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An essay on the spectral action and its relation to quantum gravity

by

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Abstract. We give a brief, critical account of Connes' spectral action principle, its physical motivation, interpretation and its possible relation to a quantum theory of the gravitational field coupled to matter. We then present some speculations concerning the quantization of the spectral action and the perspectives it might offer, most notably the speculation that the standard model, including the gauge groups and some of its free parameters, might be derived from first principles.

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1. Introduction

The question is not whether god did create the world.
The question is whether he had any choice.
`Albert Einstein`

So there is the Beauty and there is the Beast:
In the eyes of most theoretical physicists, traditionally being addicted to symmetries, general relativity is a theory of exceptional beauty. If one assumes that the field equations are of second order, then – up to two free parameters – the theory is uniquely determined by the requirements of general covariance, the motion of test particles along geodesics and the assumption that the energy-momentum tensor of the matter fields is the sole source of the gravitational field.
As compared to this, the standard model of particle physics appears as a real beast. At first glance, one might say it wears the nice suit of being based on a Yang-Mills theory. But there are many possible gauge groups, and we should really wonder why nature has chosen this particular one. Furthermore one has to spontaneously

break the symmetry by introducing the Higgs field in a rather ad hoc way, and add 48 Weyl Spinors in very specific representations of the gauge group.
Why 48 ? And why do they all sit in the trivial or the fundamental (respectively its conjugate) representation of the nonabelian gauge groups ? And why is parity broken *maximally*, which need not be the case? An ideal conception of a fundamental theory shouldn't have as many as 26 free parameters either. Only to mention the asthetic deficits of the classical theory. The consistency of the quantized theory – for example the necessary absence of anomalies – gives at least a few hints on the possible solution to these puzzles. But on the other hand it is actually not completely clear whether the theory exists in a strict, nonpertubative sense at the quantum level and if so whether it really describes all the observed phenomena like e.g. confinement correctly.
Yet, in the fairy tale the Beast has all the money, quite some power and endures all offences by its enemies completely unperturbed. The same can certainly be said about the pertubatively quantized standard model, whose phenomenological success is more than impressive. So maybe we should simply start to take it serious and ask ourselves whether it has some hidden (inner) beauty. Besides, we should never forget that the Beauty, as all beauties, is of a somewhat superficial nature: Not (yet) being quantized, it has a limited range of validity, and, in fact, even predicts its own breakdown in some regime, as is seen from the singularity theorems.
But if the Beauty and the Beast would turn out soul mates, they might be married, have children, so that general relativity finally becomes adult. It would then also be well conceivable that the appearance of the Beast is actually determined by its wife, as is the case in many marriages. Being romantic by nature, most theoretical physicists dream of such a wedlock ever since the invention of general relativity.

Noncommutative Geometry follows the above advice: It does take the standard model serious, but seeks after its hidden mathematical structure [16, 19, 14]. In the version we shall consider here it (almost completely) reformulates the standard model as part of the gravitational field on some noncommutative manifold [18]. The noncommutativity of the underlying spacetime is believed to appear via quantum effects of the gravitational field. However, at the present stage the conventional gravitational field is treated as classical, assuming that this is possible in the sense of an effective theory at presently accessible energy densities. Employing the *spectral action principle* one then obtains the *classical, ie. unquantized,* action of the standard model coupled to gravity in a way quite similar to the way one derives Einstein's equations from the assumptions given above.
Thus Noncommutative Geometry is not (yet) an approach to quantum gravity. It only takes up some heuristically derived speculations about the nature of spacetime in such a theory and adds some more support for them by revealing a plethora of unexpected and beautiful mathematical structures of the standard model. Since this program works out so smoothly there is some hope that it might now be used

as a guideline on the way to some Noncommutative Geometry approach to quantum gravity.

As concerns the present status of the spectral action principle, Thomas Schücker compared it quite appropriately with Bohr's model of the hydrogen atom. The postulates of the latter aren't completely coherent, using some classical physics and combining it with some speculative ideas about the (at that time) sought for quantum theory. Yet it reproduced Rydbergs formula and paved the way to quantum mechanics. (In particular, Bohrs postulate of quantization of angular momentum might have led people to consider the matrix representaions of $SO(3)$.) Of course, it did not explain all the experimental data known at that time, e.g. it gave the wrong value for the ground state energy of the hydrogen atom. But also these open questions it uncovered played an important role for the later development of quantum mechanics.

In view of this analogy, it would be important to state the postulates of the noncommutative description of the standard model via the spectral action principle in the same way Bohr formulated his axioms for the hydrogen atom. I have tried to do so in this article, but to tell the truth I did not succeed. It seems that there are still some points to be clarified before the goal to shape such postulates as coherent as possible can be reached. Hence I decided to give only a preliminary version, but to work out all the weak spots, so that the reader can see where improvements might still be possible. Aspects that may lead to an experimental falsification of the whole program are mentioned as well. I much more dreamed of presenting here some interesting and important questions rather than well-established results.

In order not to draw the curtain of technical details over the main conceptual issues, the article is written in a very nontechnical style. But details can be devilish and readers interested to see that things really work out the way described here are referred to the excellent reviews [9, 12]. Still, the next two sections rather deliberately use some slang from operator theory. Readers who are uneasy about this might easily skip these two sections as the explanations and ideas given there do not represent essential prerequistes for the main part of the article, which starts in 4.2. In particular, Section 3 only gives a very intuitive picture of the noncommutative world and I'm well aware that this picture may turn out completely misleading. However, if spacetime is really noncommutative then we shall have to find a physical interpretation of such spaces and in that section it is at least pointed out that a picture guided by analogy to quantum mechanics is almost surely too naive.

This paper is intended to provide some food for thought, be it in the form of critical remarks or of highly speculative ideas. Since the spectral action is intended to provide a hint for a possible approach to quantum gravity, and as this is a workshop about such routes, I felt it appropriate to add some ideas on where this Tip of the TOE [1] that is maybe revealed by Noncommutative Geometry could finally lead to. They are gathered in the last two sections.

2. Classical spectral triples

The Gelfand-Naimark-Theorem establishes a complete equivalence of locally compact topological Hausdorff spaces $\mathcal{M}$ and commutative C^*-algebras $\mathcal{A}$: Any *commutative* C^*-algebra $\mathcal{A}$ is given as the C^*-algebra of continuous functions $C(\mathcal{M})$ (with the supremum norm) on the space $\mathcal{M}$ of *irreducible representations* of $\mathcal{A}$. For commutative algebras all irreducible representations are one-dimensional. Thus, labeling the normalized basis vector of such a representation space by $|p\rangle$ $p \in \mathcal{M}$, each algebra element $f \in \mathcal{A}$ can be viewed as a function on $\mathcal{M}$ by setting

$$f(p) := \langle p|f|p\rangle.$$

The nontrivial part of the Gelfand-Naimark-Theorem then shows that this space of irreducible representations inherits a suitable topology from the norm of the algebra. In fact, this is also true if the algebra is noncommutative, so that it makes sense to speak of noncommutative C^*-algebras as noncommutative topological spaces, identifying the points with the inequivalent irreducible representations, respectively with *equivalence classes of pure states* – which is the same, as is seen via the GNS-construction.

But there's more than topology to this. Almost all geometrical structures – like for instance vector bundles or differentiable and metrical structures – can equivalently be described in the language of commutative (pre-)C^*-algebras and, even more so, this description still makes sense for noncommutative algebras. However, the generalization to the noncommutative case is in general not unique. The noncommutative de Rham calculus, to give only one example, can be defined as the dual of the space of derivations on $\mathcal{A}$ [14], on Hopf algebras also as a bicovariant differential calculus [8], or via spectral triples. For commutative algebras all these definitions are equivalent. However, if the pre-C^*-algebra $\mathcal{A}$ is noncommutative they will in general provide different answers.
Spectral triples not only provide a differential calculus and a metric on the space of pure states (irreducible representations) over the algebra. They also give rise to an effective way of computing certain differential topological invariants via the local index formula of Connes and Moscovici [15, 17]. These invariants not only appear in quantum field theory, they do so precisely in the form of the local index formula [36]. In so far, spectral triples are quite natural candidates for geometries on which a quantum theory can reside. Moreover, the language of spectral triples is manifestly covariant under automorphisms of the algebra. i.e. the noncommutative "diffeomorphisms". Therefore this language is tailored to provide an alternative description of (quantum) general relativity.
A detailed account of spectral triples is given e.g. in [13]. Here I would only like to mention some essential axioms which turn out to provide phenomenological restrictions for the noncommutative description of the standard model. I therefore found it important to point out their geometrical significance. In the following $\mathcal{A} = C^\infty(\mathcal{M})$ will always be commutative. For simplicity we shall also assume that

$\mathcal{M}$ is a compact, orientable, smooth manifold. This need not be the case: There do exist spectral triples for commutative spaces which describe geometries far beyond the realm of smooth manifolds, like for instance discrete spaces. I therefore prefer to call the manifold case "classical spectral triples".

A **real even spectral triple of dimension** d is given by the following data

$$(\mathcal{H}, D, \mathcal{A}, \gamma, J)$$

Here

- $\mathcal{H}$ is a Hilbert space.
- $\mathcal{A}$ is a pre-C^*-algebra represented on $\mathcal{H}$
- D is an unbounded, essentially selfadjoint operator on $\mathcal{H}$ such that $[D, f]$ is bounded for all $f \in \mathcal{A}$.
  Moreover we assume that the spectrum of D is discrete and that all the eigenvalues λ_n of D have only a finite degeneracy. The sum $\sum_{n=0}^{N} \lambda_n^{-d}$ is assumed to diverge logarithmically in N (the λ_n in increasing order). This defines the dimension d.
- $\gamma = \gamma^*$, $\gamma^2 = 1$ and $[\gamma, \mathcal{A}] = 0$. Moreover $D\gamma = (-1)^{d+1}\gamma D$.
- Finally, J is an antilinear isometry on $\mathcal{H}$ such that
  $$[JfJ^{-1}, g] = 0 \qquad \forall f, g \in \mathcal{A}.$$
  If $\mathcal{A}$ is commutative then it is additionally required that $JfJ^{-1} = \bar{f}$ for all $f \in \mathcal{A}$.

The data of a spectral triple are in particular required to obey the following **axioms:**

1. From the **"Order-One-Condition":**
   $$[JfJ^{-1}, [D, g]] = 0 \qquad \forall f, g \in \mathcal{A}$$
   together with the above requirements on D one infers that D is a differential operator of first order. Thus *locally*, i.e. in each point $p \in \mathcal{M}$, D can be written as:
   $$D = i\gamma^\mu \partial_\mu + \rho, \qquad [\gamma^\mu, f] = [\rho, f] = 0, \quad \forall f \in \mathcal{A}.$$
   The *locally defined* matrices γ^μ are selfadjoint and bounded. It is not clear at this point, however, whether they exist globally as sections of some bundle of matrices over $\mathcal{M}$.
2. The regularity assumption $[\sqrt{D^2}, [D, f]] \in \mathcal{B}(\mathcal{H})$ then implies that there exist scalar functions $g^{\mu\nu}$ such that
   $$\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2g^{\mu\nu} \in \mathcal{A}.$$
   For this conclusion one needs to employ the axiom of **Poincaré duality**.
3. From the axiom of **orientability** one then concludes that the $g^{\mu\nu}$ define a nondegenerate positiv definite matrix valued function on $\mathcal{M}$. Thus, the γ^μ span in each point of $\mathcal{M}$ the Clifford algebra of the correct dimension.
4. It now remains to show that the locally defined Clifford algebras (over each point $p \in \mathcal{M}$) glue together to define a (global) **spin structure over** $\mathcal{M}$. One does so with the help of a theorem of Plymen whose adaptability is (once

more) ensured by **Poincaré duality**.
Thus this axiom plays an important role in the reconstruction of the metric and the spin structure. Note that it is only shown that the $g^{\mu\nu}$ define a metrical tensor on $\mathcal{M}$ after the spin structure has been established. Without this step it would not be clear that these locally defined functions glue together appropriately and hence that they transform correctly under a change of coordinates.

5. Differential one-forms, like gauge potentials, are repreented on the Hilbert space by $\pi(f\mathrm{d}g) = f[D,g]$, and in particular $\pi(\mathrm{d}x^\mu) = i\gamma^\mu$. This works because the matrices γ^μ are now shown to transform like the basis one-forms $\mathrm{d}x^\mu$ under changes of coordinates.
6. The antilinear operator J is identified with **charge conjugation**. This is ensured by requiring relations

$$J^2 = \epsilon(d)\,\mathrm{id}, \qquad J\gamma = \epsilon'(d)\gamma J, \qquad DJ = \epsilon''(d)\,JD$$

where the signs $\epsilon(d), \epsilon'(d), \epsilon''(d) \in \{1,-1\}$ depend on the dimension according to the "spinorial chessboard". In particular, if the algebra is commutative then these relations imply that the Dirac-Operator D cannot contain a term $A = \gamma^\mu A_\mu$, i.e. an electromagnetic vector potential, as one would have $(D + A)J = \pm J(D - A)$ in such a case: Let's assume for simplicity that the vector potential is pure gauge $A = \bar{u}[D,u]$ with $u \in \mathcal{A}$ and $\bar{u} = u^{-1}$. Then, by the above condition on J we have $JuJ^{-1} = \bar{u}$ and thus

$$JAJ^{-1} = J(\bar{u}[D,u])J^{-1} = \epsilon''(d)\,u[D,\bar{u}] = -\epsilon''(d)\,u\bar{u}^2[D,u] = -\epsilon''(d)A.$$

(Similarly for generic selfadjoint one forms A.)

Remark 2.1. Note that essentially *all diffeomorphisms* φ of M are represented as unitary operators U_φ on $\mathcal{H}$, such that

$$U_\varphi\, f(p)\, U_\varphi^* = f(\varphi^{-1}(p)) \qquad \forall p \in \mathcal{M}, \quad f \in \mathcal{A}$$

and $U_\varphi X U_\varphi^* = X_\varphi$ where X is D, γ or J, and X_φ the respective image under the diffeomorphism φ.
Furthermore it is to be noted that not all unitaries on $\mathcal{H}$ correspond to such diffeomorphisms, even though that should be obvious.

As we have seen above, in the classical case $\mathcal{A} = C^\infty(\mathcal{M})$, the space of Dirac-Operators is essentially the same space as the space of all metrics over $\mathcal{M}$. The **spectral action** is defined as

$$S_\Lambda(D) = \mathrm{Tr}\left(\chi\left(\frac{D^2}{\Lambda^2}\right)\right),$$

where χ is some smooth cutoff function, cutting out the eigenvalues above 1, while $\Lambda \in \mathbb{R}$ is some scale. The function χ can be chosen such that $S_\Lambda(D)$ admits an asymptotic expansion in Λ. Note that $S_\Lambda(D)$ is **spectral invariant**, i.e. it is invariant under *all unitaries* on $\mathcal{H}$ and thus in particular under all diffeomorphisms. The same must be true for each term in the asymptotic expansion, and from dimensional reasons (D and Λ have the dimension of an energy) their repective order in the curvature is clear. Hence for the first two nontrivial orders it follows immediately that (as Λ tends to infinity):

$$S_\Lambda(D) \sim c_0\Lambda^4 \int_{\mathcal{M}} \sqrt{g}\mathrm{d}^4x \,+\, c_1\Lambda^2 \int_{\mathcal{M}} R\sqrt{g}\mathrm{d}^4x \,+\, \mathcal{O}(\Lambda^0).$$

The first two terms can be interpreted as the Einstein-Hilbert action with some cosmological constant. These are the only diffeomorphism invariant terms linear, respectively independent of the curvature. As it turns out the two constants c_0, c_1 can be chosen arbitrarily by adjusting χ appropriately. The same is true for higher orders in the curvature, which can therefore be suppressed (here). **Hence, the spectral action can be viewed as the analogue of the Einstein-Hilbert action of pure gravity**.

Remark 2.2. Honestly speaking this is not completely true: A physical action also requires the choice of initial values for the physical field – here the metric. The Einstein-Hilbert action is indeed defined as the integral of the scalar curvature over any spacetime region sandwiched by two Cauchy surfaces, with the (initial and final) values of the metric helded fixed on these Cauchy surfaces when applying the variational principle.
The spectral action, on the other hand, only reproduces the integral over all of $\mathcal{M}$. The resulting field equations therefore only state that $\mathcal{M}$ must be any Einstein manifold, but not more, as the initial values of the metric are not specified.
The reason for this unpleasant fact lies in the *spectral invariance* of the spectral action, which is much more than only diffeomorphism invariance. ("One cannot hear the shape of a drum.") The physical Einstein-Hilbert action is only diffeomorphism invariant, but it is not spectral invariant, and, in fact, that's quite fortunate: The only spectral invariant quantities one might construct out of the Dirac-Operator are its eigenvalues, which correspond to the masses of the elementary fermions. There is only a finite number of the latter. So we couldn't learn much about spacetime from the spectrum of D. But fortunately we can also measure scattering phases. [5]

We should stress however that it may be possible to reconstruct a Riemmanian space from the eigenvalues of the Dirac-Operator if the precise form of the volume element is given. To give an example, one may conclude that the underlying space is a circle if one knows that the spectrum of D is $\mathbb{Z}$, each eigenvalue occuring precisely once *and* that the volume form is given as $u^*[D,u]$, where u is some algebra element. (We need not know more about u.) A much more intriguing example is given in [35]. One may then speculate whether adding the volume form to the spectral action can lead to further progress. See the most recent [34], where this idea has been used to render the mass scale in D dynamical.

Remark 2.3. Related to the above remark, we should also stress that the spectral action only makes sense for *Riemannian* manifolds. On generic Lorentzian manifolds, when each of the eigenvalues of D is infinitely degenerate, such a cutoff trace would never exist. Even more so, the spectrum of D on generic Lorentzian manifolds is always the same, namely $\mathbb{R}$. But still we can infer most information about the geometry from the degeneracy of *a single eigenvalue of* D, i.e. the space of solutions given a fixed mass, and in addition the knowledge of the volume form [4]. We shall have to say more about this later.

3. On the meaning of noncommutativity

If $\mathcal{A}$ is commutative then all the different irreducible representations of $\mathcal{A}$ are obviously unitarily inequivalent, as they are one-dimensional:
Two representations π_1, π_2 on Hilbert spaces $\mathcal{H}_1, \mathcal{H}_2$ are said to be unitarily equivalent if there exists a unitary operator $U : \mathcal{H}_1 \to \mathcal{H}_2$ such that

$$U\,\pi_1(f)\,U^* = \pi_2(f) \qquad \forall f \in \mathcal{A}$$

and this implies $\pi_1 = \pi_2$ if $\dim(\mathcal{H}_1) = 1$.
For noncommutative algebras there do exist equivalent representations, respectively equivalent pure states. Thus, in identifying points with irreducible representations of the algebra, one has to understand the geometrical meaning of this equivalence.

First of all, note that for irreducible GNS representations such a unitary equivalence arises via **inner automorphisms of** $\mathcal{A}$, i.e. the corresponding unitaries U are elements of the algebra representations, $U \in \pi(\mathcal{A})$. A commutative algebra does not possess inner automorphisms. On the other hand, outer automorphisms, i.e. those which are not induced by algebra elements, do not give rise to equivalent *irreducible* representations. It should be stressed that this picture of noncommutative spaces is absolutely generic: A theorem of Dixmier [38] states that **a C^*-algebra is noncommutative if and only if there do exist unitarily equivalent irreducible representations**. Moreover one can formulate analogues of the Gelfand-Naimark theorem for noncommutative algebras as characterizing topological spaces together with some equivalence relation of points in many different ways (see [30] and references therein.)
Thus, as a first moral one might state that **there is a clear distinction between the inner and outer automorphisms of a noncommutative algebra: Inner automorphisms lead to equivalent irreducible representations.**

Readers may wonder how our intuition of the phase space of nonrelativistic quantum mechanics, i.e. the Heisenberg algebra $[x, p] = i$, fits into this picture. To be honest, I don't know a fully compelling answer to this question:
With a suitable regularity assumption there exists up to equivalence only one irreducible representation for this algebra – the one on $L^2(\mathbb{R})$ that we are all familiar

with. Does this mean the phase space of quantum mechanics has only one point ? Certainly the above statement implies that, given any sharp value for x, all values for p are completely indistinguishable. That is in good agreement with our intuition. But it also implies that all values for x are equivalent as well, which is certainly not consistent with the fact that we can localize an electron within a region of radius of its Compton-wavelength. Well, only with respect to the coordinate x: In a special relativistic quantum field theory every inertial observer would agree that the particle is localized in such a region. But, in view of the Unruh effect, accelerated observers would not even agree that there is only one electron. And they would not necessarily see a localized state.
The point of view of Noncommutative Geometry is completely coordinate independent. I guess that this is behind the seemingly contradictionary intuitions of Noncommutative Geometry and quantum theory: the way we currently interpret it, quantum field theory is not generally covariant – but any dynamical theory of spacetime should be. Unfortunately we still lack an interpretation of quantum theories that is in accordance with Einstein's equivalence principle.

To be more concrete, let's consider the Moyal-plane,

$$[x, y] = i\theta.$$

The noncommutativity parameter θ is then, employing the above mentioned wrong analogy to $\hbar$ in quantum mechanics, often misinterpreted as a physical observable – even though that implies that the coordinates x, y obtain a special status. It would then, for instance, no longer be allowed to rescale these coordinates by $x \to \lambda x,\ y \to \lambda y$. In quantum mechanics, due to the presence of a symplectic form, the choice of the coordinates x, p of the phase space is really canonical, p being the canonical momentum associated to the chosen coordinate x. However, there should be no obstruction to the choice of coordinates of the configuration space of a physical system. An experimental physicist should, for instance, be allowed to choose his unit of length arbitrarily. Of course, a length is actually not described by the coordinates alone, but by a metric g^{ij} where the matrix indices refer to the chosen coordinates. Note that the determinant g of this metric transforms under coordinate transformations in the same way as θ^2 (in two dimensions) would do, if we allow such transformations. Thus, the combination

$$A_p := \frac{\theta}{\sqrt{g}}$$

is an invariant in this case and could really be viewed as a physical parameter. As it has the dimension of an area, it might be viewed as the minimal area that can be resolved on such a spacetime.
As for a physical interpretation of the picture of noncommutative spaces described above, let us consider the following example:
Suppose that the configuration space Bea of a system is given by two points, $Bea = \{uty, st\}$ say. The commutative algebra of functions on Bea is then isomorphic to

the algebra of diagonal two-by-two matrices,

$$\mathcal{A} = C(Bea) = \left\{ \begin{pmatrix} f(uty) & 0 \\ 0 & f(st) \end{pmatrix} \quad | \quad f(uty), f(st) \in \mathbb{C} \right\}.$$

We now want to describe a situation where the two points, i.e. the two inequivalent representations of $C(Bea)$ are equivalent. This is achieved by adding to $C(Bea)$ a further unitary element u that interchanges the two representations, i.e.

$$u = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

It is not hard to see that the matrix u together with the matrices $e_{uty} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \in C(Bea)$ and $e_{st} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \in C(Bea)$ generate the full matrix algebra $M_2(\mathbb{C})$ of complex two-by-two matrices Up to equivalence $M_2(\mathbb{C})$ has only one irreducible representation. On the other hand, the set X of pure states of the algebra $M_2(\mathbb{C})$ is a two-sphere:

$$X = \left\{ \begin{pmatrix} \psi e^{i\sigma} \\ \sqrt{1-\psi^2} \end{pmatrix} \quad | \quad \psi \in [0,1]\,, \sigma \in [0, 2\pi] \right\}.$$

That is just the space of possible choices of bases in $\mathbb{C}^2$ up to an overall phase, i.e. the set of (mutually equivalent) representations of $M_2(\mathbb{C})$ on $\mathbb{C}^2$.
Thus, by identifying the two points we have blown up this two point set to obtain a bubble of foam. By doing so we have also introduced new degrees of freedom in our algebra $\mathcal{A}$ that we still interpret as functions on the configuration space. These new degrees of freedom, being given by off-diagonal two-by-two matrices correspond in a very precise sense (as we shall see in the next section) to **gauge degrees of freedom**. The inner automorphisms of the algebra, characterizing its noncommutativity, can, in fact, always be interpreted as gauge transformations.

4. The noncommutative description of the standard model and the physical intuition behind it

Let us now come to the main part of this article, namely the postulates governing the reformulation of the standard model of particle physics as part of the gravitational field on a certain noncommutative space. Before I describe these axioms, however, I shall first sketch its fundamental physical idea.

4.1. The intuitive idea: an effective picture of quantum spacetime at low energies

As has been shown in section 2, for classical spectral triples, $\mathcal{A} = C^\infty(\mathcal{M})$, it is not possible to add a one-form, i.e. a gauge field, to the Dirac-Operator, because of the reality condition $DJ = JD$. (We shall restrict to the case of 4 dimensions from now on, when $\epsilon''(4) = 1$.) The Dirac-Operator of real spectral triples therefore "only" describes the gravitational field.

If the algebra is noncommutative, however, one may add a term of the form $A + JAJ^{-1}$ to D, where $A = f[D, g]$ is a one-form: Now there exist A with $JAJ^{-1} \neq -A$. This is so because there then exist some $f \in \mathcal{A}$ such that JfJ^{-1} is not in $\mathcal{A}$ since JAJ^{-1} belongs to the commutant of $\mathcal{A}$, which cannot contain all of the *noncommutative* algebra $\mathcal{A}$.
Consider an unitary $u \in \mathcal{A}$, i.e. the generator of an inner automorphism. We can associate to it the unitary operator U:

$$U\psi := u\, JuJ^{-1}\, \psi =: u\psi u^*, \qquad \psi \in \mathcal{H}.$$

This then represents the inner automorphism generated by u on $\mathcal{H}$ as $U\,f\,U^* = ufu^*$ for all $f \in \mathcal{A}$. The reason why this representation is chosen that way becomes clear if one considers its action on a Dirac-Operator $D_A = D + A + JAJ^{-1}$:

$$U\, D_A\, U^* = D_{A^u}, \qquad A^u := uAu^* + u[D, u^*].$$

Hence **adding $A + JAJ^{-1}$ to D gauges the inner automorphisms of $\mathcal{A}$.** Note that $JuJ^*\psi =: \psi u^*$ can consistently be interpreted as a right action of u^* on ψ because it commutes with the left action and because J is antilinear.
Of course, the terms $A + JAJ^{-1}$ are not one-forms in a strict sense. However, if one considers all Dirac-Operators for a given algebra – i.e. all metrics – then these degrees of freedom have to be taken into account. Readers familiar with Connes' distance formula will notice that these terms do infect the metric as they do not commute with algebra elements. Let us consider an **almost commutative** algebra like $\mathcal{A} = C^\infty(\mathcal{M}) \otimes M_2(\mathbb{C})$ represented on $\mathcal{H} = L^2(\mathcal{M}, S) \otimes M_2(\mathbb{C})$, the space of square integrable spinors on $\mathcal{M}$ with values in $M_2(\mathbb{C})$. The part $M_2(\mathbb{C})$ of $\mathcal{A}$ acts by matrix multiplication from the left on this space.
As D we can take the usual Dirac-Operator on $\mathcal{M}$ – for some metric g – acting trivially on $M_2(\mathbb{C})$. J is taken to interchange the left and the right action of matrices on matrices, i.e. for $f \in C^\infty(\mathcal{M})$, $\psi \in L^2(\mathcal{M}, S)$ and $\lambda, \sigma \in M_2(\mathbb{C})$ one defines

$$J\left(\bar{f} \otimes \lambda^*\right) J^{-1} \psi \otimes \sigma = (f\psi) \otimes (\sigma\lambda)$$

In such a case the terms $A + JAJ^{-1}$ can be interpreted as one-forms over $\mathcal{M}$ with values in the Lie-algebra $\mathbf{su(2)}$ – but of course not as one-forms over the noncommutative spacetime $\mathcal{A}$.

At first glance, the above example, where the part $A + JAJ^{-1}$ of D_A that gauges the inner automorphisms can be interpreted as gauge fields on the commutative part $\mathcal{M}$ of the underlying noncommutative spacetime might not seem to be relevant for a quantum theory of gravity: The typical arguments that the semiclassical states of such a theory should correspond to *non*commutative manifolds like e.g. [3, 24, 25] do not suggest such an almost commutative algebra but much more noncommutative ones.
However, as we have seen above the appearance of gauge degrees of freedom is an absolutely generic phenomenon on noncommutative spaces and there will be even more such degrees of freedom if the spacetime is more noncommutative. But

since such a noncommutativity is expected to show up only at extremly high energy densities one might think that these degrees of freedom can not be excited in experiments that are realistic to be performed in the near future. Thus we might expect these degrees of freedom to be frozen in contemporary experiments.
On the other hand we do see some nonabelian gauge fields already at energies of the order of the Z-mass. So maybe it is not true that all these degrees of freedom which reflect the noncommutativity can only be excited at the Planck energy. After all, the Planck energy only gives some scale where we expect that quantum gravity effects can longer be neglected. It might well be possible that this is the case already at much lower energies and the above example then suggests to **interpret the strong and the weak interactions as the first shadow of the quantum corrections for the gravitational field.** In the present experiments of particle physics we do not have enough energy to resolve spacetime at scales much smaller than the Compton wavelength of the Z boson. But with this resolution we might already see a glow of the bubble of foam that replaces points: namely the two-sphere S^2 which is the set of pure states of the algebra $M_2(\mathbb{C})$.
This intuitive idea is made more precise in the following postulates for the noncommutative description of the standard model. Subsection 4.3. will then see their consequences. Yet, in order to show our colours rather than cumbersome matrix calculations, these two subsections are extremly rough and sketchy. Readers interested in the details are referred to [12, 9] As already stressed in the introduction, the postulates are not thought to be completely coherent, and many critical comments are in order. I tried to gather them in the next section.

4.2. The postulates

1. There exists an energy scale Λ up to which spacetime can effectively be described as an almost commutative geometry $C^\infty(\mathcal{M}) \otimes \mathcal{A}_F$. Here $\mathcal{M}$ denotes a four-dimensional (compact), orientable manifold, and $\mathcal{A}_F$ the finite dimensional *real* C^*-algebra
$$\mathcal{A}_F = \mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C})$$
where $\mathbb{H}$ denotes the algebra of quaternions.
In experiments with energies below Λ we will not see a deviation from this picture of the topology of spacetime.
2. At energies below the energy scale Λ it is a very good approximation to neglect the backreaction of matter on the Riemannian metric on $\mathcal{M}$. Hence we can keep this metric as a *fixed, classical* background metric that therefore need not be quantized. Accordingly we only need to take into account the "inner fluctuations" of the metric on $\mathcal{A}$, which correspond to the dynamics of the gauge degrees of freedom for the inner automorphisms of $\mathcal{A}$.
3. The spectral triple $(\mathcal{A}, \mathcal{H}, D_A, \gamma, J)$ describing the space of metrics for this situation is given by:
   - $\mathcal{H} = L^2(\mathcal{M}, S) \otimes \mathbb{C}^{90}$ where the space $\mathbb{C}^{90}$ is explained as follows:
     If one does not (yet) add right handed neutrinos to the standard model,

then there do exist 45 elementary Weyl Spinors. Here we consider the respective antiparticles separately, thus there are 90 Weyl Spinors. (We comment later on this)
The representation of $\mathcal{A}_F$ on $\mathbb{C}^{90}$ is chosen according to the action of the gauge group – i.e. the unitaries in $\mathcal{A}_F$ – on the elementary Weyl Spinors.

- The Dirac-Operator D is given schematically as:

$$D_A = D_{\mathcal{M}} + \mathbf{M}\gamma + A + JAJ^{-1},$$

where $D_{\mathcal{M}}$ denotes the Dirac-Operator on $\mathcal{M}$ corresponding the fixed background metric. $\mathbf{M}$ is the fermionic mass matrix. It contains all the masses and Kobayashi-Maskawa mixing angles. Finally A denotes an arbitrary one-form built with $D_0 = D_{\mathcal{M}} + \mathbf{M}\gamma$. Thus A parametrizes the aforementioned gauge degrees of freedom which are dynamical – unlike D_0.
- γ has eigenvalue $+1$ on the right handed fermions and -1 on the left handed ones.
- $J = C \otimes J_F$ where C denotes the charge conjugation on $\mathcal{M}$, while J_F, acting on $\mathbb{C}^{90}$ interchanges the particles with the antiparticles (as they both appear separately in $\mathbb{C}^{90}$.)

4. Only fermion fields $\psi \in \mathcal{H}$ obeying the Majorana condition $J\psi = \psi$ are physical. (Hence the double counting of particles and antiparticles is removed. It is however needed to get the quantum numbers correct.)
The action for these fermions is given as $S(\psi) = \langle \psi, D\psi \rangle$.
5. At the energy scale Λ, the action of the dynamical degrees of freedom A of D_A is invariant under all unitaries on $\mathcal{H}$.
Thus, there exists a cutoff function χ (smooth, with a Laplace-transform of rapid decay) at this scale such that

$$S_\Lambda(A) = \mathrm{Tr}\left(\chi\left(\frac{D_A^2}{\Lambda^2}\right)\right).$$

6. The effective action at energy scales below Λ is then obtained via the renormalization group flow of the pertubatively quantized action $S_\Lambda(A) + S(\psi)$ from Λ to the scale under consideration.

4.3. How such a noncommutative spacetime would appear to us

1. Under the above assumptions, the most general "inner fluctuation A of the metric" is given as

$$A = i\gamma^\mu \left(A_\mu + W_\mu + G_\mu\right) + \gamma\phi\mathbf{M}$$

where A_μ, W_μ, G_μ denote the gauge potentials for the electro-weak and strong interactions, while ϕ is the *scalar Higgs-doublet*.
The appearance of the gauge bosons is of course put in by hand, as we have chosen the algebra appropriately – considering $\mathcal{A}$ as a phenomenological

input. But the appearance of the Higgs, which comes automatically if the mass matrix $\mathbf{M}$ is nontrivial is quite surprising and really a result.

2. $10^{13}GeV \leq \Lambda \leq 10^{17}GeV$. Note that this means that there is a "big desert" in which no new physics is to be expected.
3. The spectral action $S_\Lambda(A)$ possesses an asymptotic expansion and is given at the Z-mass as

$$S_\Lambda(A) \sim \int\limits_{\mathcal{M}} \sqrt{g}\mathrm{d}^4x \left\{ \mathcal{L}_{EH}(g) + \mathcal{L}_{st.mod}(A) + \frac{9\alpha}{64\pi^2} C^{\mu\nu\rho\sigma} C_{\mu\nu\rho\sigma} + \frac{\beta}{12}|\phi|^2 R + \dots \right\}.$$

Here $\mathcal{L}_{EH}(g)$ denotes the Einstein-Hilbert action with a cosmological constant term that is to be fixed by experiments. Note that this term is viewed as a constant, since the background metric g is fixed. The same applies for the Weyl-Tensor $C^{\mu\nu\rho\sigma}$ and the scalar curvature R for g.
$\mathcal{L}_{st.mod}(A)$ denotes the complete bosonic action of the standard model of particle physics. In particular it always contains the Higgs potential that spontaneously breaks the electroweak energy. There is no freedom to eliminate this potential if the mass matrix $\mathbf{M}$ is nonvanishing.
The dots in the above formula indicate possible higher order terms in the curvature, that may be suppressed however. Note that this is not so for the terms $\frac{9\alpha}{64\pi^2} C^{\mu\nu\rho\sigma} C_{\mu\nu\rho\sigma}$ and $\frac{\beta}{12}|\phi|^2 R$ as the freedom in the choice of χ corresponding to second order terms in curvature is used to adjust the coupling constants of the standard model Lagrangean. (The precise value of the numerical factors α, β can be found in [9].)
4. At the scale Λ the coupling constants of the three interactions of the standard model obey the relation $g_{weak} = g_{strong} = \sqrt{\frac{5}{3}} g_{hypercharge}$. However under the flow of the renormalization group of the pertubatively quantized standard model one approximately obtains the experimentally measured values at the Z-mass. For that reason Λ has to lie in the range indicated above. We should remark, however, that the so obtained values are not completely consistent with the experimental values. But there are more severe problems of the spectral action anyway, as we shall see.
5. The Higgs mass m_H is constrained as $m_H = 182 \pm 20\ GeV$.

5. Remarks and open questions

5.1. Remarks

General Relativity relies on the empirical fact that the inertial and the gravitational mass are identical. Only for this reason is it possible to describe the motion of test particles as free motion along geodesics. In this subsection I shall describe the empirical facts which are necessary for the above description of the standard model as part of the gravitational field on some noncommutative space to work – well, as good or bad as it does. General relativity does not explain the equality of the gravitational and the inertial mass. Neither does the Noncommutative Geometry interpretation of the standard model explain any of the following facts on which it relies:

- The elementary fermions of the standard model only appear in the trivial, the fundamental (respectively its conjugate) representations of the gauge group. In principle it would also be possible that some fermions sit in the adjoint representation. But no other representation can occur, since the gauge groups are induced by the unitaries in some algebra.
- It is highly nontrivial that the electrical charges are assigned correctly. At first sight this would not be possible for models based on almost commutative spectral triples, as then only the $U(1)$ charges $0, \pm 1$ seem to be allowed. However the algebra $\mathcal{A}_F$ contains two $U(1)$ factors – one in $\mathbb{C}$, one in $M_3(\mathbb{C})$ – a certain combination of which drops out due to a mechanism deeply rooted in the geometrical interpretation of spectral triples [33]. The remaining $U(1)$ factor then asigns the correct weak hypercharges to the fermions, so that – after the weak symmetry is spontaneously broken – the experimentally observed electrical charges show up.
- Parity is broken maximally. In the standard model the weak interaction could couple to the left handed fermions with some strength $1 - \epsilon$ and to the right handed ones with strength ϵ. In models based on noncommutative geometry this would be impossible. Fortunately experiments tell us $\epsilon = 0$.
- Fermion masses arise in such models only via the Higgs effect. So for this description to be true, the Higgs must exist – which it hopefully (?!) does.
- The fact that the coupling constants of the standard model under the flow of the renormalization group almost meet in one point is essential: Else their derivation via the spectral action would be too bad an approximation to provide any hope that a modification of this picture of spacetime at higher energies could make contact with the precise experimental values.
- The "Order-One Condition" excludes the possibility that color symmetry is broken (if one additionally requires "S^0-reality", see [7] however). Thus, if the gluons would turn out massive, this picture of spacetime would break down.
- Finally it has to be said that there is a minor problem with the observed neutrino masses. Due to the axiom of Poincaré duality one can only add two massive neutrinos to the model. One neutrino has to remain massless. Unfortunately, it is probably impossible to detect whether all or only two neutrinos are massive, since oscillation experiments are only sensitive to mass differences, while, as it seems, only the mass of the electron neutrino could in principle be measured directly. However, on the side of noncommutative geometry it is possible to replace the axiom of Poincaré duality by other axioms. But how could we find out empirically whether we need to do so ?

5.2. Open problems, perspectives, more speculations

- It is certainly an aesthetic deficit that one has to first count particle and antiparticles as different, but then impose the Majorana condition to identify the ones with the charge conjugates of the other – even though there is something appealing about Majorana fermions.
- We should stress that spectral invariance, which is the underlying principle to construct the action $S_\Lambda(A)$ is not preserved under the renormalization group flow: It implies the constraints on the coupling constants given above and these are obviously not fullfilled at the Z-mass.
  But this breaking of spectral invariance is of course to be expected since special relativistic quantum field theories – like the standard model whose renormalization group flow is used here – are not compatible with the equivalence principle. One may hope that in a full theory of quantum gravity such a problem will not show up.
- It is not very encouraging that Λ comes out so large, implying the "big desert" [9].
- It is definitely a major drawback of this approach that the gravitational field – i.e. the metric on $\mathcal{M}$ – has to be kept fixed. The same is in fact true for the mass matrix $\mathbf{M}$ which would also be a freedom in the choice of Dirac-Operators on the noncommutative spacetime $\mathcal{A}_F$. For the latter I don't see any good physical reason why it's dynamics – if there are any – should not be visible at the Z-mass. As concerns the former, one might argue that the gravitational constant is too small, so the backreaction of matter on g can be ignored at the considered energy scales. But one has to do so, as one needs to take the renormalization group flow of the quantized theory into account, and there's as yet no way to include the gravitational field into this game.

  In the long term perspective, one has to consider the full dynamics of the Dirac-Operators D. Only then the philosophy of Noncommutative Geometry would be fully implemented.
  Note that then the fermionic mass matrices could be rendered dynamical as well. One may speculate that in this way a mechanism that dynamically generates the funny pattern of the fermion masses could be revealed [5, 6].
- At the present stage, the model still keeps many of the aesthetic defects of the standard model. In particular the gauge group has to be put in by hand. However, as shown in the series of papers ([11, 6], [31] and references given therein) with very mild irreducibility assumptions it turns out that the number of possible almost noncommutative geometries is fairly restricted. If this program succeeds, then it will be possible to infer the gauge group of the standard model from the assumption that it can be described as the inner fluctuations of the metric on an almost commutative geometry – i.e. without having to assume the algebra from the start.

- Of course, a theory of quantum gravity should be valid also at higher energies as Λ. In that case we would probably have to modify $\mathcal{A}$ however, as we expect that then the full noncommutativity of spacetime is revealed. So there is the question which algebra might replace the algebra of the standard model, but lead to the same physics at energies less than Λ. Many people have tried deformations of the Lorentz group to provide candidates for such algebras. Another rather interesting candidate for this algebra is given in [2]. I will briefly describe this idea in the next subsection. But of course, the answer to this question can only be given to us by physics, rather than by mathematical speculations:
- In [32] the author proposed a novel model for an almost commutative geometry which extends the algebra $\mathcal{A}_F$ and might exhibit interesting phenomenological consequences with respect to dark matter. This is certainly a promising route to take as it may lead to definite predictions for future experiments.
- If Noncommutative Geometry could really be valorized to an approach to quantum gravity, then of course we would also hope to understand the dynamical mechanism that leads to the noncommutative geometry of spacetime. Note that such a spacetime as the one used for the above description has a larger diffeomorphism group as commutative manifolds. In the standard model example there are for instance also the transformations of the different families of fermions into each other among the symmetries of the spectral action. If we could understand where this enlarged symmetry stems from, then we might understand why there are three families. [5]
- The spectral action is only well defined over compact *Riemmanian* manifolds. For me this has always been its major drawback. On *Lorentzian* spectral geometries one probably has to take another route to a generalization of Einsteins equations. I will come to this problem in the next section, where a proposal for a Noncommutative Geometry approach to quantum gravity will be presented.

  It is sometimes said that this problem is not important as one may always "Wick-rotate" from the Riemannian to the Lorentzian case. As concerns gravity that's just wrong: Wick rotation requires the existence of a timelike Killing vector to be unambiguos. So it is only possible if we severely restrict the allowed space of metrics, which is certainly not in the spirit of the equivalence principle.
- The prediction of the Higgs mass might turn out a spectacular virtue of the theory of course. However, it is not unlikely that the Higgs turns out to be lighter than $180 GeV$. I wouldn't consider this a real problem, though. For me this prediction has always had a similar flavor as the false prediction of the ground state energy of the hydrogen atom by the Bohr model. In fact, as we have seen above, the real lessons about the standard model we can hope to learn from Noncommutative Geometry concern its particle content

rather than the spectral action, which is not defined for the physically realistic Lorentzian manifolds anyway.

5.3. Comparision of the intuitive picture with other approaches to Quantum Gravity

One could actually keep this story very short: As yet the effective picture of spacetime at the Z-mass that Noncommutative Geometry assumes is not shown to be in contradiction to the expectations indicated by any of the prominent approaches to quantum gravity. Mainly because most other approaches start at the Planck-energy and don't have definite predictions at the Z-mass yet, of course. Noncommutative Geometry, on the other hand, follows a bottom-up strategy and, as we have pointed out above, will now have to refine its picture by taking corrections into account that will become relevant at energies higher than Λ. So it could be viewed as a complementary approach that may well turn out to be equivalent to one (or all) of the other approaches:

- String theory leads quite naturally to an effective description via noncommutative gauge theories at lower energies [23, 10]. Moreover String theory predicts that there is an infinite tower of elementary particles, thus there are many degrees of freedom which are frozen at the Z-mass. This is very much analogous to the picture that I described in section 3: If we believe that spacetime is not only almost commutative but "really noncommutative" then we would expect many more degrees of freedom than only the gauge bosons of the standard model to show up.
  As a side remark – not meant too seriously – one could add that String theory replaces four dimensional spacetime $\mathcal{M}$ by $\mathcal{M} \times CY$ where CY is an appropriately chosen compact six-dimensional Calabi-Yau manifold. Noncommutative Geometry replaces $C^\infty(\mathcal{M})$ by $C^\infty(\mathcal{M}) \otimes \mathcal{A}_F$. But $\mathcal{A}_F$ does of course not correspond to extra dimensions. We only observe it via the gauge bosons of the standard model and hence we need not invent any mechanism to hide it. This is so, because the geometrical concepts behind the spectral action are formulated in a completely background independent way. Unfortunately, in order to make contact with the real world, the spectral invariance has to be sacrified at the present stage.
  Hence, both theories, String theory and Noncommutative Geometry, still have to come up with a truely background independent formulation, before one can compare them more sensibly with each other, and, much more importantly, with reality.
- Besides the heuristic arguments in [3], quite recently many different approaches to quantum gravity have found clues for a noncommutativity of the spacetimes corresponding to semi-classical states of these theories, see [24, 25] for nice examples. In particular the nontrivial dimension spectrum seen independently in [26, 27] is a generic feature of noncommutative spectral triples [15, 17]

- As concerns Loop Quantum Gravity, there will be many remarks in the next chapter. See however [20] for an alternative approach to combine Loop Quantum Gravity with Noncommutative Geometry.
- Finally I would like to mention the noncommutative approach to quantum gravity advocated in [2]: Here the basic idea is to consider a noncommutative algebra that is constructed via the frame-bundle E over $\mathcal{M}$ and its structure group, i.e. the local Lorentz-transformations. To me this seems to provide a very natural candidate for a noncommutativity of spacetime, not only because it is obviously related to the algebra found in [24, 25]: If noncommutativity is interpreted as appearance of equivalence relations among points of spacetime, and if the dynamical coupling of quantum matter to geometry leads to such a noncommutativity, then one would expect that these equivalence relations are related to the fundamental principles underlying this coupling. Thus, it is well conceivable that the noncommutativity of spacetime is related to local changes of frames, i.e. the local Lorentz transformations.

6. Towards a quantum equivalence principle

The noncommutative description of the standard model certainly has many very appealing features. However, the appropriateness of the spectral action appears somewhat problematic: It is only available over (noncommutative) Riemannian manifolds and it is not compatible with initial conditions for field equations. Moreover its guiding principle – spectral invariance – is not fullfilled in the standard model alone. That problem could of course be overcome once the quantization of the gravitational field is achieved. In that case the metric might be considered as a dynamical degree of freedom, which is not really the case at the present stage.
Last not least, one would actually not want to put in by hand the noncommutativity of spacetime. Rather one would like to infer the dynamical mechanism that leads to this noncommutativity in a theory of quantum matter coupled to geometry. Concrete proposals for such mechanisms have been made in [3, 24, 25]. However, while the first one of these arguments is not yet in accordance with the equivalence principle, the other two are still restricted to low dimensional models, when gravity only possesses topological degrees of freedom. What would really be required first is a background independent formulation of quantum theories in the spirit of General Relativity.

6.1. Globally hyperbolic spectral triples

The approach for such a formulation proposed in [21, 22] requires not only a notion of *Lorentzian* spectral triples but also of causal structures and a generalization of the Cauchy problem for the Dirac equation to such noncommutative spacetimes. In this subsection the logic behind this notion shall be briefly sketched.
A **Lorentzian spectral triple (Lost) over** $\mathcal{A}$ has the data

$$\mathrm{L} = (\mathcal{A}, \mathcal{H}, D, \beta, \gamma, J).$$

There is one new ingredient: the fundamental symmetry $\beta = -\beta^*$, $\beta^2 = -1$, which can be viewed as a timelike one-form. In fact it is required that there exist algebra elements $f^{(i)}, g^{(i)}$ such that $\beta = \sum_i f^{(i)}[D, g^{(i)}]$ (time-orientability). Most of the axioms of spectral triples remain unchanged except that the Lorentzian Dirac-Operator D is, of course, no longer selfadjoint but β-symmetric,

$$D^* = \beta D \beta \,,$$

on the common domain of D and $\beta D \beta$ which is required to be dense in $\mathcal{H}$. Another difference is that we now allow for nonunital algebras $\mathcal{A}$, corresponding to *non-compact* manifolds. If $\mathcal{A}$ is the algebra of smooth functions of compact support on some smooth manifold $\mathcal{M}$ then the Losts over $\mathcal{A}$ correspond to Lorentzian metrics and spin structures over $\mathcal{M}$.

An important remark is in order here: Given the other data, the choice of β (and thus D^*) is not uniquely determined by the axioms. That should not be the case anyway, as β involves the choice of a timelike direction in the commutative case and there is no preferred such direction in general. Indeed it can be shown that two Losts which differ only by the choice of β are unitarily equivalent. We shall denote the set of all admissable β given the remaining data of $\mathbf{L}$ by $\mathbf{B_L}$. To any such β we associate the symmetric operator

$$\partial_\beta = \frac{1}{2}\{D, \beta\} = \frac{1}{2}\beta(D - D^*).$$

In the classical case $\mathcal{A} = C_c^\infty(\mathcal{M})$ one can prove that ∂_β is a derivation on the algebra, i.e $[\partial_\beta, f] \in \mathcal{A}$ for all $f \in \mathcal{A}$.

We call a Lost *timelike foliated* if ∂_β is essentially selfadjoint for all choices of β and if there exists an unitary algebra element u such that $\beta = u^*[D, u]$. Moreover it is required that $u^\kappa \in \mathcal{A}$, $\forall \kappa \in \mathbb{R}$.

Needless to say that these conditions ensure in the classical case that one may reconstruct a foliation of $\mathcal{M} = \mathbb{R} \times \Sigma$ along the timelike direction specified by β. But due to the required essential selfadjointness the ∂_β will – even in the noncommutative case – give rise to one-parameter groups $U_\tau = e^{i\partial_\beta \tau}$ of unitatries on $\mathcal{H}$. We denote by

$$\mathcal{T}_{\mathrm{L}} := \{e^{i\partial_\beta \tau} \,|\, \beta \in \mathbf{B_L}\}$$

the set of all these "time-flows". As a matter of fact, the set $\mathcal{T}_{\mathrm{L}}$ can now be used for all Losts to

- Define the noncommutative analogue of the space $\mathcal{H}_c^\infty$ of smooth spinors of compact support.
- Given two distributions $\xi, \eta \in (\mathcal{H}_c^\infty)'$ one can characterize whether the support of η lies in the causal future/past of the support of ξ with the help of $\mathcal{T}_{\mathrm{L}}$.
- One may even reformulate the wavefront sets of such distributions by employing the elements of $\mathcal{T}_{\mathrm{L}}$.

A **globally hyperbolic spectral triple (ghyst)** is now simply defined as a timelike foliated Lost for which there exist **uniquely determined advanced and retarded**

propagators $E_\pm : \mathcal{H}_c^\infty \to (\mathcal{H}_c^\infty)'$.
Here a propagator E is defined as a map $E : \mathcal{H}_c^\infty \to (\mathcal{H}_c^\infty)'$ such that $D\,E(\psi) = \psi$ in a distributional sense:

$$\langle E(\psi), D\varphi\rangle = \langle \psi, \varphi\rangle \qquad \forall \psi, \varphi \in \mathcal{H}_c^\infty.$$

Such propagators are called advanced (respectively retarded) if the support of $E(\psi)$ is contained in the causal future (respectively past) of that of ψ. Given an advanced propagator E_+ and a retarded one E_- one can construct all the solutions for the Cauchy-problem for D with the help of $G = E_+ - E_-$.

Remark 6.1. We still have to work out the precise conditions under which a time-like foliated Lost is a ghyst. An even more urgent and important open problem is to generalize the concept of **isometric embeddings** of globally hyperbolic space-times into each other to the noncommutative case. In the classical case this can be done by replacing the unitaries representing the diffeomorphisms on $\mathcal{H}$ by appropriate partial isometries. However this notion turns out to restrictive if $\mathcal{A}$ is noncommutative, as is seen in noncommutative examples for ghysts.

6.2. Generally covariant quantum theories over spectral geometries

An important virtue of Noncommutative Geometry that has not been mentioned so far is that it provides a mathematical language ideally tailored to reconstruct (non-commutative) spacetimes which approximate the semiclassical states of a background independent quantum theory. We do not yet have such a theory, of course, and accordingly the definition of "semiclassical" states for such a theory is far from clear. But however such a theory may look like: it should provide observables to reconstruct spacetime, and hence there should exist a map from a suitable subalgebra of observables to globally hyperbolic spectral triples (ghysts) or some variant therof. Moreover this map should be covariant with respect to diffeomorphisms and respect **locality**, i.e. the fact that observables localized at spacelike separated regions commute.

In view of arguments like the one presented in [3], it has to be expected that the image of this map does not contain classical, i.e. commutative, manifolds. But there are probably also many noncommutative ghysts which are not realized in a quantum theory of gravity coupled to matter.

Now suppoose we can invert the above map. The inverse would map ghysts to algebras of observables. It then follows that classical ghysts cannot lie in the domain of this map, as else the map would produce a theory that cannot exist according to [3]– namely a theory of quantized gravity coupled to quantum matter for which there does exist a semiclassical state corresponding to a commutative manifold on which the matter fields reside. If one attempts to show that a full quantum theory of gravity coupled to matter implies a noncommutativity of spacetime one may therefore adopt the following strategy [21]:

1. **Reformulate local quantum theory as a generally covariant map** *Obs* **from the category of ghysts to the category of algebras**. Thus, *Obs* is required to be a covariant tensor functor between these categories. Note that the above

requirements on *Obs* imply that $Obs(\mathcal{N})$ is a subalgebra of $Obs(\mathcal{M})$ whenever $\mathcal{N}$ can be embedded as a globally hyperbolic submanifold of $\mathcal{M}$. In particular every diffeomorphism φ of manifolds induces an algebra homomorphism α_φ on the image of *Obs*.

2. **Require as a further constraint on *Obs* that there exists a causal dynamical law.** This is achieved by demanding that for any submanifold $\mathcal{N}$ of $\mathcal{M}$ which contains a *full Cauchy surface* of $\mathcal{M}$ it follows that $Obs(\mathcal{N}) = Obs(\mathcal{M})$, i.e. it is sufficient to know the restriction of observables to one Cauchy surface to know them everywhere on $\mathcal{M}$.
3. **Demand now that geometry and matter are dynamically coupled by requiring the existence of a diffeomorphism invariant state for *Obs*.** Diffeomorphism invariance of states ω here means that, with the above notation
$$\omega \circ \alpha_\varphi = \omega \qquad\qquad \forall \varphi.$$
4. **Show that the domain of a map *Obs* meeting all the above requirements contains only certain noncommutative ghysts.**

This is of course a very (over- ?)ambitious program, that I could sketch only very roughly here. A complete definition of generally covariant quantum theories and some remarks concerning its applicability to quantum gravity can be found in [37] and the contribution of Klaus Fredenhagen to the present volume. I should stress that it is far from clear whether the last two points of the above program are well-defined. In particular we have not yet shown that there exists any nontrivial functor *Obs* for which there exists a diffeomorphism invariant state. However for Loop Quantum Gravity there does exist a uniqely defined diffeomorphism invariant state [28, 29] and we hope to show that this provides an example for such a functor. Even more so, we have not shown that the existence of such a state really implies diffeomorphism invariance – we only have some indications for this so far. Moreover, as mentioned above, there are still many technical and conceptual difficulties with the definition of the category of ghysts that we have to overcome in the zeroth step of the program.

Nevertheless, once these problems are overcome, we hope that we may uncover the dynamical mechanism that leads to the noncommutativity of spacetime. At least it is to be expected that the required existence of a diffeomorphism invariant state is such a strong demand that it excludes many (commutative) ghysts from the domain of the corresponding functor. Thus in order to clarify this question it will be essential to understand in which way the morphisms of the category of ghysts $\mathcal{M}$ act on the states over the algebras $Obs(\mathcal{M})$ associated to it.

Note that, if it succeeds, the above program can really be viewed as a quantum version of the postulates that lead to Einstein's equations: Namely diffeomorphism invariance, the motion of test particles along geodesics and the existence of second order field equations.

This then leads us back to the spectral action. It is not the idea of the above program to produce a diffeomorphism invariant classical action and then to quantize it, but rather to directly infer the “Quantum Einstein equations” from first principles. Only in that way one may also hope that the dynamical mechanism leading to the noncommutativity of spacetime can be uncovered – if there exists such a mechanism. If we are able to reveal the precise nature of this noncommutativity it will turn out whether it can really be approximated at the Z-mass by the almost commutative geometry underlying the standard model.

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