wedge (\mathbf{m}^\delta \wedge (\Delta \mathbf{m}^\delta - \delta^{-1} \nabla u^\delta)) \tag{88}$$

in $\Omega_\delta \times (0, T)$, which is equivalent to

$$R_{\mathbf{m}^\delta} \frac{\partial \mathbf{m}^\delta}{\partial t} = \Delta \mathbf{m}^\delta + |\nabla \mathbf{m}^\delta|^2 \mathbf{m}^\delta - \frac{1}{\delta} (\mathbf{1} - \mathbf{m}^\delta \otimes \mathbf{m}^\delta) \nabla u^\delta. \tag{89}$$

Here we use the abbreviation

$$R_{\mathbf{m}^\delta} X = \tilde{\alpha} X + \tilde{\gamma} \mathbf{m}^\delta \wedge X,$$

and

$$\tilde{\alpha} = \frac{\hat{\alpha}}{\hat{\alpha}^2 + \hat{\gamma}^2}, \quad \tilde{\gamma} = \frac{\hat{\gamma}}{\hat{\alpha}^2 + \hat{\gamma}^2}.$$

We normally use the form (89) of the equation. We impose homogeneous Neumann boundary data again, that is,

$$\frac{\partial \mathbf{m}^\delta}{\partial x_3} = 0 \quad \text{in } \Omega \times \{0, \delta\} \times [0, T), \tag{90}$$

$$\frac{\partial \mathbf{m}^\delta}{\partial \nu} = 0 \quad \text{on } \partial\Omega \times (0, \delta) \times [0, T). \tag{91}$$

This is the time scale where we expect the development of stationary boundary vortices in the limit $\delta \searrow 0$, in analogy to the results of the previous section. To study the dynamical behavior of the vortices, on the other hand, we need to rescale the time axis by the factor $\log\log\frac{1}{\delta}$ (accelerating the time by this factor). The equation then becomes

$$\frac{R_{\mathbf{m}^\delta}\frac{\partial \mathbf{m}^\delta}{\partial t}}{\log\log\frac{1}{\delta}} = \Delta \mathbf{m}^\delta + |\nabla \mathbf{m}^\delta|^2 \mathbf{m}^\delta - \frac{1}{\delta}(\mathbf{1} - \mathbf{m}^\delta \otimes \mathbf{m}^\delta)\nabla u^\delta \quad \text{in } \Omega_\delta \times (0, T). \tag{92}$$

The boundary conditions remain of the form (90), (91).

We want to reproduce the asymptotic theory of the previous section for the model given by the energy E_δ and the equations (84), (89), and (92). There are several additional difficulties here, however, that we have to overcome. First, the target space for our maps is now $\mathbb{S}^2$, not $\mathbb{S}^1$, which means that $\mathbf{m}^\delta$ can no longer be represented by a single phase function. The curvature of $\mathbb{S}^2$ also has the consequence that we have to consider equations with nonlinear terms involving first derivatives of $\mathbf{m}^\delta$. Together with the fact that our domains Ω_δ are now three-dimensional, this means that we must expect solutions of the equations with singularities. To simplify the presentation of the results, we always assume here that we have smooth solutions; but without this assumption, regularity is an issue that requires extra care.

The most important new aspect of this model, however, is the nonlocal operator appearing in the magnetostatic energy. This is at first a major impediment to using the methods from the theory of Ginzburg-Landau vortices, for these methods require pointwise comparisons between integrands of the lower order energy terms. To overcome this difficulty, we compare E_δ with another functional that has only local terms, namely

$$F_\delta(\mathbf{m}^\delta) = \frac{1}{2\delta}\int_{\Omega_\delta}\left(|\nabla \mathbf{m}^\delta|^2 + \frac{(m_3^\delta)^2}{\delta}\right) d\mathbf{x} + \frac{\log\frac{1}{\delta}}{2\delta}\int_{\partial\Omega\times(0,\delta)} (m^\delta \cdot \nu)^2 \, d\mathcal{H}^2.$$

Here $\mathcal{H}^k$ denotes the k-dimensional Hausdorff measure. The functional F_δ can be thought of as the three-dimensional equivalent of

$$G_\delta(\mathbf{m}^\delta) = \frac{1}{2}\int_{\Omega}\left(|\nabla \mathbf{m}^\delta|^2 + \frac{(m_3^\delta)^2}{\delta}\right) dx + \frac{1}{2}\log\frac{1}{\delta}\int_{\partial\Omega} (m \cdot \nu)^2 \, d\mathcal{H}^1,$$

where for the latter functional, we consider $\mathbf{m}^\delta \in H^1(\Omega, \mathbb{S}^2)$.

The connection to the theory discussed earlier is obvious. The connection to the theory of Ginzburg-Landau vortices becomes even more apparent when one observes that $(m_3^\delta)^2 = 1 - |m^\delta|^2$ (since $\mathbf{m}^\delta$ has values in the unit sphere), recovering thus an integrand in the first integral of G_δ that is similar to the one used in most works on Ginzburg-Landau vortices. Less obvious is the connection between E_δ and F_δ. Before we give any rigorous arguments, we look at this question heuristically. The magnetostatic energy seeks to minimize the divergence of $\chi_{\Omega_\delta}\mathbf{m}^\delta$, which consists of two parts: the divergence of $\mathbf{m}^\delta$ in the interior of Ω_δ on the one hand, and the distribution given by the perpendicular part of $\mathbf{m}^\delta$ on $\partial\Omega_\delta$ on the other hand. The latter further splits into two parts according to the natural decomposition of the boundary into $\Omega \times \{0, \delta\}$ and $\partial\Omega \times (0, \delta)$. The part coming from $\operatorname{div} \mathbf{m}^\delta$ now gives a contribution to the magnetostatic energy which is of the same order as the exchange energy. Both of the other parts correspond to one of the terms in F_δ.

The next few lemmas give a more precise description of the relation between these functionals.

Lemma 6. *For $\delta \in (0, e^{-e}]$ and $\mathbf{m}^\delta \in H^1(\Omega_\delta, \mathbb{R}^3)$, the inequality*

$$\begin{aligned}\|\nabla u^\delta\|^2_{L^2(\mathbb{R}^3)} &\leq C\sqrt{\delta}\|\nabla \mathbf{m}^\delta\|^2_{L^{4/3}(\Omega_\delta)} + C\|m_3^\delta(\,\cdot\,,0)\|^2_{L_{4/3}(\Omega)} \\ &\quad + C\int_{\partial\Omega} \|\chi_{(0,\delta)} m^\delta(x,\,\cdot\,)\cdot\nu(x)\|^2_{H^{-1/2}(\mathbb{R})} d\mathcal{H}^1(x)\end{aligned}$$

holds for a constant C that depends only on Ω. Here $u^\delta \in H^1(\mathbb{R}^3)$ is the function determined by (82).

Proof. We have

$$\begin{aligned}\int_{\mathbb{R}^3} |\nabla u^\delta|^2\, d\mathbf{x} &= \int_{\Omega_\delta} \nabla u^\delta \cdot \mathbf{m}^\delta\, d\mathbf{x} \\ &= \left(\int_{\Omega\times\{\delta\}} - \int_{\Omega\times\{0\}}\right) u^\delta m_3^\delta\, dx + \int_{\partial\Omega}\int_0^\delta u^\delta m^\delta \cdot \nu\, dx_3\, d\mathcal{H}^1 \\ &\quad - \int_0^\delta \int_\Omega u^\delta \operatorname{div} \mathbf{m}^\delta\, dx\, dx_3.\end{aligned}$$

Now we use the continuous trace operators $H^1(\mathbb{R}^3) \to L^4(\Omega)$ for every slice $\Omega \times \{x_3\}$ and the continuous trace operator $H^1(\Omega) \to L^2(\partial\Omega, H^{1/2}(-1,1))$ to estimate the traces of u in these spaces. Furthermore, an integration of $\frac{\partial m_3^\delta}{\partial x_3}$ along vertical lines gives

$$\|m_3^\delta(\,\cdot\,,0) - m_3^\delta(\,\cdot\,,\delta)\|^2_{L^{4/3}(\Omega)} \leq \sqrt{\delta}\|\nabla \mathbf{m}\|^2_{L^{4/3}(\Omega_\delta)},$$

and the desired estimate then follows from the Hölder inequality. □

Lemma 7. *For $\delta \in (0, e^{-e}]$, let $m^\delta \in H^1(\Omega, \mathbb{S}^1)$ and $\mathbf{m}^\delta(x, x_3) = (m^\delta(x), 0)$. Then*

$$\|\nabla u^\delta\|_{L^2(\mathbb{R}^3)} \leq C\delta^2 \left(\|\nabla m^\delta\|^2_{L^{4/3}(\Omega)} + \log\frac{1}{\delta}\,\|m^\delta \cdot \nu\|^2_{L^2(\partial\Omega)}\right)$$

for a constant C that depends only on Ω.

Proof. A direct computation shows that the characteristic function $\chi_{(0,\delta)}$ of the interval $(0,\delta)$ satisfies

$$\|\chi_{(0,\delta)}\|_{H^{-1/2}(\mathbb{R})} \leq c\delta^2 \left(1 + \log \frac{1}{\delta}\right)$$

for a certain constant c which is independent of δ. The claim now follows directly from Lemma 6. □

Proposition 17. *There exists a constant C, dependent only on Ω, such that*

$$\inf_{H^1(\Omega_\delta,\mathbb{S}^2)} E_\delta \leq \pi \log\log \frac{1}{\delta} + C$$

for $\delta \in (0, e^{-e}]$.

Proof. We construct a map $m^\delta \in H^1(\Omega, \mathbb{S}^1)$ with two standard vortices centered at two different points on the boundary, similarly as in the proof of Proposition 7 (with $\epsilon = 1/\log\frac{1}{\delta}$). For $\mathbf{m}^\delta(x, x_3) = (m^\delta(x), 0)$, the estimates of Proposition 7, together with Lemma 7, give a bound for $E_\delta(\mathbf{m}^\delta)$ of the desired form. □

In fact the quantity $\pi \log\log \frac{1}{\delta}$ gives also a lower bound for the infimum of E_δ in $H^1(\Omega_\delta, \mathbb{S}^2)$ up to a constant. That is, it determines the asymptotic behavior of this infimum. The proof of the lower estimate is technically more involved; we therefore give only a sketch of the proof here.

Lemma 8. *For $\delta \in (0, e^{-e}]$ and $\mathbf{m}^\delta \in H^1(\Omega_\delta, \mathbb{S}^2)$, the inequality*

$$\int_{\Omega_\delta} (m_3^\delta)^2 \, d\mathbf{x} \leq C \left[\delta \int_{\Omega_\delta} |\nabla m_3^\delta|^2 \, d\mathbf{x} + \int_{\mathbb{R}^3} |\nabla u^\delta|^2 \, d\mathbf{x} + \delta^2\right] \tag{93}$$

holds for a constant C that depends only on Ω.

Sketch of the proof. We test (82) with a function $\phi \in C^{0,1}(\mathbb{R}^3)$ which is defined on $\Omega \times \mathbb{R}$ by

$$\phi(x, x_3) = \begin{cases} 0, & \text{if } x_3 \leq 0 \text{ or } x > 2\delta, \\ \int_0^{x_3} m_3(x,s)\, ds & \text{if } 0 < x_3 \leq \delta, \\ (2 - x_3/\delta) \int_0^\delta m_3(x,s)\, ds & \text{if } \delta < x_3 \leq 2\delta, \end{cases}$$

and extended suitably to $\mathbb{R}^3$. We recover the left-hand side of (93) as one of the terms in the resulting equation (after an integration by parts). All other terms can then be estimated with standard methods. □

For $s \geq 0$ we now define the sets

$$\begin{aligned} V_s &= \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) < s\}, & V_s^\delta &= V_s \times (0,\delta), \\ \Gamma_s &= \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) = s\}, & \Gamma_s^\delta &= \Gamma_s \times (0,\delta). \end{aligned}$$

We fix $s_0 > 0$ such that Γ_s is a smooth curve for every $s \in (0, 2s_0]$. Moreover, we define

$$\kappa(\delta) = \frac{1}{\log \frac{1}{\delta}}.$$

Lemma 9. *There exists a constant C, depending only on Ω, such that for every $\delta \in (0, e^{-e}]$ and every $s \in [0, s_0]$ with $s \leq \kappa(\delta)$, the inequality*

$$\log \frac{1}{\delta} \int_{\Gamma_s^\delta} (m^\delta \cdot \nu)^2 \, d\mathcal{H}^2 \quad \leq \quad C \int_{V_{\kappa(\delta)}^\delta} \left(|\nabla \mathbf{m}^\delta|^2 + \frac{(m_3^\delta)^2}{\delta} \right) d\mathbf{x} + \frac{C}{\delta} \int_{\mathbb{R}^3} |\nabla u^\delta|^2 \, d\mathbf{x} + C\delta \tag{94}$$

is satisfied for any $\mathbf{m}^\delta \in H^1(\Omega_\delta, \mathbb{S}^2)$.

Sketch of the proof. The idea is to test (82) with a suitably constructed function ϕ satisfying $\phi = m^\delta \cdot \nu$ on Γ_0^δ and supported on a $\kappa(\delta)$-neighborhood of Γ_0^δ. An integration by parts on one side of the resulting equation then yields, among other terms, the left-hand side of (94) for $s = 0$. A careful estimate of the other terms gives the required inequality for Γ_0^δ. To obtain the corresponding inequality for other values of s, integrate the derivative of $(m^\delta \cdot \nu)^2$ along rays in the direction of $-\nu$. □

Proposition 18. *For any $K \in \mathbb{R}$ there exists a constant C, depending only on Ω and K, such that the following holds. Suppose $\delta \in (0, e^{-e}]$ and $s \in [0, s_0]$. If $\mathbf{m}^\delta \in H^1(\Omega_\delta, \mathbb{S}^2)$ satisfies*

$$E_\delta(\mathbf{m}^\delta) \leq \pi \log \log \frac{1}{\delta} + K, \tag{95}$$

then

$$\frac{1}{\delta} \int_{\Omega_\delta} \left(|\nabla m_3^\delta|^2 + \left| \frac{\partial \mathbf{m}^\delta}{\partial x_3} \right|^2 + \frac{(m_3^\delta)^2}{\delta} \right) d\mathbf{x} + \frac{1}{\delta} \int_{V_{\kappa(\delta)}^\delta} |\nabla \mathbf{m}^\delta|^2 \, d\mathbf{x} + \frac{1}{\delta^2} \int_{\mathbb{R}^3} |\nabla u^\delta|^2 \, d\mathbf{x} + \frac{\log \frac{1}{\delta}}{\delta} \int_{\Gamma_s^\delta} (m^\delta \cdot \nu)^2 \, d\mathcal{H}^2 \quad \leq \quad C. \tag{96}$$

Sketch of the proof. With the arguments from Section 3, combined with similar arguments from the theory of Ginzburg-Landau vortices, applied to slices of the form $\Omega \backslash V_s \times \{t\}$, we obtain the estimate

$$\frac{1}{2\delta} \int_{\Omega_\delta \backslash V_s^\delta} \left(\left| \frac{\partial m^\delta}{\partial x_1} \right|^2 + \left| \frac{\partial m^\delta}{\partial x_2} \right|^2 + \frac{(m_3^\delta)^2}{\delta} \right) d\mathbf{x} + \frac{\log \frac{1}{\delta}}{\delta} \int_{\Gamma_s^\delta} (m^\delta \cdot \nu)^2 \, d\mathcal{H}^2 \quad \geq \quad \pi \log \log \frac{1}{\delta} - C_1 \tag{97}$$

for a certain constant C_1 that depends only on Ω. Combining this with Lemma 8, Lemma 9, and (95), we obtain the desired inequality. □

Proposition 19. *There exists a constant C, dependent only on Ω, such that*

$$\inf_{H^1(\Omega_\delta, \mathbb{S}^2)} E_\delta \geq \pi \log \log \frac{1}{\delta} - C$$

for $\delta \in (0, e^{-e}]$.

Proof. Choose a minimizer $\mathbf{m}^\delta$ of E_δ in $H^1(\Omega_\delta, \mathbb{S}^2)$, then (95) holds for a certain constant K by Proposition 17. Thus $\mathbf{m}^\delta$ satisfies (96) and (97), and the claim follows. □

Proposition 17 and Proposition 18 together describe the asymptotic behavior of the minimal energy as $\delta \searrow 0$ up to an additive constant, and it is the behavior we expect also for the functionals F_δ or G_δ. Moreover, if $\mathbf{m}^\delta$ is independent of x_3, we can estimate each term in E_δ by a combination of terms in G_δ, and vice versa, according to Lemmas 6–9. This already relates the asymptotic regime studied here to the model used in Section 3. We discover more similarities, however, when we study the asymptotic behavior of critical points of E_δ or solutions of the Landau-Lifshitz equations (89) and (92)

If we have a family of solutions $\mathbf{m}^\delta$ of one of the variational problems associated to E_δ, it is natural to apply variants of the usual arguments from the theory of Ginzburg-Landau vortices to the slices $\Omega \times \{s\}$ with $0 < s < \delta$. More precisely, we use arguments like those discussed in the previous section for the behavior near $\partial\Omega$, and arguments from the theory of Bethuel, Brezis, and Hélein [4, 5] for the behavior in the interior of Ω. This is the key element in the proofs of each of the results that follow. We omit a detailed presentation of these proofs (since similar arguments have been discussed earlier), but we give a brief discussion of some additional arguments that are needed in each case. For the complete proofs, the reader is referred to [34, 35, 36].

Theorem 11. *For $\delta \in (0, e^{-e}]$, suppose $\mathbf{m}^\delta \in C^\infty(\Omega, \mathbb{S}^2)$ are stable critical points of E_δ, i.e., solutions of (84) satisfying the boundary conditions (85) and (86), such that (87) holds for every $\psi \in C^\infty(\overline{\Omega}_\delta, \mathbb{R}^3)$. Suppose further that there exists a number K such that*

$$E_\delta(\mathbf{m}^\delta) \leq \pi \log\log \frac{1}{\delta} + K$$

for every δ. Then there exist a sequence $\delta_k \searrow 0$, two distinct points $x^1, x^2 \in \partial\Omega$, and a map $\mathbf{m} = (m, 0) \in W^{1,1}(\Omega, \mathbb{S}^1 \times \{0\})$ with $m \cdot \nu = 0$ on $\partial\Omega$, such that the following holds.

(i) *For any $p < 2$, the sequence $\{\overline{\mathbf{m}}_{\delta_k}\}$ converges weakly in $W^{1,p}(\Omega, \mathbb{R}^3)$ to $\mathbf{m}$. The convergence also holds weakly in $H^1(\Omega', \mathbb{R}^3)$ for every $\Omega' \subset \Omega$ with $\overline{\Omega'} \subset \overline{\Omega} \backslash \{x^1, x^2\}$.*

(ii) *The limit map m satisfies*

$$\Delta m + |\nabla m|^2 m - \nabla u_0 + (m \cdot \nabla u_0) = 0 \quad \text{in } \Omega, \tag{98}$$

where u_0 is the trace on $\Omega \times \{0\}$ of the function determined by (83).

(iii) *If $\mathbb{R}^2$ is identified with the complex plane $\mathbb{C}$ by $z = x_1 + ix_2$ (and similarly $z^1 = x_1^1 + ix_2^1$ and $z^2 = x_1^2 + ix_2^2$ for $x^1 = (x_1^1, x_2^1)$ and $x^2 = (x_1^2, x_2^2)$), then m has the representation*

$$m(z) = \frac{z - z^1}{|z - z^1|} \frac{z - z^2}{|z - z^2|} e^{i\theta(z)}$$

for a function $\theta \in C^0(\overline{\Omega})$ which solves

$$\Delta\theta = m_1 \frac{\partial u_0}{\partial x_2} - m_2 \frac{\partial u_0}{\partial x_1} \quad \text{in } \Omega. \tag{99}$$

Thus at least at the lowest possible energy level, we observe the development of two boundary vortices in the limit. Note also that the limit equation (98) is formally the Euler-Lagrange equation for the (formal) functional

$$\frac{1}{2}\int_{\Omega} |\nabla m|^2 \, dx + \frac{1}{2}\int_{\mathbb{R}^3} |\nabla u|^2 \, d\mathbf{x},$$

where u denotes the function in $H^1(\mathbb{R}^3)$ defined by (83). It turns out, however, that this quantity is identically infinite. It can be replaced by a functional involving the Dirichlet energy of θ, and then (98) truly becomes an Euler-Lagrange equation, but we omit the details here.

The stability condition (87) is needed in the proof of this theorem in order to estimate the Dirichlet energy of $\mathbf{m}^\delta$ in small cylinders of the form

$$(B_{\mu\sqrt{\delta}}(x) \cap \Omega) \times (0, \delta),$$

where $\mu > 0$ is a fixed constant. This allows to use estimates from the regularity theory of harmonic maps and to conclude that

$$|\nabla \mathbf{m}^\delta| \leq \frac{C}{\sqrt{\delta}}$$

for a certain constant C which is independent of δ. Apart from the fact that such a gradient estimate is normally used in the theory of Ginzburg-Landau vortices, it is in this context also important for another reason: It means that $\mathbf{m}^\delta$ varies very little in the third direction if δ is small, for the thickness of Ω_δ is small compared with $|\nabla \mathbf{m}^\delta|$. This permits to work on a suitable slice $\Omega \times \{s\}$ and pretend that the domain is two-dimensional for much of the proof. The previously mentioned arguments then give (i) and the representation of m in (iii).

To derive the limit equation (98), we first take the vector product with $\mathbf{m}^\delta$ on both sides of (84), which gives

$$\delta \operatorname{div}(\mathbf{m}^\delta \wedge \nabla \mathbf{m}^\delta) = \mathbf{m}^\delta \wedge \nabla u^\delta = 0 \quad \text{in } \Omega_\delta.$$

This form of the equation has the advantage that it does no longer explicitly contain the term $|\nabla \mathbf{m}^\delta|^2 \mathbf{m}^\delta$ (which would be difficult to handle with the weak convergence that we have). In particular it is then possible to pass to the limit and to show that m satisfies

$$\operatorname{div}(m_1 \nabla m_2 - m_2 \nabla m_1) = m_1 \frac{\partial u_0}{\partial x_2} - m_2 \frac{\partial u_0}{\partial x_1} \quad \text{in } \Omega.$$

This equation is exactly (99) if m is represented by θ as in (iii). Finally, the equation is also equivalent to (98).

Theorem 12. *For $T \in (0, \infty]$ and $\delta \in (0, e^{-e}]$, suppose $\mathbf{m}^\delta \in C^\infty(\Omega_\delta \times [0, T), \mathbb{S}^2)$ satisfy the Landau-Lifshitz equations (92) with boundary conditions (90), (91). Also suppose that the initial data*

$$\hat{\mathbf{m}}^\delta(\mathbf{x}) = \mathbf{m}^\delta(\mathbf{x}, 0)$$

satisfy

$$E_\delta(\hat{\mathbf{m}}^\delta) \leq \pi \log \log \frac{1}{\delta} + K$$

and

$$|\nabla \hat{\mathbf{m}}^\delta| \leq \frac{K}{\sqrt{\delta}}$$

in Ω_δ for a constant K that is independent of δ. Then there exist a sequence $\delta_k \searrow 0$, a map $\mathbf{m} = (m, 0) \in L^\infty([0,T), W^{1,1}(\Omega, \mathbb{S}^1 \times \{0\}))$ with $m \cdot \nu = 0$ on $\partial\Omega \times [0,T)$, and two distinct points $x^1, x^2 \in \partial\Omega$, such that the following holds.

(i) *The sequence $\{\overline{\mathbf{m}}^{\delta_k}\}$ converges weakly* in $L^\infty([0,T), W^{1,p}(\Omega, \mathbb{R}^3))$ to $\mathbf{m}$ for any $p < 2$, and also weakly* in $L^\infty([0,T), H^1(\Omega', \mathbb{R}^3))$ for any $\Omega' \subset \Omega$ with $\overline{\Omega'} \subset \overline{\Omega} \backslash \{x^1, x^2\}$.*

(ii) *The limit map solves*

$$\tilde{\alpha} \frac{\partial m}{\partial t} = \Delta m + |\nabla m|^2 m - \nabla u_0 + (m \cdot \nabla u_0) m \quad \text{in } \Omega \times (0,T), \qquad (100)$$

where $u_0(\,\cdot\,, t)$ is the trace on $\Omega \times \{0\}$ of the function determined by (83) for almost every fixed $t \in [0,T)$.

(iii) *It is of the form*

$$m(z,t) = \frac{z - z^1}{|z - z^1|} \frac{z - z^2}{|z - z^2|} e^{i\theta(z,t)},$$

where $\theta \in C^0(\overline{\Omega} \times [0,T))$ is a solution of

$$\tilde{\alpha} \frac{\partial \theta}{\partial t} = \Delta \theta - m_1 \frac{\partial u_0}{\partial x_2} + m_2 \frac{\partial u_0}{\partial x_1} \quad \text{in } \Omega \times (0,T).$$

(Here we use an identification of $\mathbb{R}^2$ with $\mathbb{C}$ as in the previous theorem.)

This is the time scale where we have stationary boundary vortices. The limit equation (100) is formally the L^2-gradient flow for the formal functional mentioned earlier (up to a constant). It is also the true gradient flow for a related functional.

It is interesting here to compare the limit equation (100) with the original Landau-Lifshitz equation, especially if the latter is in the form (88). We had originally a gyromagnetic term with coefficient $\hat{\gamma}$ and a damping term with coefficient $\hat{\alpha}$. The gyromagnetic term has vanished in the thin-film limit (as it is to be expected when $\mathbf{m}$ remains in the plane $\mathbb{R}^2 \times \{0\}$). We still have a damping term, but the damping coefficient is now

$$\frac{1}{\tilde{\alpha}} = \hat{\alpha} + \frac{\hat{\gamma}^2}{\hat{\alpha}}.$$

Thinking of $\hat{\gamma}$ as a fixed constant and of $\hat{\alpha}$ as small in comparison, we are in the seemingly paradox situation that decreasing the damping coefficient $\hat{\alpha}$ accelerates the dynamics in the thin-film limit. This phenomenon has already been discovered by formal computations by W. E and C. García-Cervera [15].

With the gradient estimate that we impose on the initial data, we can use the same methods as in the proof of Theorem 11 to obtain the same development of boundary vortices for $\hat{\mathbf{m}}^\delta$ that we have found for stable critical points. To prove this for times $t > 0$, we need slightly different arguments. Here we calculate how the energy develops locally with time, that is, for a function $\eta \in C_0^\infty(\mathbb{R}^3)$ with $\eta = 0$ in a neighborhood of $\{x^1, x^2\} \times [0, \delta]$, we calculate

$$\frac{d}{dt}\left(\int_{\Omega_\delta} \eta |\nabla \mathbf{m}^\delta|^2 \, d\mathbf{x} + \frac{1}{\delta}\int_{\mathbb{R}^3} \eta |\nabla u^\delta|^2 \, d\mathbf{x}\right). \tag{101}$$

We find that away from the vortex center points x^1, x^2, the energy increases at most by a constant that is independent of δ in bounded time intervals. We can then again use arguments from the theory of Ginzburg-Landau vortices for fixed times $t > 0$.

Theorem 13. *Under the conditions of Theorem 12, but with (89) replaced by (92), there exist a sequence $\delta_k \searrow 0$, two curves $x^1, x^2 \in C^{0,1/2}([0,T), \partial\Omega)$, and a map*

$$\mathbf{m} = (m, 0) \in \bigcap_{p<2} L^\infty([0,T), W^{1,p}(\Omega, S^1 \times \{0\}))$$

with $m \cdot \nu = 0$ on $\partial\Omega \times [0,T)$, such that the following holds.

(i) *For any $p < 2$ and any $q < \infty$, the sequence $\{\overline{\mathbf{m}}^{\delta_k}\}$ converges weakly in $L^q([0,T), W^{1,p}(\Omega, \mathbb{R}^3))$ to $\mathbf{m}$.*

(ii) *For almost every $t \in [0,T)$ and every $\Omega' \subset \Omega$ with $\overline{\Omega'} \subset \overline{\Omega} \backslash \{x^1(t), x^2(t)\}$, the map $m(\,\cdot\,, t)$ belongs to $H^1(\Omega', \mathbb{S}^1)$.*

(iii) *The equation*

$$\Delta m + |\nabla m|^2 m - \nabla u_0 + (m \cdot \nabla u_0) m = 0 \quad \text{in } \Omega \times (0,T)$$

holds, where $u_0(\,\cdot\,, t)$ is the trace on $\Omega \times \{0\}$ of the function determined by (83) for almost every fixed $t \in [0,T)$.

(iv) *The map m is of the form*

$$m(z,t) = \frac{z - z^1(t)}{|z - z^1(t)|} \frac{z - z^2(t)}{|z - z^2(t)|} e^{i\theta(z,t)},$$

where $\theta \in C^0(\overline{\Omega} \times [0,T])$ is a solution of

$$\Delta\theta = m_1 \frac{\partial u_0}{\partial x_1} - m_2 \frac{\partial u_0}{\partial x_2} \quad \text{in } \Omega \times (0,T).$$

In contrast to the situation of Theorem 12, we now have moving boundary vortices. This means in particular that when we calculate the evolution of the localized energy in (101), the vortex centers may enter the support of η after some time. For this reason, the estimates we obtain are not quite as good as before, and the type of convergence we find is weaker. On the other hand, an analysis of the energy increase over a fixed time interval permits to estimate the distance that the vortex centers have moved in this time (since most of the micromagnetic energy is concentrated in the vortex centers). This way we obtain the Hölder continuity of x^1 and x^2.

Finally, comparing Theorem 13 with the results of Section 3, especially Theorem 9, we see that one statement is missing here: We do not have any information about the law that governs the motion of the vortices. There is no obvious reason why the model discussed here should have a significantly different behavior in this respect, but the technical difficulties mentioned earlier make it hard to carry over the arguments from the simpler model. This problem thus remains open.

Acknowledgment. The authors are grateful for the support of SPP 1095 through the grants De 772/1-1, De 772/1-2 and Mu 1067/6-3. They would like to thank A. DeSimone and S. Müller for many stimulating discussions at the MPI for Mathematics in the Sciences, and the latter in particular for his valuable suggestions and comments on this work.

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