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by

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STABILITY OF CALDERÓN INVERSE CONDUCTIVITY PROBLEM IN THE PLANE *

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Abstract

It is proved that, in two dimensions, Calderón inverse conductivity problem is stable when the conductivities are Hölder continuous with any exponent $\alpha > 0$. The approach is based on reducing the conductivity equation to a complex Beltrami equation as in Astala-Päivärinta proof of the uniqueness in Calderón problem for L^∞ conductivities.

1 Introduction

We consider a bounded domain Ω in $\mathbb{C}$ with connected complement. A well known problem that A.P. Calderón proposed was the determination of an isotropic L^∞ conductivity coefficient γ on Ω from boundary measurements.

These measurements are given by the Dirichlet to Neumann map defined for a function f on $\partial\Omega$ as the Neumann value $\Lambda_\gamma(f) = \gamma\frac{\partial}{\partial\nu}u$, where u is the solution of the Dirichlet boundary value problem

$$\begin{cases} \nabla\cdot(\gamma\nabla u) = 0 \\ u_{\partial\Omega} = f \end{cases} \tag{1.1}$$

and $\frac{\partial}{\partial\nu}$ denotes the outer normal derivative. The Dirichlet to Neumann map

$$\Lambda_\gamma : H^{1/2} \to H^{-1/2} \tag{1.2}$$

can be defined for such general domain and conductivities as

$$\langle\Lambda_\gamma(f), \varphi_0\rangle = \int_\Omega \gamma\nabla u\cdot\nabla\varphi \tag{1.3}$$

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where $\varphi \in W^{1,2}(\Omega)$ is such that $\varphi_{\partial\Omega} = \varphi_0$. The uniqueness of the inverse problem, that means the injectivity of the map

$$\gamma \to \Lambda_\gamma$$

has been completely solved in the two dimensional case by K. Astala and L. Päivärinta in [7] improving previous results [27] and [34]. In higher dimension the known results require some extra a priori regularity on γ, basically some control on $\frac{3}{2}$ derivatives of γ, see [33], [13], [29] and [16].

A relevant question (specially in applications) is the stability of the inverse problem, that is the continuity of the inverse map

$$\Lambda_\gamma \to \gamma.$$

For dimension $n > 2$ the known results are due to Alessandrini [2], [3], who proved stability in for $\gamma \in W^{2,\infty}$. In the planar case, $n = 2$ the situation is different. Liu proved stability for conductivities in $W^{2,p}$ with $p > 1$ in [24]. Recently, stability was obtained $\gamma \in \mathcal{C}^{1+\alpha}$ with $\alpha > 0$ in [10].

In this paper we prove that Hölder regularity of γ is enough to obtain stability:

Theorem 1.1. *Let Ω be a Lipschitz domain in the plane. Let $\gamma = \gamma_1, \gamma_2$ two planar conductivities satisfying*

- *(I) Ellipticity: $\|\frac{1-\gamma}{1+\gamma}\|_{L^\infty} < \kappa < 1$*
- *(II) α-regularity: $\gamma \in \mathcal{C}^\alpha(\bar{\Omega})$ with $\alpha > 0$ and with a priori bound $\|\gamma\|_{\mathcal{C}^\alpha} < \Lambda_0$.*

Then there exists a non decreasing continuous function $V : \mathbb{R} \to \mathbb{R}$ with $V(0) = 0$ such that

$$\|\gamma_1 - \gamma_2\|_{L^\infty(\Omega)} \leq V(\|\Lambda_{\gamma_1} - \Lambda_{\gamma_2}\|_{H^{1/2}(\partial\Omega) \to H^{-1/2}(\partial\Omega)}) \tag{1.4}$$

The function V can be taken

$$V(\rho) = C \log(\rho)^{-a}. \tag{1.5}$$

Where C and a depend on κ, α and Λ_0. An expression for a is given in (3.33).

Observe that the usual ellipticity condition for γ is

$$1/C \leq \gamma(x) \leq C,$$

for a certain constant $C > 0$. This is equivalent to condition (I). We use this formulation because the dilation $\mu = \frac{1-\gamma}{1+\gamma}$ will be the coefficient of a Beltrami equation to be introduced later.

An example, due to Alessandrini [2], shows that some extra regularity in γ is necessary to obtain stability. Alessandrini gives non continuous conductivities in L^∞ such that $\|\Lambda_1 - \Lambda_2\|_{H^{1/2}\to H^{-1/2}} \to 0$, meanwhile $\|\gamma_1 - \gamma_2\|_{L^\infty} = 1$. Namely, if we denote by $B_{r_o} = \{x \in \mathbf{R}^2, |x| < r_o\}$ the ball centered at the origin with radius r_o, take $\Omega = B_1$ the unit ball in $\mathbf{R}^2$, $\gamma_1 = 1$ and $\gamma_2 = 1 + \chi_{B_{r_o}}$, then $\|\gamma_1 - \gamma_2\|_{L^\infty(\Omega)} = 1$, but $\|\Lambda_{\gamma_1} - \Lambda_{\gamma_2}\|_{H^{\frac{1}{2}}\to H^{-\frac{1}{2}}} \leq 2r_o \to 0$ as $r_0 \to 0$. The details are given in [2]. Observe that in this case these conductivities actually lie in $W^{\epsilon,2}$ for any $\epsilon < \frac{1}{2}$. This example suggests that some control on the modulus of continuity of the conductivities is necessary to prove stability in the L^∞ norm. A natural choice is the Hölder continuity condition (II).

The stability we obtain is just logarithmic. Unfortunately, an argument of A.Mandache [26] shows that even for C^∞ conductivities this must be the case. Therefore the regularity of γ is just reflected on the constant a, see (3.33). An interesting problem is to determine what additional conditions on γ would imply a better stability, like Lipschitz or Hölder stability. Some answers in this direction are given in [4].

Finally, the Lipschitz regularity of the domain Ω is needed to recover the boundary values of γ and then to reduce the problem to $\Omega = \mathbb{D}$ and conductivities compactly supported there, see Section 6.

Since the foundational paper of Calderón [18] there has been an intensive research on this problem [22],[23]. It turns out that dimension $n = 2$ is very different from higher dimensions and special techniques have been developed to treat this case. There has been several approaches to the Calderón problem in the plane, all of which are based on the construction of the approximated exponential solutions, the so called complex geometric optics solutions, which have asymptotics e^{ikz} depending on the complex frequency k.

The first approach reduces the conductivity equation (1.1) to the Schrödinger equation

$$\Delta v + qv = 0,$$

where $q(x) = \frac{\Delta \gamma^{1/2}}{\gamma^{1/2}}$. This works in any dimensions, however in the plane the problem no is longer overdetermined and it can not be treated in the same way. Local uniqueness was obtained in [34] by using, as in higher dimensions, only bounds for the geometric optics solutions for large frequencies, but requiring the potential to be sufficiently close to 1 in $W^{3,\infty}$. Nachman [27] obtained uniqueness for conductivities in $W^{2,p}$, for $p > 1$, by studying the so called scattering transform $\mathbf{t}(\mathbf{q}, \mathbf{k})$ of the potential q. This transform is given essentially by the behavior at infinity of the complex geometric optics solutions. Using this method Liu, [24], obtained stability for γ with the same regularity.

The next approach is due to Brown and Uhlmann [17]. They use the inverse scattering method of Beals and Coifmann [11] for 2×2 matrices

used before in the study of well posedness of some non linear systems. The conductivity equation is reduced to the $\partial\overline{\partial}$ system

$$\begin{pmatrix} \overline{\partial} & 0 \\ 0 & \partial \end{pmatrix} \begin{pmatrix} v \\ w \end{pmatrix} = Q(x) \begin{pmatrix} v \\ w \end{pmatrix}$$

where

$$Q(x) = \begin{pmatrix} 0 & q(x) \\ \overline{q}(x) & 0 \end{pmatrix}$$

and

$$q(x) = -\frac{1}{2}\partial(\log\gamma) = -\frac{\partial\gamma^{1/2}}{\gamma^{1/2}}.$$

This reduction has the advantage of requiring only one derivative of the conductivity. Introducing the scattering transform of the matrix Q, $S(Q,k)$, the uniqueness is obtained for conductivities in $W^{1,p}$ for $p > 2$. In [10] it was shown that it is possible to quantify this approach and obtain stability for $\gamma \in \mathcal{C}^{1+\alpha}$.

The last approach is that of Astala-Päivärinta. The conductivity equation is then reduced to a complex Beltrami type equation

$$\partial_{\overline{z}} f = \mu\overline{\partial_z f},$$

where $\mu = \frac{\gamma-1}{\gamma+1}$ has also its correspondent scattering transform $\tau(\mu,k)$.

These three approaches share essentially the same philosophy. First the Dirichlet to Neumann data determine the corresponding scattering transform. Second the complex geometric optics are solutions in the k variable to the so called $\overline{\partial}$-equation which depends on the scattering transform. If there is uniqueness for this equation, the problem is solved. For stability one looks first for an explicit formula relating the difference of Dirichlet to Neumann maps to the difference of scattering transforms. Then one needs that the corresponding $\overline{\partial}$-equation in the k variable enjoys suitable a priori estimates.

The lack of regularity for μ in L^∞ makes the situation very delicate and topological arguments in both variables z and k are needed. In [7], the authors use also the special structure of the Beltrami equations to obtain a subexponential decay of the geometric optic solutions which depends strongly on the given conductivity μ. However, this decay is not enough to guarantee uniqueness for the corresponding equation in the k variable. This is overcome in [7] by combining the information in the z and k variables at the same time and making again use of the topological degree. This approach have proved to be effective also when dealing with the anisotropic case, see [9].

We use the approach of [7] to prove our stability Theorem 1.1. As in previous works in stability, our task is to replace qualitative uniqueness arguments by a priori estimates. For the first step, from the Dirichlet to

Neumann map to the scattering transform, we only need the conductivities to satisfy the ellipticity condition. We accomplish this step in two manners, the first by recovering the geometric optics solutions in the exterior of $\mathbb{D}$ by mean of non homogeneous Beltrami equations. The second manner is by showing an explicit formula for the difference of scattering transforms, which is valid for conductivities just in L^∞ and which might be of independent interest. The second step, from the scattering transform to the coefficients, is more complex since we need to achieve a priori estimates from topological arguments. The proof is much more involved than in the uniqueness case and, in particular, it needs a uniform control on the assymptotics in k for the geometric optics solutions, which we call uniform subexponential growth. This is one of the points at which the Hölder continuity of γ is essential.

A more precise description of the paper is as follows: In section 2 we obtain some a priori estimates for solutions of non homogeneous Beltrami equations and collect some results from [7] which are essential to our work. In section 3 we prove some additional properties of the geometric optic solutions in the k and z variables. In the z variable we obtain Schauder type estimates and uniform lower bounds for the Jacobians. In the k variable we obtain an uniform decay as $k \to \infty$. We need to prove as well that the solutions are uniformly Lipschitz with respect to k.

In section 4 we study the stability from the Dirichlet to Neumann map to the scattering transform. This section does not require the $\mathcal{C}^\alpha$- regularity of γ_j. In section 5 we prove the stability from scattering transform to the geometric optic solutions. It is needed to remark that from section 2 to section 5 we study the particular case $\Omega = \mathbb{D}$ and conductivities to be 1 in a neighborhood of $\partial\Omega$. The Theorem 1.1 is proved in section 6. The proof includes the reduction to the particular case of $\Omega = \mathbb{D}$ and compactly supported conductivities, see Proposition 6.1 We make this reduction in a different and more direct way than in previous works on the Calderón problem. Finally, for the convenience of the reader we include an appendix with regularity results for Beltrami type equation with Hölder continuous dilatations μ.

Let us finish the introduction with some further comments on the result and possibilities for future research. As opposite to [10], in the first step we do not need a global control of the scattering transform, which in the mentioned work was essential to assure uniqueness of the $\overline{\partial}$- equation in the second step. This gives Hölder type stability from the Dirichlet to Neumann map to local L^∞norm of the scattering transform. The logarithmic stability appears in the second part from the scattering transform to the coefficients.

Concerning future research, let us remark that besides the example of Alessandrini, the lack of regularity of the solutions for just L^∞ conductivities yields another reason why it would not be expectable to obtain L^∞ stability. In fact, for a given $\kappa < 1$ there exists a conductivity γ satisfying the ellipticity condition $\frac{\gamma-1}{\gamma+1} < \kappa$ such that the corresponding solution

satisfies that for every disc B in Ω

$$\int_B |\nabla u|^{1+\frac{1}{\kappa}} = \infty.$$

For a reference see [5], [19]. Under this viewpoint it is plausible that L^p stability holds for much more irregular conductivities, say in some Sobolev space. We will investigate this in a forthcoming work.

The scattering transform gives a non linear Fourier transform of the unknown coefficient. Each approach extends the previous transform to a wider class of functions. To see the relation between the Schrödinger and the Beals-Coifman scattering transform, see [28]. It will be of interest to see how the different definitions of scattering transform relate and which are the properties of this non linear Fourier transform. In this direction some results related to Plancherel identity were obtained in [15].

Notation:

Differential operators:

$\partial_{\overline{z}} u = \frac{1}{2}(\partial_x + i\partial_y)$ and $\partial_z u = \frac{1}{2}(\partial_x - i\partial_y)$

Spaces:

$W^{\alpha,p}(\mathbb{C}) = \{f : \|[(1+|\cdot|^2)^{\frac{\alpha}{2}} \widehat{f}(\cdot)]\check{}\|_{L^p} < \infty\}$, $C^\alpha(\Omega) = \{f : \|f\|_{L^\infty} + \sup_{x,y\in\Omega} \frac{|f(x)-f(y)|}{|x-y|^\alpha} < \infty\}$, $H^1(\Omega) = W^{1,2}(\Omega)$, $H^1_0 = W^{1,2}_0(\Omega)$ $H^{1/2}(\partial\Omega) = H^1(\Omega) \setminus H_0(\Omega)$,

We define the L^p-difference of a function f by

$$\omega_p(f)(y) = \|f(\cdot + y) - f(\cdot)\|_{L^p(\mathbb{C})}. \tag{1.6}$$

Then the L^p modulus of continuity of f is given for $t > 0$ by

$$\boldsymbol{\omega}_p(f)(t) = \sup_{|y|<t} \omega_p(f)(y). \tag{1.7}$$

Constants: We remark that C or a denote constants which may change at each occurrence. We will indicate the dependence of the constants on parameters κ, Γ,... , by writing $C = C(\kappa, \Gamma, ...)$

Finally, $e_k(z)$ denotes the unitary $e^{ikz+i\bar{k}\bar{z}}$ and

$$\rho = \|\Lambda_{\gamma_1} - \Lambda_{\gamma_2}\|_{H^{1/2}\to H^{-1/2}}.$$

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2 Preliminaries

Our starting point is the work of K. Astala and L. Päivärinta [7]. We will make an overview of their beautiful use of quasiconformal mapping methods in the Calderón Inverse problem, and we will collect some facts, to be used in our proof of stability. Some of the results that we remark here have been only implicitly stated in their work.

The relation between elliptic equations and quasiconformal mappings generalizes the fact that a harmonic function is the real part of a holomorphic mapping. That is, given a real solution u in $W^{1,2}$ of the elliptic equation

$$\nabla \cdot (\gamma \nabla u) = 0, \tag{2.1}$$

one can find a real v, unique modulo constants, such that $f = u + iv$ is a solution of the $\mathbb{R}$-linear Beltrami equation

$$\partial_{\bar{z}} f = \mu \overline{\partial_z f} \tag{2.2}$$

where the distortion or Beltrami coefficient μ is given for $z \in \mathbb{C}$ as

$$\mu(z) = (1 - \gamma)/(1 + \gamma) \tag{2.3}$$

The ellipticity condition is then equivalent to the existence of $\kappa < 1$ such that

$$\|\mu\|_{L^\infty} < \kappa. \tag{2.4}$$

For the detailed argument see [7, Lemma 2.1]. Then, Astala and Paivärinta worked with equation (2.2) instead of with equation (2.1). They were able to prove the existence of complex geometric optic solutions $f(z, k)$, control their asymptotics when $k \to \infty$, find the appropriate equations in the k variable and conclude their proof of uniqueness by an ingenious argument, combining the behaviour of the complex geometric optic solutions in the k and z variables.

In subsection 2.1 we will recall results in the theory of Beltrami equations required to treat this type of equations and also we will prove some further properties needed in our work. Then, in subsection 2.2 we will gather all the results from [7] needed for our approach.

2.1 Beltrami Equations

The theory of Beltrami equations and planar quasiregular and quasiconformal mappings generalizes many aspects of classical geometric function theory and is they key for a very rich theory of planar elliptic systems. We refer to the classical monographs [1], [21],[25], [30],[36]. See also the recent [6] were the last advances in the planar theory are collected (or proved).

Key tools in this theory are the following operators:

The solid Cauchy transform P,

$$P(g) = -\frac{1}{\pi}\int_{\mathbb{C}} \frac{g(w)}{w-z} dm(w). \tag{2.5}$$

The Beurling transform T,

$$T(g) = -\frac{1}{\pi}\int_{\mathbb{C}} \frac{g(w)}{(w-z)^2} dm(w). \tag{2.6}$$

The Beltrami operator B,

$$B = I - \mu T - \nu \overline{T} \tag{2.7}$$

where

$$|\mu(z)| + |\nu(z)| < \kappa.$$

The basic properties of these operators can be found in [7, section 3]. Several of our arguments are based on the following estimates for non homogeneous Beltrami type of differential inequalities. The next theorem extends Proposition 3.3 in [7] to the nonhomogeneous case.

Theorem 2.1. *Let $\kappa < 1$, $M > 0$ and $2 < p < \infty$. Let $F \in W^{1,2}_{loc}(\mathbb{C})$ satisfy the differential inequality,*

$$|\partial_{\bar{z}} F| \leq \chi_{\mathbb{D}}(\kappa |\partial_z F| + M|F| + E). \tag{2.8}$$

with $E \in L^2(\mathbb{D})$. Suppose in addition that

$$\lim_{z\to\infty} F(z) = 0 \tag{2.9}$$

Then it holds that

a) There exists a constant C_1 depending on (p, κ) such that

$$\|F\|_{L^p(\mathbb{C})} \leq e^{C_1(1+M)} \|E\|_{L^2(\mathbb{D})} \tag{2.10}$$

b) Suppose that $E \in L^p(\mathbb{D})$. Then there exists constant C_2 depending on p such that

$$\|F\|_{L^\infty(\mathbb{C})} + \|F\|_{W^{1,p}(\mathbb{C})} \leq e^{C_2(1+M)} \|E\|_{L^p(\mathbb{D})} \tag{2.11}$$

Proof of theorem 2.1:

We reformulate the differential inequality (2.8) as a Beltrami type equation. Namely, there exists complex valued functions $\nu : \mathbb{D} \to \mathbb{D}(0, \kappa), \gamma : \mathbb{D} \to \mathbb{D}(0, M)$ and $E \in L^2(\mathbb{D}, \mathbb{C})$ such that

$$\overline{\partial_z} F = \chi_{\mathbb{D}}(-\nu \partial_z F + \gamma F + E). \tag{2.12}$$

Formally we declare, $\eta = P(I - \nu T)^{-1}(\gamma)$, $g = e^\eta$, $s = P(I - \nu T)^{-1}(Ee^{-\eta})$ and $H = sg$. Then we have that

1. $\overline{\partial_z}\eta - \nu\partial_z\eta = \gamma$.
2. $\overline{\partial_z}g - \nu\partial_z g = \gamma g$.
3. $\overline{\partial_z}s - \nu\partial_z s = Ee^{-\eta}$
4. $\overline{\partial_z}H - \nu\partial_z H = \gamma H + E$

Let us check that we can use all these operators. Since $\gamma \in L^\infty(\mathbb{D})$, $(I - \nu T)^{-1}(\gamma) \in L^p(\mathbb{D})$ for every

$$1 + \kappa < p < 1 + \frac{1}{\kappa}. \tag{2.13}$$

Since for $p > 2$, $P : L^p(\Omega) \to W^{1,p}(\mathbb{C})$ is bounded, $\eta \in W^{1,p}(\mathbb{C})$ for every $2 < p < 1 + \frac{1}{\kappa}$. Moreover, for $p > 2$, $L^\infty(\mathbb{C}) \hookrightarrow W^{1,p}(\mathbb{C})$ and hence we have that $\eta \in L^\infty(\mathbb{C})$ with

$$\|\eta\|_{L^\infty(\mathbb{C})} \leq CM. \tag{2.14}$$

Thus, g is a well defined solution of (2) and $Ee^{-\eta} \in L^2(\mathbb{D})$. Then, $(I - \nu T)^{-1}(Ee^{-\eta}) \in L^2(\mathbb{D})$ and s is in $L^p(\mathbb{C}) \cap VMO$ for every $p > 2$ with the estimate

$$\|s\|_{L^p(\mathbb{C})} \leq C(p)\|E\|_{L^2(\mathbb{D})}e^{CM}. \tag{2.15}$$

In particular, $s \in W^{1,2}_{loc}(\mathbb{C})$ and since $\eta \in W^{1,p}(\mathbb{C})$, $H \in W^{1,2}_{loc}(\mathbb{C})$ as well. Moreover, (2.15) implies that

$$\lim_{z\to\infty} H(z) = 0. \tag{2.16}$$

Now consider the function $R = F - H$. Clearly it solves the homogeneous equation

$$\overline{\partial_z}R - \nu\partial_z R = \gamma R. \tag{2.17}$$

and by (2.16) and the assumptions on F

$$\lim_{|z|\to\infty} R(z) = 0. \tag{2.18}$$

Now, we argue as in [7, Proposition 3.3]. Consider $G = e^{-\eta}R$. It follows from (2.17) that G solves

$$\overline{\partial_z}G - \partial_z G = 0. \tag{2.19}$$

Being $G \in W^{1,2}_{loc}$, this implies that G is quasiregular mapping. Since $\eta \in L^\infty(\mathbb{C})$ and R satisfies (2.18) it follows that $\lim_{|z|\to\infty} G(z) = 0$. But then, the representation theorem of quasiregular mappings and Liouville theorem imply that $G = 0$. Thus $R = 0$ as well and,

$$F = H = se^{\eta}. \tag{2.20}$$

With this representation of F, (2.10) follows from (2.15) and (2.14). If $E \in L^p(\mathbb{D})$ for $p > 2$ we have that

$$\|\partial_{\bar{z}}(s)\|_{L^p(\mathbb{D})} \leq C_p e^M \|E\|_{L^p(\mathbb{D})}$$

and arguing exactly as for η, we conclude that $s \in W^{1,p}(\mathbb{C})$ and therefore in L^∞. This yields the required estimate (2.11) for H and thus for F. □

2.2 Astala-Päivärinta approach

The strategy of Astala and Päivärinta's work is the construction of the approximate complex exponential solution of the elliptic equation, used in previous works, through some appropriate solutions of the Beltrami equation. This allows then to avoid any a priori regularity assumption on the coefficient γ.

The first step is to find a map associated to the equation (2.2), which contains the information of the Dirichlet to Neumann map Λ_γ. This is done by the Hilbert transform $\mathcal{H}_\mu$ with respect to the distortion μ. For u real as above we define $\mathcal{H}_\mu : H^{1/2}(\partial\mathbb{D}) \to H^{1/2}(\partial\mathbb{D})$ as

$$\mathcal{H}_\mu(u_{|\partial\mathbb{D}}) = v_{|\partial\mathbb{D}} \tag{2.21}$$

Since

$$\mathcal{H}_\mu \circ \mathcal{H}_{-\mu} u = \mathcal{H}_{-\mu} \circ \mathcal{H}_\mu = -u + \frac{1}{2\pi}\int_{\partial\mathbb{D}} u, \tag{2.22}$$

it is natural to extend $\mathcal{H}_\mu$ $\mathbb{R}$-linearly by defining

$$\mathcal{H}_\mu(iu) = i\mathcal{H}_{-\mu}(u).$$

We have (Proposition 2.3 in [7])

Proposition 2.2. *The Dirichlet to Neumann map Λ_γ uniquely determines $\mathcal{H}_\mu$, $\mathcal{H}_{-\mu}$ and $\Lambda_{\gamma^{-1}}$. For u and v real valued, the following identity holds*

$$\partial_T \mathcal{H}_\mu(u + iv) = \Lambda_\gamma(u) + i\Lambda_{\gamma^{-1}}(v). \tag{2.23}$$

Furthermore, for two given conductivities γ_1 and γ_2 we have

$$\|\mathcal{H}_{\mu_1} - \mathcal{H}_{\mu_2}\|_{H^{1/2}(\partial\mathbb{D}) \to H^{1/2}(\partial\mathbb{D})} \leq C\|\Lambda_{\gamma_1} - \Lambda_{\gamma_2}\|_{H^{1/2}(\partial\mathbb{D}) \to H^{-1/2}(\partial\mathbb{D})}. \tag{2.24}$$

Above, by an abuse of language, we have identified u and v with their restrictions to $\partial\mathbb{D}$.

The new Hilbert transforms give rise to the corresponding Riesz projections onto the μ Hardy spaces on ∂D. Namely consider the operators

$$\begin{aligned} P_\mu(g) &= \frac{1}{2}(I + i\mathcal{H}_\mu)(g) + \frac{1}{2}\oint_{\partial\mathbb{D}} g ds, \\ Q_\mu(g) &= \frac{1}{2}(I - i\mathcal{H}_\mu(g)) - \frac{1}{2}\oint g ds \end{aligned} \tag{2.25}$$

Then P_μ is the analogous of the Riesz projections in the following sense.

Lemma 2.3 ([8]). *Let $g \in H^{1/2}(\partial\mathbb{D})$. Then*

i) $P_\mu(g) + Q_\mu(g) = I$

ii) $P_\mu^2 = P_\mu$.

iii) $P_\mu(g) = g \iff g = f_{|\partial\mathbb{D}}$ *for some* $f \in W^{1,2}$ *with*

$$\partial_{\overline{z}} f_\mu = \mu \overline{\partial_z f_\mu}. \tag{2.26}$$

iv) The range of P_μ consists of boundary values of solutions to (2.26).

The following theorem gives the existence and properties of the approximate complex exponential solutions of the equation (2.2). We look for perturbations of the solutions e^{ikz} of the equation when $\mu = 0$. It relies on the following lemma.

Lemma 2.4. *Let $2 < p < p_\kappa = 1 + \frac{1}{\kappa}$, $\alpha \in L^\infty$ supported on $\mathbb{D}$ and $|\nu(z)| \leq \kappa\chi_{\mathbb{D}}$ for almost every $z \in \mathbb{D}$. Then, the operator $K = K(\nu, \alpha)$ defined by*

$$K(g) = P(I - \nu\overline{T})^{-1}(\alpha\bar{g}) \tag{2.27}$$

as well as $(I - K)^{-1}$ are bounded operators from $L^p(\mathbb{C}) \to L^p(\mathbb{C})$. Also $K : L^p \to W^{1,p}$

Theorem 2.5. *For each $k \in \mathbb{C}$ and each $2 < p < 1 + 1/\kappa$ there exists a unique solution $f_\mu \in W^{1,p}_{loc}$ of (2.2) such that*

$$f_\mu(z, k) = e^{ikz} M_\mu(z, k) \tag{2.28}$$

with

$$M_\mu(z, k) = 1 + O(1/z) \text{ as } z \to \infty \tag{2.29}$$

and

$$f_\mu(z, 0) = 1 \tag{2.30}$$

$$M_\mu(z, k) = e^{\eta(z,k)} \tag{2.31}$$

$$\Re\left(\frac{M_\mu(z, k)}{M_{-\mu}(z, k)}\right) > 0 \tag{2.32}$$

The solution in the above theorem is constructed as

$$M_\mu(z, k) = (1 + \omega_\mu(z, k)) \text{ where } \omega_\mu = (I - K)^{-1}(K(\chi_{\mathbb{D}})), \tag{2.33}$$

for a suitable choice of α. The function $\omega(z, k)$ is actually in $W^{1,p}$ for $2 < p < 1 + 1/\kappa$. In fact we have the following estimate.

Proposition 2.6. *Let $\omega(z,k) = M_\mu(z,k) - 1$. Then $\omega(z,k) \in W^{1,p}(\mathbb{C}, dz)$ is a solution to the equation*

$$\partial_{\overline{z}}\omega - \mu e_{-k}\overline{\partial_z \omega} = -i\overline{k}\mu e_{-k}(\overline{\omega} + 1). \tag{2.34}$$

Moreover, there exists a constant $C(\kappa, p)$ such that for every $2 < p < 1 + \frac{1}{\kappa}$

$$\|\omega\|_{W^{1,p}(\mathbb{C},dz)} \leq e^{C(1+|k|)} \tag{2.35}$$

Proof: The fact that ω is a $W^{1,p}(\mathbb{C}, dz)$ solution to (2.34) follows from the definition and properties of the operator K with $\alpha = -i\overline{k}\mu e_{-k}$. Once we know that ω satisfies equation (2.34) the estimate (2.35) follows from Theorem 2.1. □

The other key ingredient in the proof of uniqueness is the so called $\bar{\partial}_k$-equation which we state as

Theorem 2.7. *Let us define*

$$F_+(z,k) = \frac{1}{2}(M_\mu + M_{-\mu}) \tag{2.36}$$

$$F_-(z,k) = \frac{ie_{-k}}{2}(\overline{M_\mu} - \overline{M_{-\mu}}) \tag{2.37}$$

Then the functions $k \to F_\pm(z,k)$ has continuous derivatives with values in $W^{1,p}(\mathbb{C})$ in the norm sense. They satisfy

$$\partial_{\overline{k}} F_+(z,k) = \tau(\mu,k)e_{-k}(z)\overline{F_-(z,k)} \tag{2.38}$$

$$\partial_{\overline{k}} F_-(z,k) = \tau(\mu,k)e_{-k}(z)\overline{F_+(z,k)} \tag{2.39}$$

Where the scattering transform $\tau(\mu,\cdot)$, defined as

$$\overline{\tau(\mu,k)} = \frac{1}{4\pi i}\int_{\mathbb{D}} \partial_{\overline{z}}(M_\mu(z,k) - M_{-\mu}(z,k))dm(z) \tag{2.40}$$

satisfies $|\tau(\mu,k)| < 1$.

We need to work with solutions of the original elliptic equation (2.1). The following definitions, [7, Section 1], give an extension to the case $\gamma \in L^\infty$ of the so called complex geometric optics solutions of Nachman and Sylvester-Uhlmann:

$$u_\gamma(z,k) = Ref_\mu(z,k) + iImf_{-\mu}(z,k) \tag{2.41}$$

We also need the solutions of the γ^{-1}-conductivity equation $\nabla\cdot(\gamma^{-1}\nabla u) = 0$ defined as

$$-u_{\gamma^{-1}}(z) = Imf_\mu - iRef_{-\mu} \tag{2.42}$$

We collect some of the properties of u_γ.

Proposition 2.8. *The complex geometric optics solutions $u = u_\gamma(z, \cdot)$ satisfy*

$$u(z,k) = e^{ikz}(1 + R(z,k)) \text{ with } R(z,k) \in W^{1,p}(\mathbb{C}, dz) \tag{2.43}$$

$$u(z,k) = e^{\delta(z,k)} \tag{2.44}$$

with

$$\delta(z,k) = ik(z + O(1/z)) \text{ for fixed } k \text{ as } z \to \infty \tag{2.45}$$

and

$$\delta(\cdot,k)(\mathbb{C}) = \mathbb{C} \text{ for } k \neq 0. \tag{2.46}$$

Moreover, $u(z,\cdot) = u$ satisfies the equation

$$\partial_{\bar{k}} u = -i\tau(\mu,k)\bar{u} \tag{2.47}$$

This proposition follows from the previous results for f_μ. In fact (2.43) is a consequence of (2.33) and the definition (2.41) with

$$R(z,k) = \omega_\mu + \omega_{-\mu} + e_{-k}(\overline{\omega_\mu - \omega_{-\mu}}). \tag{2.48}$$

(2.44) follows from (2.32). (2.45) follows from (2.29). Equation (2.46) from (2.45) and a homotopy argument. Finally (2.47) is a consequence of (2.38) and (2.39).

To end this section we remark Theorem 7.1 in [7].

Theorem 2.9. *Let us define $f_{\lambda\mu}$ the solution of Theorem 2.5 with μ substituted by $\lambda\mu$ where $\lambda \in \partial\mathbb{D}$. Then $f_{\lambda\mu}$ admits a representation*

$$f_{\lambda\mu}(z,k) = e^{ik\phi_\lambda(z,k)} \tag{2.49}$$

where, for fixed $k \in \mathbb{C} \setminus 0$ and $\lambda \in \partial\mathbb{D}$, the function $\phi_\lambda(\cdot,k) : \mathbb{C} \to \mathbb{C}$ is a quasiconformal homeomorphism that satisfies

$$\phi_\lambda(z,k) = z + O(1/z) \text{ as } z \to \infty \tag{2.50}$$

and

$$\bar{\partial}\phi_\lambda(z) = -\frac{\bar{k}}{k}\lambda\mu(z)e_{-k}(\phi_\lambda(z,k))\overline{\partial\phi_\lambda(z,k)} \tag{2.51}$$

3 Further properties of the complex geometric optic solutions.

In this section we study several properties of f_μ and $u = u_\gamma(z,k)$ when we assume the coefficient μ to be in the Hölder class $\mathcal{C}^\alpha$ for some fixed $\alpha > 0$. We start by stating a more precise asymptotic expansion of the "logarithmic" functions of the geometric optics solutions in the k variable and finally we prove some facts about their derivatives in the k variable. We also analyze the extra regularity that we gain in the z variable.

3.1 Uniform subexponential growth of f_μ

If $\mu \in L^\infty$. It is proved in [7] that the Jost functions satisfy that

$$M_\mu = e^{ik(z-\varphi_\mu(z,k))},$$

where

$$z - \varphi_\mu(z,k) \to 0$$

uniformly in $z \in \mathbb{C}$ as $k \to \infty$. However the convergence depends on μ. We will prove that this uniformity can be extended to all conductivities μ whose Hölder norm is bounded by some Λ_0. We start with some elementary properties of Hölder functions.

Lemma 3.1. *Let $\mu \in \mathcal{C}^\alpha$, supported on $\mathbb{D}(0,R)$, $\beta < \alpha$ and $1 \leq p < \infty$. Then*

i) There exists a constant $C = C(p,R)$ such that

$$\|\mu\|_{W^{\beta,p}} \leq \frac{C}{\alpha-\beta}\|\mu\|_{C^\alpha}. \tag{3.1}$$

ii) Let $f \in W^{\alpha,2}(\mathbb{C})$ then there exists a constant $C = C(R)$ such that

$$\|\mu f\|_{W^{\beta,2}} \leq \frac{C}{\alpha-\beta}\|\mu\|_{C^\alpha}\|f\|_{W^{\beta,2}} \tag{3.2}$$

Proof:

We first prove the second claim. Recall that by [32, Section 3.5.2, chapter 5] we have

$$\|\mu f\|_{W^{\alpha,2}} \sim \int \frac{\boldsymbol{\omega}_2(\mu f)(t)^2}{t^{-2(1+\alpha)}} + \|\mu f\|_{L^2}, \tag{3.3}$$

where $\boldsymbol{\omega}_2$ denotes the modulus of continuity, defined in the introduction. It is easy to see that

$$\boldsymbol{\omega}_2(\mu f)(t) \leq \|f\|_{L^2}\|\mu\|_{C^\alpha}t^{2\alpha} + \|\mu\|_{L^\infty}\boldsymbol{\omega}_2(f)(t). \tag{3.4}$$

Thus, the claim follows by plugging (3.4) into (3.3) and integrating.

The first claim with $p = 2$ follows by taking φ a compactly supported function which is 1 on the support of μ. If $p \neq 2$ the claim is a consequence of sufficient conditions, in terms of the L^p-modulus of continuity, for a function to be in $W^{\beta,p}$. See [32, Section 3.5.2 chapter V] □

Lemma 3.2. *Let $f \in W^{\beta,2}(\mathbb{C})$. Then for $R_0 > 1$*

$$\|\chi_{\{|(.)|>R_0\}}\hat{f}\|_{L^2} \leq CR_0^{-\beta}.$$

Proof:

$$\int_{|\xi|>R_0} |\hat{f}|^2 \leq R_0^{-2\beta} \int (1+|\xi|^2)^{\beta} |\hat{f}|^2 \leq R_0^{-2\beta} \|f\|_{W^{\beta,2}}.$$

□

Our task is to control the asymptotics of the exponent functions of

$$f_{\lambda\mu} = e^{ik\phi_\lambda(z,k)}, \tag{3.5}$$

with respect to the norm in the Hölder class $\mathcal{C}^\alpha$ of μ. From Theorem 2.9, the function $\phi_\lambda(z,k)$ in (3.5) for each k fixed in $\mathbb{C}\setminus\{0\}$, and $\lambda \in \partial\mathbb{D}$, is a quasiconformal homeomorphism on $\mathbb{C}$ that satisfies the non linear equation (2.51) and for fixed k

$$\phi_\lambda(z,k) = z + O_k(1/z) \text{ as } z \to \infty \tag{3.6}$$

We start with the solution of the linear equation related to 2.51.

Proposition 3.3. *Let $\psi(z,k)$ be the solution in $W^{1,2}_{loc}$ of the equation*

$$\partial_{\bar{z}}\psi(z) = -\frac{\bar{k}}{k}\lambda\mu(z)e_{-k}(z)\partial_z\psi(z) \tag{3.7}$$

such that

$$\psi(z,k) = z + O(1/z) \text{ as } z \to \infty \tag{3.8}$$

Let $R, \alpha, \Lambda_0 > 0$, $0 < \kappa < 1$. Then, there exists constants $a > 0$ and $C > 0$ depending on these parameters such that for any $z \in \mathbb{C}$, $\lambda \in \partial\mathbb{D}$ and any μ which satisfies $\|\mu\|_{\mathcal{C}^\alpha} < \Lambda_0$ and $|\mu| < \kappa\chi_{\mathbb{D}(0,R)}$ it holds that,

$$|\psi(z,k) - z| \leq C|k|^{-a}. \tag{3.9}$$

An expression for a is given in (3.33).

We may reduce to $R = 1$. Recall that by the Cauchy formula we have that

$$\psi(z,k) - z = C\int_{\mathbb{D}} \Phi(w,z)\partial_{\bar{z}}\psi(w,k)dm(w), \tag{3.10}$$

where $\Phi(w,z) = \frac{\chi_{\mathbb{D}}}{w-z}$. In the next lemmas we analyze more in detail the two factors $\Phi(w,z)$ and $\partial_{\bar{z}}\psi$. We will write $K_z(w) = \frac{1}{z-w}$.

Lemma 3.4. *Let $1 < q < 2$ and let $s < \frac{2-q}{2q}$. Then the following properties hold:*

i) There exists $C = C(q,R)$ such that for every $z \in \mathbb{C}$, $\|K_z\|_{L^q(\mathbb{D}(0,R))} \leq C$.

ii) There exists $C = C(q,s)$ such that for every $z \in \mathbb{C}$

$$\boldsymbol{\omega}_q(\Phi_z)(t) \leq Ct^s. \tag{3.11}$$

Proof: The first claim follows by direct calculation. For the second, take $q < \tilde{q} < 2$. Then, if $t < 1$ it holds that

$$\boldsymbol{\omega}_q^q(\Phi_z)(t) \leq \boldsymbol{\omega}_q^q(K_z)(t) + \|K_z\|^q_{L^{\tilde{q}}(\mathbb{D}(0,2))} \boldsymbol{\omega}_{\frac{\tilde{q}}{\tilde{q}-q}}(\chi_{\mathbb{D}})(t). \tag{3.12}$$

Now on one hand, for example by [1, page 86] there exists C independent of z such that

$$\boldsymbol{\omega}_q^q(K_z)(t) \leq C|t|^{2-q}. \tag{3.13}$$

On the other for $p \geq 1$, a direct calculation yields that

$$\boldsymbol{\omega}_p(\chi_{\mathbb{D}})(t) = t^{\frac{1}{p}}. \tag{3.14}$$

Since for $1 < q < \tilde{q} < 2$, $2 - q > \frac{\tilde{q}-q}{\tilde{q}}$, plugging (3.13) and (3.14) into (3.12) we obtain that

$$\boldsymbol{\omega}_q(\Phi_z)(t) \leq C(q,\tilde{q})t^{\frac{\tilde{q}-q}{\tilde{q}q}}.$$

By taking the limit when $\tilde{q} \to 2$ we prove the second claim. □

The uniform control on the modulus of continuity of $\Phi_z = \Phi(z,\cdot)$ translates into a uniform control of the speed of converge of a suitable mollification of Φ_z.

Lemma 3.5. *Let $\delta > 0$. Then there exists a function $\Phi_\delta(z,w)$ such that for every $1 < q < 2$ the following properties hold:*

i) For every $s < \frac{2-q}{2q}$ there exists $C = C(s,q)$ such that

$$\|\Phi_\delta(z,w) - \Phi_z(\omega)\|_{L^q} \leq C\delta^s$$

ii) There exists $C = C(q)$ such that

$$\|\Phi_\delta(z,\cdot)\|_{L^2} \leq C(q)\delta^{1-\frac{2}{q}}$$

iii) Let $\delta R_0 > 1, m > 0$. Then there exists $C(q,m)$ such that

$$\|\widehat{\Phi_\delta(z,\cdot)}\|_{L^2(\mathbb{C}\setminus\mathbb{D}(R_0))} \leq C(q,m)\delta^{1-\frac{2}{q}}(R_0\delta)^{-m}.$$

Proof:

Take a compactly supported smooth function $\phi(w) = \phi_0(|w|)$ and let us consider the mollification $H_\delta(z,w)$ defined by

$$\widehat{H_\delta}(z,\cdot)(\xi) = \widehat{\Phi}(\xi,z)\widehat{\phi}(\delta\xi).$$

The first claim follows from lemma 3.4, since,

$$\|\Phi_z(\cdot) - \Phi_\delta(z,\cdot)\|_{L^q} \leq \int_{\mathbb{C}} \omega_q(\Phi_z)(w)\phi_\delta(w)dw$$

$$\leq \int \boldsymbol{\omega}_q(\Phi_z)(|w|)|\phi_\delta(w)|dw \leq \int \boldsymbol{\omega}_q(\Phi_z)(\delta|y|)|\phi_0(|y|)|dy \leq C(s)\delta^s,$$

Where $\phi_\delta(w) = \delta^{-2}\phi(\delta^{-1}w)$.

For the second, using Plancherel, Hölder and Hausdorff-Young inequalities and lemma 3.4 we obtain, for $1/q - 1/p = 1/2$, that

$$\|\Phi_\delta\|_{L^2} \leq \|\Phi_z\|_{L^q}\||\hat{\phi}(\delta\chi)\|_{L^p} \leq C\delta^{1-\frac{2}{q}}.$$

For the third claim write again

$$\begin{aligned}\|\widehat{\Phi_\delta}\|_{L^2(|\xi|>R_0)} &\leq \|\Phi_z\|_{L^q}\||\hat{\phi}(\delta\xi)\|_{L^p(|\xi|>R_0)} \\ &\leq \|\Phi_z\|_{L^q}\delta^{1-2/q}\|\hat{\phi}(\xi)\|_{L^p(|\xi|>\delta R_0)}\end{aligned}$$

Since $\hat{\phi}$ is rapidly decreasing for any m there exists $C(m)$ such that

$$\|\hat{\phi}(\xi)\|_{L^p(|\xi|>\delta R_0)} \leq C(m)(\delta R_0)^{-m}$$

from where (iii) follows. □

Next, we analyze $\partial_{\bar{z}}\psi$. We use a decomposition Lemma as in [7]. However, the extra Hölder regularity allows us to simplify the proof obtaining a uniform subexponential decay.

Lemma 3.6. *Let $\psi(z,k)$ the solution of (3.7) and (3.8). Choose $2 < p$ such that*

$$\kappa\|T\|_{L^p\to L^p} = \kappa_1 < 1.$$

Given $n_0 \in \mathcal{N}$, we can decompose $\bar{\partial}\psi = g + h$, where $g = g_{\lambda\mu}(z,k)$ and $h = h_{\lambda\mu}(z,k)$ satisfy

i) $\sup_{k\in\mathbb{C}} \|h(.,k)\|_{L^p} < C(\kappa,p)(\kappa\|T\|_{L^p})^{n_0}$

ii) $\sup_{k\in\mathbb{C}} \|g(.,k)\|_{L^p} < C_0 = C_0(\kappa)$

iii) Let $|k| \geq 2R_0$ and $\beta < \alpha$. Then there exists a universal constant C such that, $\int_{|\xi|<R_0} |\hat{g}|^2 \leq (C\frac{\Lambda_0}{\alpha-\beta})^{n_0+1}|k|^{-\beta}$

Proof:
The function $\partial_{\bar{z}}\psi(z,k)$ is the solution in L^p of the equation

$$(I + \frac{\bar{k}}{k}\lambda\mu(z)e_{-k}(z)T)\partial_{\bar{z}}\psi(z,k) = -\frac{\bar{k}}{k}\lambda\mu(z)e_{-k}(z).$$

We may solve the equation by the Born series

$$\partial_{\bar{z}}\psi(z,k) = \sum_{n=0}^{\infty}(-\frac{\bar{k}}{k}\lambda\mu(z)e_{-k}(z)T)^n(-\frac{\bar{k}}{k}\lambda\mu e_{-k}) \tag{3.15}$$

We take h to be the tail

$$h(z,k) = \sum_{n=n_0}^{\infty}(-\frac{\bar{k}}{k}\lambda\mu(z)e_{-k}(z)T)^n(-\frac{\bar{k}}{k}\lambda\mu e_{-k}).$$

Then (i) holds easily.

For this choice, $g(z,k)$ has the form

$$g(z,k) = \sum_{n=0}^{n_0} G_n(z,k), \tag{3.16}$$

where

$$G_n(z,k) = (-\frac{\bar{k}}{k}\lambda\mu(z)e_{-k}(z)T)^n(-\frac{\bar{k}}{k}\lambda\mu e_{-k}) \tag{3.17}$$

Then g satisfies condition (ii):

$$\|g(.,k)\|_{L^p} \leq \pi^{1/p}\frac{\kappa_1}{1-\kappa_1} = C_0(\kappa).$$

Only (iii) remains to be proved. We write $G_n(z,k)$ as

$$G_n = (\frac{-\bar{k}}{k}\lambda)^{n+1}e_{-(n+1)k}f_n, \tag{3.18}$$

where

$$\begin{cases} f_n(z) = \mu T_n\mu T_{n-1}\mu...\mu T_1(\mu), \text{ for } n>0 \\ f_0 = \mu. \end{cases} \tag{3.19}$$

The operator $T_j = e_{jk}Te_{-jk}$ is a Fourier multiplier with unimodular symbol $(\xi - jk)/(\overline{\xi - jk})$. Therefore, $\|T_j\|_{W^{\alpha,2}\to W^{\alpha,2}} = 1$. Combining this with Lemma 3.1 we arrive at

$$\|f_n\|_{W^{\beta,2}} \leq (C\frac{\Lambda_0}{\alpha-\beta})^{n+1}. \tag{3.20}$$

Thus,

$$\int_{|\xi|<R_0}|\hat{g}|^2 = \sum_{n=0}^{n_0}\int_{|\xi|<R_0}|\hat{f_n}(\xi-(n+1)k)|^2.$$

If $|k| \geq 2R_0$ this is bounded by

$$\sum_{n=0}^{n_0}\int_{|\xi|>(n+1)|k|-R_0}|\hat{f_n}|^2 \leq (C\frac{\Lambda_0}{\alpha-\beta})^{n_0+1}\sum_{n=0}^{n_0}((n+1)|k|-R_0)^{-\beta},$$

where the last inequality follows from Lemma 3.2. Therefore

$$\int_{|\xi|<R_0} |\hat{g}|^2 \leq (C\frac{\Lambda_0}{\alpha-\beta})^{n_0+1}|k|^{-\beta}(n_0+1)^{1-\beta} \leq (\frac{C\Lambda_0}{\alpha-\beta})^{n_0+1}|k|^{-\beta}.$$

The proof is concluded . □

Proof of Proposition 3.3:
By (3.10) and Lemma 3.6,

$$\psi(z,k) - z = C\int_{\mathbb{D}} \Phi(w,z)(g+h)dm(w). \tag{3.21}$$

Let $2 < p$ be such that

$$\kappa\|T\|_{L^p\to L^p} < 1, \tag{3.22}$$

and let q be its dual. Then, Hölder's, Lemma 3.4 and Lemma 3.6 imply that,

$$|\int h\Phi(w,z)| \leq \|\Phi(w,z)\|_{L^q}\|h\|_{L^p} \leq C(\kappa,q)(\kappa\|T\|_{L^p})^{n_0+1} < \frac{\epsilon}{4}. \tag{3.23}$$

The last inequality follows for any $n_0 \in \mathcal{N}$ such $n_0 \geq \frac{\log(\epsilon C)}{\log(\kappa\|T\|_{L^p\to L^p})}$, where $C = C(\kappa,p) > 0$.

On the other hand by Lemma 3.5 for every $s < \frac{2-q}{2q}$

$$|\int g\Phi| \leq |\int g\Phi_\delta| + \|g\|_{L^p}\|\Phi-\Phi_\delta\|_{L^q} \leq |\int g\Phi_\delta| + C(\kappa,s,q)\delta^s \leq |\int g\Phi_\delta| + \frac{\epsilon}{4} \tag{3.24}$$

where the last inequality is obtained if $\delta \leq C(\kappa,s,q)\epsilon^{\frac{1}{s}}$.

It only remains to estimate the term $\int g\Phi_\delta$. We do this on the Fourier transform side:

$$|\int g\Phi_\delta| \leq |\int_{|\xi|<R_0} \hat{g}\hat{\Phi}_\delta| + |\int_{|\xi|\geq R_0} \hat{g}\hat{\Phi}_\delta|.$$

But, by Lemma 3.5 if $R_0 \geq \delta^{-1} \geq C(\kappa,s,q)\epsilon^{\frac{-1}{s}}$

$$|\int_{|\xi|\geq R_0} \hat{g}\hat{\Phi}_\delta| \leq \|g\|_{L^2}\|\hat{\Phi}_\delta\|_{L^2(|\chi|\geq R_0)} \leq C(q,m)\delta^{1-\frac{2}{q}}(R_0\delta)^{-m} < \frac{\epsilon}{4}$$

the last inequality being if we further require $R_0 \geq C(q,m)\epsilon^{-\frac{m-1+2q+s}{sm}}$. Thus, by taking the limit when $m \to \infty$, we see, for s fixed in the open range $s < \frac{2-q}{2q}$, that there exists $C(\kappa,s,q)$ such that if $R_0 \geq C(\kappa,s,q)\epsilon^{-\frac{1}{s}}$

$$|\int_{|\xi|\geq R_0} \hat{g}\hat{\Phi}_\delta| \leq \frac{\epsilon}{4}. \tag{3.25}$$

Finally, we need to estimate $|\int_{|\xi|<R_0}\hat{g}\hat{\Phi}_\delta|$, where R_0, δ and n_0 have been fixed satisfying $R_0 \geq C(\kappa,s,q)\epsilon^{-\frac{1}{s}}$, $\delta \leq C(\kappa,s,q)\epsilon^{\frac{1}{s}}$ and $n_0 \geq \frac{\log(\epsilon C)}{\log(\kappa\|T\|_{L^p\to L^p})}$. By Hölder

$$|\int_{|\xi|\leq R_0}\hat{g}\hat{\Phi}_\delta| \leq (\int_{|\xi|\leq R_0}|\hat{g}|^2)^{\frac{1}{2}}\|\Phi_\delta\|_{L^2} \tag{3.26}$$

We want to apply Lemma 3.5 and Lemma 3.6(iii). Thus, we need to assume that

$$|k| \geq 2R_0 \geq C(\kappa,s,q)\epsilon^{-\frac{1}{s}}. \tag{3.27}$$

We obtain that

$$|\int_{|\xi|\leq R_0}\hat{g}\hat{\Phi}_\delta| \leq \left(C\frac{\Lambda_0}{\alpha-\beta}\right)^{1/2(n_0+1)}|k|^{-\beta/2}\delta^{1/2-1/q} \tag{3.28}$$

which, after plugging in the optimal values of δ and n_0, is dominated by $C(\kappa,\alpha,q)\epsilon^{-C(\kappa,q)\log(\frac{\Lambda_0}{\alpha-\beta}+e)-1/s(1/q-1/2)}|k|^{-\beta/2}$. Now we impose this quantity to be bounded by $\frac{\epsilon}{4}$ which can be attained if

$$|k| \geq C\epsilon^{-(1/\beta)(2+C(\kappa,q)\log(\frac{\Lambda_0}{\alpha-\beta}+e)+1/s(2/q-1))} \tag{3.29}$$

to arrive at

$$|\int_{|\xi|\leq R_0}\hat{g}\hat{\Phi}_\delta| \leq \frac{\epsilon}{4} \tag{3.30}$$

as desired. Putting together (3.27) and (3.29), we have proved the following claim: For any $1<q<2$ such that $p=\frac{q}{q-1}$ satisfies $\kappa\|T\|_{L^p\to L^p}<1$, any $s<\frac{2-q}{2q}$, there exists positive constants C and b depending on κ, Λ_0, α, s and q such that

$$|k| \geq C\epsilon^{-b} \tag{3.31}$$

implies $|\phi(z,k)-z| \leq \epsilon$. The constant b above is given by

$$\max\{\frac{1}{s}, \frac{2+C\log(\frac{\Lambda_0}{\alpha-\beta}+e)+1/s(2/q-1)}{\beta}\}$$

Therefore we have obtained the claim of the proposition with

$$a = 1/b = \min\{s, \beta\left(2+C\log(\frac{\Lambda_0}{\alpha-\beta}+e)+1/s(2/q-1)\right)^{-1}\} \tag{3.32}$$

By choosing appropriated s,q and β, a possible value of the exponent is

$$a = \min\{\frac{\delta(\kappa)}{4}, \frac{\alpha}{2}(6+C(\kappa)\log(2\Lambda_0/\alpha+e))^{-1}\} \tag{3.33}$$

□

Unlike in [7], the corresponding Theorem for the nonlinear equation follows easily thanks to the uniformity of the estimates in the case C^α.

Theorem 3.7. *Let ϕ_λ be the solution of (2.51) which satisfies (3.6). Let $R, \alpha, \Lambda_0 > 0$, $0 < \kappa < 1$. Then, there exists constants a and C depending on these parameters such that for any $z \in \mathbb{C}$, $\lambda \in \partial\mathbb{D}$ and any μ which satisfies $\|\mu\|_{\mathcal{C}^\alpha} < \Lambda_0$ and $|\mu| < \kappa\chi_{\mathbb{D}(0,R)}$ it holds that,*

$$|\phi_\lambda(z,k) - z| \leq C|k|^{-a} \tag{3.34}$$

Proof: We observe as in [7] that since the estimates are uniform in z it is equivalent to prove similar asymptotics for the inverse function. $\psi_\lambda = \phi_\lambda^{-1}$. This satisfies the equation

$$\bar{\partial}\psi_\lambda(z) = -\frac{\bar{k}}{k}\lambda\mu(\psi_\lambda(z))e_{-k}(z)\partial\psi_\lambda(z),$$

under the condition

$$\psi_\lambda(z,k) = z + O_k(1/z) \text{ as } z \to \infty$$

The theorem is a corollary of the proposition 3.3. We just need to prove that the coefficient $\nu(z) = \mu(\psi_\lambda(z,k))$ satisfies:
(a) It is in an a Hölder Class $\mathcal{C}^\beta$ for $\beta = \beta(\alpha, \kappa)$.
(b) Its support is contained in $4\mathbb{D}$.
These conditions are satisfied, since (b) follows from $\frac{1}{4}$-Koebe theorem and (a) follows from

$$\|\nu\|_{\mathcal{C}^{\alpha\beta}} \leq \|\mu\|_{\mathcal{C}^\alpha}\|\psi_\lambda\|^\alpha_{\mathcal{C}^\beta} \leq \Lambda_0\|\psi_\lambda\|^\alpha_{W^{1,p}} \leq C(\Lambda_0)$$

where $\beta = 1 - \frac{2}{p}$, and $p < 1 + \frac{1}{\kappa}$. The Sobolev norm is on $4\mathbb{D}$ and is bounded by a constant depending of κ by the normalization (3.6) ([1]). □

3.2 Uniform subexponential growth of u

Theorem 3.8. *Let u be the solution of the equation*

$$div(\gamma\nabla u) = 0$$

constructed as $u(z) = \Re f_\mu + i\Im f_{-\mu}$, where μ, supported on $\mathbb{D}$ is such that $\|\mu\|_\infty < \kappa$ and $\|\mu\|_{\mathcal{C}^\alpha} < \Lambda_0$. Then there exists $a = a(\kappa, \alpha, \Lambda_0)$ and $C = C(\kappa, \alpha, \Lambda_0)$ such that for every $z \in \mathbb{C}$, we may write

$$u(z,k) = e^{ik(z+\epsilon_z(k))}, \text{ where } |\epsilon_z(k)| \leq C|k|^{-a} \tag{3.35}$$

Remark: A similar estimate can be proved for the solution $u = u_{\gamma^{-1}}$ of

$$div(\gamma^{-1}\nabla u) = 0,$$

given in 2.42
Proof: We can write

$$u = f_\mu \left(1 + \frac{f_\mu - f_{-\mu}}{f_\mu + f_{-\mu}}\right)^{-1} \left(1 + \frac{\overline{f_\mu} - \overline{f_{-\mu}}}{f_\mu + f_{-\mu}}\right) \tag{3.36}$$

Let

$$\alpha(z) = \frac{f_\mu - f_{-\mu}}{f_\mu + f_{-\mu}}.$$

Then Theorem 3.7 implies that we may reduce the proof of (3.35) to prove

$$|\alpha(z)| \leq 1 - e^{-k\epsilon}, \tag{3.37}$$

where $|\epsilon(z,k)| \leq C|k|^{-a}$. In [7, Lemma 8.2] it is shown that estimating α reduces to estimate $f_{\lambda\mu}$ for $\lambda \in S^1$. Since in our case the control of $f_{\lambda\mu}$ is uniform in our class of μ,$z \in \mathbb{C}$ and $\lambda \in S^1$, (3.37) follows. □

3.3 Regularity of the complex geometric optic solutions

Theorem 3.9. *Let β be such that $0 < \beta < \alpha$. There exists constant $C_1 = C_1(\kappa, \Lambda_0, |k|, \beta)$, and $C_2 = C_2(\kappa, \Lambda_0, |k|, \beta) > 0$ such that for any μ with $|\mu| \leq \chi_{\mathbb{D}}\kappa$ and $\|\mu\|_{\mathcal{C}^\alpha} \leq \Lambda_0$*

i)

$$\|f_\mu(\cdot, k)\|_{\mathcal{C}^{1+\beta}(\mathbb{D}, dz)} \leq C_1 \tag{3.38}$$

ii)

$$\inf_{z \in \mathbb{D}} |J_{f_\mu}(z,k)| \geq C_2. \tag{3.39}$$

Where J_{f_μ} is the Jacobian in the z variable of f_μ

The proof is based on the following Theorem (see [30] thms II.5.2 and II.5.47)

Theorem 3.10. *Let μ such that $|\mu(z)| \leq \kappa < 1$ and $\mu \in \mathcal{C}^\alpha(G)$ for a domain $G \subset \mathbb{C}$ with $\|\mu\|_{\mathcal{C}^\alpha} < \Lambda_0$. Given a function ω such that:*

- *(i) ω is a solution in $W^{1,p}_{loc}(G)$, $p > 2$, of the equation*

$$\partial_{\overline{z}}\omega = \mu\overline{\partial_z \omega} \tag{3.40}$$

- *(ii) For any $G' \subset\subset G$, there exists $K = K(G')$ such that*

$$\sup_{z \in G'} |\omega(z)| \leq K. \tag{3.41}$$

Then the following properties hold:

- *(I) If β is such that $0 < \beta < \alpha$ and $D \subset\subset G'$. Then there exists a $\Lambda_1 = \Lambda_1(\kappa, K, D, \beta)$ such that*

$$\|\partial_{\overline{z}}\omega\|_{\mathcal{C}^\beta(D)} + \|\partial_z\omega\|_{\mathcal{C}^\beta(D)} \leq \Lambda_1. \tag{3.42}$$

- *(II) Assume that ω is a quasiconformal homeomorphism in $\mathbb{C}$. Then for any $D \subset\subset \mathbb{C}$, there exists a constant $J = J(K, \kappa, \Lambda_0, D)$ such that*

$$\inf_{z\in D}(|\partial_z w(z)|^2 - |\partial_{\overline{z}} w(z)|^2) = \inf_{z\in D}(J_\omega(z)) > J. \tag{3.43}$$

This theorem will be proved in the Appendix.

Proof of Theorem 3.9:

(i) is an immediate consequence of Theorem 3.10, once we prove the existence of the constant $K(2\mathbb{D})$ in (3.41) for $f_\mu(\cdot, k)$. This follows from (k is fixed) $f_\mu(z, k) = e^{ik\phi(z,k)}$, where $\phi(z, k) = z+\epsilon(z, k)$ with $\epsilon(z, k)$ uniformly bounded for fixed k. The uniform bound only depends on k and κ.

The same argument as in (i) proves that

$$\inf_{z\in 2\mathbb{D}} |f_\mu(z, k)| > c = c(k, \kappa) > 0$$

Then (ii) follows from the same property for the exponent $\phi(z, k)$. From Theorem 2.9, $\phi(z, k)$ is a κ-quasiconformal homeomorphism. The property is a consequence of (II) in theorem 3.10. □

3.4 Growth of the k derivatives of the Complex geometric optic solutions

In this section we study the behavior of the complex geometric optic solutions respect to the k variable. It is proved in Theorem 2.5 that the Jost functions $M_\mu(k, z)$ are C^∞-smooth in the k variable. As with many other properties, for stability it is needed a quantitative version of this fact. Bounds are provided by the fact that the derivatives in the k plane of $M_\mu(k, z)$ solve corresponding non homogeneous Beltrami equations. The results in this section just assume that $|\mu| \leq \kappa\chi_{\mathbb{D}}$.

Lemma 3.11. *Let $2 < p < p_k$, $e_t \in S^1$ and $h(z, k) = \partial_{e_t}\omega_\mu(z, k)$. Then $h(z, k) \in W^{1,p}(\mathbb{C})$ and satisfies the equation*

$$\partial_{\overline{z}} h - \mu e_{-k}\overline{\partial_z h} = \mu(\gamma h + E). \tag{3.44}$$

Here

$$\gamma = -i\overline{k}e_{-k}$$

and E is an error term given by

$$E = \partial_{e_t} e_{-k}\overline{\partial}_z\omega + (\overline{\omega} + 1)(-i)\partial_{e_t}(\overline{k}e_{-k}).$$

Proof:

We start by finding a unique solution $h \in W^{1,p}$ of equation (3.44). Since $\omega \in W^{1,p}(\mathbb{C}, dz)$ and μ has compact support , it follows that $E \in L^p(\mathbb{C})$ and $\gamma \in L^\infty$. Declare $\nu = \mu e_{-k}$. We will find h with the help of the operators K introduced in Lemma 2.4. Recall that for any L^∞ functions ν, α supported in $\mathbb{D}$ and $|\nu| \leq \kappa$ the operator $K(\nu, \alpha) : L^p(\mathbb{C}) \to W^{1,p}(\mathbb{C})$ was defined by

$$K(g) = P(I - \nu\overline{S})^{-1}(\alpha g).$$

Since by Lemma 2.4, $(I - K)$ is invertible in L^p we can find $h \in L^p$ solving the equation

$$h - K(\nu, \mu\gamma)(h) = K(\nu, \mu)(E). \tag{3.45}$$

Since for $\eta \in L^p$, $\partial_{\overline{z}} P(\eta) = \eta$ if we take the distributional $\partial_{\overline{z}}$ derivative in (3.45), we obtain that

$$\partial_{\overline{z}} h = (I - \nu S)^{-1}(\mu\gamma h + \mu E). \tag{3.46}$$

Now h and E are in $L^p(\mathbb{C})$ for the required range of exponents. By the boundedness of the Beltrami operators, $\partial_{\overline{z}} h \in L^p(\mathbb{C})$. By means of the Beurling transform we achieve that $h \in W^{1,p}(\mathbb{C})$. Moreover, if h solves (3.46) then h solves (3.44) as well.

To see that in fact $h = \partial_{e_t}\omega$, we observe that equation (3.44) is obtained by formally differentiating in the k variable respect to the t direction the equation satisfied by ω (2.34). Standard arguments using differences quotients show that $h = \partial_{e_t}\omega$ as desired. □

Now we implement our knowledge of estimates for non homogeneous Beltrami equations to achieve the required control on the derivatives of the Jost functions and of the scattering transform.

Theorem 3.12. *There exists a constant $C = C(\kappa, p)$ such that for $2 < p < 1 + \frac{1}{\kappa}$ it holds that*

$$\|\nabla_k M_\mu(k, z)\|_{W^{1,p}(dz)} \leq e^{(1+C)|k|}. \tag{3.47}$$

For every $k \in \mathbb{C}$

$$|\nabla_k \tau(\mu, k)| \leq e^{(1+C)|k|} \tag{3.48}$$

Proof:

Observe that independently of the direction $e_t \in S^1$

$$\|\gamma(z, k)\|_{L^\infty(\mathbb{D}, dz)} \leq |k| \tag{3.49}$$

and for $2 < p < 1 + \frac{1}{\kappa}$ by Proposition 2.6

$$\|E\|_{L^p(\mathbb{D}, dz)} \leq 6|k|\|\omega\|_{W^{1,p}(\mathbb{D}, dz)} \leq Ce^{C|k|} \tag{3.50}$$

Thus, the claim follows from equation (3.44) and Theorem 2.1. Now, from the formula

$$\overline{\tau(\mu,k)} = \frac{1}{4\pi i}\int_{\mathbb{D}} \partial_{\overline{z}}(M_\mu - M_{-\mu})dz\wedge d\overline{z}$$

we can take derivatives with respect to k under the integral sign and conclude by Holder's inequality. □

4 Stability from the Dirichlet to Neumann map to the scattering transform

The main result in this section is Corollary 4.5. We obtain it in two different ways. In section 4.1, following the lines of [7] [8], we recover with stability the values of the geometric optics solution in the exterior of $\mathbb{D}$. In section 4.2 we prove a formula relating the differences of scattering transforms to the differences of Dirichlet to Neumann maps. Similar expressions were essential in previous works about stability, See [2] , [27], [24] and [10]. Let us remark that in both subsections no extra regularity on γ is required.

4.1 Values of M_μ on $\mathbb{C}\setminus\mathbb{D}$

Theorem 4.1. *Let $2 < p < \infty$. Then there exists an uniform constant $c = c(\kappa, p)$ such that if μ_1, μ_2 are complex with $|\mu_1|, |\mu_2| < \chi_{\mathbb{D}}$. Then,*

$$\|M_{\mu_1} - M_{\mu_2}\|_{L^p(\mathbb{C}\setminus\mathbb{D})} \leq e^{c|k|}\rho$$

where

$$\rho = \|\Lambda_{\gamma_1} - \Lambda_{\gamma_2}\|_{H^{1/2}\to H^{-1/2}}.$$

The proof of Theorem 4.1 is decomposed in the following Lemmas.

Lemma 4.2. *There exists $C = C(\kappa)$, such that for any pair f_{μ_1}, f_{μ_2} there exists two other functions $f_1, f_2 \in H^{1/2}(\partial\mathbb{D})$ such that:*

1. $(f_{\mu_1} - f_{\mu_2})_{|\partial\mathbb{D}} = f_1 + f_2$
2. $\|f_2\|_{H^{1/2}} \leq C\rho\|f_{\mu_2}\|_{H^{1/2}}$
3. $f_1 = g|_{\partial\mathbb{D}}$, *with g such that*

$$\partial_{\overline{z}} g = \mu_1\overline{\partial g}$$

Proof:

The claim follows from lemma 2.26. Let $g = f_{\mu_1} - f_{\mu_2}$. We set $f_1 = P_{\mu_1}(g)$, $f_2 = Q_{\mu_1}(g)$. Claims (1) and (3) are straightforward. For claim (2) observe that $Q_{\mu_1}(f_{\mu_1}) = 0$ and

$$Q_{\mu_1}(f_{\mu_2}) = Q_{\mu_2}(f_{\mu_2}) + i(\mathcal{H}_{\mu_2} - \mathcal{H}_{\mu_1})(f_{\mu_2}) = i(\mathcal{H}_{\mu_2} - \mathcal{H}_{\mu_1})(f_{\mu_2})$$

Therefore

$$\|f_2\|_{H^{\frac{1}{2}}} \leq \|f_{\mu_2}\|_{H^{1/2}} \|\mathcal{H}_{\mu_2} - \mathcal{H}_{\mu_1})\|_{H^{1/2} \to H^{1/2}} \leq \rho \|f_{\mu_2}\|_{H^{\frac{1}{2}}}$$

as desired.

Lemma 4.3. *There exists a constant $C(\kappa)$ such that*

$$\|f_\mu\|_{H^{\frac{1}{2}}(\partial\mathbb{D})} \leq e^{C|k|}$$

Proof: From (2.35) we control $\|\omega\|_{H^1(\mathbb{D})}$. Thus, the claim follows from the trace lemma.

Lemma 4.4. *Let $E : H^{1/2}(\partial\mathbb{D}) \to W^{1,2}(\mathbb{D})/W_0^{1,2}(\mathbb{D})$ be the extension operator. Let us define $G : \mathbb{C} \to \mathbb{C}$ by*

$$G(z,k) = (f_{\mu_1} - f_{\mu_2})(1 - \chi_{\mathbb{D}}) + g + E(f_2)\chi_D.$$

Then G solves the equation

$$\partial_{\overline{z}} G - \mu_1 \overline{\partial_z} G = (\partial_{\overline{z}} E(f_2) - \mu_1 \overline{\partial_z} E(f_2))\chi_D. \tag{4.1}$$

Furthermore let $G_0 = e^{-ikz} G$ and p be the right hand side of (4.1), i.e

$$p(f_2) = (\partial_{\overline{z}} E(f_2) - \mu_1 \overline{\partial_z} E(f_2))\chi_D.$$

Then

$$\partial_{\overline{z}} G_0 = e^{-(ikz + i\overline{k}\overline{z})} \mu_1 \overline{\partial_z} G_0 - ik\mu_1 \overline{G_0} + e^{-(ikz)} p(f_2) \tag{4.2}$$

Proof of the Theorem 4.1 Let G_0 as in Lemma 4.4. Then G_0 satisfies the differential inequality

$$|\partial_{\overline{z}} G_0| \leq \kappa |\partial_z G_0| + \gamma |G_0| + h \tag{4.3}$$

with $\kappa = \|\mu_1\|_{\infty,\mathbb{D}}$, $\gamma = k\kappa\chi_D$, $h = |p(f_2)|e^{|k|}$. In particular we see that $\gamma \in L^\infty(\mathbb{C})$ and is compactly supported. Therefore we can apply Theorem 2.1 to obtain the bound

$$\|G_0\|_{L^p(\mathbb{C})} \leq e^{C|k|} \|p\|_{L^2},$$

Finally we conclude the proof observing that on one hand, $\|p(f_2)\|_{L^2} \leq \|f_2\|_{H^{1/2}(\partial\mathbb{D}} \leq e^{c|k|}\rho$. where we have used Lemma 4.2 and Lemma 4.3. On the other hand $G_0(1 - \chi_{\mathbb{D}}) = e^{-ikz}(f_{\mu_1} - f_{\mu_2})(1 - \chi_{\mathbb{D}}) = M_{\mu_1} - M_{\mu_2}$. □

Corollary 4.5. *For $i = 1, 2$, let $\mu_i$be as in Theorem 4.1 and let $\tau(\mu_i, k)$ be the corresponding scattering transforms defined in (2.40). Then there exists an uniform constant $c = c(\kappa)$ such that for every $k \in \mathbb{C}$,*

$$|\tau(\mu_1, k) - \tau(\mu_2, k)| \leq c\rho e^{c|k|} \tag{4.4}$$

Proof:

The scattering transform $\tau(\mu, k)$ can be also defined in terms of the asymptotics in z of M_μ. Namely

$$M_\mu - M_{-\mu} = \frac{\tau(\mu, k)}{z} + O(\frac{1}{|z|^2}).$$

Therefore, we have that

$$D(z, k) = (M_{\mu_1} - M_{-\mu_1} - M_{\mu_2} + M_{-\mu_2})(z, k) = \frac{\tau(\mu_1, k) - \tau(\mu_2, k)}{z} + O(\frac{1}{|z|^2}).$$

Now $D(z, k)$ is analytic in $\mathbb{C} \setminus \mathbb{D}$. Thus, for every $r > 1$ it holds that

$$|\tau(\mu_1, k) - \tau(\mu_2, k)| = |\int_{\partial \mathbb{D}_r} D(z, k)dz|.$$

Integrating this expression respect to r yields that

$$\begin{aligned} |\tau(\mu_1, k) - \tau(\mu_2, k)| &\leq \frac{1}{2} \int_1^2 \int_{\partial \mathbb{D}_r} |D(z, k)||dz|dr \\ &\leq \int_{1 \leq |z| \leq 2} |D(z, k)|dz \\ &\leq c\|D(z, k)\|_{L^p(\mathbb{C} \setminus \mathbb{D})} \leq \rho e^{c|k|}. \end{aligned}$$

The last inequality follows from Theorem 4.1. The corollary is proved. □

4.2 An explicit formula.

We will denote the dependencies by writing $u(\mu, k, z)$, etc. The scattering transform of μ is defined by

$$\tau(\mu, k) = \frac{i}{4\pi} \int_{\mathbb{D}} \partial_z(\bar{M}(\mu, z, k) - \bar{M}(-\mu, z, k))dm(z). \tag{4.5}$$

Since $f_\mu(z, k) = e^{ikz} M_\mu(z, k)$ we also have

$$\tau(\mu, k) = \frac{i}{4\pi} \int_{\mathbb{D}} \partial_z(e^{i\bar{k}\bar{z}}(\bar{f}_\mu - \bar{f}_{-\mu}))dm(z). \tag{4.6}$$

Theorem 4.6. *Let γ_1 and γ_2 be L^∞ conductivities which are identically 1 in a neighborhood of $\mathbb{C} \setminus \mathbb{D}$ and μ_1 and μ_2 the corresponding Beltrami coefficients. We have that*

$$\tau(\mu_1, k) - \tau(\mu_2, k) = \frac{-1}{8\pi\bar{k}} \int_{\partial\mathbb{D}} \bar{u}(\mu_1, z, -k)(\Lambda_\gamma - \Lambda_{\gamma_2}) u(\mu_2, z, k) d\sigma \quad (4.7)$$

where $u(\mu_i, z, k) = u_{\gamma_i}$ is the complex geometric optics solution given by (2.41).

To prove the theorem we need an alternative formula for the scattering transform.

Lemma 4.7.

$$\tau(\mu, k) = \frac{-1}{4\pi\bar{k}} \int_{\partial\mathbb{D}} \bar{z} e^{i\bar{k}\bar{z}} \partial_{\bar{z}} u d\sigma \quad (4.8)$$

Proof:

Since the normal at any point $z \in \partial\mathbb{D}$ can be identified with z, by Green Formula

$$\tau(\mu, k) = \frac{i}{4\pi} \int_{\partial\mathbb{D}} \bar{z} e^{i\bar{k}\bar{z}} (\bar{f}_\mu - \bar{f}_{-\mu}) d\sigma. \quad (4.9)$$

Notice that for $z \in \mathbb{D}$ the tangential derivative is

$$\partial_T(e^{i\bar{k}\bar{z}}) = \bar{k}\bar{z} e^{i\bar{k}\bar{z}}.$$

Thus, by integration by parts it follows that

$$\tau(\mu, k) = \frac{-i}{4\pi\bar{k}} \int_{\partial\mathbb{D}} e^{i\bar{k}\bar{z}} \partial_T (\bar{f}_\mu - \bar{f}_{-\mu}) d\sigma \quad (4.10)$$

Now we write $f_\mu - f_{-\mu}$ in terms of u and its Hilbert transform:

$$f_\mu - f_{-\mu} = \Re(u) + i\mathcal{H}_\mu(\Re(u)) + \mathcal{H}_\mu(\Im(u)) - i\Im(u)$$

Thus,

$$\partial_T(\bar{f}_\mu - \bar{f}_{-\mu}) = \partial_T(\Re(u)) - i\Lambda_\gamma(\Re(u)) + \Lambda_\gamma(\Im(u)) + i\partial_T(\Im(u)).$$

Since $\gamma = 1$ on $\partial\mathbb{D}$, we have that (in the weak sense) $\Lambda_\gamma(\varphi) = \partial_\nu(\varphi)$ for every $\varphi \in H^{\frac{1}{2}}(\partial\mathbb{D})$. Hence,

$$\partial_T(\bar{f}_\mu - \bar{f}_{-\mu}) = \partial_T(u) - i\partial_\nu(u) \quad (4.11)$$

Since,

$$\bar{z}\partial_{\bar{z}} u = \partial_\nu u + i\partial_T u,$$

(4.11) implies that

$$i\partial_T(\bar{f}_\mu - \bar{f}_{-\mu}) = \bar{z}\partial_{\bar{z}} u,$$

which plugged into (4.10) yields the claim. □

Proof of the Theorem: The theorem will follow from the following claim.
Claim

$$\begin{aligned}&\int_{\partial\mathbb{D}} ze^{ikz}\partial_z\bar{u}(\mu_1,z,-k)d\sigma - \int_{\partial\mathbb{D}} \bar{z}e^{i\bar{k}\bar{z}}\partial_{\bar{z}}\bar{u}(\mu_2,z,k)d\sigma = \\ &\frac{1}{2}\int_{\partial\mathbb{D}} \bar{u}(\mu_1,z,-k)(\Lambda_{\gamma_1}-\Lambda_{\gamma_2})u(\mu_2,z,k)d\sigma.\end{aligned} \tag{4.12}$$

Assume the Claim, then take $\gamma_1 = \gamma_2$ to obtain

$$\tau(\mu_1,k) = \frac{-1}{4\pi\bar{k}}\int_{\partial\mathbb{D}} ze^{ikz}\partial_z\bar{u}(\mu_1,z,-k)d\sigma$$

and hence (4.7).
Proof of Claim: From Proposition 2.8, we have

$$e^{ikz} = u(\mu_2,z,k) - e^{ikz}R(\mu_2,k,z) \tag{4.13}$$

$$e^{i\bar{k}\bar{z}} = \bar{u}(\mu_1,z,k) - e^{i\bar{k}\bar{z}}\bar{R}(\mu_1,z,-k), \tag{4.14}$$

where $R(\mu_1,k,z) \in W^{1,p}(\mathbb{C})$ and $R(\mu_1,k,z) = O(\frac{1}{|z|})$. Inserting these expressions in the first member of (4.12) we obtain

$$\int_{\partial\mathbb{D}} ze^{ikz}\partial_z\bar{u}(\mu_1,z,-k)d\sigma - \int_{\partial\mathbb{D}} \bar{z}e^{i\bar{k}\bar{z}}\partial_{\bar{z}}\bar{u}(\mu_2,z,k)d\sigma =$$

$$\int_{\partial\mathbb{D}} u(\mu_2,z,k)z\partial_z\bar{u}(\mu_1,z,-k)d\sigma - \int_{\partial\mathbb{D}} \bar{u}(\mu_1,z,-k)\bar{z}\partial_{\bar{z}}u(\mu_2,z,k)d\sigma -$$

$$\int_{\partial\mathbb{D}} e^{ikz}R(\mu_2,z,k)z\partial_z\bar{u}(\mu_1,z,-k)d\sigma + \int_{\partial\mathbb{D}} e^{i\bar{k}\bar{z}}\bar{R}(\mu_1,z,-k)\bar{z}\partial_{\bar{z}}u(\mu_2,z,k)d\sigma.$$

We divide the proof of the claim in two Lemmata:

Lemma 4.8.

$$\int_{\partial\mathbb{D}} u(\mu_2,z,k)z\partial_z\bar{u}(\mu_1,z,-k)d\sigma - \int_{\partial\mathbb{D}} \bar{u}(\mu_1,z,-k)\bar{z}\partial_{\bar{z}}u(\mu_2,z,k)d\sigma =$$

$$\frac{1}{2}\int_{\partial\mathbb{D}} \bar{u}(\mu_1,z,-k)(\Lambda_{\gamma_1}-\Lambda_{\gamma_2})u(\mu_2,z,k)d\sigma.$$

Lemma 4.9.

$$-\int_{\partial\mathbb{D}} e^{ikz}R(\mu_2,z,k)z\partial_z\bar{u}(\mu_1,z,-k)d\sigma + \int_{\partial\mathbb{D}} e^{i\bar{k}\bar{z}}\bar{R}(\mu_1,z,-k)\bar{z}\partial_{\bar{z}}u(\mu_2,z,k)d\sigma = 0.$$

Proof of Lemma 4.8: Since

$$\bar{z}\partial_{\bar{z}} = 1/2\partial_\nu + i/2\partial_T$$

and

$$z\partial_z = 1/2\partial_\nu - i/2\partial_T,$$

we have, integrating by parts

$$\int_{\partial\mathbb{D}} u(\mu_2,z,k)z\partial_z\bar{u}(\mu_1,z,-k)d\sigma - \int_{\partial\mathbb{D}} \bar{u}(\mu_1,z,-k)\bar{z}\partial_{\bar{z}}u(\mu_2,z,k)d\sigma =$$

$$\frac{1}{2}\int_{\partial\mathbb{D}} u(\mu_2,z,k)\Lambda_{\gamma_1}\bar{u}(\mu_1,z,-k) - \frac{1}{2}\int_{\partial\mathbb{D}} \bar{u}(\mu_1,z,k)\Lambda_{\gamma_2}u(\mu_2,z,-k) -$$

$$\frac{i}{2}\int_{\partial\mathbb{D}} \partial_T(u(\mu_1,z,-k)u(\mu_2,z,k))d\sigma$$

The last integral vanishes from the fundamental theorem of Calculus. Since Λ_γ is selfadjoint the lemma follows. □

Proof of Lemma 4.9: Let $\mathbb{D}_R$ be the disc of radius R centered at the origin. Integration by parts in $\mathbb{D}_R \setminus \mathbb{D}$ gives

$$-\int_{\partial\mathbb{D}} e^{ikz}R(\mu_2,z,k)z\partial_z\bar{u}(\mu_1,z,-k)d\sigma + \int_{\partial\mathbb{D}} e^{i\bar{k}\bar{z}}\bar{R}(\mu_1,z,-k)\bar{z}\partial_{\bar{z}}u(\mu_2,z,k)d\sigma =$$

$$\int_{\partial\mathbb{D}_R} \frac{z}{|z|}e^{ikz}R(\mu_2,z,k)\partial_z\bar{u}(\mu_1,z,-k)d\sigma - \int_{\partial\mathbb{D}_R} \frac{\bar{z}}{|z|}e^{i\bar{k}\bar{z}}\bar{R}(\mu_1,z,-k)\partial_{\bar{z}}u(\mu_2,z,k)d\sigma +$$

$$\int_{\mathbb{D}_R\setminus\mathbb{D}} \Big(-\partial_{\bar{z}}(e^{ikz}R(\mu_2,z,k)\partial_z\bar{u}(\mu_1,z,-k)) + \partial_z(e^{i\bar{k}\bar{z}}\bar{R}(\mu_1,z,-k)\partial_{\bar{z}}u(\mu_2,z,k))\Big)\,dm \tag{4.15}$$

Since u is harmonic on the exterior of $\mathbb{D}$, $\partial_{\bar{z}}\partial_z u = 0$, this together with expressions (4.13) and (4.14) gives

$$\partial_{\bar{z}}(e^{ikz}R(\mu_2,z,k)\partial_z\bar{u}(\mu_1,z,-k)) = \partial_{\bar{z}}u(\mu_2,z,k)\partial_z\bar{u}(\mu_1,z,-k) =$$

$$\partial_z(e^{i\bar{k}\bar{z}}\bar{R}(\mu_1,z,-k)\partial_{\bar{z}}u(\mu_2,z,k)),$$

Hence the last term in (4.15) vanishes. To finish the proof we estimate

$$\int_{\partial\mathbb{D}_R} \frac{z}{|z|}e^{ikz}R(\mu_2,z,k)\partial_z\bar{u}(\mu_1,z,-k)d\sigma = o(R) \tag{4.16}$$

and

$$\int_{\partial\mathbb{D}_R} \frac{\bar{z}}{|z|}e^{i\bar{k}\bar{z}}\bar{R}(\mu_1,z,-k)\partial_{\bar{z}}u(\mu_2,z,k)d\sigma = o(R). \tag{4.17}$$

ar $R \to \infty$. We know from (2.48) that

$$R(\mu_1,z,k) = \omega_{\mu_1} + \omega_{-\mu_1} + e_{-k}(\overline{\omega_{\mu_1} - \omega_{-\mu_1}})) \in W^{1,p}(\mathbb{C}).$$

We also have, since $\bar{f}_{\mu_1}$ is antianalytic on the exterior of $\mathbb{D}$,

$$\partial_z \bar{u}(\mu_1, z, k) = 1/2(\partial_z \bar{f}_{\mu_1} + \partial_z \bar{f}_{-\mu_1} + \partial_z f_{\mu_1} - \partial_z f_{-\mu_1}) = 1/2(\partial_z f_{\mu_1} - \partial_z f_{-\mu_1})$$

$$= 1/2 e^{-ikz} \partial_z (\tilde{w}_{\mu_{1_1}} - \tilde{w}_{\mu_2})$$

Since $\partial_z(\tilde{w}_{\mu_1} - \tilde{w}_{\mu_2})$ is analytic in the exterior of $\mathbb{D}$ and is in $L^p(\mathbb{C})$, it decays as $O(1/|z|)$. Hence

$$|\int_{\partial \mathbb{D}_R} \frac{z}{|z|} e^{ikz} R(\mu_2, z, k) \partial_z \bar{u}(\mu_1, z, -k) d\sigma| \leq$$

$$C(\int_{\partial \mathbb{D}_R} |R(\mu_2, z, k)|^p)^{1/p} (\int_{\partial \mathbb{D}_R} \frac{1}{|z|^q})^{1/q} \leq C R^{-1/p} \|R(\mu_2, \cdot, k)\|_{L^p(\partial \mathbb{D}_R)}$$

From the trace theorem

$$\|R(\mu_2, \cdot, k)\|_{L^p(\partial \mathbb{D}_R)} \leq C \|R\|_{W^{1,p}(\mathbb{C})}$$

and (4.16) is proved. (4.17) can be proved in a similar way.

Remark 4.10. It follows from (4.8) that for every $k \in \mathbb{C}$

$$|\tau(\mu_1, k) - \tau(\mu_2, k)| \leq \frac{1}{|k|} \rho \|u(\mu_1)\|^2_{H^{\frac{1}{2}}(\partial \mathbb{D})}. \tag{4.18}$$

On the other hand $\tau(\mu_i, 0) = 0$ and the Lipschitz bound given in Theorem 3.12 imply that for every k

$$|\tau(\mu_i, k)| \leq e^{c(1+|k|)} |k| \tag{4.19}$$

Thus, we can use (4.19) for $|k| \leq \rho^{\frac{1}{2}}$ and (4.18) for $|k| \leq \rho^{\frac{1}{2}}$ to obtain the Hölder stability

$$|\tau(\mu_1, k) - \tau(\mu_2, k)| \leq \rho^{\frac{1}{2}} e^{c(1+|k|)}. \tag{4.20}$$

Comparing this expression with Corollary 4.5 seems to indicate that there is room to improvement in (4.4) for small $|k|$.

5 Stability of the complex geometric optics solutions $u(z, k)$.

In this section we consider two Beltrami coefficients μ_1 μ_2. Throughout the section we will use $j = 1, 2$ and assume $|\mu_j| \leq \kappa \chi_{\mathbb{D}}$ and that $[\mu_j]_{C^\alpha} \leq \Lambda_0$. We will assume that $\rho = \|\Lambda_{\gamma_1} - \Lambda_{\gamma_2}\| \leq \frac{1}{2}$, which is not a loss of generality as shown in the next section. The constant C may change at each occurrence, if the change involves new parameters we will write them explicitly.

Theorem 5.1. *Let us denote $u_j(z,k) = u_{\gamma_j}(z,k)$, given in (2.41). Then for each k there exists a constant C such that*

$$\|u_1(z,k) - u_2(z,k)\|_{L^\infty(\mathbb{D},dz)} \leq \frac{C(k)}{|\log(\rho)|^a}. \tag{5.1}$$

The exponent $a = a(\kappa, \Lambda_0, \alpha)$ is given in (3.35).

Remark: A similar estimate can be proved for the solutions $u_{\gamma^{-1}}$ of

$$div(\gamma^{-1}\nabla u) = 0,$$

which are constructed as $u = i\Re f_{-\mu} - \Im f_\mu$, see (2.42).

From Proposition 2.8, we know that for a conductivity γ_j the corresponding geometric optics solutions to the conductivity equation $u_j = u_{\gamma_j}(z,k)$ satisfies the pseudoanalytic equation in the k variable

$$\partial_{\overline{k}} u_j = -i\tau_j \overline{u_j}, \tag{5.2}$$

and can be written as

$$u_j(z,k) = e^{\delta_j(z,k)}. \tag{5.3}$$

Moreover it follows from Theorem 3.8 that under our regularity conditions on μ the exponent functions have asymptotics

$$\delta_j(z,k) = ik(z + v_j(z,k)) \quad \text{for fixed } k \tag{5.4}$$

$$\delta_j(z,k) = ik(z + \epsilon_j(z,k)) \text{ for fixed } z \tag{5.5}$$

where $v_j(z,k)$ is in L^∞ and $|\epsilon_j(z,k)| \leq C|k|^{-a}$.

The proof of Theorem 5.1 is inspired by the uniqueness proof in [7]. First we notice that the arguments leading to the uniqueness of the so-called transport matrix in [7], which satisfies a Beltrami type equation in k, can be applied directly to the functions u, that satisfy the pseudoanalytic equation (5.2). This allows us to avoid the use of the transport equation.

It is not known whether uniqueness might be derived directly from the equation (5.2) in k. However, there is additional information to exploit, since the u_j are functions of z, k with controlled asymptotics in both variables. The asymptotics in z imply that the functions $\delta_j(z,k)$ have range $\mathbb{C}$, see Proposition 2.8. From the asymptotics in k and the equation (5.2) we obtain that if $\tau_{\mu_1} = \tau_{\mu_2}$, for $z \neq w$ and $k \neq 0$, then $\delta_1(z,k) \neq \delta_2(w,k)$. Uniqueness follows then easily from these two facts. For stability we need also the first fact, which we restate as Proposition 5.2. However, we will obtain an appropriate quantitative version in Proposition 5.3 below.

Proposition 5.2 ([7])**.** *The functions $\delta_j(.,k)$ for $k \neq 0$ have range $\mathbb{C}$.*

Proposition 5.3. *Let $g(z,w,k) = \delta_1(z,k) - \delta_2(w,k)$. Then there exists a constant C such that if $k \neq 0$ and $|z-w| \geq \frac{C}{|\log(\rho)|^a}$, then $g(z,w,k) \neq 0$. In particular, this gives that the conditions $g(z,w,k) = 0$ and $k \neq 0$ imply that $|z-w| \leq C|\log(\rho)|^{-a}$. The constant $a = a(\kappa, \Lambda_0) > 0$ is given in (3.35).*

We postpone the proof of Proposition 5.3 and prove first Theorem 5.1.

Proof of Theorem 5.1: Let $z \in \mathbb{D}$. We want to estimate

$$|u_1(z,k) - u_2(z,k)|.$$

We write $u_j(z,k) = e^{\delta_j(z,k)}$. Since δ_1 is onto there exists $\omega \in \mathbb{C}$ such that $\delta_1(\omega,k) = \delta_2(z,k)$ and hence $u_1(\omega,k) = u_2(z,k)$. Moreover, by Proposition 5.3 $|\omega - z| \leq C|\log(\rho)|^{-a}$. Then by Theorem 3.9 u_1 is Lipschitz in $\mathbb{D}(0,2)$ with constant $C(|k|)$. Thus,

$$|u_1(z,k) - u_2(z,k)| = |u_1(z,k) - u_1(\omega,k)| \leq C(|k|)C|\log(\rho)|^{-a},$$

and we are done. □

Let us turn to the proof of Proposition 5.3, which is the heart of the matter. From (5.4) we can write for $\lambda = z - w$

$$g(z,w,k) = i\lambda k + k\epsilon(k), \tag{5.6}$$

where $|\epsilon(k)| \leq 2C|k|^{-a}$. We split the proof of Proposition 5.3 in several Lemmas. First we find the equation satisfied by g in the k variable.

Lemma 5.4. *g satisfies the equation*

$$\partial_{\overline{k}} g = \sigma g + E. \tag{5.7}$$

Where, if τ denotes the scattering transform,

$$\|\sigma(z,\cdot)\|_{L^\infty(\mathbb{C})} \leq 2\|\tau(\mu_2)\|_{L^\infty(\mathbb{C})} \leq 2 \tag{5.8}$$

and E and its derivatives satisfy

$$|E(k)| \leq \rho e^{C(1+|k|)}, \quad |DE(k)| \leq e^{C(1+|k|)} \tag{5.9}$$

Proof: Since

$$\partial_{\overline{k}}\delta_j = \partial_{\overline{k}}(\log u_j) = \frac{\partial_{\overline{k}} u_j}{u_j} = -i\tau(\mu_j,k)\frac{\overline{u}_j}{u_j} = -i\tau(\mu_j,k)e^{\overline{\delta_j}-\delta_j}$$

then the $\partial_{\overline{k}}$ derivative of the function g satisfies the equation

$$\partial_{\overline{k}} g = -i(\tau(\mu_1) - \tau(\mu_2))e^{\overline{\delta_1}-\delta_1} - i\tau(\mu_2)(e^{\overline{\delta_1}-\delta_1} - e^{\overline{\delta_2}-\delta_2})$$

which can be written as

$$\partial_{\overline{k}} g = \sigma g + E \tag{5.10}$$

where

$$\sigma = -i\tau_{\mu_2}(k)[\frac{e^{\overline{\delta_1}-\delta_1} - e^{\overline{\delta_2}-\delta_2}}{\delta_1 - \delta_2}]$$

$$E(k) = -i(\tau_{\mu_1} - \tau_{\mu_2})e^{\overline{\delta_2}-\delta_2}.$$

Since the function $e^{i\theta}$ is globally Lipschitz for $\theta \in \mathbb{R}$, we can bound the coefficients of this equation as $|\sigma| \leq 2|\tau(\mu_2)| < 2$ and since, as we know from the previous section (4.4) $|\tau(\mu_1) - \tau_(\mu_2)| \leq \rho e^{c|k|}$, we have also $|E(k)| \leq \rho e^{ck}$, where $\rho = \|\Lambda_{\sigma_1} - \Lambda_{\sigma_2}\|$ only depending on $\|\mu\|_\infty \leq \kappa < 1$. Finally, since $\tau(\mu_1), \tau(\mu_2), \delta_1, \delta_2$ are C^∞ functions on the k variable with bounds given by Theorem 3.12, (5.9) follows. □

In the following lemma we use the equation (5.7) to decompose g suitably. Let a and C given in (3.35). Then we define a function $R : \mathbb{C} \to R$ by

$$R(\lambda) = \begin{cases} |\frac{\lambda}{4C}|^{-\frac{1}{a}} \text{ if } |\lambda| \leq 4C \\ 1 \text{ otherwise} \end{cases} \tag{5.11}$$

This choice guarantees for $|k| \geq R(\lambda)$ that in (5.6) we have $|\epsilon(k)| \leq \frac{|\lambda|}{2}$. We emphasize that R depends only on Λ_0 and κ.

Lemma 5.5. *There exists complex valued functions $\eta = \eta(z, w, k)$ and $S = S(z, w, k)$ such that the function*

$$F = e^{-\eta} g - S \tag{5.12}$$

is analytic for $k \in \mathbb{D}(0, R(\lambda))$ and for any $\theta < 1$ there exists a constant C such that

$$\|\eta\|_{L^\infty(\mathbb{C},dk)} \leq CR(\lambda) \tag{5.13}$$

$$\|S\|_{L^\infty(\mathbb{C},dk)} \leq \rho R(\lambda) e^{R(\lambda)} \leq \rho e^{CR(\lambda)} \tag{5.14}$$

$$\|\nabla S\|_{L^\infty(\mathbb{C},dk)} \leq \rho^\theta e^{CR(\lambda)} \tag{5.15}$$

Proof: By Lemma 5.4 for fixed z and w, g satisfies in k the equation

$$\partial_{\overline{k}} g = \sigma g + E. \tag{5.16}$$

Let us consider the disc $\mathbb{D}(0, R)$ centered at the origin 0 and radius R. Let us define

$$\eta = P(\sigma \varphi_R)$$

$$S = P(e^\eta E \varphi_R)$$

where P is the Cauchy transform and $\varphi_R \in C_0^\infty(\mathbb{D}(0, 2R))$ is a cut-off function such that $\varphi_R = 1$ on $\mathbb{D}(0, R)$. Then the equation $\partial_{\overline{k}} g = \sigma g + E$ is equivalent to

$$\partial_{\overline{k}}(e^{-\eta} g - S) = 0$$

Therefore by Weyl's lemma, the function

$$F(\omega, z, k) = e^{-\eta} g - S$$

is holomorphic in $\mathbb{D}(0, R)$. Now, since $\eta = P(\sigma\varphi_R)$,

$$|\eta(k)| = |P(\sigma\varphi_R)| \leq |\int_{\mathbb{D}_{2R}} \frac{\sigma(t)}{k-t} dt| \leq \int_{\mathbb{D}_{2R}} \frac{2}{|k-t|} dt \leq 8\pi R. \tag{5.17}$$

Moreover, since the Beurling transform T maps $L^p \to L^p$, we have that

$$\|\partial_{\overline{k}} \eta\|_{L^p} \ \leq C(p)\|\sigma\varphi_R\|_{L^p} \ \leq C(p)\pi R^{\frac{2}{p}}. \tag{5.18}$$

Now we turn to S. For the L^∞ norm, since $S = P(e^{-\eta} E\varphi_R)$ it follows that

$$|S(k)| \leq |\int_{\mathbb{D}_{2R}} \frac{e^{-\eta(t)} E(t)}{k-t} dt| \leq \rho e^{c(1+R)} e^{8\pi R} \int_{\mathbb{D}_{2R}} \frac{dt}{|k-t|} \leq \rho e^{CR} \rho,$$

where we have used (5.9). To bound the derivatives of S recall that since $S = P(e^{-\eta} E\varphi_R)$,

$$\partial_{\overline{k}} S = e^{-\eta} E\varphi_R.$$

Now, let us first notice that by (5.17), (5.9) and the compact support of φ_R

$$\|e^{-\eta} E\varphi_R\|_{L^\infty(\mathbb{C})} \leq e^{4\pi R} \rho e^{CR}. \tag{5.19}$$

Direct application of the boundedness of T would yield L^p(or BMO) bounds for $\partial_k S$. In particular for $1 < p < \infty$

$$\|\partial_k S\|_{L^p} = \|T(e^{-\eta} E\varphi_R)\|_{L^p} \leq C\|e^{-\eta} E\varphi_R\|_{L^p} \leq \rho R^{2/p} e^{CR} \leq \rho e^{CR}. \tag{5.20}$$

However we need an L^∞ bound. The crucial observation is that combining estimates (5.17), (5.18), (5.19) with (5.9), we obtain that for every $1 < p < \infty$

$$\|e^{-\eta} E\varphi_R\|_{W^{1,p}(\mathbb{C})} \leq R^{\frac{2}{p}} (e^{4\pi R} e^{CR} + \rho C(p)\pi) \leq e^{CR}, \tag{5.21}$$

where $C = C(p, \kappa)$. Since T is a Calderón -Zygmund integral and also a Fourier multiplier it preserves the spaces $W^{1,p}$ and hence

$$\|\partial_k S\|_{W^{1,p}} \leq e^{CR}. \tag{5.22}$$

Now we intend to use an interpolation argument to combine estimates (5.20) and (5.22). Let $0 \leq \theta \leq 1$ and $1 < p < \infty$. Then, for example, by complex interpolation ([12]) or by the wavelets characterization of Sobolev spaces (see [20]) we have that

$$\|\partial_k S\|_{W^{\theta,p}} \leq \rho^{(1-\theta)} e^{\theta CR}.$$

Finally choosing $\theta > \frac{2}{p}$, $W^{\theta,p} \hookrightarrow L^\infty$ and estimate (5.15) follows with any $\theta < 1$. The proof is concluded. □

Remark Let us go back for a moment to the uniqueness proof in [7]. If $\rho = 0$, $S = 0$ and we would have $g = e^\eta F$. Furthermore by the asymptotic behaviour of g in (5.6) if $z \neq \omega$

$$F(k) \sim (z - \omega) k e^\eta \text{ in } \partial\mathbb{D}(0, R(\lambda)) \tag{5.23}$$

Thus, by the argument principle F has a unique 0. Since the zeros of g are those of F the proof of uniqueness is concluded. This strategy faces two obstacles in the stability setting. First, to have something like (5.23) we need $|z - \omega|$ to be sufficiently big in comparison to the size of S (and hence to the size of ρ). The second obstacle is how to pass information from F to $g = e^\eta(F + S)$.

The next proposition handles the first obstacle:

Proposition 5.6. *Let a given in (3.35). Then there exists a constant C_1 such that if $|\lambda| \geq C_1 |\log(\rho)|^{-a}$, then $F(w, z, k) = 0$ only when $k = 0$.*

Proof: We start by proving that $F(k)$ can not vanish in the set $|k| \leq R(\lambda)$ where it is holomorphic.

We will characterize the zeros of F by proving that F is homotopic in the k variable to $e^{-\eta}\lambda k$ in $\partial\mathbb{D}(0, R(\lambda))$, where $R(\lambda)$ was defined in Lemma 5.5. Let $|k| = R(\lambda)$, and $0 \leq t \leq 1$, then

$$\begin{aligned} |tF(k) + (1-t)e^{-\eta} i\lambda k| &= |te^{-\eta}(g - i\lambda k) + e^{-\eta} i\lambda k - tS| \\ &\geq e^{-\|\eta\|_\infty} |\frac{\lambda}{2} k| - \max_{|k|=R} |S(k)|. \end{aligned} \tag{5.24}$$

Suppose that (5.24) is strictly positive. Then $\deg(F, \mathbb{D}(0, R(\lambda)), 0) = \deg(e^\eta \lambda k, \mathbb{D}(0, R(\lambda)), 0) = 1$ and being holomorphic, F would have a unique zero at $k = 0$. Therefore, the proof of the proposition will be finished if the following claim holds:

Claim: Under the assumptions of the proposition

$$4(\max_{|k|=R} |S(k)|) e^{\|\eta\|_\infty} \leq |\lambda R(\lambda)|. \tag{5.25}$$

To see that this is the case, we observe that Lemma 5.5 implies that

$$4(\max_{|k|=R} |S(k)|) e^{\|\eta\|_\infty} \leq \rho e^{CR} e^{CR} = e^{CR} \rho.$$

Thus, to attain (5.25) it suffices that

$$\rho \leq |\lambda| e^{-CR(\lambda))}. \tag{5.26}$$

Since $\rho \leq \frac{1}{2}$ and for an appropriate constant C we have that $|\lambda| \geq e^{-CR(\lambda)}$. Therefore the claim follows for a suitable constant C_1.

Finally if $|k| \geq R(\lambda)$, $|F(k)| \geq e^{-CR}|\frac{\lambda k}{2}| - \rho e^{CR}$ which again by (5.26) never vanishes. □

To handle the second obstacle, how to pass information from F to $g = e^{\eta}(F + S)$, it turns out that since F is analytic, from its asymptotics we gain more precise information that just the number of zeroes. Namely, in the following two lemmas we prove that, near the origin, it behaves basically as λk.

Lemma 5.7. *Let λ be such that $|\lambda| = |z - w| \geq C_1 |\log \rho|^{-a}$, for C_1 given in Proposition 5.6. There exists a constant $M_0 > 0$ such that any function $F(w, z, k)$ as in Lemma 5.5, can be written as*

$$F(w, z, k) = \lambda k e^{\nu(k)}, \tag{5.27}$$

for some function $\nu(k) = \nu(w, z, k)$ which on $|k| \leq R(\lambda)$ is holomorphic and satisfies

$$|\nu(k)| \leq M_0 R(\lambda). \tag{5.28}$$

Proof: The function $F(w, z, k) = e^{-\eta} g - S$ is analytic in $\mathbb{D}(0, R)$, therefore we might use the maximum principle. We will use the bounds

$$\|\eta\|_{L^\infty(\mathbb{D}(0,R))} \leq CR(\lambda) \tag{5.29}$$

$$\|S\|_{L^\infty(\mathbb{D}(0,R))} \leq \rho e^{CR(\lambda)}. \tag{5.30}$$

On one hand, if $|k| = R(\lambda)$, $g(w, z, k) = i\lambda k + k\epsilon(k)$ with $|\epsilon(k)| < |\lambda|/2$ and we obtain (see the claim in the proof of Proposition 5.6)

$$|F(k)| \leq \frac{3}{2}|\lambda||k| e^{\|\eta\|_\infty} + |S(k)| \leq |\lambda| R(\lambda) e^{M_0 R(\lambda)}$$

On the other hand F is analytic and from Proposition 5.6, only vanishes at $k = 0$ for $|\lambda| > \frac{C_1}{|\log(\rho)|^a}$. Then for such a λ, $\frac{F(k)}{k}$ is holomorphic as well and $\frac{F(k)}{k} \neq 0$ for every k. Therefore there exists a $\nu(k)$ analytic, such that

$$F(k) = \lambda k e^{\nu(k)}.$$

Moreover, by the maximum principle

$$\sup_{k \in \mathbb{D}(0,R)} |\frac{F(k)}{k}| = \sup_{|k|=R} |\frac{F(k)}{k}| \leq |\lambda| e^{M_0 R(\lambda))}$$

This proves that $|\nu(k)| \leq CR$ if $|k| < R$. □

Let us denote

$$\mathcal{F}_\lambda = \{F \in \mathcal{H}(\mathbb{D}(0, R(\lambda)) : F(k) = \lambda k e^{\nu(k)}, |\nu(k)| < M_0 R(\lambda)\}. \tag{5.31}$$

The next lemma describes two key properties of the class $\mathcal{F}_\lambda$.

Lemma 5.8. *Let $F \in \mathcal{F}_\lambda$, then there exists a constant d such that*

i) $F^{-1}(\mathbb{D}(0,\delta)) \subset \mathbb{D}(0, \frac{\delta e^{M_0 R}}{|\lambda|})$.

ii) $\inf_{|k|<d} |F'(k)| > \frac{1}{2}|\lambda| e^{-M_0 R(\lambda)}$.

Proof: Part i) follows directly from the definition of $\mathcal{F}_\lambda$. Let k be such that $|F(k)| < \delta$. Then,

$$|F(k)| = |\lambda k e^{\nu(k)}| < \delta \Rightarrow |k| < |\frac{\delta e^{-\nu(k)}}{\lambda}| \leq \frac{\delta e^{M_0 R}}{|\lambda|}.$$

For part (ii) the definition of $\mathcal{F}_\lambda$ implies that

$$|F'(k)| = |\lambda e^{\nu(k)} + \lambda k e^{\nu(k)} \nu'(k)| \geq e^{-M_0 R} |\lambda| |1 + \nu'(k) k|. \tag{5.32}$$

Let $|k| \leq d_1 \leq \frac{R(\lambda)}{2}$. Since $\nu(k)$ is analytic we can use Cauchy integral formula to estimate

$$|\nu'(k)| \leq |\frac{1}{2\pi i} \int_{\partial \mathbb{D}(0,R)} \frac{\nu(\omega)}{(\omega - k)^2}| \leq \frac{4M_0(R)R}{R^2} = 4M_0$$

Then $|\nu'(k)k| < 4M_0|k| < 1/2$ if $|k| < \frac{1}{8M_0}$ and this implies

$$|1 + \nu'(k)k| \geq \frac{1}{2}. \tag{5.33}$$

We define $d = \inf\{d_1, \frac{1}{8M_0}\}$.

Inserting (5.33) into (5.32) yields that for $|k| \leq d$, $|F'(k)| > \frac{1}{2}|\lambda| e^{-M_0 R(\lambda)}$. □

The idea to conclude the proof comes from the fact that a linear function λk can not be intersected twice by a function S, which is sufficiently small in $W^{1,\infty}$ and with $S(0) = 0$. By Lemma 5.8 one expects the same for $F \in \mathcal{F}_\lambda$, to see this we show first that if $(-S)$ and F meet they must do it in a neighborhood of the origin.

Lemma 5.9. *Let $H = F + S$. Then there exists C_2 such that if $|\lambda| > C_2 |\log \rho|^{-a}$ then the set*

$$Z(H) = \{k : H(k) = 0\}$$

is contained in $\mathbb{D}(0,d)$, d is given in the previous Lemma.

Proof: If k is a zero of H, then $F(k) = -S(k)$. But we know from Lemma 5.5 that

$$\|S(k)\|_{L^\infty(\mathbb{C}, dk)} \leq \rho e^{CR}.$$

Thus, it follows that

$$Z(F+S) \subset F^{-1}(\mathbb{D}(0, \rho e^{CR})) \subset \mathbb{D}(0, \frac{\rho}{|\lambda|} e^{CR})$$

where the last inequality follows from Lemma 5.8 part i). Therefore we need λ to satisfy that

$$\rho \leq d|\lambda| e^{-CR} \tag{5.34}$$

which can be attained as (5.26).

□

Next, we prove that the jacobians DF and $(-DS)$ can not meet near 0.

Proposition 5.10. *There exists C_3 such that for $|\lambda| > C_3 |\log(\rho)|^{-a}$, $\det DH(k) \neq 0$ for every $|k| < d$*

Proof: Choose first $|\lambda| > C_2 |\log(\rho)|^{-a}$. We have

$$\det DH(k) = |\partial_k H|^2 - |\partial_{\overline{k}} H|^2 \geq |F'|^2 - |DS|^2.$$

Let $|k| < d$. Then, Lemma 5.8 ii) and Lemma 5.5 condition (5.15) imply that

$$\det DH(k) \geq |\lambda| e^{-M_0 R(\lambda)} - \rho^{2\theta} e^{CR(\lambda)}, \tag{5.35}$$

which is not zero if $\rho^{2\theta} < |\lambda| e^{-(M_0 + C) R(\lambda)}$. By choosing a new constant C_3 the Proposition is proved. □

Let us put all our knowledge together and conclude by a degree argument:

End of the proof of Proposition 5.3: Choose $\lambda \geq \frac{C_3}{|\log(\rho)|^a}$. By Lemma 5.9 the zeros of $g = e^{\eta} H$ belong also to $\mathbb{D}(0, d)$. Now, since

$$g(w, z, k) = i\lambda k + \epsilon(k) k$$

for $|k| = R(\lambda)$, $g(w, z, k)$ is homotopic to λk, then

$$\deg(g, \mathbb{D}(0, R(\lambda)), 0) = 1$$

and since $g = e^{\eta}(F + S)$ with e^{η} continuous

$$\deg(F + S, \mathbb{D}(0, R(\lambda)), 0) = 1$$

From Lemma 5.9 the zeroes of $H = F + S$ are in $\mathbb{D}(0, d)$ where $\det(H) \neq 0$. Thus, since $H \in C^1$ we can express the Brouwder degree by the formula

$$1 = \deg(H, \partial B(0, R(\lambda)), 0) = \sum_{k_i \in Z(H)} \mathrm{Ind}(H, k_i) = \sum_{k_i \in Z(H)} sign \det DH(k_i).$$

Suppose that there exists more that one zero. This would mean, that there exists $k_i \in Z(H) \setminus \{0\}$ such that $\det DH(k_i) < 0$ and, by continuity of the determinant, there would exists $t \in (0, 1)$ such that $\det(H(tk_2)) = 0$. We arrive to a contradiction with proposition 5.10. Thus if $|\lambda| > C_3 |\log \rho|^{-a}$ there is a unique zero of g and the proof is concluded with $C = C_3$ □

6 The proof of Theorem 1.1

Let γ_j be two conductivities satisfying the ellipticity (I) and regularity conditions (II) in Theorem 1.1. We start by showing that there is no loss of generality assuming that the $\gamma_j - 1$ are compactly supported in Ω and that $\Omega = \mathbb{D}$. The key point is the stable recovery of the values of the conductivities on $\partial\Omega$ from the Dirichlet to Neumann map. This is the content of the following proposition which follows easily from [14]. We would like to thank R. Brown for this personal communication. The theorem is stated for hypothesis adapted to our conductivities.

Proposition 6.1 ([14])**.** *Let Ω be a Lipschitz domain and γ_1, γ_2 two conductivities in $\mathcal{C}(\bar{\Omega})$ Then there exists $C = C(\Omega, \kappa)$ such that*

$$\|\gamma_1 - \gamma_2\|_{L^\infty(\partial\Omega)} \leq C\|\Lambda_{\gamma_1} - \Lambda_{\gamma_2}\|_{H^{1/2}(\partial\Omega)\to H^{-1/2}(\partial\Omega)} \tag{6.1}$$

The stable boundary recovery has been studied in several works, for instance for $\gamma_j \in W^{1,p}$, some $p > 2$ and Ω a Lipschitz domain, see [3] and [27] and for continuous conductivities and smooth domains in [35]. By a combination of Proposition 6.1 and the bilinear weak formulation of the Dirichlet to Neumann map we prove that we can reduce to the case where $\gamma_j - 1$ compactly supported in $\mathbb{D}$.

Theorem 6.2. *Let Ω be a Lipschitz domain, $\Omega \subset\subset \mathbb{D}$. Let us denote*

$$\rho = \|\Lambda_{\gamma_1} - \Lambda_{\gamma_2}\|_{H^{1/2}(\partial\Omega)\to H^{-1/2}(\partial\Omega)}$$

There exist extensions $\tilde{\gamma}_j$ of γ_j such that

$$\|\gamma_j\|_{\mathcal{C}^\alpha(\mathbb{C})} < C\Lambda_0 \tag{6.2}$$

$$\|\Lambda_{\tilde{\gamma}_1} - \Lambda_{\tilde{\gamma}_2}\|_{H^{1/2}(\partial\mathbb{D})\to H^{-1/2}(\partial\mathbb{D})} \leq C\rho \tag{6.3}$$

$$supp(\tilde{\gamma}_j - 1) \subset \mathbb{D}. \tag{6.4}$$

Proof: We use the Whitney extension operator $\mathcal{E}_0$, see [32, page 174]. We consider the closed set $F = \partial\Omega \cup (\mathbb{C} \setminus \mathbb{D}(r_0))$ for some $r_0 < 1$ such that $\Omega \subset\subset \mathbb{D}(r_0)$ and take the functions, defined on F, given by

$$f_j(z) = \begin{cases} \gamma_j \text{ for } z \in \partial\Omega \\ 1 \text{ for } z \in \mathbb{C} \setminus \mathbb{D}(r_0). \end{cases}$$

We take the Whitney extensions $\mathcal{E}_0(f_j)$ and define the extended conductivities as

$$\tilde{\gamma}_j(z) = \begin{cases} \gamma_j(z) \text{ for } z \in \Omega \\ \mathcal{E}_0(f_j) \text{ for } z \in \mathbb{C} \setminus \Omega \end{cases} \tag{6.5}$$

The condition (6.2) follows from the continuity of Whitney extension on $\mathcal{C}^\alpha$, $0 < \alpha < 1$.

From the linearity of Whitney extension we have

$$\|\tilde{\gamma}_1 - \tilde{\gamma}_2\|_{L^\infty(\mathbb{D}\setminus\Omega)} = \|\mathcal{E}_0(f_1 - f_2)\|_{L^\infty(\mathbb{D}\setminus\Omega)} \leq \|\gamma_1 - \gamma_2\|_{L^\infty(\partial\Omega)} \tag{6.6}$$

By (6.1) we have

$$\|\tilde{\gamma}_1 - \tilde{\gamma}_2\|_{L^\infty(\mathbb{D}\setminus\Omega)} \leq C\rho. \tag{6.7}$$

Now we are in conditions to prove (6.3). Let $\varphi_0 \in H^{\frac{1}{2}}(\partial\mathbb{D})$ and let $\tilde{u_j} \in H^1(\mathbb{D})$, for $j = 1, 2$, be the solutions to

$$\begin{cases} \nabla \cdot (\tilde{\gamma_j}\nabla\tilde{u_j}) = 0 \\ \tilde{u_j}|_{\partial\mathbb{D}} = \varphi_0. \end{cases} \tag{6.8}$$

Consider u_2 be the solution to

$$\begin{cases} \nabla \cdot (\gamma_2\nabla u_2) = 0 \\ u_2|_{\partial\Omega} = \tilde{u_1} \end{cases} \tag{6.9}$$

and declare

$$\tilde{v_2} = u_2\chi_\Omega + \tilde{u_1}\chi_{\mathbb{D}\setminus\Omega}.$$

The idea is that if $\rho = 0$, in fact $\tilde{v_2} = \tilde{u_2}$. Thus, it is natural to conceive that $\int_{\mathbb{D}} |\nabla(\tilde{v_2} - \tilde{u_2})|^2$ might be controlled in terms of ρ. In fact,

$$\int_{\mathbb{D}} |\nabla(\tilde{v_2} - \tilde{u_2})|^2 \leq c\int_{\mathbb{D}} \tilde{\gamma_2}\langle\nabla(\tilde{v_2} - \tilde{u_2}), \nabla(\tilde{v_2} - \tilde{u_2})\rangle = \int_{\mathbb{D}} \tilde{\gamma_2}\langle\nabla\tilde{v_2}, \nabla(\tilde{v_2} - \tilde{u_2})\rangle.$$

Adding and subtracting $\tilde{\gamma_1}\nabla\tilde{u_1}$ we get,

$$\begin{aligned} \int_{\mathbb{D}} |\nabla(\tilde{v_2} - \tilde{u_2})|^2 \leq & \left|\int_{\mathbb{D}} \langle\tilde{\gamma_1}\nabla\tilde{u_1}, \nabla(\tilde{v_2} - \tilde{u_2})\rangle\right| \\ & + \left|\int_\Omega \langle\gamma_1\nabla\tilde{u_1} - \gamma_2\nabla u_2, \nabla(\tilde{v_2} - \tilde{u_2})\rangle\right| \\ & + \left|\int_{\mathbb{D}\setminus\Omega} (\tilde{\gamma_2} - \tilde{\gamma_1})\langle\nabla\tilde{v_2}, \nabla(\tilde{v_2} - \tilde{u_2})\rangle\right|. \end{aligned} \tag{6.10}$$

The first term vanishes by (6.8). For the second we use the definition of ρ. Namely,

$$\begin{aligned} & \left|\int_\Omega \langle\gamma_1\nabla\tilde{u_1} - \gamma_2\nabla u_2, \nabla(\tilde{v_2} - \tilde{u_2})\rangle\right| \\ = & \left|\langle(\Lambda_{\gamma_1} - \Lambda_{\gamma_2})(\tilde{u_1}_{|\partial\Omega}), (\tilde{v_2} - \tilde{u_2})_{|\partial\Omega}\rangle\right| \\ & \leq \rho\|\tilde{v_2} - \tilde{u_2}\|_{H^1(\Omega)}\|\tilde{u_1}\|_{H^1(\mathbb{D})} \\ & \leq \rho\|\nabla(\tilde{v_2} - \tilde{u_2})\|_{L^2(\mathbb{D})}\|\tilde{u_1}\|_{H^1(\mathbb{D})} \end{aligned} \tag{6.11}$$

Here the Dirichlet to Neumann mappings Λ_{γ_j} are taken on $\partial\Omega$. Finally, from (6.7),

$$\int_{\mathbb{D}\setminus\Omega}(\tilde{\gamma_2}-\tilde{\gamma_1})\langle\nabla\tilde{v_2},\nabla(\tilde{v_2}-\tilde{u_2})\rangle \leq \|\tilde{\gamma_2}-\tilde{\gamma_1}\|_{L^\infty(\mathbb{D}\setminus\Omega)}\|\nabla\tilde{v_2}\|_{L^2(\mathbb{D}\setminus\Omega)}\|\nabla(\tilde{v_2}-\tilde{u_2})\|_{L^2(\mathbb{D})}$$
$$\leq \rho\|\nabla(\tilde{v_2}-\tilde{u_2})\|_{L^2(\mathbb{D})}\|\tilde{u_1}\|_{H^1(\mathbb{D})}. \tag{6.12}$$

Then (6.10), (6.11), (6.12) together yield that

$$\left(\int_{\mathbb{D}}|\nabla(\tilde{v_2}-\tilde{u_2})|^2\right)^{1/2} \leq \rho\|\tilde{u_1}\|_{H^1(\mathbb{D})} \leq \rho\|\varphi_0\|_{H^{\frac{1}{2}}(\partial\mathbb{D})}. \tag{6.13}$$

Our last task is to compare the Dirichlet to Neumann mappings on $\partial\mathbb{D}$, $\Lambda_{\tilde{\gamma_j}}$. Let $\varphi_0,\psi_0\in H^{\frac{1}{2}}(\partial\mathbb{D})$ and $\psi\in H^1(\partial\mathbb{D})$ an extension of ψ_0. Then

$$\langle(\Lambda_{\tilde{\gamma_1}}-\Lambda_{\tilde{\gamma_2}})(\varphi_0),\psi_0\rangle = \int_{\mathbb{D}}\langle\tilde{\gamma_1}\nabla\tilde{u_1}-\tilde{\gamma_2}\nabla\tilde{u_2},\nabla\psi\rangle \tag{6.14}$$

Now we want to add and subtract $(\gamma_2\chi_\Omega+\tilde{\gamma_1}\chi_{\mathbb{D}\setminus\Omega})\nabla\tilde{v_2}$. Since we have

$$\begin{aligned}|\int_{\mathbb{D}}\langle\tilde{\gamma_1}\nabla\tilde{u_1}-(\gamma_2\chi_\Omega+\tilde{\gamma_1}\chi_{\mathbb{D}\setminus\Omega}))\nabla\tilde{v_2},\nabla\psi\rangle| &= |\int_\Omega|\langle\gamma_1\nabla\tilde{u_1}-\gamma_2\nabla u_2,\nabla\psi\rangle| \\ &= |\langle(\Lambda_{\gamma_1}-\Lambda_{\gamma_2})(u_{1|\partial\Omega}),\psi_{|\partial\Omega}\rangle| \leq \rho\|\tilde{u_1}\|_{H^1(\mathbb{D})}\|\psi\|_{H^1(\mathbb{D})} \\ &\leq \rho\|\varphi_0\|_{H^{\frac{1}{2}}(\partial\mathbb{D})}\|\psi_0\|_{H^{\frac{1}{2}}(\partial\mathbb{D})}\end{aligned} \tag{6.15}$$

We can obtain

$$\begin{aligned}|\langle(\Lambda_{\tilde{\gamma_1}}-\Lambda_{\tilde{\gamma_2}})(\varphi_0),\psi_0\rangle| &\leq \rho\|\tilde{u_1}\|_{H^1(\mathbb{D})}\|\psi\|_{H^1(\mathbb{D})} \\ &+|\int_{\mathbb{D}}\langle(\gamma_2\chi_\Omega+\tilde{\gamma_1}\chi_{\mathbb{D}\setminus\Omega})\nabla\tilde{v_2}-\tilde{\gamma_2}\nabla\tilde{u_2},\nabla\psi\rangle| \leq\end{aligned} \tag{6.16}$$

The second term is majorized by

$$|\int_\Omega\langle\gamma_2\langle\nabla(\tilde{v_2}-\tilde{u_2}),\nabla\psi\rangle|+|\int_{\mathbb{D}\setminus\Omega}\langle\tilde{\gamma_2}\nabla\tilde{u_2}-\tilde{\gamma_1}\nabla\tilde{u_1},\nabla\psi\rangle|, \tag{6.17}$$

which using (6.7) and (6.13) is controlled by

$$C\rho\|\tilde{u_1}\|_{H^1(\mathbb{D})}\|\psi\|_{H^1(\mathbb{D})} \leq C\rho\|\varphi_0\|_{H^{\frac{1}{2}}(\partial\mathbb{D})}\|\psi_0\|_{H^{\frac{1}{2}}(\partial\mathbb{D})}. \tag{6.18}$$

Therefore we arrive to

$$|\langle(\Lambda_{\tilde{\gamma_1}}-\Lambda_{\tilde{\gamma_2}})(\varphi_0),\psi_0\rangle| \leq C\rho\|\varphi_0\|_{H^{\frac{1}{2}}(\partial\mathbb{D})}\|\psi_0\|_{H^{\frac{1}{2}}(\partial\mathbb{D})}, \tag{6.19}$$

which is equivalent to (6.3). The proof is concluded

□

We now come back to the complex geometric optic solutions arising from μ_j compactly supported in $\mathbb{D} = \Omega$. So far we have obtained the stability of the complex geometric optics solutions, this suffices for uniqueness. But for stability we need to go further and obtain the stability of the derivatives of the functions f_μ, since the Beltrami coefficients are

$$\mu = \partial_{\overline{z}} f_\mu / \overline{\partial_z f_\mu}.$$

By using an interpolation argument we extend the stability in the L^∞ norm proved in Theorem 5.1 to stability in the $W^{1,\infty}$ norm. We still require $\rho \leq \frac{1}{2}$. This requirement will be removed at the end of the proof.

Proposition 6.3. *Let f_{μ_1}, f_{μ_2} be the complex geometric optic solutions. Then there exist constants $a = a(\kappa, \Lambda_0, \alpha)$ and $C(|k|) = C(\kappa, \Lambda_0, \alpha, |k|)$ such that*

$$\|f_{\mu_1}(z,k) - f_{\mu_2}(z,k)\|_{W^{1,\infty}(\mathbb{D},dz)} \leq C(|k|)|\log \rho|^{-a}$$

Proof: Let us start by noticing that, by (2.41), (2.42), the stability of the geometric optic solutions for both Beltrami and conductivity equations are equivalent. Thus, Theorem 5.1 implies that

$$\|f_{\mu_1}(z,k) - f_{\mu_2}(z,k)\|_{L^\infty(\mathbb{D},dz)} \leq C|\log(\rho)|^{-a}. \tag{6.20}$$

On the other hand by Theorem 3.9 we know that for $\epsilon < \alpha - 1$

$$\|f_{\mu_1}(z,k) - f_{\mu_2}(z,k)\|_{C^{1+\epsilon}(\mathbb{D},dz)} \leq C(|k|) \tag{6.21}$$

Let us consider the function $U = (f_{\mu_1}(z,k) - f_{\mu_2}(z,k))\varphi$ where $\varphi \in C_0^\infty(\mathbb{D}(0,2))$ is a cut-off function $\varphi\chi_{\mathbb{D}} = 1$. Then for every $1 < p < \infty$, (6.21) and (6.20) imply that

$$\|U\|_{W^{1+\epsilon,p}(\mathbb{C},dz)} \leq C(|k|) \quad \text{and,} \ \|U\|_{L^p(\mathbb{C},dz)} \leq C(|k|)|\log \rho|^{-a}, \tag{6.22}$$

which calls for an interpolation argument. By interpolation between L^p and $W^{1+\epsilon,p}$ we can obtain estimates for $\|U\|_{W^{\theta(1+\epsilon),p}}$ and consequently for $\|DU\|_{W^{\theta(1+\epsilon)-1,p}}$. If we further require that $\theta(1+\epsilon) - 1 > \frac{2}{p}$, by Sobolev embedding we also control U in L^∞. Namely under these conditions we have

$$\begin{aligned} \|DU\|_{L^\infty(\mathbb{D},dz)} &\leq C\|DU\|_{W^{\theta(1+\epsilon)-1,p}(\mathbb{C},dz)} \leq C\|U\|_{W^{\theta(1+\epsilon),p}(\mathbb{C},dz)} \\ &\leq \|U\|_{L^p}^{1-\theta}\|U\|_{W^{1+\epsilon,p}}^{\theta} \leq |\log(\rho)|^{-a(1-\theta)}C(|k|). \end{aligned} \tag{6.23}$$

Since we can take any $1 < p < \infty$ the estimate is valid for any $\theta < \frac{1}{1+\epsilon}$. □

Proof of Theorem 1.1

By Theorem 6.2 we can reduce to conductivities γ_j such that $\gamma_j - 1$ are compactly supported in $\Omega = \mathbb{D}$. It is enough to control the difference of the Beltrami coefficients μ_j, since $\gamma = \frac{\mu+1}{1-\mu}$ implies that

$$\|\gamma_1 - \gamma_2\|_{L^\infty(\mathbb{D})} \leq \frac{4}{1-\kappa^2}\|\mu_1 - \mu_2\|_{L^\infty(\mathbb{D})}. \tag{6.24}$$

We can also assume that $\rho \leq \frac{1}{2}$. Otherwise $\rho \geq \frac{1}{4\kappa}\|\mu_1 - \mu_2\|_{L^\infty(\mathbb{D})}$ and the claim follows. Therefore Proposition 6.3 can be applied. We only need the complex geometric optics solutions with fixed $k = 1$. Thus in the rest of the proof we take $f_\mu(z) = f_\mu(z, 1)$. The stability of the Beltrami coefficients is reduced to the stability of the derivatives. Since, from the definition of the Beltrami coefficients,

$$\|\mu_1 - \mu_2\|_{L^\infty(\mathbb{D})} = \|\partial_{\overline{z}} f_{\mu_1}/\overline{\partial_z f_{\mu_1}} - \partial_{\overline{z}} f_{\mu_2}/\overline{\partial_z f_{\mu_2}}\|_{L^\infty(\mathbb{D})}. \tag{6.25}$$

Now we use the regularity of the solutions. By Theorem 3.9 there exists a uniform constant $m > 0$ depending on Λ_0 such that for every μ

$$\inf_{\mathbb{D}} |\partial_z f_\mu| \geq m,$$

and by (6.21) there exists another uniform constant

$$\max_{\mathbb{D}} |Df_\mu| \leq M$$

Thus,

$$\|\partial_{\overline{z}} f_{\mu_1}/\overline{\partial_z f_{\mu_1}} - \partial_{\overline{z}} f_{\mu_2}/\overline{\partial_z f_{\mu_2}}\|_{L^\infty(\mathbb{D})} \leq \frac{M}{m}\|Df_{\mu_1} - Df_{\mu_2}\|_{L^\infty(\mathbb{D})} \leq C|\log(\rho)|^{-a} \tag{6.26}$$

and the theorem is proved. □

7 Appendix. Proof of theorem 3.10:

For the convenience of the reader we give a proof of

Theorem 7.1. *Let μ such that $|\mu(z)| \leq \kappa < 1$ and $\mu \in \mathcal{C}^\alpha(G)$ for a domain $G \subset \mathbb{C}$ with $\|\mu\|_{\mathcal{C}^\alpha} < \Lambda_0$. Given a function ω such that:*

- *(i) ω is a solution in $W^{1,p}_{loc}(G)$, $p > 2$, of the equation*

$$\partial_{\overline{z}}\omega = \mu\overline{\partial_z \omega} \tag{7.1}$$

- *(ii) For any $G' \subset\subset G$, there exists $K = K(G')$ such that*

$$\sup_{z \in G'} |\omega(z)| \leq K. \tag{7.2}$$

Then the following properties hold:

- *(I) If β is such that $0 < \beta < \alpha$ and $D \subset\subset G'$. Then there exists a $\Lambda_1 = \Lambda_1(\kappa, K, D, \beta)$ such that*

$$\|\partial_{\overline{z}}\omega\|_{\mathcal{C}^\beta(D)} + \|\partial_z \omega\|_{\mathcal{C}^\beta(D)} \leq \Lambda_1. \tag{7.3}$$

- *(II) Assume that ω is a quasiconformal homeomorphism in $\mathbb{C}$. Then for any $D \subset\subset \mathbb{C}$, there exists a constant $J = J(K, \kappa, \Lambda_0, D)$ such that*

$$\inf_{z\in D}(|\partial_z w(z)|^2 - |\partial_{\bar{z}} w(z)|^2) = \inf_{z\in D}(J_\omega(z)) > J. \tag{7.4}$$

Proof: Let us start with the proof of (I). Let be d, to be chosen later on, with the condition $2d < d(D, \partial_z G')$. Then for $z_0 \in D$ and $0 < |z - z_0| < 2d$ take

$$g(z) = g(z; z_0) = \frac{\omega(z) - \omega(z_0) - \mu(z_0)(\overline{\omega(z) - \omega(z_0)})}{z - z_0}$$

This g is a solution in $0 < |z - z_0| < 2d$ of the equation

$$\partial_{\bar{z}} g = \mu_0 \overline{\partial_z} g + B_0, \tag{7.5}$$

where

$$\mu_0(z) = \mu_0(z; z_0) = \frac{(\mu(z) - \mu(z_0))\bar{z}}{(1 - \mu(z_0)\overline{\mu(z)})z} \tag{7.6}$$

and

$$B_0 = \frac{(\mu(z) - \mu(z_0))}{(1 - \mu(z_0)\overline{\mu(z)})z} \tag{7.7}$$

We extend μ_0 and B_0 by zero in $\mathbb{C} \setminus \mathbb{D}(z_0, 2d)$.

Now, in order to choose $d = d(\Lambda_0, \kappa, \alpha)$, consider $C(\alpha) = \|T\|_{L^{q_\alpha} \to L^{q_\alpha}}$ where $q_\alpha = 2/(1 - \alpha)$, and $\epsilon < 1/C(\alpha)$ and then, by the uniform Hölder condition for any μ in the hypothesis of the theorem, take $2d$ such that for any $|z - z_0| < 2d$

$$|\frac{\mu(z) - \mu(z_0)}{1 - \kappa^2}| < \epsilon \tag{7.8}$$

This choice of d makes

$$\sup_{z,z_0} |\mu_0(z; z_0)| \leq \epsilon$$

From the Hölder condition we also have $B_0(\cdot; z_0) \in L^q$, for $q < q_\alpha$, with norm uniformly bounded with respect to $z_0 \in D$ and depending only on Λ_0, κ and q.

From the representation theorem for solution of equation (7.5) we can write

$$g(z) = e^{s(z)} f(F(z)), \text{ for } 0 < |z - z_0| < 2d.$$

Where $|s(z)| < S_1$ and $|s(z) - s(z')| \leq S_2|z - z'|^\beta$, with $\beta < \alpha$ because of (7.8) and the choice of ϵ. Here S_1 and S_2 only depend on κ_1 and Λ_0.

The function $t = F(z)$ is a Beltrami normalized one to one solution on $\mathbb{C}$ such that

$$|t - t'| = |F(z) - F(z')| \leq S_3(R, \kappa_1, \Lambda_0)|z - z'|^\beta, \tag{7.9}$$

for β as above and $z, z' \in \mathbb{D}(R)$. Also the inverse function $z = F^{-1}(t)$ satisfies a similar estimate

$$|z - z'| = |F^{-1}(t) - F^{-1}(t')| \leq S_4(R, \kappa_1, \Lambda_0)|t - t'|^\beta \tag{7.10}$$

for β as above and $z, z' \in \mathbb{D}(2R)$.

The function f is analytic at least on $F(0 < |z - z_0| < 2d)$. If we write $f(t) = e^{-s(F^{-1}(t))} g(F^{-1}(t))$ we see that for $t_0 = F^{-1}(z_0)$

$$|f(t)| \leq e^{S_1} \frac{1+\kappa}{1-\kappa} |\frac{\omega(F^{-1}(t)) - \omega((F^{-1}(t_0))}{F^{-1}(t) - F^{-1}(t_0)}| \leq C|t - t_0|^{-1/\beta} \tag{7.11}$$

in a neighborhood of t_0. The last inequality follows from (7.9) and the boundedness of ω.

It follows that t_0 is not an essential singularity. We may use the argument principle to see that t_0 is a removable singularity. In fact, let us consider a oriented circle $\Gamma = \{|t - t_0| = r\}$ for small r such that f has no zeroes on $\{0 < |t - t_0| < r\}$. Then, since ω and F^{-1} preserve the orientation and F^{-1} is a homeomorphism, we have that the variation of the argument along Γ

$$\Delta_\Gamma \log f(t) = \Delta_\Gamma \log \frac{\omega(z) - \omega(z_0)}{z - z_0} \geq 0 \tag{7.12}$$

where we understand $z = F^{-1}(t)$. Hence $t = t_0$ is not a pole of f and this shows that g can be extended to a solution of (7.5) on $|z - z_0| < 2d$.

From (7.9) and (7.10) there exist d' and d_0 such that $\mathbb{D}(t_0, 2d') \subset F(\mathbb{D}(z_0, d))$ and $F^{-1}(\mathbb{D}(z_0, d')) \subset \mathbb{D}(z_0, d_0)$.

From the maximum principle and (7.11) we have

$$|f(F(z))| \leq e^{S_1} C(\kappa) \frac{2K(G')}{d} \tag{7.13}$$

for $z \in \mathbb{D}(z_0, d)$ and hence

$$|f(t)| \leq e^{S_1} C(\kappa) \frac{2K(G')}{d} = K_1 \tag{7.14}$$

for $t \in \mathbb{D}(t_0, 2d')$.

From Cauchy formula we have

$$|f'(t)| \leq K_1/d' \tag{7.15}$$

for $t \in \mathbb{D}(t_0, d')$.

Then, from the representation, we have

$$\begin{aligned} |g(z) - g(z_0)| &= |e^{s(z)} f(F(z)) - e^{s(z_0)} f(F(z_0))| \\ &\leq e^{S_1} K_1 |s(z) - s(z_0)| + e^{S_1} \frac{K_1}{d'} |z - z_0|^\beta \leq K_3 |z - z_0|^\beta, \end{aligned}$$

if $|z-z_0|<d_0$. Hence

$$|\omega(z)-\omega(z_0)-\mu(z_0)(\overline{\omega(z)-\omega(z_0)})-g(z_0)(z-z_0)|\leq K_3|z-z_0|^{1+\beta} \quad (7.16)$$

This is equivalent to

$$w(z)=w(z_0)+\frac{\mu(z_0)\overline{g(z_0;z_0)}}{1-|\mu(z_0)|^2}(\overline{z-z_0})+\frac{g(z_0;z_0)}{1-|\mu(z_0)|^2}(z-z_0)+O(|z-z_0|^{1+\beta}) \quad (7.17)$$

From the above expression, ω is differentiable for every $z_0\in D$ and it satisfies the equation everywhere, moreover, for any $z_0\in D$

$$|\partial_{\bar{z}}\omega(z_0)|<|\partial_{\bar{z}}\omega(z_0)|\leq C=C(\Lambda_0,\kappa,K(G')). \quad (7.18)$$

It remains to prove the Hölder condition. To attain this take z_1 and z_2 in D, with $|z_1-z_2|<d_0$, then from (7.18)

$$|w(z_1)-w(z_2)+\partial_{\bar{z}}\omega(z_1)(\overline{z_2-z_1})+\partial_z\omega(z_1)(z_2-z_1)|\leq K_3|z-z_0|^{1+\beta} \quad (7.19)$$

Changing the roles of z_1 and z_2 in (7.19), using both estimates and the triangle inequality we obtain

$$||\partial_z\omega(z_1)-\partial_z\omega(z_2)|-|\partial_{\bar{z}}\omega(z_1)-\partial_{\bar{z}}\omega(z_2)||\leq K_4|z_1-z_2|^{\beta}.$$

Then using the equation we have

$$\begin{aligned}|\partial_{\bar{z}}\omega(z_1)-\partial_{\bar{z}}\omega(z_2)| &= |\mu(z_1)\partial_z\omega(z_1)-\mu(z_2)\partial_z\omega(z_2)|\\ &\leq |\mu(z_1)(\partial_z\omega(z_1)-\partial_z\omega(z_2))|+|\mu(z_1)-\mu(z_2)||\partial_z\omega(z_2)|\end{aligned}$$

Hence

$$(1-\kappa)|\partial_z\omega(z_1)-\partial_z\omega(z_2)|\leq K_4|z_1-z_2|^{\beta}+C\Lambda_0|z_1-z_2|^{\alpha}$$

where C is the constant in (7.18).

We have proved the β-Hölder condition for $\beta<\alpha$ in $\mathbb{D}(z_0,d_0)$ for any $z_0\in D$. Since D is compact a covering argument extends the condition to D, with constants now depending also on D itself. This finishes (I).

If ω is an homeomorphism as in (II), we have equality in (7.12), for Γ the circle $|t-t_0|=r$ and any $r<2d$ (otherwise $\omega(F(\Gamma))$ would have selfintersections). Then there are no zeros of f in $|t-t_0|<2d$. From the condition on ω, there exists a constant $d_1>0$, only depending on κ, such that

$$d_1=\min\{|\omega(z)-\omega(z_0)|:|z-z_0|=d\}.$$

Hence, from the minimum principle,

$$|f(F(z_0))|\geq e^{-S_1}\frac{1-\kappa}{1+\kappa}\frac{d_1}{d}=K_5,$$

and

$$|g(z_0; z_0)| > e^{-S_1} K_5$$

Now from (7.17) we have, for any $z_0 \in D$,

$$|\partial_z \omega(z_0)| = |\frac{g(z_0; z_0)}{1 - |\mu(z_0)|^2}| \geq J > 0.$$

□

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